Two-Toned Tilings and Compositions of Integers

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Abstract. Following the article "Combinatorics of Two-Toned Tilings" by Benjamin, Chinn, Scott, and Simay [1], this paper introduces a function to count tilings of length $r + n$ that use any number of white tiles (of length between 1 and n) and exactly r identical red squares. We explore a number of combinatorial identities and generalizations of this concept, along with some connections to generalized Fibonacci numbers and applications to compositions of integers.

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I. Introduction

One of the simpler topics studied in combinatorics is the sequence of Fibonacci numbers, introduced by Leonardo de Pisa, an influential Italian mathematician from the Middle Ages who is commonly known by the name "Fibonacci". This sequence of numbers appears frequently in nature - for example, in the family trees of honeybees, the number of petals in some flowers, the spiral patterns of seed heads and pine cones, the arrangement of leaves on stems, and so on. Fibonacci numbers also show up in the meter of Sanskrit poetry and in art, music, and architecture. The Fibonacci numbers are defined recursively by setting $F_1 = F_2 = 1$ and letting $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. One easily proved fact, which we will later establish, is that the Fibonacci number F_{m+1} counts the number of tilings of an m by 1 rectangle (hereafter called an m - strip) using only squares and dominoes.

At the other extreme, one of the more subtle topics studied in combinatorics is how to count the number of partitions of an integer (i.e., the number of ways of writing some positive integer n as a sum of positive integers). For instance, 7 is the number of integer partitions of 5, since 5 may be written as any of the following seven expressions

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ \text{and}\ 1+1+1+1+1.$$

Note that the order of the summands is disregarded, so 3 + 2 is considered equivalent to 2 + 3 as an integer partition of 5. It is very difficult to obtain a closed-form formula for the number of integer partitions. Famously, mathematicians Hardy and Ramanujan proved in 1918 that the number of partitions of n approaches $\frac{1}{4n\sqrt{3}} \exp\left(\pi \sqrt{\frac{2n}{3}}\right)$ as $n \to \infty$.

Various questions concerning integer partitions have been explored over the centuries. For example, if we agreed to distinguish sums by the order of their terms, then 5 + 2 and 2 + 5 would be considered different from each other, and these two "ordered partitions" of 7 would be counted separately. Such an ordered partition is called a **composition** of n. We may want to count the compositions of n with exactly p parts (i.e., summands) of size k. For instance, there are four compositions of 7 with exactly three parts of size 2:

$$2+2+2+1,\ 2+2+1+2,\ 2+1+2+2,\ \text{and}\ 1+2+2+2.$$

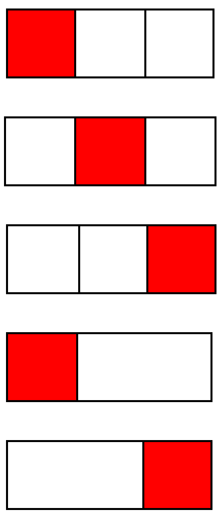
This project explores these ideas further, examining the article "Combinatorics of two-toned tilings" by Benjamin, Chinn, Scott, and Simay [1]. Their paper introduces several interesting identities involving tilings, including relationships with generalized Fibonacci numbers and compositions of integers.

II. Two-toned Tilings

For nonnegative integers r and n, let the function $a(r, n)$ count the number of "two-toned tilings" of a strip of length $r + n$ consisting of exactly r red squares and any number of white tiles of any length (from 1 to n). To abbreviate, we will call these (r, n) - tilings. The possible (1, 2) - tilings are

R11, 1R1, 11R, R2, 2R,

where R signifies a red square and a number signifies a white tile of that length. Visually, these tilings may be represented as follows.



This enumeration demonstrates that $a(1, 2) = 5$. On the other hand, $a(2, 1) = 3$ since

RR1, R1R, and 1RR

are all the possible (2, 1) - tilings. In general, $a(r, 1) = r + 1$ since there are $r + 1$ potential positions for the white tile (in this case a square) and red squares must fill the other positions.

We begin with initial conditions and a recurrence relation for $a(r, n)$.

Identity 1: For $r \geq 0$, the number of (r, n) - tilings satisfies

$$a(r, 0) = 1.$$

For $n \geq 1$,

$$a(0, n) = 2^{n-1}.$$

For $r, n \geq 1$,

$$a(r, n) = a(r-1, n) + 2a(r, n-1) - a(r-1, n-1).$$

Proof : If there are no white tiles, all r tiles must be red squares, so $a(r, 0) = 1$. If there are no red squares, we may tile an n - strip with white tiles by deciding whether or not to end a tile at every cell except the final one (which *must* end a tile). Thus $a(0, n) = 2^{n-1}$, and we note that this expression also corresponds very naturally to the number of compositions (i.e., ordered partitions) of the positive integer n.

To count (r, n) - tilings for positive values of r and n, we condition on the way the (r, n) - tiling ends. If it ends with a red square, there are $a(r-1, n)$ ways to tile the previous $r-1+n$ cells. If it ends with a white square, there are $a(r, n-1)$ ways to tile the previous $r+n-1$ cells. If it ends with a white tile of length greater than 1, we may obtain it from an $(r, n-1)$ - tiling that ends in a white tile by lengthening the last tile by 1. There are $a(r, n-1) - a(r-1, n-1)$ of these (all $(r, n-1)$ - tilings except those ending in a red square). Since these are the only possibilities, we have $a(r, n) = a(r-1, n) + 2a(r, n-1) - a(r-1, n-1)$. ▍

Using Identity 1, we can fill in a table of $a(r, n)$ for values of r and n between 0 and 5, inclusive. When $n = 0$, we have $a(r, 0) = 1$, which gives us every entry of the first column of the table. When $r = 0$ and $n \geq 1$, we have $a(0, n) = 2^{n-1}$, which gives us the rest of the first row of the table. We also know $a(r, 1) = r + 1$, which gives us the rest of the second column. For each remaining entry, we use $a(r, n) = a(r-1, n) + 2a(r, n-1) - a(r-1, n-1)$, which corresponds to that entry's North neighbor, plus twice its West neighbor, minus its Northwest neighbor. From above, we know $a(1, 2) = 5$ and $a(2, 1) = 3$, so we use the recursion to calculate $a(2, 2)$: $a(1, 2) + 2a(2, 1) - a(1, 1) = 5 + 2(3) - 2 = 9$. Then $a(3, 2) = 9 + 2(4) - 3 = 14$, and so on.

r \ ***n***	0	1	2	3	4	5
0	1	1	2	4	8	16
1	1	2	5	12	28	64
2	1	3	9	25	66	168
3	1	4	14	44	129	360
4	1	5	20	70	225	681
5	1	6	27	104	363	1182

Table 1: Two-toned tilings $a(r, n)$

III. Combinatorial Identities

The Fibonacci numbers **1**, **1**, **2**, **3**, **5**, **8**, … appear as sums of the diagonals of Pascal's triangle, indicated in color below.

$$
\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & \mathbf{1} & & \mathbf{1} & & & & \\
 & & & \mathbf{1} & & \mathbf{2} & & \mathbf{1} & & & \\
 & & \mathbf{1} & & \mathbf{3} & & \mathbf{3} & & 1 & & \\
 & \mathbf{1} & & \mathbf{4} & & 6 & & 4 & & 1 & \\
\mathbf{1} & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

In a similar manner, our next identity relates sums of the diagonals of the table for two-toned tilings (Table 1) to the odd-indexed Fibonacci numbers. As an example,

$$a(0,2) + a(1,1) + a(2,0) = 2 + 2 + 1 = 5 = F_5.$$

Identity 2: For $n \geq 0$,

$$\sum_{r=0}^{n} a(r, n-r) = F_{2n+1}.$$

Proof: The left side enumerates the two-toned tilings of length n with r red squares where r may vary from 0 to n.

We wish to define a map $\widehat{g}$ between the set of two-toned tilings of length n and the set of tilings of a strip of length $2n$ using white squares and dominoes. To do so, we first define a map g which is the restriction of $\widehat{g}$ to the individual tiles belonging to a two-toned tiling of length n. Specifically, let the function g map each red square to a white domino (i.e., $\mathrm{R} \rightarrow 2$) and each white tile of length $k \geq 1$ to $k-1$ white dominoes, preceded and followed by a white square (i.e., $k \rightarrow 122...21$ where there are $k-1$ copies of 2). In particular, g maps each white square to two white squares (i.e., $1 \rightarrow 11$). To construct $\widehat{g}$, apply g to each tile of a two-toned tiling from left to right until reaching the end, concatenating the results. For any two-toned tiling $y_1 y_2 ... y_h$ of length n where the y_i represent individual tiles, we have $\widehat{g}(y_1 y_2 ... y_h) = g(y_1) g(y_2) ... g(y_h)$. The function

$\widehat{g}$ effectively takes any two-toned tiling, doubles its length and changes it all to white, using only squares and dominoes. We will show that $\widehat{g}$ is a bijection.

To show $\widehat{g}$ is onto, we will begin with a "white tiling" of a $2n$ - strip using white squares and dominoes and identify the two-toned tiling which is its preimage under $\widehat{g}$. Given an arbitrary white tiling, its leftmost block of tiles must consist of one of the following and have the given preimage:

- some number d of dominos preceded by zero white squares. The preimage is d red squares (i.e., RR...R $\rightarrow$ 22...2 where the number of R's and 2's are the same).
- an even number $2t$ of consecutive white squares for some $t \geq 1$. The preimage is t consecutive white squares.
- an odd number $2t + 1$ of consecutive white squares, $t \geq 0$, followed by $d \geq 1$ white dominoes and another white square. (Note that the tiling of a $2n$ - strip cannot consist solely of an odd number white squares since $2n$ is even.) The preimage is t consecutive white squares, then a white tile of length $d + 1$. For example, if we have one white square followed by $d \geq 1$ white dominoes and another white square, the preimage is a white tile of length $d + 1$ (i.e., $d + 1 \rightarrow$ 122...21 with d copies of 2).

After finding the preimage of the leftmost block of tiles, move on to the next leftmost block of tiles and find its preimage. Each such block will be of even length and its preimage will be a two-toned tiling of half its length and will be uniquely determined by the rules above. Since we may deconstruct every possible tiling of a $2n$ - strip into such blocks with the specified preimages, we have shown $\widehat{g}$ is onto.

Next, we prove g is one-to-one. Suppose two distinct two-toned tilings Y and Z of length n are mapped by $\widehat{g}$ to the same tiling of the $2n$ - strip; i.e., $\widehat{g}(Y) = \widehat{g}(Z)$. Let $Y = y_1y_2...y_h$ and $Z = z_1z_2...z_j$ where $y_1, \ldots, y_h, z_1, \ldots, z_j \, \varepsilon \, \{1, 2, \ldots, n-r, R\}$. By way of contradiction, assume $Y \neq Z$. Then there exists some first tile k in which Y and Z differ, so that $y_k \neq z_k$ and $y_m = z_m$ for all m such that $1 \leq m < k$. But then $g(y_k) \neq g(z_k)$, which implies $\widehat{g}(Y) \neq \widehat{g}(Z)$, a contradiction. So $\widehat{g}$ is one-to-one and thus a bijection.

Finally, we want to show that F_{2n+1} counts the set of all tilings of a $2n$ - strip using white squares and dominoes, which will complete the proof of the identity. Let t_m represent the number of tilings of an m - strip using white tiles of length 1 or 2. To count these, we condition on the rightmost tile. If it is a white square, there are t_{m-1} ways to tile the remaining $m - 1$ cells. If it is a white domino, there are t_{m-2} ways to tile the remaining $m - 2$ cells. Thus for $m \geq 2$, we have proved the recurrence relation $t_m = t_{m-1} + t_{m-2}$. This is identical to the Fibonacci recurrence $F_m = F_{m-1} + F_{m-2}$. Next, compare the initial conditions. We know $F_1 = F_2 = 1$. A strip with no length has the "empty tiling" and a 1-strip may be tiled only with a white square, so

$t_0 = t_1 = 1$. A 2-strip may be tiled using two squares or one domino, so $t_2 = 2$. So $t_m = F_{m+1}$ for any $m \geq 0$, which shows that the number t_{2n} of tilings of a $2n$-strip, is F_{2n+1}. ▮

The next three identities involve binomial coefficients and arise from constructing our two-toned tilings in various manners. First, suppose we want to count two-toned tilings by conditioning on the number of white tiles.

Identity 3: For $r \geq 0$ and $n \geq 1$,

$$a(r, n) = \sum_{j=1}^{n} \binom{n-1}{j-1}\binom{j+r}{r}.$$

Proof: The left side counts the number of (r, n) - tilings. We may also count these tilings by conditioning on j, the number of white tiles, where $1 \leq j \leq n$. For each value of j, we first determine the lengths of the white tiles that comprise length n. Since the j^{th} tile must end at the n^{th} cell, there are $\binom{n-1}{j-1}$ ways to choose $j-1$ other cells of the remaining $n-1$ cells where a white tile ends. We now have white tiles $w_1, w_2, \ldots, w_j$. Thus $\binom{n-1}{j-1}$ counts compositions of n with exactly j summands.

Next, we intersperse r red squares among the white tiles in their given order. Altogether, there are $j + r$ positions for tiles, so $\binom{j+r}{r}$ ways to choose where to put the r red squares. By the product property, we have $\binom{n-1}{j-1}\binom{j+r}{r}$ two-toned tilings of length $r + n$ with exactly j white tiles. Alternatively, in the second step, we may consider the $j + 1$ gaps before and after the j white tiles as locations to which we may assign r red squares in $\left(\!\binom{j+1}{r}\!\right)$ ways. Then

$$\left(\!\binom{j+1}{r}\!\right) = \binom{j+1+r-1}{r} = \binom{j+r}{r},$$

giving the same product as before. ▮

Now suppose we want to count two-toned tilings by conditioning on the number of locations among the red squares where white tiles occur.

Identity 4: For $r \geq 0$ and $n \geq 1$,

$$a(r, n) = \sum_{j=1}^{r+1} \binom{r+1}{j}\binom{n-1}{j-1} 2^{n-j}.$$

Proof: Again, the left side enumerates the number of (r, n) - tilings. Another way to count these is to begin with r red squares and choose from the $r + 1$ gaps before and after these red squares exactly j regions in which to place white squares. (Since $n \geq 1$, there must be at least 1 such region and at most $r + 1$ regions, so $1 \leq j \leq r + 1$.) We may choose j regions in $\binom{r+1}{j}$ ways. Next, select a subset $\{x_1, x_2, \ldots, x_{j-1}\}$ of $\{1, 2, \ldots, n - 1\}$ such that $1 \leq x_1 < x_2 < \ldots < x_{j-1} < n$. There are $\binom{n-1}{j-1}$ such subsets. Each element x_i of a particular subset gives the partial sum of white squares placed in the first i regions; i.e., put x_1 white squares in region 1, $x_i - x_{i-1}$ white squares in region i for $2 \leq i \leq j - 1$, and $n - x_{j-1}$ white squares in region j. Finally, decide which white squares to attach or link together with the square on their left. There are n white squares, j of which are leftmost in their region, so 2^{n-j} ways to decide whether or not to attach the remaining $n - j$ white squares to their left neighbor. By the product principle, $\binom{r+1}{j}\binom{n-1}{j-1} 2^{n-j}$ is the number of (r, n) - tilings with exactly j regions where white tiles occur, which we sum over all possible values of j. ▮

The final identity concerning two-toned tilings gives a similar expression, but with a factor of 2 before the summation.

Identity 5: For $r \geq 0$ and $n \geq 1$,

$$a(r, n) = 2^{n-r-1} \sum_{j=0}^{r} \binom{r+1}{j}\binom{r-j+n}{n}.$$

Proof: We make use of three different types of two-toned tilings. First, let T be the total set of all (r, n) - tilings. Then $|T| = a(r, n)$. Next, let S represent the set of all (r, n) - tilings that consist solely of squares. Since the only question is which of the $r + n$ squares are red, $|S| = \binom{r+n}{r}$. Finally, let D represent square-only (r, n) - tilings in which we may place red “dividers” on any boundaries not adjacent to a white square (i.e., before cell 1 and/or after cell $r + n$ if it is red, and/or between two consecutive red squares). Then D is a “decorated” version of S, and we say the dividers are “white-averse”. For example, if $r = 8$ and $n = 4$, one element of D is the tiling |RR*w*R|R|RR*ww*R*w*R| with red dividers before the first, fifth, and sixth cells, and after the twelfth cell; another is R|R*w*RRR|R*ww*R*w*R| with red dividers before the second and seventh cells, and after the twelfth cell.

To count tilings in D, let’s condition on j, the number of red dividers. If j were allowed to be $r + 1$, none of the gaps before and after the r red squares could have any white squares, since the dividers are white-averse. But $n \geq 1$, so there are at most r red dividers, and $0 \leq j \leq r$. We claim the number of square-only (r, n) - tilings with exactly j red dividers is $\binom{r+1}{j}\binom{r-j+n}{n}$. To see this, start with r red squares and place red dividers in j of the $r + 1$ gaps before and after them in $\binom{r+1}{j}$ ways. Since no white square may be next to a divider, there remain $r + 1 - j$

regions where we may distribute n white squares in $\left(\!\binom{r+1-j}{n}\!\right) = \binom{r+1-j+n-1}{n} = \binom{r-j+n}{n}$ ways. So

$$|D| = \sum_{j=0}^{r} \binom{r+1}{j} \binom{r-j+n}{n}.$$

We still need to show that $a(r, n) = 2^{n-r-1}|D|$. For $k \geq 0$, let S_k be the set of tilings in S with exactly k boundaries not adjacent to any white square. For example, if $r = 8$ and $n = 4$, the tiling RR*w*RRRR*ww*R*w*R has $k = 6$. Similarly, we may define respective subsets T_k and D_k of T and D with k non-white boundaries. Since D_k differs from S_k in that we may choose whether or not to place red dividers at any of the non-white boundaries, $|D_k| = 2^k |S_k|$. Note that if $r = 0$ or all red squares are surrounded on both sides by white squares, we have $k = 0$. Just as j, the number of red dividers, is bounded by r, so is k, the number of non-white boundaries where we may put red dividers, bounded by r.

Next, let v = the number of boundaries between adjacent white squares. Then v is a function of k. Given some fixed k, we may turn a tiling in S_k into a tiling in T_k by deciding at each white/white boundary whether or not to join the 2 squares together to make a longer tile. For instance, the tiling RR*w*RRRR*ww*R*w*R in S_6 has $v = 1$ so creates 2 tilings in T_6: one with all squares as shown and the other with a white domino in place of *ww*. So $|T_k| = 2^v |S_k|$.

In S_k, there are $n + r$ tiles and a total of $n + r + 1$ boundaries. We have four types of boundaries as we read a tiling in S_k from left to right: red/red, red/white, white/red, and white/white. The "walls" before cell 1 and after cell $n + r$ are colored red. After the left wall and after each of the r red squares, there is either a red/red or a red/white boundary. Since k of these are red/red, that leaves $r + 1 - k$ red/white boundaries. Similarly, before each of the r red squares and before the right wall, there is either a red/red or a white/red boundary. Since k of these are red/red, that leaves $r + 1 - k$ white/red boundaries. Summing all four types of boundaries, we have $k + 2(r + 1 - k) + v = n + r + 1$. Solving for v, we get $v = n - r - 1 + k$. By substitution, $|T_k| = 2^{n-r-1+k}|S_k| = 2^{n-r-1+k}(|D_k| / 2^k) = 2^{n-r-1}|D_k|$. We want $|T|$, the total number of two-toned (r, n) - tilings. This is just the sum over all nonnegative k values of two-toned (r, n) - tilings with k red/red boundaries: $a(r, n) = |T| = \sum_{k \geq 0} |T_k| = \sum_{k \geq 0} 2^{n-r-1}|D_k| = 2^{n-r-1} \sum_{k \geq 0} |D_k| = 2^{n-r-1}|D|$. ∎

IV. Generalizations

Next, we restrict ourselves to two-toned tilings that end with at least s white tiles. These so-called (r, n, s) - tilings of length $r + n + s$ have r red squares and white tiles of total length

$n + s$. So s has a dual purpose, contributing both to the total length of white tiles and also specifying the minimum number of white tiles with which the tiling ends. For example, consider the possible (1, 2, 1) - tilings:

R111, R12, R21, R3, 1R11, 1R2, 2R1, and 11R1.

So $a(1, 2, 1) = 8$. Note that when $s = 0$, the number of (r, n, s) - tilings is equivalent to the number of (r, n) - tilings; i.e., $a(r, n, 0) = a(r, n)$.

Identity 6: For $r, n \geq 0$ and $s \geq 1$,

$$a(r, n, s) = \sum_{j=0}^{n} a(r, j, s-1).$$

Proof: The left side is the number of (r, n, s) - tilings. Let j be the length of the last tile (white of positive length since $s \geq 1$). Now $n + s$ is the total length of white tiles. Since our tiling ends with at least s white tiles, there are at least $s - 1$ cells used in the other "ending" white tiles, so j has maximum length $n + s - (s-1) = n + 1$. If we remove the final tile of length j, the preceding $r + n + s - j$ cells may be tiled in $a(r, n+1-j, s-1)$ ways. So $\sum_{j=1}^{n+1} a(r, n+1-j, s-1) = a(r, n, s)$. Now when $j = 1$, $n+1-j = n$, and when $j = n + 1$, $n+1-j = 0$, so we may replace the second argument $n+1-j$ with j and index from 0 to n: $a(r, n, s) = \sum_{j=0}^{n} a(r, j, s-1)$. ∎

For example, to calculate $a(1, 2, 2)$, we need to find some initial values. Note that the only (1, 0, 1) - tiling is R1, so $a(1, 0, 1) = 1$; R11, R2 and 1R1 are the only (1, 1, 1) - tilings, so $a(1, 1, 1) = 3$. Using Identity 6 and our previous work, we get $a(1, 2, 2) = a(1, 0, 1) + a(1, 1, 1) + a(1, 2, 1) = 1 + 3 + 8 = 12$. We verify this result by listing all (1, 2, 2) - tilings:

R1111, R112, R121, R211, R22, R13, R31, 1R111, 1R12, 1R21, 2R11, and 11R11.

Identity 7: For $r \geq 1$ and $n, s \geq 0$,

$$a(r, n, s) = \sum_{j=0}^{n} a(r-1, n-j, s+j)$$

Proof: The left side is the number of (r, n, s) - tilings. For $0 \leq j \leq n$, suppose $s + j$ gives the exact number of white tiles with which any two-toned tiling of length $r + n + s$ ends. Then those $s + j$ white tiles are immediately preceded by a red square which, if removed, leaves a tiling of length $(r-1)+n+s$ that ends with at least $s+j$ white tiles. We may count the number of such tilings by $a(r-1,\ n-j,\ s+j)$. Summing over all possible values of j yields the identity. ▌

Identity 8: For $r, s \geq 0$ and $n \geq 1$,

$$a(r, n, s) = \sum_{j=0}^{n} \binom{n+s-1}{j+s-1}\binom{r+j}{r}.$$

Proof: The left side is the number of (r, n, s) - tilings. We may construct such a tiling by starting with exactly $j+s$ white tiles, where j may vary from 0 (since all white tiles may be at the end) to n (since $n+s$ is the total length of white tiles, and each could be a square). The total length of white tiles is $n+s$ cells, and the last cell must end a tile, so we may choose the $j+s-1$ other cells where white tiles end in $\binom{n+s-1}{j+s-1}$ ways. Now we place the red squares. Because the tiling must end with at least s white tiles, there are $j+1$ locations in which we may put the r red squares, so $\left(\!\binom{j+1}{r}\!\right) = \binom{r+j+1-1}{r} = \binom{r+j}{r}$ ways to do this. Applying the product principle and summing over all values of j yields the identity. ▌

Consider the case in which r and s are equal.

Identity 9: For $r \geq 0$ and $n \geq 1$,

$$a(r, n, r) = \binom{n+r}{r}\frac{n+2r}{n+r}2^{n-1}.$$

Proof: First, we find an equivalent expression for part of the right side using the distributive property and the definition of binomial coefficients: $\binom{n+r}{r}\frac{n+2r}{n+r} = \binom{n+r}{r}1 + \binom{n+r}{r}\frac{r}{n+r}$ $= \binom{n+r}{r} + \frac{(n+r)!}{r!\,(n+r-r)!}\,\frac{r}{n+r} = \binom{n+r}{r} + \frac{(n+r-1)!}{(r-1)!\,n!} = \binom{n+r}{r} + \binom{n+r-1}{r-1}$. So now we will prove the equivalent equation $a(r, n, r) = \left[\binom{n+r}{r} + \binom{n+r-1}{r-1}\right]2^{n-1}$.

The left side counts two-toned tilings of length $n+2r$ that have exactly r red squares and end with at least r white tiles. We transform this type of tiling into another by replacing each of the r red squares one by one, from left to right, with the final r white tiles, and then coloring these pink. Since all r red squares are replaced, this new tiling is of length $n+r$ and consists of white and pink tiles, exactly r of which are pink. Because we may undo these steps to get back the original two-toned tiling, these white and pink tilings are also counted by $a(r, n, r)$.

Next, we show another way to count the white and pink tilings just described. Let X, an r-subset of $\{1, 2, \ldots, n+r\}$, name the r cells in which pink tiles begin. We may choose this subset X in $\binom{n+r}{r}$ ways. Consider whether the very first tile is pink. Case 1: If the first of r pink tiles begins in cell 1, then 1 belongs to X and there are $\binom{n+r-1}{r-1}$ ways to choose the remaining elements of X. We may decide whether each of the remaining n cells begins a white tile in 2^n ways. Case 2: If the first pink tile does not begin in cell 1, there are $\binom{n+r}{r} - \binom{n+r-1}{r-1}$ ways to choose X and so designate the r cells in which pink tiles begin. Cell 1 must begin a white tile, so there remain $n-1$ other cells which may or may not begin a white tile in 2^{n-1} ways. Summing these two cases gives us $\binom{n+r-1}{r-1} 2^n + [\binom{n+r}{r} - \binom{n+r-1}{r-1}] 2^{n-1}$, from which we factor out 2^{n-1}: $[\binom{n+r-1}{r-1} 2 + \binom{n+r}{r} - \binom{n+r-1}{r-1}] 2^{n-1} = [\binom{n+r}{r} + \binom{n+r-1}{r-1}] 2^{n-1}$. ∎

We now introduce the k^{th} order Fibonacci numbers. Given initial conditions $F_n^{(k)} = 0$ for $n \le 0$ and $F_1^{(k)} = 1$, we define the n^{th} term of the k^{th} order Fibonacci sequence recursively for $n \ge 2$:

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \ldots + F_{n-k}^{(k)}.$$

To illustrate,

$$\begin{aligned} F_2^{(3)} &= F_1^{(3)} + F_0^{(3)} + F_{-1}^{(3)} = 1 + 0 + 0 = 1; \\ F_3^{(3)} &= F_2^{(3)} + F_1^{(3)} + F_0^{(3)} = 1 + 1 + 0 = 2; \\ F_4^{(3)} &= F_3^{(3)} + F_2^{(3)} + F_1^{(3)} = 2 + 1 + 1 = 4; \\ F_5^{(3)} &= F_4^{(3)} + F_3^{(3)} + F_2^{(3)} = 4 + 2 + 1 = 7; \text{ etc.} \end{aligned}$$

Thus the 3^{rd} order Fibonacci sequence begins 1, 1, 2, 4, 7, …. Note that the 2^{nd} order Fibonacci sequence, in which each term is the sum of the previous two, is simply the familiar Fibonacci sequence 1, 1, 2, 3, 5, …. The next identity relates k^{th} order Fibonacci numbers and two-toned tilings.

Identity 10: For $n, k \ge 1$,

$$F_{n+1}^{(k)} = \sum_{r=0}^{\lfloor \frac{n}{k+1} \rfloor} (-1)^r a(r, n-r(k+1), r).$$

Proof: Fix $k \ge 1$ and let $t_n^{(k)}$ be the number of tilings of an n-strip using only white tiles of length k or less. Condition on the length of the last tile. We see that $t_n^{(k)} = t_{n-1}^{(k)} + t_{n-2}^{(k)} + \ldots + t_{n-k}^{(k)}$. This recurrence matches that of the k^{th} order Fibonacci numbers. Considering the initial conditions, if $k \ge 1$ there is one way to tile a strip of zero length and one way to tile a 1-strip, i.e., $t_0^{(k)} = 1 = F_1^{(k)}$ and $t_1^{(k)} = 1 = F_2^{(k)}$. Thus $t_n^{(k)} = F_{n+1}^{(k)}$.

Now we interpret the right side of the identity. We begin with all two-toned tilings of length $n - rk + r$ that have exactly r red squares and end with at least r white tiles. These are counted by $a(r, n-r(k+1), r)$. Given such a tiling, if we lengthen each of its last r tiles by k, then replace its r red squares with the elongated white tiles one by one from left to right, the tiling is longer by r copies of k and shorter by r, so its length is now $n - rk + r + rk - r = n$. This white tiling of length n is guaranteed to have tiles longer than k in the r positions that originally had red squares. To count white tilings of length n with tiles of maximum length k, we apply the inclusion/exclusion principle. The greatest allowable number of "too long" tiles (length $k+1$ or more) is $\lfloor \frac{n}{k+1} \rfloor$ since this would give us a total length of $(k+1)\lfloor \frac{n}{k+1} \rfloor \leq n$. So r varies from 0 to $\lfloor \frac{n}{k+1} \rfloor$. When $r = 0$, $(-1)^0 a(0, n-0(k+1), 0) = a(0, n, 0) = a(0, n)$, which counts tilings of an n-strip using white tiles of any length up to n. By Identity 1, $a(0, n) = 2^{n-1}$. From this total number of white tilings using tiles of lengths up to n, we subtract the white tilings where $r = 1$, meaning those with at least one tile longer than k. Then we add back all white tilings where $r = 2$, meaning those with at least two tiles longer than k. We continue to alternately subtract and add in this manner, giving the sum on the right side of the identity. ▮

Corollary: For $n, k \geq 1$,

$$F^{(k)}_{n+1} = \sum_{r=0}^{\lfloor \frac{n}{k+1} \rfloor} (-1)^r \binom{n-rk}{r} \frac{n-rk+r}{n-rk} 2^{n-rk-r-1}.$$

Proof: Start with Identity 10 and use Identity 9 to transform the right side:

$$F^{(k)}_{n+1} = \sum_{r=0}^{\lfloor \frac{n}{k+1} \rfloor} (-1)^r a(r, n-r(k+1), r) = \sum_{r=0}^{\lfloor \frac{n}{k+1} \rfloor} (-1)^r \binom{n-r(k+1)+r}{r} \frac{n-r(k+1)+2r}{n-r(k+1)+r} 2^{n-r(k+1)-1}$$

$$= \sum_{r=0}^{\lfloor \frac{n}{k+1} \rfloor} (-1)^r \binom{n-rk}{r} \frac{n-rk+r}{n-rk} 2^{n-rk-r-1}. \quad ▮$$

V. Applications to Compositions

A **composition** of n is an ordered list of positive integers that sum to n. For example, the compositions of 4 are

1111, 112, 121, 211, 22, 13, 31, and 4;

there are eight of them. A composition of n corresponds to an uncolored tiling of an n-strip with tiles of length 1 to n, where each summand is represented as a tile of positive integer length. Let $L(k, n)$ be defined as the number of compositions of n in which at least 1 copy of the summand k appears. Similarly, define $L_p(k, n)$ as the number of compositions of n in which at least p copies of the summand k appear. For example, from the list above, we see that $L(2, 4) = 4$ and $L_2(2, 4) = 1$.

Identity 11: For $n, k \geq 1$, $L(k, n) = \sum_{j \geq 1} (-1)^{j-1} a(j, n-jk)$.

Proof: Now $a(j, n-jk)$ counts two-toned tilings of length $j + n - jk$ with exactly j red squares. If we replace each red square with a pink tile of length k, the tiling shrinks by j and grows by jk so is of length n. This corresponds to a composition of n with j or more instances of the summand k, j of which are written in pink. For example, $a(1, n-k)$ stands for all compositions of n where k is the first summand (and possibly others), or k is the second summand (and possible others), and so on. However, this is an overcount since a composition with more than one summand k will be counted in each of the positions at which k occurs. To correct this, we use the inclusion-exclusion principle. Since j denotes the minimum number of occurrences of the summand k, j has a value of at least 1. Note that once $j > \lfloor \frac{n}{k} \rfloor$ then $n-jk < 0$ and $a(j, n-jk) = 0$, so the sum is finite. ∎

Identity 12: For $p, n, k \geq 1$, $L_p(k, n) = \sum_{j \geq p} (-1)^{j-p} \binom{j-1}{p-1} a(j, n-jk)$.

Proof: By modifying the inclusion-exclusion principle somewhat, as shown in *Proofs that really count* [2], we may count the number of ways a property occurs at least p times. Indexing over values of $j \geq p$, we multiply each unsigned summand of Identity 11 by $(-1)^{j-p} \binom{j-1}{p-1}$ to get the more generalized formula in Identity 12. Notice that when $p = 1$, this reduces to Identity 11. ∎

Let $E_p(k, n)$ be defined as the number of compositions of n with exactly p copies of the summand k. For example, the compositions of 4 with exactly 2 copies of the summand 1 are 112, 121, and 211, so $E_2(1, 4) = 3$.

Identity 13: For $p, n, k \geq 1$, $E_p(k, n) = \sum_{j \geq p} (-1)^{j-p} \binom{j}{p} a(j, n-jk)$.

Proof: If we want the compositions of n in which the summand k appears exactly p times, we may start with the compositions of n in which the summand k appears at least p times and subtract the compositions of n in which the summand k appears at least $p + 1$ times. That is,

$$
\begin{aligned}
E_p(k, n) &= L_p(k, n) - L_{p+1}(k, n) \\
&= \sum_{j \geq p} (-1)^{j-p} \binom{j-1}{p-1} a(j, n-jk) - \sum_{j \geq p+1} (-1)^{j-p-1} \binom{j-1}{p} a(j, n-jk) && \text{by Identity 12} \\
&= \sum_{j \geq p} (-1)^{j-p} \binom{j-1}{p-1} a(j, n-jk) + \sum_{j \geq p+1} (-1)^{j-p} \binom{j-1}{p} a(j, n-jk) && \text{by exponent rules} \\
&= \sum_{j \geq p} (-1)^{j-p} \binom{j-1}{p-1} a(j, n-jk) + \sum_{j \geq p} (-1)^{j-p} \binom{j-1}{p} a(j, n-jk) && \text{since } \binom{p-1}{p} = 0
\end{aligned}
$$

$$= \sum_{j \geq p} (-1)^{j-p} \left[\binom{j-1}{p-1} + \binom{j-1}{p} \right] a(j, n-jk) \qquad \text{by the distributive property}$$

But $\binom{j-1}{p-1} + \binom{j-1}{p} = \frac{(j-1)!}{(p-1)!\,(j-p)!} + \frac{(j-1)!}{p!\,(j-p-1)!} = \frac{p\,(j-1)!}{p\,(p-1)!\,(j-p)!} + \frac{(j-p)\,(j-1)!}{(j-p)\,p!\,(j-p-1)!} = \frac{j\,(j-1)!}{p!\,(j-p)!} = \binom{j}{p}$,

which is known as Pascal's identity [4]. This yields $E_p(k, n) = \sum_{j \geq p} (-1)^{j-p} \binom{j}{p} a(j, n-jk)$. ▮

VI. Conclusion

In this paper, we used a recurrence relation to build a table for the number of (r, n) - tilings and showed how diagonal rows of this table summed to odd Fibonacci numbers. We explored combinatorial identities based on how the two-toned tilings were constructed − for example, conditioning on the number of white tiles, the number of locations of white tiles, and the number of white tiles at the end of a tiling. We related two-toned tilings to generalized Fibonacci numbers and, finally, to integer compositions. We noted that compositions of n correspond to $(0, n)$ - tilings, and we explored formulas for counting the number of compositions of n with at least or exactly a certain number of copies of the summand k.

Compositions of integers may be used in other fields besides mathematics for modeling sequences that are subject to certain constraints. For instance, a geneticist might use integer compositions to model DNA sequences with a particular mutation. A computer programmer may seek the number of binary sequences of a certain length with ones in some minimum number of places. A scheduler in production may want to determine the number of ways to split up the hours of the workweek into shifts of certain lengths, and how to distribute workers' breaks so production progresses smoothly. Thus we may begin to imagine how the final three identities might have practical application, beyond stimulating our intellect.

VII. References

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