OPTIMAL STRATEGIES FOR HAT GAMES

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Abstract. Following the article "Yet Another Hat Game" by M. Paterson and D. Stinson, this paper introduces Ebert's Hat Game and a variation called 'Hats-on-a-line'. We examine optimal strategies for these two games and then introduce a new hat game which is a hybrid of these two. We conclude by providing an optimal strategy for the new game and presenting the combinatorial argument that proves optimality.

Under the direction of
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A Math 501 Project
Submitted in partial fulfillment of the
requirements for the degree of

Master of Science in Mathematics
at
Portland State University

Date: March 10, 2012.

Contents

1. Introduction

1.1. **What is a Hat Game?** What comes to your mind when you think of a 'hat game'? There are several possible definitions for a 'hat game', but in this paper, we will view a hat game as a specific type of strategy puzzle that involves finding the 'best' outcome for a given set of parameters. Now, these particular puzzles are called 'hat games' since all of them are placed in a context where we have a given number of players, and we place hats of varying colors on their heads. These types of games are sometimes also called 'gnome hat games' since often the players are said to be gnomes.

1.2. **Variations of These Games.** Within these games there are several parameters that can be varied. There can be variations in the rules pertaining to many aspects of the game, including

- number of players
- number of hat colors
- visual information available to players
- auditory information available to players
- random uniform vs. non-uniform hat distributions
- rule of how hat colors are chosen (ie, based on fair coin flip vs. not fair coin flip)
- sequential vs. simultaneous guessing
- ability to pass or guess vs. being required to guess
- desired results (ie, majority guess correct, no one guess incorrect, highest probability to win for a specified definition of a win, etc...)
- adversarial vs. non-adversarial settings
- types of strategies allowed, etc.

So, with all of these possible variations in the games, one can imagine how wide an assortment of games can be defined.

1.3. **Overview of This Paper.** Throughout this paper I define three types of hat games and give their solutions with brief justifications. Then a fourth game, which is a hybrid of two of the first, will be defined, and a given strategy for this new game will be proven to be optimal.

2. Ebert's Hat Game

2.1. **Explanation and History.** In 1998 Todd Ebert – a PhD student at the University of California at Santa Barbara – proposed the following problem in his PhD thesis 'Applications of recursive operators to randomness and complexity.'

Three people walk into a room and each has a hat placed on their head. The hats can be one of two possible colors (black or white), based on the outcome of a flip of a fair coin. The rules are as follows:

- *No player can see their own hat color, but each person can see the hat color of every other person.*
- *No communication is allowed between players, except for a strategy-planning meeting before they walk into the room.*
- *Each player can either guess their hat color or pass.*
- *All players will guess simultaneously.*

The group wins the game if at least one person guesses correctly and no one guesses incorrectly.

The question to answer is, what is the best strategy to give the group the highest possible probability of winning?

2.2. **A Sample Game.** A sample of how this game works is as follows.

Let the three players be Alice, Bob, and Carl. Suppose Alice walks into the room and has a black hat placed on her head, Bob also receives a black hat, and Carl receives a white hat. They then must each either guess a color or pass. (See Table 1 in the Appendix for all possible hat configurations. This sample game is Row 2.) Recall that, to win, at least one person must guess, and guess correctly.

So, Alice has 3 choices (black, white, or pass), as do Bob and Carl. By a simple counting argument, this gives a total of $3^3 = 27$ possible guess configurations, as each person's guess is independent (see Table 2 for all guess configurations). For the given configuration of hat colors, which guess configurations correspond to winning outcomes?

As can be verified by the reader, the seven guess configurations below are precisely the winning configurations out of the 27 possible:

$$PPW,\ PBP,\ BPP,\ BBP,\ BPW,\ PBW,\ BBW.$$

So, with no game strategy (in other words, if the players all guess or pass randomly so that each guess configuration is equally likely), the probability that the group wins is $\frac{7}{27} \approx 25.9\%$. A simple symmetry argument shows that this probability of winning is independent of the given hat configuration. So the question is: *can we do better?* The answer to this is: *absolutely!*

Consider what would happen if we restrict the strategy so that only one person guesses and the other two pass. Then there are only 6 possible guess configurations (choose the person who guesses, 3 choices, then they choose a color to guess, 2 choices). Again for the given configuration of hat colors, now which guesses are winning outcomes? As can be verified by the reader, the three guess configurations below are precisely the winning configurations out of the 6 possible:

$$PPW,\ PBP,\ BPP.$$

Notice, we have increased the chance of winning to $\frac{3}{6} = 50\%$. Again, a simple symmetry argument shows that this probability of winning is independent of the given hat configuration. So this strategy is better than no strategy at all. But can we do better yet? Intuitively we might think not, but in fact we can!

2.3. **An Optimal Strategy.** For the solution to this problem, there are a few things to notice.

First, if only one random or designated person guesses, we have a 50% chance of a win, as seen above. If more people guess, it would seem that this could only decrease the probability that the group will win. After all, if one person guessed correctly, each additional person to guess must also guess correctly. Further, if one person guesses incorrectly, it doesn't matter what the other people guessed.

Now, since nobody can see their own hat, any strategy we come up with will still have each player guessing correctly with a probability of 50%. But notice one important fact: multiple wrong guesses by players are no worse than a single player guessing wrong. So roughly speaking, the strategies we want to strive for should have any wrong guesses of the players concentrated in the fewest number of guess configurations.

To illustrate, consider the following solution. Let Alice guess 'white' if both Bob and Carl are wearing black, Alice will guess 'black' if they are both wearing white, and Alice will 'pass' if they are wearing opposite colors. Likewise for Bob and Carl. Then a table of our guesses looks as follows:

configuration	Alice	Bob	Carl
BBB	W	W	W
BBW	P	P	W
BWB	P	W	P
WBB	W	P	P
BWW	B	P	P
WBW	P	B	P
WWB	P	P	B
WWW	B	B	B

We can see that, within each column, each player guesses exactly 4 times and 2 of those guesses will be right. So we are consistent with the fact that each player has a 50% chance of guessing correctly when they guess. However, from this strategy, we can see that the group wins in every row except the top and bottom rows! So, by following this strategy, the group wins with probability $\frac{6}{8} = 75\%$! This is much better than the 50% that we had previously! Why does this work?

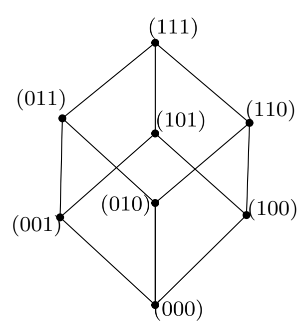
2.4. **Introduction to Hamming codes.** The strategy above is inspired by an object in coding theory known as a *Hamming code*. There are several definitions we need to proceed here.

For any positive integer n, the n-dimensional *hypercube*, denoted $\{0,1\}^n$, is defined to be a graph with 2^n vertices, each of which is a binary sequence of length n. Two such vertices are joined by an edge (and are said to be *adjacent*) whenever they differ in exactly one location.

For example, in the hypercube $\{0,1\}^5$, the binary sequence 10010 would be adjacent to each of the following:

$$00010,\ 11010,\ 10110,\ 10000,\ 10011.$$

Every vertex of $\{0,1\}^n$ has degree n (there are n vertices adjacent to each vertex). Also note that the hat configurations correspond naturally to the vertices of the hypercube, while the number of players will correspond to the dimension of the hypercube. The 3-dimensional hypercube is depicted below.



Another definition we need is that of Hamming codes. To a coding theorist, a *Hamming code* is a particular 1-error-correcting code that can be defined using modular arithmetic on a set of binary sequences. To keep things simple, however, we can view a *code* simply as a subset of the vertices of a hypercube. The vertices that are elements of a code are called *codewords*. For a code to be *1-error-correcting*, we simply require that every vertex of the hypercube is either equal to, or is distance one from, a unique codeword. An example of a 1-error-correcting code in the 3-dimensional cube above is the set $\{(000),(111)\}$.

Hamming codes are a well-known class of codes that exist for hypercubes of certain dimensions, and we need only remark that they have the following properties:

- The minimum distance between any two codewords is 3.
- No two codewords are adjacent to each other, and any other vertex of $\{0,1\}^n$ is adjacent to exactly one codeword.
- Hamming codes only exist for $\{0,1\}^n$ where $n = 2^m - 1$ for some integer $m \geq 2$.
- Hamming codes in $\{0,1\}^n$ have 2^{n-m} codewords.

When $m = 2$, the Hamming code has (000) and (111) as the codewords.
When $m = 3$ there are 16 codewords:

0000000, 0001111, 0010110, 0100101, 1000011, 1001100, 1010101, 1100110,

0101010, 0110011, 0011001, 1110000, 1101001, 1011010, 0111100, 1111111

Now, lets apply these ideas to our game.

2.5. **Justifying the Strategy with Hamming Codes.** Suppose we are playing the hat game with $n = 2^m - 1$ players, so that a Hamming code exists in the corresponding hypercube. The vertices of the hypercube correspond to the possible hat configurations.

If our given configuration is not a codeword in the Hamming code, then there is a unique codeword adjacent to it. This implies that there is exactly one person who will see a possible codeword configuration. (This will be the person corresponding to the location that differs from the codeword). For our strategy, this person will guess the color that would *not* make the configuration be a codeword, and everyone else will pass. This ensures that if we do **not** have a codeword, only one person will guess (correctly) and the rest will pass (resulting in a win).

On the other hand, if our configuration **is** a codeword, then everyone will see a possible codeword configuration, so everyone will follow the same strategy and guess the color that would *not* make the configuration be a codeword (incorrectly). This results in a loss.

So the probability that this strategy results in a loss is only $1/2^m$.

Notice that the strategy allows us to confine all of our incorrect guesses to the small set of only 2^{n-m} codeword configurations. So every player guesses wrong in the same configurations as every other player and no other configurations.

Staying with our 3-player game example, if black is called 0 and white 1, then 111 and 000 are the codewords in the Hamming code. Also, since Alice has 0, Bob has 0 and Carl has 1, we have received 001 as our given configuration. Thus only Carl sees a possible codeword configuration, so only Carl will guess, and he will be correct.

As a second example, if we received 111 as our configuration, then each player would guess 0 and everyone would be wrong.

In summary, this strategy wins anytime the configuration is not a codeword and loses any time it is a codeword. This solution, while ingenious, only works when the number of players is one less than a power of 2, since it can be shown that Hamming codes only exist for these numbers.

3. Hats On a Line

3.1. **Explanation and History.** This next game is known as the 'Hats on a line' game. This game has n players standing in a line and, for ease, we denote player i by P_i and we let P_1 be the player at the back of the line, and P_n be the player at the front of the line.

Each player has a hat (black or white) placed on their head based on the outcome of a flip of a fair coin. The rules are as follows:

- *No player can see their own hat color.*
- *Player P_i can see the hat color of player P_j for all $j > i$. Note that P_1 can see everyone's hat color except his own.*
- *Players will guess in sequential order ($P_1, P_2, P_3 \ldots$, etc.) and all players can hear the guesses made before them.*

- *Players may have a strategy meeting prior to the game.*
- *Each player must make a guess (no passing).*

Given this setup, we want a strategy that will maximize the number of correct guesses.

With no strategy, or utilization of auditory advantages, each player has a 50% chance of guessing correctly. Ideally, we would want to develop a strategy that would allow all the players to always guess correctly. This is not possible, since player P_1 – having no extra information – has only a 50% chance of guessing correctly. However, if we can come up with a strategy that allows P_1 to give the rest of the players some useful information when he/she guesses, then perhaps only P_1 has a 50% chance of guessing correctly, while $P_2, \ldots, P_n$ might fare better.

3.2. **Solution.** The solution is as follows. The players will first agree that one color, say black, is denoted as 0 and the other, white, is denoted as 1.

Now, player P_1 will look at the colors of the hats of $P_2, \ldots, P_n$ and count how many white hats he/she sees and announce a guess of 0 ("black") if the parity is even or 1 ("white") if the parity is odd.

Then, player P_2 hears this guess and knows the parity of white hats (including his/her own, and excluding player P_1). So if P_2 sees the opposite parity of hats on players $P_3, \ldots, P_n$ than P_1 indicated, then P_2 knows (correctly) that he/she is wearing white. Similarly, if P_2 sees the same parity as what P_1 indicated, then P_2 knows correctly that he/she is wearing black. So player P_2 now announces his/her (correct) guess.

Then P_3 can likewise deduce his/her hat color from the knowledge given by P_1 and the additional information provided by P_2. This can continue in this way, and all players except P_1 will have a 100% chance of being correct!

3.3. **Justifying Solution with Modular Arithmetic.** The solution to this game uses nothing but a simple notion of modular arithmetic and the auditory advantages allowed by the game. If any player is not paying attention, this strategy would be useless.

A more formal way to state the solution is as follows.
Let black be denoted as 0 and white be denoted as 1. Also, let c_i denote the color of P_i's hat implying $c_i \in \{0, 1\}$. Then because player P_1 can see $c_2, \ldots, c_n$, player P_1 will make a guess of

$$g_1 = \sum_{i=2}^{n} c_i \pmod 2.$$

Notice that P_1 still has a 50% chance of guessing correctly, but has provided the rest of the players with the information they need to guess correctly! To see this, we know P_2 hears g_1 and can see $c_3, \ldots, c_n$. Thus P_2 can calculate

$$c_2 = g_1 - \sum_{i=3}^{n} c_i \pmod 2$$

indicating P_2 will guess c_2 and be correct. Then P_3 will calculate

$$c_3 = g_1 - c_2 - \sum_{i=4}^{n} c_i \pmod 2$$

and guess c_3 correctly, but he better have heard c_2's correct guess! In general player P_j (for $j \neq 1$) hears the correct guesses $c_2, \ldots, c_{j-1}$ and can see the values of $c_{j+1}, \ldots, c_n$ and has the parity of hats from g_1 allowing P_j to compute

$$c_j = g_1 - \sum_{i \in \{2,\ldots,n\} \setminus \{j\}} c_i \pmod 2.$$

3.4. **A Sample Game.** To see how this works, consider 5 players. Suppose that the configuration of hats is BBWBW. In this example, P_1 would guess black(0). Then, P_2 knows there are an even number of white hats, and as P_2 sees an even number of white hats, P_2 would guess black(0) correctly. Similarly, P_3 knows there was an even number of white hats, and that P_2 is not one of them. Since P_3 sees an odd number of white hats, he knows his hat must be white. So P_3 would guess, correctly, white(1). Now P_4 knows there was an even number of white hats by P_1's guess, and P_2's guess didn't change that, but P_3's guess forced the number of white hats on P_4 and P_5 to be odd. So, since P_4 sees an odd number of white hats, P_4 knows correctly that his hat is black(0). Finally, P_5 knows that there was an even number to start, and an odd number of players between P_2 and P_4 guessed white, so P_5's hat must be white. This outcome would yield the pleasant outcome that everyone guesses correctly!

Note that if the configuration was WBWBW, the same pattern would have been followed and $P_2, \ldots, P_5$ would have still guessed correctly, but P_1 would have guessed incorrectly. Thus, this strategy can only have an incorrect guess by P_1.

3.5. **Generalization to q Colors.** The above solution can be generalized to handle a game which allows any number q of hat colors, by all calculations being taken mod q instead of mod 2. This still only uses auditory advantages and modular arithmetic (just mod q instead of mod 2).

To understand how this more general game works with our solution, consider the following. Certainly, player P_1 always has a $\frac{1}{q}$ chance of winning. When P_1 guesses, however, they are giving the other players the sum of the hat colors (mod q) for players $P_2 \ldots P_n$. Thus, each player P_j can sum the hats they see and the hats already guessed. Then, taking this total away from the total sum of hat colors uniquely determines what color their own hat is.

This is a simple strategy to guarantee everyone (but player P_1) wins, and keeps P_1's chances of winning at $\frac{1}{q}$. The players just better hope that P_1 will agree to their strategy and can add mod q, otherwise they are all in trouble!

4. Majority Hat Game

4.1. **Explanation and History.** In 1992, prior to Ebert's paper, Aspenes, Beigel, Furst and Rudich proposed the following game.

Three people walk into a room and each have a hat of two possible colors (black or white) placed on their head based on the outcome of a flip of a fair coin. The rules are as follows:

- *No player can see their own hat color but each person can see the hat color of every other person.*
- *No communication is allowed between players except for a strategy planning meeting before they walk into the room.*
- *Each player must guess their hat color, no passing allowed.*
- *Players will all guess simultaneously.*

The group wins the game if the majority of the players guess correct.

The question to answer is: what is the best strategy to maximize the probability of winning?

4.2. **The Solution.** Let the players again be Alice, Bob and Carl. Let Alice pick the opposite color of Bob's hat, Bob pick the opposite color of Carl's hat and Carl pick the opposite color of Alice's hat. Then the following table shows all possible configurations of hat colors and the outcome of each game using this strategy.

configuration	*Alice*	*Bob*	*Carl*	*outcome*
BBB	W	W	W	*lose*
BBW	W	B	W	*win*
BWB	B	W	W	*win*
WBB	W	W	B	*win*
BWW	B	B	W	*win*
WBW	W	B	B	*win*
WWB	B	W	B	*win*
WWW	B	B	B	*lose*

Here again we see a high probability of winning, due to a concentration of multiple incorrect guesses into just 2 cases (the top and bottom rows). Indeed, we get a $\frac{3}{4} = 75\%$ chance of winning.

4.3. **Generalizing the Solution.** The strategy above can be generalized to any number of players for which a Hamming code exists. The following formulation, due to Elwyn Berlekamp, is based on a clever orientation of the edges of the hypercube $\{0,1\}^n$ in the case of $n = 2^m - 1$ players.

To describe the strategy in general, notice that each hat configuration corresponds to a vertex of $\{0,1\}^n$. Relative to a given configuration (vertex), each player can be associated with a specific edge of $\{0,1\}^n$ that is incident with that vertex. For example, if the configuration is 10110, then player 1 corresponds to the $(10110, \mathbf{0}110)$ edge, player 2 to the $(10110, 1\mathbf{1}110)$ edge, player 3 to the $(10110, 10\mathbf{0}10)$ edge, and so on. In this way, each of the n players corresponds to one of the edges incident with the configuration vertex.

Using the Hamming code in $\{0,1\}^n$, we will direct the graph as follows. Let every edge incident with one of the 2^{n-m} codewords be directed away from the codeword. Originally, each vertex of the hypercube is incident with $2^m - 1$ edges. But without the edges we have just oriented, the set of non-codeword vertices induce a subgraph in which each vertex is incident with $2^m - 2$ edges. As such, the remainder of the edges can be oriented along an Eulerian circuit. (Or, if it is not connected, at least the remaining edges can be partitioned and oriented into a disjoint union of directed cycles.) Specifically, we have the following.

Lemma 4.1. *Removing the codeword vertices and their incident edges from $\{0,1\}^n$ leaves an even Eulerian graph on each connected component.*

Proof. Every vertex of $\{0,1\}^n$ has degree n. Removing the 2^{n-m} codewords and their incident edges leaves $2^n - 2^{n-m}$ vertices all with degree $n - 1 = (2^m - 1) - 1 = 2^m - 2 = 2(2^{m-1} - 1)$. By Euler's theorem, since every vertex has even degree, it follows that each connected component of the remaining graph has an Eulerian circuit. □

Thus, we direct any edge that is not incident with a codeword by orienting the Eulerian circuits that remain after the removal of the codewords and their incident edges. This completes the orientation of the edges of the hypercube.

Now the strategy can be described as follows. Given any configuration, recall that each player corresponds to an edge in the hypercube. The player will then guess their hat color by choosing the color determined by the endpoint toward which their directed edge points. For example, if player 1's edge $(10110, 00110)$ was directed towards 10110, then player 1 would guess 1 rather than 0.

Lemma 4.2. *The number of correct guesses for a given configuration is equal to the in-degree of the corresponding vertex.*

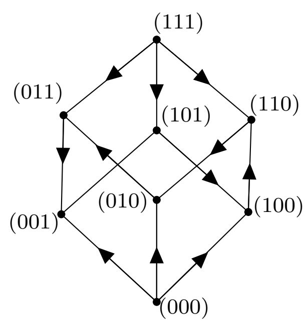
Proof. The in-degree of any vertex is the number of edges incident with it and that are directed towards it. Given a configuration, the edges incident with that vertex correspond to the players in the hat game for that configuration. So by the way we defined the guessing strategy, the number of correct guesses for a given configuration is the same as the in-degree of its corresponding vertex. □

Notice that, by the way we have oriented the graph, each codeword has out-degree n and in-degree 0. So any codeword configuration will lead to 0 correct guesses. Also, each non-codeword has in-degree $2^{m-1} = \frac{n+1}{2}$ and out-degree of $2^{m-1} - 1 = \frac{n-1}{2}$. Thus, for any non-codeword, there are $\frac{n+1}{2}$ correct guesses and $\frac{n-1}{2}$ incorrect guesses, and we have majority of the guesses correct. Since there are 2^{n-m} codewords, this yields a

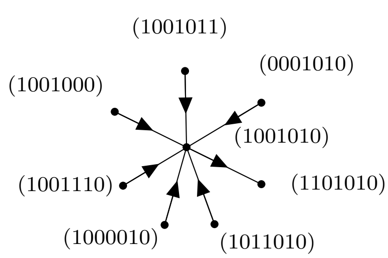
$$1 - \frac{2^{n-m}}{2^n} = 1 - \frac{1}{2^m} = \frac{2^m - 1}{2^m} = \frac{n}{n+1}$$

probability of winning the game. So for this strategy, we have a $\frac{n}{n+1}$ chance of winning the game, just like in Ebert's game!

4.4. **A Sample Game.** For a sample game to illustrate this strategy, let us look at the 3-player game with hat configuration BBW. Letting B be denoted by 1, and W by 0, we get the 110 vertex of $\{0,1\}^3$. Notice that Berlekamp's orientation of the hypercube relative to the Hamming code $\{(000),(111)\}$ agrees with the simple 'guess the opposite of your neighbors hat color' strategy! The directed graph appears below. Notice that the edges are oriented away from the codewords, and the remaining edges are oriented along an Eulerian circuit. For the given configuration BBW, the guess configuration would be WBW, which is a win.



A statement must be made that this 'guess the opposite of your neighbor' strategy does not always agree with the optimal Hamming code strategy for different numbers of players. For example, consider the 7-player game with hat configuration 1001010. In this example, all but one of the arrows in $\{0,1\}^7$ will be facing the 1001010 vertex for the 'guess the opposite of your neighbors hat' color strategy as seen below. Notice however, this will never correspond to a directed $\{0,1\}^7$ as defined, since the construction using a Hamming code always results in an equal in-degree and out-degree for the edges coming from the Eulerian cycles.



5. The New Hats-On-A-Line Game

5.1. **Explanation and History.** Finally, we come to the topic this paper is based on. This final game is a combination of Ebert's hat game and the Hats on a line game. This game has n players standing in a line, and we again denote player i by P_i, letting P_1 be the player at the back of the line and P_n be the player at the front of the line.

Each player has a hat of any one of $q \geq 2$ colors placed on their head, with each color having even probability of being picked. The rules are as follows:

- *No player can see their own hat color.*
- *Player P_i can see the hat color of player P_j for all $j > i$. Note that P_1 can see everyone's hat color (except his own).*
- *Players will guess in sequential order ($P_1, P_2, P_3, \ldots$, etc.), and all players can hear the guesses made before them.*
- *Players may have a strategy meeting prior to the game.*
- *Each player may either make a guess or pass.*

To win this game, no player can guess incorrectly and at least one player must guess correctly.

We are looking for a strategy that will maximize the probability of winning.

5.2. **The Solution.** We will call the solution to this game 'The Strategy' which is defined as follows:

Let black be one of the q hat colors. For each player P_i for $1 \leq i \leq n$, if P_i sees a black hat then they pass, otherwise they guess black. In other words, the first person not to see a black hat guesses black. After anyone has guessed black, the rest of the players will pass.

First we need the following lemma that will help to show this solution is optimal.

Lemma 5.1. *It is sufficient to only consider strategies where exactly one player guesses.*

Proof. With any strategy, if one person guesses, they have a $\frac{1}{q}$ chance of guessing correctly. If this first person guesses correctly, any additional guesses only decrease the chances of winning, reducing the probability of winning from $\frac{1}{q}$ to $\frac{1}{q} \times (\frac{1}{q})^k$, where k is the number of additional guesses. If the first person guessed incorrectly, the game is already lost, and it does not matter how many additional people guess. So any optimal strategy can be modified by insisting that nobody guess after a first guess has been made, and such a modification will not decrease the probability of winning. □

5.3. **Proof of Optimality.** To prove 'The Strategy' is optimal over all possible strategies, we must show it has the maximal success probability. To accomplish this, we do three things:

(1) determine the winning probability of 'The Strategy',
(2) show that any optimal strategy is a what we call a *restricted strategy*, and
(3) show that no restricted strategy has a success probability greater than 'The Strategy'.

In the context of this game, we are defining a *restricted strategy* to be any strategy where every player except possibly player 1 will either pass or guess correctly. So for a restricted strategy, only player 1 can have a chance of guessing incorrectly.

Following the example of the original hats-on-a-line game, we let $(c_1, c_2, \ldots, c_n)$ denote the hat configuration of the game.

Theorem 5.2. *For the New Hats-on-a-Line Game with q colors and n players, 'The Strategy' has a success probability equal to $1 - (\frac{q-1}{q})^n$.*

Proof. The probability that P_1 sees no black hats is $(\frac{q-1}{q})^{n-1}$. This is because the probability that each person, other than P_1, is not wearing black is $\frac{q-1}{q}$, and since each are independent, we just multiply the $n-1$ of them together. According to 'The Strategy', player P_1 would then guess black and win with probability $\frac{1}{q}$. But if P_1 passes, then there is a black hat in the row, and each person will pass until the last person in the row wearing a black hat. This player sees no more black hats in front of them, and therefore knows that his/her hat is black. Thus if P_1 passes they always win the game. Hence the overall probability to win is:

$$\Pr(P_1 \text{ guess correct}) \times \Pr(P_1 \text{ sees no black}) + \Pr(\text{win if } P_1 \text{ pass}) \times \Pr(P_1 \text{ pass}).$$

This equals

$$\begin{aligned}
\frac{1}{q} \times \left(\frac{q-1}{q}\right)^{n-1} + 1 \times \left(1 - \left(\frac{q-1}{q}\right)^{n-1}\right) &= \left(\frac{q-1}{q}\right)^{n-1}\left(\frac{1}{q} - 1\right) + 1 \\
&= \left(\frac{q-1}{q}\right)^{n-1}\left(\frac{1-q}{q}\right) + 1 \\
&= \left(\frac{q-1}{q}\right)^{n-1}\left(-\frac{q-1}{q}\right) + 1 \\
&= 1 - \left(\frac{q-1}{q}\right)^{n}.
\end{aligned}$$

Thus, the probability to win using 'The Strategy' is $1 - \left(\frac{q-1}{q}\right)^{n}$. □

Theorem 5.3. *Any optimal (1-guess) strategy for the New Hats-on-a-line Game is a restricted strategy.*

Proof. For the sake of contradiction, suppose for the New Hats-on-a-line Game there exists a non-restricted optimal strategy, $\mathcal{S}$. For any configuration $(c_1, c_2, \ldots, c_n)$ it is true that P_1 will either pass or guess. If P_1 guesses, then everyone else will pass (since by theorem above we are only considering the strategies where exactly one person guesses) and the group will win with a $\frac{1}{q}$ chance. If P_1 passes, however, then the outcome of the game depends on the $(n-1)$-tuple $(c_2, \ldots, c_n)$. Since P_1 knows the strategy being used by all players, P_1 knows which of the $(n-1)$-tuples $(c_2, \ldots, c_n)$ will result in an incorrect guess, and cause the group to lose. Let $\mathcal{F}$ be the set of configurations $(c_2, \ldots, c_n)$ in which a player other than P_1 will guess incorrectly. We know, since $\mathcal{S}$ is not restricted, that $\mathcal{F} \neq \emptyset$. Create a new strategy $\mathcal{S}'$ by modifying $\mathcal{S}$ as follows:

(1) If $(c_2 \ldots, c_n) \in \mathcal{F}$, then P_1 should guess an arbitrary color.
(2) If $(c_2 \ldots, c_n) \notin \mathcal{F}$, then proceed as in $\mathcal{S}$.

This new strategy, $\mathcal{S}'$, is a restricted strategy since P_1 passes with $\mathcal{S}'$ if and only if $(c_2, \ldots, c_n)$ is a configuration in which $P_2, \ldots, P_n$ all guess correctly. Comparing $\mathcal{S}$ and $\mathcal{S}'$ it should be seen that they only differ for cases in which $(c_2 \ldots, c_n) \in \mathcal{F}$. If $(c_2 \ldots, c_n) \in \mathcal{F}$, then $\mathcal{S}'$ has probability of $\frac{1}{q}$ of winning. On the other hand, if $(c_2 \ldots, c_n) \in \mathcal{F}$, then $\mathcal{S}$ has probability 0 of winning if P_1 passes and $\frac{1}{q}$ if P_1 guesses. Let $k = |\mathcal{F}|$. The probability for any strategy to win is:

$$(\text{Probability win given } (c_2 \ldots, c_n) \notin \mathcal{F}) + \ (\text{Probability win given } (c_2 \ldots, c_n) \in \mathcal{F})$$

Now since $\mathcal{S}$ and $\mathcal{S}'$ differ only when $(c_2 \ldots, c_n) \in \mathcal{F}$, only the second term in the above sum changes. For $\mathcal{S}$ we have:

$$(\text{Probability win given } (c_2 \ldots, c_n) \in \mathcal{F}) = \ (\text{prob } P_1 \text{ guesses}) \ \times \left(\frac{1}{q}\right) < \frac{1}{q},$$

noting that since $\mathcal{S}$ is not restricted, there must be a configuration in which a player other than P_1 guesses (incorrectly); and since exactly 1 player guesses, this means P_1 does not always guess. For $\mathcal{S}'$ we have

$$(\text{Probability win given } (c_2 \ldots, c_n) \in \mathcal{F}) = \frac{1}{q}.$$

Therefore, $\mathcal{S}'$ has a higher probability of winning than $\mathcal{S}$, which contradicts $\mathcal{S}$ being optimal. So there does not exist an optimal strategy that is not restricted. □

Finally we will show that no restricted strategy can beat 'The Strategy'.

Theorem 5.4. *The maximum success probability for any restricted (1-guess) strategy for the New Hats-on-a-line Game with q hat colors and n players is* $1 - \left(\frac{q-1}{q}\right)^n$.

Proof. Let $\mathcal{S}$ be an optimal restricted (1-guess) strategy. Define three sets A, B, and C as follows:

$A =$ the set of $(n-1)$-tuples $(c_2, \ldots, c_n)$ for which P_1 guesses
$B =$ the set of $(n-1)$-tuples $(c_2, \ldots, c_n)$ for which P_1 passes and P_2 guesses correctly
$C =$ the set of $(n-1)$-tuples $(c_2, \ldots, c_n)$ for which P_1 and P_2 pass.

Then, every $(n-1)$-tuples is in exactly one of the above sets, and since the total number of $(n-1)$-tuples is q^{n-1}, this implies

$$|A| + |B| + |C| = q^{n-1}. \tag{1}$$

Now, create new multisets A', B', and C' by removing the first coordinate c_2 from each of the $(n-1)$-tuples in A, B and C. For example, if BBWWB was a configuration in A, then BWWB would be a configuration in A'. Also, if WBWWB was a configuration in A, then BWWB would occur twice in A', which is why we need to allow A', B', and C' to be multisets. With these definitions, the following statements are true.

(i) $B' \cap C' = \emptyset$. This holds, since the $(n-2)$-tuples $(c_3, \ldots, c_n)$ where P_1 passes that have P_2 guess correctly are in B', and the ones that have P_2 pass are in C'. Since an $(n-2)$-tuple cannot have P_2 both guess and pass, these multisets cannot share any elements.
(ii) $A' \cap C' = \emptyset$. To see this let $(c_2, \ldots, c_n) \in A$ be arbitrary. This implies P_1 will guess, and no other player will guess correctly since $\mathcal{S}$ is a 1-guess strategy. Notice though, $(c_2, \ldots, c_n) \in A$ implies by definition $(c_3, \ldots, c_n) \in A'$. If $(c_3, \ldots, c_n) \in C'$ also then this implies one of the players $P_3, \ldots, P_n$ will guess correctly which is a contradiction. Thus $(c_3, \ldots, c_n) \notin C'$ if $(c_3, \ldots, c_n) \in A'$.
(iii) For each element $(c_3, \ldots, c_n) \in B'$ there are exactly $q-1$ occurrences of $(c_3, \ldots, c_n) \in A'$ since only one of the q possible c_2's yields a correct guess by P_2.

Define another restricted strategy $\mathcal{S}'$ on only players $P_2, \ldots P_n$ by modifying $\mathcal{S}$ as follows:

(1) If $(c_3, \ldots, c_n) \in A' \cup B'$ let P_2 guess arbitrarily. If $(c_3, \ldots, c_n) \in C'$ let P_2 pass.
(2) Let $P_3, \ldots, P_n$ follow $\mathcal{S}$.

Now, as the location of the $(n-2)$-tuples are not affected by this modification it implies that P_2 passes for the same $(n-2)$-tuples in $\mathcal{S}'$ as they did in $\mathcal{S}$. Which, if we recall, were the $(n-2)$-tuples such that one of $P_3, \ldots, P_n$ will guess correct. Thus by definition $\mathcal{S}'$ is indeed restricted.

Let ω_n be the maximum number of winning $(n-1)$-tuples, in an optimal restricted strategy, for which P_1 passes. We want to show:

$$\omega_n \leq q^{n-1} - (q-1)^{n-1}. \tag{2}$$

To show this we use induction on n. For $n = 2$ this is clearly true, since the maximum number of $(n-1)$-tuples for which P_1 passes is 1. This is the case since P_1 will only pass if P_2 is wearing a specific color hat. Hence $\omega_n \leq 1$. Assume equation (2) for values less than n. Then we have a few facts from the information known. First, from (iii) above, we get

$$|A| \geq (q-1)|B|. \tag{3}$$

Second, since $\mathcal{S}'$ is restricted for $n-1$ players, the inductive hypothesis yields

$$|C| \leq q\omega_{n-1}. \tag{4}$$

Finally, since $\mathcal{S}$ is optimal, it is the case that

$$|B| + |C| = \omega_n. \tag{5}$$

Thus using (1), (3), (4), and (5), we can conclude that

$$\begin{aligned}
\omega_n &= |B| + |C| \qquad \text{by (5)} \\
&= q^{n-1} - |A| \qquad \text{by (1)} \\
&\leq q^{n-1} - (q-1)|B| \qquad \text{by (3)} \\
&= q^{n-1} - (q-1)(\omega_n - |C|) \qquad \text{by (5)} \\
&\leq q^{n-1} - (q-1)\omega_n + q(q-1)\omega_{n-1} \qquad \text{by (4)}
\end{aligned}$$

Solving for ω_n in the above inequality we get,

$$\begin{aligned}
\omega_n &\leq q^{n-2} + (q-1)\omega_{n-1} \\
&\leq q^{n-2} + (q-1)(q^{n-2} - (q-1)^{n-2}) \qquad \text{(Inductive Hypothesis)} \\
&= q^{n-1} - (q-1)^{n-1}.
\end{aligned}$$

Therefore, (2) is true.

Lastly, use (2) to calculate the success probability of $\mathcal{S}$.

$$\begin{aligned}
\text{Prob } \mathcal{S} \text{ wins} &= [\text{Prob } P_1 \text{ passes}] + (\text{Prob } P_1 \text{ guesses } \times \text{ Prob } P_1 \text{ guesses correctly}) \\
&= [\text{Prob } P_1 \text{ passes}] + [(1 - (\text{Prob } P_1 \text{ passes})) \times \text{ Prob } P_1 \text{ guesses correctly}] \\
&= [\text{Prob } P_1 \text{ passes}] + [(1 - (\text{Prob } P_1 \text{ passes})) \times \frac{1}{q}] \\
&= [\text{Prob } P_1 \text{ passes}] + \frac{1}{q} - \frac{1}{q} (\text{Prob } P_1 \text{ passes}) \\
&= \frac{1}{q} + [\text{Prob } P_1 \text{ passes}] (1 - \frac{1}{q}) \\
&\leq \frac{1}{q} + \frac{\omega_n}{q^{n-1}} \times \left(1 - \frac{1}{q}\right) \\
&\leq \frac{1}{q} + \left(\frac{q^{n-1} - (q-1)^{n-1}}{q^{n-1}}\right) \times \left(1 - \frac{1}{q}\right) \\
&= \frac{1}{q} + \frac{q^{n-1}}{q^{n-1}} - \frac{(q-1)^{n-1}}{q^{n-1}} - \frac{q^{n-1}}{q^n} + \frac{(q-1)^{n-1}}{q^n} \\
&= 1 - \frac{(q-1)^{n-1}}{q^{n-1}} + \frac{(q-1)^{n-1}}{q^n} \\
&= 1 - \left(\frac{q(q-1)^{n-1}}{q^n} - \frac{(q-1)^{n-1}}{q^n}\right) \\
&= 1 - \frac{(q-1)^n}{q^n} \\
&= 1 - \left(\frac{q-1}{q}\right)^n
\end{aligned}$$

Hence,

$$\text{Prob } \mathcal{S} \text{ wins} \leq 1 - \left(\frac{q-1}{q}\right)^n.$$

□

Thus, we have shown that 'The Strategy' is an optimal strategy, since it is restricted and has the same probability of winning as the maximum probability.

5.4. **A Sample Game.** In this section we will consider 3 'sample' games. One will have a fixed number of hat colors (q), one will have a fixed number of people (n) and the third will have a fixed number of both hat colors and people. Within each sample game we will show that the game has the same probability as 'The Strategy' and then give a specific example.

5.4.1. *Game 1:* Let us arbitrarily fix the number of hat colors q to be 2 and let the number of people be an arbitrary n.

Claim 5.5. *The maximum success probability for any strategy for the New Hats-on-a-line Game with 2 hat colors and n players is $1 - (\frac{1}{2})^n$.*

Proof. By induction on n:
For our base case, let $n = 1$. Then any guess by P_1 is correct with probability $\frac{1}{2} = 1 - \frac{1}{2}$. Assume the claim is true for all number of players up to n. Suppose there are c configurations of $n-1$ hats for which P_1 guesses.
case 1: $c \geq 1$

There are c cases where P_1's guess is correct with probability $\frac{1}{2}$. Thus the probability P_1 guesses wrong is

$$(\text{Prob } P_1 \text{ guesses}) \times (\text{Prob } P_1 \text{ guesses wrong}) \ = \frac{c}{2^{n-1}} \times \frac{1}{2} = \frac{c}{2^n} \geq \frac{1}{2^n}.$$

Implying that the probability that P_1 guesses correct is at most $1 - 2^{-n}$
case 2: $c = 0$
If P_1 always passes then the probability of winning is that of the $n-1$ player case which by induction is $1 - 2^{-n+1}$.
In either case we get that the probability of winning is $\max\{1 - 2^{-n}, 1 - 2^{-n+1}\} = 1 - 2^{-n}$ □

For an example of this case, let the configuration be $BBBWBBWWWB\ldots$ Then the first player would pass, as another player P_i for $i > 1$ is wearing black, and by our strategy we would win. This can be seen to be a maximum probability, as the probability that any configuration loses is the probability that our configuration is of the form $XWWW\ldots$, where $X \in \{B, W\}$, multiplied by the probability that P_1 guesses wrong;

$$\left(\frac{1}{2}\right)^{n-1} \times \left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{n}.$$

So the probability of winning is $1 - \left(\frac{1}{2}\right)^{n} = \frac{2^n - 1}{2^n}$, which is the same as 'The Strategy' with $q = 2$.

5.4.2. *Game 2:* For this game we fix the number of players n to be 2 and let the number of hat colors q be arbitrary.

Claim 5.6. *The maximum success probability for any strategy for the New Hats-on-a-line Game with q hat colors and 2 players is $1 - (\frac{q-1}{q})^2 = \frac{2q-1}{q^2}$.*

Proof. Suppose P_1 makes a guess of their hat color if P_2 is wearing $\{c_1, \ldots, c_r\}$ for a specified r of the q possible colors. When P_1 makes a guess, they guess correctly with probability $\frac{1}{q}$. Consider two cases,

case 1: $r = q$
If $r = q$ then we get an overall success probability of $\frac{1}{q}$ by how this is defined.

case2: $r < q$
If this is the case, then P_1 passes with probability of $\frac{q-r}{q}$. Now, given that P_1 passes, then P_2 guesses correctly with probability of $\frac{1}{q-r}$ since P_1 passing tells P_2 they are not wearing one of the specified r colors, indicating they are wearing one of the other $q - r$ colors. This gives an overall success probability of

$$(\text{probability } P_1 \text{ guesses}) \times (\text{probability } P_1 \text{ guesses correctly}) +$$
$$(\text{probability } P_1 \text{ passes}) \times (\text{probability } P_2 \text{ guesses correctly}).$$

This is equivalent to

$$\frac{r}{q} \times \frac{1}{q} + \frac{q-r}{q} \times \frac{1}{q-r} = \frac{r}{q^2} + \frac{1}{q}.$$

Now, $\frac{r}{q^2} + \frac{1}{q} = \frac{r+q}{q^2}$ is maximized when $r = q - 1$ since $1 \leq r \leq q - 1$. Thus, since $\frac{2q-1}{q^2} > \frac{1}{q}$, this case is the optimal strategy. □

Hence, for an example, suppose the configuration is BW where B, W are in the set of possible hat colors. Then by 'The Strategy', if W (white) is the predetermined color then P_1 will pass and

P_2 will guess white, indicating a win. If W (white) was not the agreed-on color, then P_1 will guess, and guess correctly with probability $\frac{1}{q}$. Thus

$$\begin{aligned}
&(\text{probability } P_1 \text{ guesses}) \times (\text{probability } P_1 \text{ guesses correctly})+ \\
&(\text{probability } P_1 \text{ passes}) \times (\text{probability } P_2 \text{ guesses correctly}). \\
&= \left(\frac{q-1}{q}\right) \times \left(\frac{1}{q}\right) + \frac{1}{q} \times 1 \\
&= \frac{q-1}{q^2} + \frac{q}{q^2} = \frac{2q-1}{q^2} \\
&= \frac{q^2-q^2+2q-1}{q^2} = \frac{q^2-(q^2-2q+1)}{q^2} \\
&= 1 - \frac{(q-1)^2}{q^2} = 1 - \left(\frac{q-1}{q}\right)^2.
\end{aligned}$$

Thus this has the same probability as 'The Strategy' with $n = 2$.

5.4.3. *Game 3:* Suppose $q = 2$ and $n = 4$. Then we have 16 possible configurations.

$Configuration$	player who guesses	player's guess
$BBBB$	4	B
$BBBW$	3	B
$BBWB$	4	B
$BWBB$	4	B
$WBBB$	4	B
$BBWW$	2	B
$BWBW$	3	B
$BWWB$	4	B
$WBBW$	3	B
$WBWB$	4	B
$WWBB$	4	B
$WWWB$	4	B
$WWBW$	3	B
$WBWW$	2	B
$BWWW$	1	B
$WWWW$	1	B

Notice that the only wrong guess here is the $WWWW$ configuration. So this game wins with this strategy with probability $\frac{15}{16}$. Is this equivalent to 'The Strategy' probability? 'The Strategy' probability is $1 - \left(\frac{2-1}{2}\right)^4 = 1 - \frac{1}{2^4} = 1 - 1\frac{1}{16} = \frac{15}{16}$. Hence they are the same probabilities, and our strategy is in fact optimal.

6. **Conclusion**

Due to all the possible variations, as stated in the introduction, it should be obvious that there is not one specific solution method for **all** hat games. We saw several different games within this paper and some had solutions based on different topics. As we saw in this paper, there are connections between Hamming codes in the hypercube and strategies for some hat games. In these results, the coding theory supports the study of the hat games. But, as often happens in mathematics, a connection between topics can be exploited in both directions. It is reasonable to hope that in some future work, strategies for hat games might give us some crucial insight into the world of error-correcting codes or other branches of mathematics. Some of these hat games do not seem to have practical applications. However, that does not mean they don't! Even if their only application is to let people enjoy a good puzzle and use their brains! Hat games are a fun and entertaining math puzzle!

7. Appendix

Alice	*Bob*	*Carl*
B	*B*	*B*
B	*B*	*W*
B	*W*	*B*
W	*B*	*B*
W	*W*	*B*
W	*B*	*W*
B	*W*	*W*
W	*W*	*W*

TABLE 1. configurations for Ebert's Hat game

Alice	*Bob*	*Carl*	*Alice*	*Bob*	*Carl*
B	*B*	*B*	*B*	*B*	*B*
B	*B*	*W*	*B*	*B*	*B*
B	*W*	*B*	*B*	*B*	*B*
W	*B*	*B*	*B*	*B*	*B*
W	*W*	*B*	*B*	*B*	*B*
W	*B*	*W*	*B*	*B*	*B*
B	*W*	*W*	*B*	*B*	*B*
W	*W*	*W*	*B*	*B*	*B*
B	*W*	*W*	*B*	*B*	*B*
B	*W*	*W*	*B*	*B*	*B*
B	*W*	*W*	*B*	*B*	*B*
B	*W*	*W*	*B*	*B*	*B*
B	*W*	*W*	*B*	*B*	*B*
B	*W*	*W*			

TABLE 2. guess configurations for Ebert's Hat game

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