Explorations in Recursion with John Pell and the Pell Sequence

Recurrence Relations and their Explicit Formulas

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John Pell (1611-1685)
John Pell (1611-1685) An “obscure” English Mathematician

• Part of the 17th century intellectual history of England and of Continental Europe.

• Pell was married with eight children, taught math at the Gymnasium in Amsterdam, and was Oliver Cromwell’s envoy to Switzerland.

• Pell was well read in classical and contemporary mathematics.

• Pell had correspondence with Descartes, Leibniz, Cavendish, Mersenne, Hartlib, Collins and others.

• His main mathematical focus was on mathematical tables: tables of squares, sums of squares, primes and composites, constant differences, logarithms, antilogarithms, trigonometric functions, etc.
John Pell (1611-1685) An “obscure” English Mathematician

- Many of Pell’s booklets of tables and other works do not list himself as the author.

- Did not publish much mathematical work. Is more known for his activities, correspondence and contacts.

- Only one of his tables was ever published (1672), which had tables of the first 10,000 square numbers.

- His best known published work is, “An Introduction to Algebra”. It explains how to simplify and solve equations.

- Pell is credited with the modern day division symbol and the double-angle tangent formula.

- Pell is best known, only by name, for the Pell Sequence and the Pell Equation.
John Pell (1611-1685) An “obscure” English Mathematician

- Division Symbol: \( \div \)

- Double-Angle Tangent Formula:
  \[
  \tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}
  \]

- Pell Sequence:
  \[
  p_n = 2p_{n-1} + p_{n-2} \quad p_0 = 1, p_1 = 2, n \geq 2
  \]

- Pell Equation:
  \[
  x^2 + 2y^2 = \pm 1
  \]
John Pell (1611-1685) An “obscure” English Mathematician

• Both the Pell Sequence and the Pell Equation are erroneously named after him.

• Euler, after reading John Wallis’s “Opera Mathematica”, mistakenly gave credit to Pell for the Pell Equation.

• He had constant financial trouble throughout his life and was twice imprisoned for unpaid debts.

• In summary, Pell seemed easily distracted, had multiple projects going on at once, and many unfinished projects. Not a well known mathematician because of lack of publishing and the desire to remain anonymous.

• Despite all this, he dedicated much of his life to mathematics and therefore is recognized as a minor figure in the history of mathematics.
The Pell Sequence

- Defined by the recurrence relation:

\[ p_n = 2p_{n-1} + p_{n-2} \quad p_0 = 1, p_1 = 2, n \geq 2 \]

- The first few terms of the Pell Sequence are:

\[ 1,2,5,12,29,70,168,408,\ldots \]

\[ p_2 = 2p_{2-1} + p_{2-2} = 2p_1 + p_0 = 2(2) + 1 = 5 \]

\[ p_3 = 2p_{3-1} + p_{3-2} = 2p_2 + p_1 = 2(5) + 2 = 12 \]

\[ p_4 = 2p_{4-1} + p_{4-2} = 2p_3 + p_2 = 2(12) + 5 = 29 \]

\[ \text{etc} \ldots \]
The Pell Sequence

• One solution to the recurrence relation is:

\[ p_n = \frac{\sqrt{2}}{4} \left[ \left( 1 + \sqrt{2} \right)^n - \left( 1 - \sqrt{2} \right)^n \right] \quad \forall n \geq 1 \]

• Here is a second solution to the recurrence relation:

\[ p_n = \sum_{\substack{i,j,k \geq 0 \ \text{i,j,k} \geq 0 \ i + j + 2k = n}} \frac{(i+j+k)!}{i!j!k!} \]
The Pell Sequence

• Here is how to find the first term in the Pell Sequence using the second solution:

\[ p_0 = 1 \]
\[ i + j + 2k = n \]
\[ i + j + 2k = 0 \]
\[ (i,j,k) \]
\[ (0,0,0) \]
\[ \frac{(0+0+0)!}{0!0!0!} = \frac{1}{1} = 1 \]
\[ p_0 = 1 \]

• Now, it is your turn!
Verification of the Pell Sequence

• Let \( p_n \) count the number of ways to fill an \( n \) foot flagpole.

• There are red, white, and blue flags.

\[
\text{red} = i, \text{blue} = j, \text{white} = k \quad p_0 = 1, p_1 = 2, n \geq 2
\]

• Red and blue flags are each 1 feet tall and white flags are 2 feet tall.

• If all flags are blue or red or any combination of the 2, then the possibilities are:

\[
3^6 = 729
\]
Verification of the Pell Sequence

• Consider for all cases which flag is at the top of the flagpole.

• Case 1: If a blue flag is on top then anything underneath is:

  \[ p_{n-1} \]

• Case 2: If a red flag is on top then anything underneath is:

  \[ p_{n-1} \]

• Case 3: If a white flag is on top then anything underneath is:

  \[ p_{n-2} \]

• The cases yield the desired recurrence relation which is the Pell Sequence:

  \[ p_n = 2p_{n-1} + p_{n-2} \]
Verification of the Pell Sequence

• Here are some examples on a case-by-case basis:

• 1) There is one way to fill a zero-foot flagpole if all flags are zero feet tall.

\[ n = 0 \rightarrow p_0 = 1 \rightarrow (i, j, k) \rightarrow (0, 0, 0) \rightarrow i + j + 2k = 0 + 0 + 2(0) = 0 \]

• 2) There are 2 ways to fill a 1-foot flagpole with either a blue or red flag

\[ n = 1 \rightarrow p_1 = 2 \rightarrow (i, j, k) \rightarrow (1, 0, 0) \rightarrow i + j + 2k = 1 \rightarrow 1 + 0 + 2(0) = 1 \]
\[ or \rightarrow (0, 1, 0) \rightarrow 0 + 1 + 2(0) = 1 \]

• 3) There are 5 ways to fill a 2-foot flagpole:

\[ n = 2 \rightarrow p_2 = 5 \rightarrow (i, j, k) \rightarrow (2, 0, 0), (0, 2, 0), (0, 0, 1)(1, 1, 0), (1, 1, 0) \rightarrow i + j + 2k = 2 \]
\[ red = i, j = blue, k = white \]
Properties of the Pell Sequence

- Here is the Pell Sequence recurrence relation and the first few terms.

\[ p_n = 2p_{n-1} + p_{n-2} \quad p_0 = 1, p_1 = 2, n \geq 2 \]

\[ 1, 2, 5, 12, 29, 70, 169, 408, \ldots \]

- Sometimes the sequence begins with zero.

- Here is one solution to the Pell Sequence.

\[ p_n = \frac{\sqrt{2}}{4} \left[ \left(1 + \sqrt{2}\right)^n - \left(1 - \sqrt{2}\right)^n \right], \forall n \geq 1 \]

- The only triangular Pell number is 1.

- For a Pell number to be prime, the index needs to be prime.
Properties of the Pell Sequence

• The only Pell numbers that are cubes, squares or any other higher power are:

0, 1, 144

• The Pell Numbers can be represented geometrically with the “Silver Rectangle”. The ratio of length to width is length “y” and width 1.

• When 2 squares with the side equal to the width are taken out of the rectangle, what remains has the same ratio of length to width as the original rectangle.

• Here is an algebraic representation:

\[ \frac{y}{1} = \frac{1}{y - 2} \rightarrow y^2 - 2y - 1 = 0 \rightarrow y = \left(1 + \sqrt{2}\right) \]
Properties of the Pell Sequence

Figure 1
Properties of the Pell Sequence
Properties of the Pell Sequence

• The generating function for the Pell Sequence is:

\[
\frac{1}{1 - 2x - x^2} = \sum_{n=1}^{\infty} P_n x^n
\]

• The Pell numbers can be generated by the matrix:

\[
M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}
\]

• Identities of the Pell Sequence can produce Pythagorean Triples and square numbers.
Properties of the Pell Sequence

• The proportion $\sqrt{2} : 1$ or $\frac{99}{70}$ is used in paper sizes A3, A4 and others.

• The Pell Numbers are the denominators of the fractions that are the closest rational approximations to the $\sqrt{2}$

$$1, 3, 7, 17, 41, 99, 29, 70, \ldots$$

• The sum of the numerator and the denominator of the previous term is the denominator of the current term.
Properties of the Pell Sequence

• The numerator of the current fraction is the sum of the numerator and 2 times the denominator of the previous fraction.

\[
\begin{align*}
\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots \\
1', 2', 5', 12', 29', 70', 2', 5', 29', 70', 12', 2'
\end{align*}
\]

• Alternating fractions determine approximations closer and closer to the \(\sqrt{2}\)

\[
\begin{align*}
\frac{1}{1}, \frac{7}{5}, \frac{41}{29}, \ldots < \sqrt{2}, \frac{99}{70}, \frac{17}{12}, \frac{3}{2}
\end{align*}
\]
Properties of the Pell Sequence

• There is a relationship between the Pell Sequence and the Pell Equation.

• The Pell Equation is defined:

\[ x^2 + 2y^2 = \pm 1 \]

• and, if

\[ x = p_{n+1} - p_n \quad y = p_n \]

• Then \( x \) and \( y \) will satisfy the Pell Equation.
Properties of the Pell Sequence

• Example:

\[ p_2 = 5 \rightarrow x = p_{2+1} - p_2 \rightarrow y = p_2 \]
\[ x = p_3 - p_2 \rightarrow y = p_2 \]
\[ x = 12 - 5 = 7 \rightarrow y = 5 \]
\[ x^2 + 2y^2 = \pm 1 \]
\[ 7^2 + 2(5)^2 \rightarrow 49 - 50 = -1 \]
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Introduction to recurrence relations

• A sequence of numbers can be defined recursively by what is known as a recurrence relation.

• The sequence of numbers:

\[1, 2, 5, 12, 29, 70, 169, 408, \ldots\]

• can be defined with the recurrence relation:

\[p_n = 2p_{n-1} + p_{n-2}\]

• The first few terms are known as the initial conditions of the sequence.

\[p_0 = 1, p_1 = 2, n \geq 2\]
Introduction to Recurrence Relations

• The numbers in the list are the terms of the sequence.

\[ p_0 = 1, p_1 = 2, p_2 = 5, etc \ldots \]

• A "solution" to the recurrence relation is:

\[ p_n = \frac{\sqrt{2}}{4} \left[ (1 + \sqrt{2})^n - (1 - \sqrt{2})^n \right] \forall n \geq 1 \]

• This is also known as an "explicit" or "closed-form" formula.
4 techniques for solutions to recurrence relations:
Guess and check with the Principle of Mathematical Induction

• Guess and check with the Principle of Mathematical Induction.

• Consider the sequence defined by:

\[ a_n = 2a_{n-1} + 1 \quad a_1 = 1 \quad n \geq 2 \]

• The first few terms in the sequence can be computed as follows:

\[
\begin{align*}
  a_1 &= 1 \\
  a_2 &= 2a_{2-1} + 1 = 2a_1 + 1 = 2(1) + 1 = 3 \\
  a_3 &= 2a_{3-1} + 1 = 2a_2 + 1 = 2(3) + 1 = 7 \\
  a_4 &= 2a_{4-1} + 1 = 2a_3 + 1 = 2(7) + 1 = 15 \\
  a_5 &= 2a_{5-1} + 1 = 2a_4 + 1 = 2(15) + 1 = 31 \\
  a_6 &= 2a_{6-1} + 1 = 2a_5 + 1 = 2(31) + 1 = 63 
\end{align*}
\]
4 techniques for solutions to recurrence relations:
Guess and check with the Principle of Mathematical Induction

• From this data we can notice a pattern and guess a formula:

\[ a_1 = 2^1 - 1 = 1 \]
\[ a_2 = 2^2 - 1 = 3 \]
\[ a_3 = 2^3 - 1 = 7 \]
\[ a_4 = 2^4 - 1 = 15 \]
\[ a_5 = 2^5 - 1 = 31 \]
\[ a_6 = 2^6 - 1 = 63 \]
\[ \therefore a_n = 2^n - 1, \forall n \geq 1 \]

• Use induction to prove \( a_n = 2^n - 1 \) holds for all \( n \geq 1 \)
4 techniques for solutions to recurrence relations:
Guess and check with the Principle of Mathematical Induction

• Proof: (i) Base cases: For

\[ n = 1 \rightarrow a_n = 2^n - 1 \rightarrow a_1 = 2^1 - 1 = 1. \]

• (ii) induction step:

Assume \( a_n = 2^n - 1 \) is true, then \( a_{n+1} = 2^{n+1} - 1 \) is true. Then

\[ a_{n+1} = 2a_{(n+1)-1} + 1 \rightarrow 2a_n + 1 \rightarrow 2(2^n - 1) + 1 \]
\[ \rightarrow 2^{n+1} - 2 + 1 \rightarrow 2^{n+1} - 1 \]

• Therefore by induction \( a_n = 2^n - 1 \) holds for all \( n \geq 1 \)
4 techniques for solutions to recurrence relations:  
The Characteristic Polynomial

• Consider the recurrence relation: \( a_n = -5a_{n-1} + 6a_{n-2} \) \( a_0 = 5, a_1 = 19, n \geq 2 \)

\[
\begin{align*}
a_n &= -5a_{n-1} + 6a_{n-2} \\
a_n + 5a_{n-1} - 6a_{n-2} &= 0 \\
x^2 + 5x - 6 &\rightarrow (x - 1)(x + 6) = 0 \\
x_1 &= -6, x_2 = 1 \\
a_n &= c_1(x_1^n) + c_2(x_2^n) \\
a_n &= c_1(-6^n) + c_2(1^n) \\
a_0 &= 5 \rightarrow 5 = c_1(-6^0) + c_2(1^0) \\
5 &= c_1 + c_2 \rightarrow equation 1 \\
a_1 &= 19 \rightarrow 19 = c_1(-6^1) + c_2(1^1) \\
19 &= -6c_1 + c_2 \rightarrow equation 2
\end{align*}
\]
4 techniques for solutions to recurrence relations:
The Characteristic Polynomial

- Multiplying equation 1 by 6 and adding equation 1 to equation 2 yields:

\[ c_1 = -2, c_2 = 7 \]
\[ a_n = -2(-6^n) + 7(1^n) \Rightarrow a_n = -2(-6^n) + 7, \forall n \geq 0 \]
4 techniques for solutions to recurrence relations: Generating Functions

Consider the recurrence relation: \( a_n = 2a_{n-1} \quad a_0 = 1, n \geq 1 \)

Solution: →

\[
\begin{align*}
f(x) &= \sum_{n=0}^{\infty} a_n x^n \\
f(x) &= a^0 x^0 + \sum_{n=1}^{\infty} (2a_{n-1}) x^n \\
f(x) &= 1 + 2 \sum_{n=1}^{\infty} (a_{n-1}) x^n \\
f(x) &= 1 + 2x \sum_{n=0}^{\infty} (a_{n-1}) x^{n-1} \\
f(x) &= \sum_{n=0}^{\infty} a_n x^n \Rightarrow f(x) = 1 + 2xf(x) \\
f(x) - 2xf(x) &= 1 \rightarrow f(x)(1 - 2x) = 1 \\
f(x) &= \frac{1}{1 - 2x} \rightarrow f(x) = \sum_{n=0}^{\infty} (2x)^n \rightarrow f(x) = \sum_{n=0}^{\infty} 2^n x^n \\
\therefore a_n &= 2^n \rightarrow \forall n \geq 0.
\end{align*}
\]
4 techniques for solutions to recurrence relations: Linear Algebra

- Solve the recurrence relation:
  \[ a_{n+1} = 3a_n - 2a_{n-1} \quad \text{with} \quad a_0 = -4, a_1 = 0, n \geq 1 \]

- Solution:
  \[
  v_n = A^n \bullet v_{n-1} \\
  \begin{bmatrix}
  a_{n+1} \\
  a_n
  \end{bmatrix} =
  \begin{bmatrix}
  3 & -2 \\
  1 & 0
  \end{bmatrix}
  \begin{bmatrix}
  a_n \\
  a_{n-1}
  \end{bmatrix}
  \\
  A^n v_0 = A^n \begin{bmatrix}
  a_1 \\
  a_0
  \end{bmatrix} = A^n \begin{bmatrix}
  0 \\
  -4
  \end{bmatrix}
  \]
4 techniques for solutions to recurrence relations:
Linear Algebra

• Next is the characteristic polynomial of $A$ by the diagonalization of $A$

$$\begin{align*}
(A - \lambda I) &= \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3 - \lambda & -2 \\ 1 & -\lambda \end{bmatrix} \\
\det(A - \lambda I) &= (3 - \lambda)(-\lambda) - (-2)(1) \\
\lambda^2 - 3\lambda + 2 &= 0 \rightarrow (\lambda - 2)(\lambda - 1) \rightarrow \lambda_1 = 1, \lambda_2 = 2 \\
D &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}
\end{align*}$$
4 techniques for solutions to recurrence relations:
Linear Algebra

• The Eigen vectors of A are: $\lambda_1$ and $\lambda_2$ The Eigen space for A is:

$$
\begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}
$$

• To find the Eigen space for $\lambda_1 = 1$ we have:

$$(A - \lambda I)x = 0$$

$$
\begin{bmatrix}
3 - \lambda_1 & -2 \\
1 & -\lambda_1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = 0 \rightarrow
\begin{bmatrix}
2 & -2 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = 0
$$

$2x_1 - 2x_2 = 0$

$x_1 - x_2 = 0$
4 techniques for solutions to recurrence relations: Linear Algebra

• Where \( x_2 = t_1 \) is free and \( x_1 = x_2 = t_1 \) and:

\[
x = \begin{bmatrix} t_1 \\ t_1 \\ t_1 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

• To find the Eigen space for \( \lambda_2 = 2 \) we have:

\[
(A - \lambda I)x = 0
\]

\[
\begin{bmatrix} 3 - \lambda_2 & -2 \\ 1 & -\lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0
\]

\[
x_1 - 2x_2 = 0
\]

\[
x_1 - 2x_2 = 0
\]
4 techniques for solutions to recurrence relations: 
Linear Algebra

- Where \( x_2 = t_2 \) is free and \( x_1 = 2x_2 = 2t_2 \) and:

\[
x = \begin{bmatrix} 2t_2 \\ t_2 \end{bmatrix} = t_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]

- Then we will write the matrices \( P, P^{-1}, D \) to solve for A:

\[
P = \begin{bmatrix} t_1 & t_2 \\ t_1 & t_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}
\]

\[
D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}
\]
4 techniques for solutions to recurrence relations: 
Linear Algebra

Solution:

\[
P^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \rightarrow \frac{1}{(1)(1)-(-2)(1)} \begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix}
\]

\[
P^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}
\]

\[
P^{-1} AP = D \Rightarrow A = PDP^{-1}
\]

\[
v_n = A^n v_0 = (PDP^{-1})v_0 = (PDP^{-1}) \begin{bmatrix} 0 \\ -4 \end{bmatrix}
\]

\[
P^{-1}v_0 = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}
\]

\[
PD^n = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix} = \begin{bmatrix} 1 & 2^{n+1} \\ 1 & 2^n \end{bmatrix}
\]

\[
v_n = PD^n \cdot P^{-1}v_0
\]

\[
v_n = \begin{bmatrix} 1 & 2^{n+1} \\ 1 & 2^n \end{bmatrix} \begin{bmatrix} -8 \\ 4 \end{bmatrix} = \begin{bmatrix} -8 + 4(2^{n+1}) a_{n+1} \\ -8 + 4(2^n) a_n \end{bmatrix}
\]

\[
\therefore a_n = -8 + 4(2^n) \rightarrow a_n = 4\left[-2 + \left(2^n\right)\right] \forall n \geq 0.
\]
Curriculum for Instructors and Students

- The curriculum consists of 8 lessons: Introduction to Recurrence Relations, Characteristic Polynomial, Checking Explicit Formulas, Guess and Check with Induction, Pell Sequence, Tower of Hanoi, Generating Functions, Linear Algebra.

- Each lesson has a lesson plan, student handout, instructor solutions, and lesson reflection. In the case of the Tower of Hanoi models were made.

- All lessons were done except for Generating Functions and Linear Algebra due to time constraints and students lacking prerequisites.

- The unit was done with my high school Advanced Algebra 2 class with mostly 10th and 11th grade students with a few 12th and 9th grade students. The unit was done January 2011.
Curriculum for Instructors and Students

• A chapter on recursive sequences in their Advanced Algebra 2 book was done before the curriculum. It contained arithmetic and geometric sequences, writing recursive formulas, shifted geometric sequences - (concept of a limit), graphs of sequences, application problems.

• Students had the most success with Introduction to Recurrence Relations, Characteristic Polynomial, Pell Sequence and Tower of Hanoi.

• Students had the least success with Checking the Explicit Formula, and Guess and Check with Induction.

• Here are some examples of student work which are contained within the student handouts.
5) \( a_{n+1} = 7a_n - 10a_{n-1} \) given \( a_0 = 10 \) and \( a_1 = 29 \)

\[
\begin{align*}
X^2 - 7 + 10 & = (X - 5)(X - 2) \\
\end{align*}
\]

\[
\begin{align*}
a_n &= C_1(-5^n) + C_2(-2^n) \\
a_0 &= C_1(-5^0) + C_2(-2^0) \\
10 &= C_1 + C_2 \\
a_1 &= C_1(-5^1) + C_2(-2^1) \\
29 &= -5C_1 + 2C_2 \\
\end{align*}
\]

\[
\begin{align*}
10 &= 3 + C_2 \\
29 &= 5C_1 + 2C_2 \\
20 &= -2C_1 - 2C_2 \\
9 &= 3C_1 \\
C_1 &= 3 \\
C_2 &= 7 \\
\end{align*}
\]

\[
\begin{align*}
a_n &= 3(5^n) + 7(2^n) \\
\end{align*}
\]
b) Use the characteristic polynomial technique to solve this recurrence relation.

\[ P_n - 2P_{n-1} - P_{n-2} = 0 \]

\[ x^2 - 2x - 1 = 0 \]

\[ x_1 = 1 + \sqrt{2} \]
\[ x_2 = 1 - \sqrt{2} \]

\[ 1 + \sqrt{2} = (1 + \sqrt{2})C_1 + (1 + \sqrt{2})C_2 \]
\[ (-) \quad 2 = (1 + \sqrt{2})C_1 + (1 - \sqrt{2})C_2 \]
\[ \sqrt{2} - 1 = (2\sqrt{2})C_2 \]
\[ \frac{2 - \sqrt{2}}{4} = C_2 \]

\[ P_n = C_1 (1 + \sqrt{2})^n + C_2 (1 - \sqrt{2})^n \]

\[ P_0 = C_1 (1 + \sqrt{2})^0 + C_2 (1 - \sqrt{2})^0 \]
\[ 1 = C_1 + C_2 \]
\[ 1 = C_1 + C_2 \times 1 + \sqrt{2} \]

\[ P_1 = C_1 (1 + \sqrt{2}) + C_2 (1 - \sqrt{2}) \]
\[ 2 = (1 + \sqrt{2})C_1 + C_2 (1 - \sqrt{2}) \]
\[ \frac{2 + \sqrt{2}}{4} = C_1 \]

\[ P_n = \frac{2 + \sqrt{2}}{4} (1 + \sqrt{2})^n + \frac{2 - \sqrt{2}}{4} (1 - \sqrt{2})^n \]
Alternate Pell Formula – Student Work

\[ P_2 = 5 \quad \text{true?} \]

\[ P_2 \rightarrow i + j + 2k = 2 \]
\[ \begin{align*}
(1, 1, 0) \\
(0, 0, 1) \\
(2, 0, 0) \\
(0, 2, 0)
\end{align*} \]

\[ \frac{(1+1+0)!}{1! \cdot 1! \cdot 0!} + \frac{(0+0+1)!}{0! \cdot 0! \cdot 1!} + \frac{(2+0+0)!}{2! \cdot 0! \cdot 0!} + \frac{(0+2+0)!}{0! \cdot 2! \cdot 0!} \]

\[ \frac{2}{1} \quad \frac{1}{1} \quad \frac{2}{2} \quad \frac{2}{2} \]

\[ \frac{1}{2} \quad \frac{1}{1} \quad \frac{2}{2} \quad \frac{2}{2} \]

\[ P_2 = 5 \checkmark \]
5) \( a_{n+1} = 7a_n - 10a_{n-1} \) given \( a_0 = 10 \) and \( a_1 = 29 \)

\[
a_n = \frac{3}{5}(5^n) + \frac{21}{6}(2^n)
\]

\[
\frac{3}{5}(5^n) + \frac{21}{6}(2^n) = 7\left[\frac{3}{5}(5^n) + \frac{21}{6}(2^n)\right] - 10\left[\frac{3}{5}(5^n) + \frac{21}{6}(2^n)\right]
\]

\[
\frac{3}{5}(5^n) + \frac{21}{6}(2^n) - 7\left[\frac{3}{5}(5^n) + \frac{21}{6}(2^n)\right] + 10\left[\frac{3}{5}(5^n) + \frac{21}{6}(2^n)\right] = 0
\]

\[
5^{n+2} - \frac{2}{5}(5^n) + \frac{21}{6}(2^n) - 7\left[\frac{3}{5}(5^n) + \frac{21}{6}(2^n)\right] + 10\left[\frac{3}{5}(5^n) + \frac{21}{6}(2^n)\right] = 0
\]

\[
5^{n+2} - 2^{n+2} + 15 + 14 - 7(3 + 7) + 16\left[\frac{3}{5} + \frac{21}{6}\right] = 0
\]

\[
5^{n+2} - 2^{n+2} + 29 - 70 + 41 = 0
\]

\[
5^{n+2} - 2^{n+2} = 0
\]
1) Prove that the sum of \( n \) consecutive positive odd integers is \( n^2 \). In other words prove that \( 1 + 3 + 5 + \ldots + (2n - 1) = n^2 \)

**Base Case:** \( n = 1 \)

\[
2 \cdot 1 - 1 = 1^2
\]

\[
2 - 1 = 1^2
\]

\[
1 = 1
\]

\[
W.T.S \quad (2k-1) + 2(k+1) - 1 = k^2 + 2(k+1) - 1
\]

\[
(2k-1) + 2(k+1) - 1 = k^2 + 2(k+1) - 1
\]

\[
= k^2 + 2k + 2 - 1
\]

\[
= k^2 + 2k + 1
\]

\[
= (k+1)(k+1)
\]

\[
= (k+1)^2
\]
Tower of Hanoi – Student Work

\[\begin{array}{ccc}
A & B & C \\
\text{start} & 1, 2 & \star & \star \\
1 & 2 & 1 \\
2 & \star & 1 & 2 \\
3 & \star & \star & 1 & 1 & 2 \\
\end{array}\]

- h = 2, moves = 3

\[\begin{array}{ccc}
A & B & C \\
\text{start} & 1, 2, 3 & \star & \star \\
1 & 2, 3 & \star & 1 \\
2 & 3 & 2 & 1 \\
3 & 3 & 1, 2 & \star \\
4 & \star & 1, 2 & 3 \\
6 & 1 & 2 & 3 \\
7 & \star & \star & 1, 2, 3 \\
\end{array}\]

\[a_0, a_1, a_2, a_3, a_4, a_5\]

- \[0, 1, 3, 7, 15, 31\]
- \[+1 + 2 + 4 + 8 + 16\]

\[B. 2^{an} = 2^{an-1} + 1\]

- recurrence relation

\[C. 2^{an} = 2^n - 1\]

- explicit formula
15) Write a recurrence relation for the following sequences. Use $a_1$ for the first term in the sequence

a) $1, 2, 3, 5, 8, 13, ..., 21, 34, 55, 89$

$$u_n = u_{n-1} + u_{n-2}$$

$$u_3 = u_2 + u_1 = 2$$

b) $1, 4, 9, 16, ..., 25, 36, 49, 64, 81$

$$a_n = n^2$$

c) $1, 2, 6, 15, ..., 24, 120, 720$

$$a_n = n!$$

d) $4, 1, 3, -2, 5, -7, 12, -19, 31, ..., a_{n-2} - a_{n-1} = a_n$

$$a_1 = 4$$

$$a_2 = -1$$

$$n \geq 3$$
Curriculum for Instructors and Students

• Summary of Curriculum:

• Overall it went well, sometimes painful and sometimes beauty

• Small class of 24 students, many smart and motivated students, I have known many of them since 6th grade.

• Summary of M.S.T. 501 project:

• It took about 9-12 months, summer 2010 getting ideas, fall-winter 2010-2011 doing math, winter-spring 2010 paper and power point.