The Gossip Problem

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Introduction

The motivation for this paper came from a basic idea. Math at secondary and lower college levels is mostly continuous mathematics, and the situations involved seem to be developed by -- and directed towards -- adult engineers. I feel like there is a void to fill with discrete mathematics.

This is not to say we need to throw the entire k12 math curriculum on its head. Students need to understand functions, and linear and parabolic functions pave the way for later more challenging ones. But I remember being a student who thought math was boring. If I hadn't just stayed in math class because I was good at the subject, I would never have discovered all the other things that we can investigate using mathematics.

One of the first questions I became familiar with as a math teacher was the classic (and dreaded) one: "When am I ever going to use this?" As I have developed, however, I have gone through several iterations of answers. When I started, I was impressed with my ability to connect our current mathematics to later classes and uses. As I gained more experience, though, I realized that I was not so much giving helpful information as unintentionally intimidating the students into silence by using a lot of language they didn't understand. But really this is not a question we should have to answer. We don't normally think of students in art, or band, or P.E. asking when they are ever going to need to know how to draw, play the drums, or just play. These things are intrinsically rewarding. I think that by giving students some recreational yet useful discrete mathematics at an early age, we can start to show them why learning math is worthwhile.

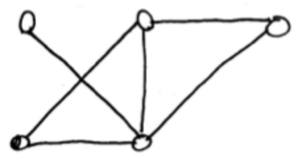
The issue here is that the math curriculum is currently designed around goals that people between the ages of 20-40 want it to have. And again, largely this makes sense since we want students to get a foundation upon which we can later teach them engineering, science, programming, or higher level math. But we also need to make math relevant for their current lives. Using iPhones in our word problems is a superficial fix. We need to think about what captures students' attention.

One of the things that I feel captures students' interest is gossip. They all want to know, and they all want to be the one who knows. Finding ways to bring mathematics into this natural interest could engage students and make them see mathematics in a slightly different light. While finding out how many conversations must take place for every person in a group to know a scandal isn't necessarily much more practical, it is relatable and different from much of the math they are normally exposed to. Much of the math looks more obviously like puzzles they are used to solving for fun than the math they are required to learn.

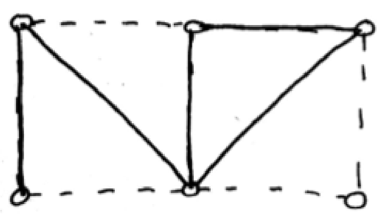
In this 501 project, I investigated a theorem from Graph Theory known as the Gossip Problem. The statement is simple and yet the proof is highly non-trivial. This exploration was a challenge for me and gave me plenty of opportunities to try and recall many of the basic concepts of the subject. Inspired by what I learned, I developed some activities to try and bring some of the fun ideas into a form that is accessible to pre-calculus students. In this paper I will describe both the exploration and the curriculum items in detail.

Terminology

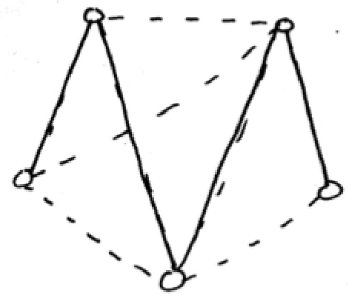
A **graph** G is made up of a finite set, V, of **vertices** and a collection of 2-element subsets of V called **edges**. In the drawing below, the 5 points are the vertices and the 6 lines connecting them are the edges.



We say two vertices are **adjacent** when they are joined by an edge. A **subgraph** of a graph G is a graph whose vertices and edges all belong to G. In the figure below, the 6 vertices and 5 bold edges form a subgraph of the entire 9-edge graph.



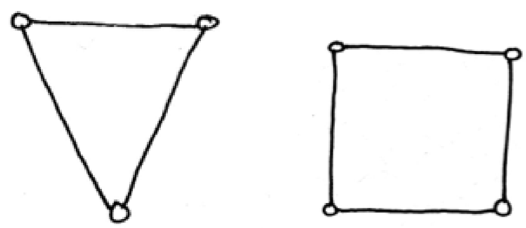
A **path** is a graph whose vertices can be listed in such a way that vertices are adjacent if and only if they are consecutive on the list. A path is understood to have no vertices or edges that are repeated.



The solid line denotes a path in the graph G.

We are particularly interested in situations where paths appear as subgraphs of other graphs, as indicated in the figure above.

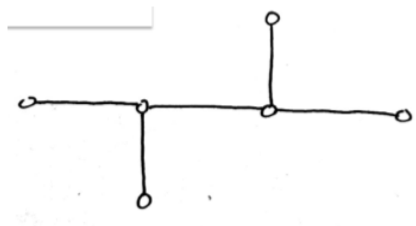
A **cycle** is a closed path. In other words, a cycle is a path whose endpoints are the same. A **k-cycle** is a cycle with k vertices (and k edges).



3-cycle 4-cycle

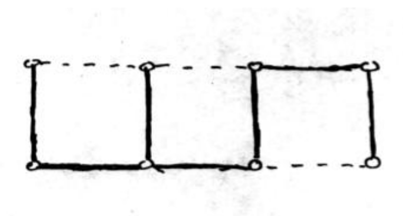
A **walk** is an alternating list of vertices and edges, where an edge comes between two vertices iff they are its endpoints. A walk is more general than a path, in that it is allowed to repeat vertices and edges. A graph is **connected** if every pair of vertices is joined by a walk.

A **tree** is a connected graph with no cycles. (A graph with no cycles is said to be acyclic.)



A tree T.

A **spanning tree** for a graph G is any tree that is subgraph of G containing every vertex.

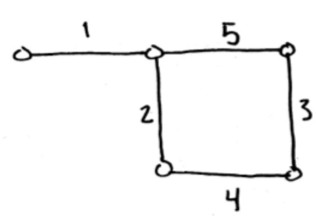


A spanning tree T on a graph G

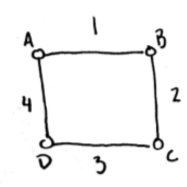
Introduction to the Gossip Problem

Imagine we are trying to transmit information throughout a group of people. We assume that everyone knows one part of a story, and the goal is for everyone to know the entire story. In order to represent this situation, we can let people be vertices in a graph and let edges between them be calls that they make. For the basic version of the problem, we assume that a call only goes between two people (no conference calls). After a call is made, both parties know all the information from the other caller. The crux of the gossip problem, also known as the telephone problem, is this: How many calls must be made for n people to allow everyone to know the entire story?

In order to describe the sequence in which the calls should be made, we introduce the corresponding notion for graphs. An **ordered graph** is any graph with a numbering on its edges.

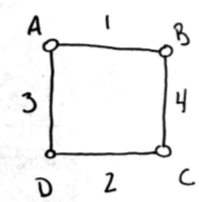


The edges between vertices are numbered to represent the order that the calls take place. This is because the order of calls matters. For example, consider the 4-cycle numbered as indicated below. Notice that there is no increasing path between two vertices D to B for this numbering:



This is important because it means that B never hears D's information. As information moves from D, D's call with A is A's last call, and D's call with C is C's last call, so there is never a call to B telling D's information.

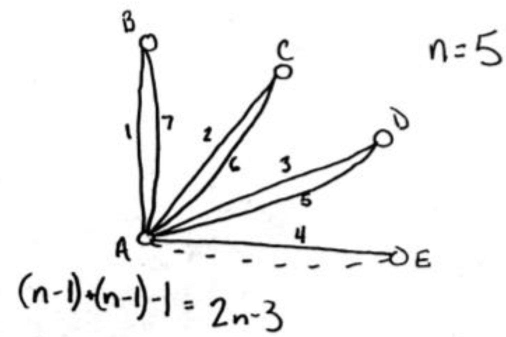
The following numbering does much better:



Here we can see that information can now pass from any person to any other person. In contrast to the previous example, we can see that C's call to B happens after her call with D, so B does hear D's information.

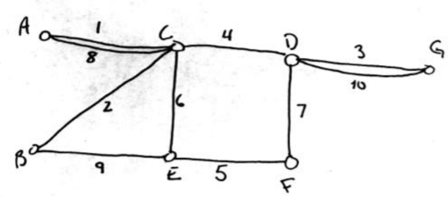
We can generalize this intuition of whether information can travel between any two people into the definition of a desirable type of ordered graph known as a gossip scheme. A **gossip scheme** is an ordered graph where there is an increasing path from any vertex to any other. Our first numbering of the 4-cycle above is not a gossip scheme, while the second attempt is.

One method (that we can be certain always works) to create a gossip scheme for n people is called the "busy body" method. In this construction, we simply have every person call one person, a "busy body", and then have that person call everyone back. This results in 2n-3 calls. There are n-1 calls in, and n-1-1 calls back out, since the last call in doesn't need to be called back. The following is an example for n=5:



So we have a scheme that will always give us 2n-3 calls for n people. But is this number of calls the fewest possible for n people? Is there no gossip scheme with fewer edges?

It turns out we can get further savings if we have every person call one of four people, then have those people share their information with each other and finally call everyone back, as so:



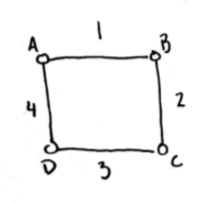
The key here is that this particular ordering on a four cycle is highly efficient:

In fact it is the most efficient ordering we can get. The main theorem of this paper establishes precisely this claim. Indeed, in the next section, we prove that 2n-4 is the lower bound on edges for any gossip scheme with n vertices. The basic idea is that if we have a gossip scheme with fewer calls, then we would have to have better savings on some larger n. So if we could have a scheme with 5 calls for 5 people, or 7 calls for 6 people, then we could have a better scheme. It turns out this is impossible.

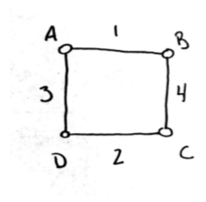
Proof of the Main Result

In this section, we will see why the gossip scheme described earlier cannot be improved. We begin with a relevant definition.

The **NOHO property** stands for **N**o **O**ne **H**ears their **O**wn information. In a graph this means no one's information is propagated back to him or her. We formalize this by saying there is no increasing closed walk in our ordered graph:



The above graph doesn't satisfy the NOHO property since we have a 1234 path around the loop. A can't hear C or D's info without hearing her own.



This graph does satisfy the NOHO property since there is no increasing path around the loop. No one has to hear their own to hear anyone else's information.

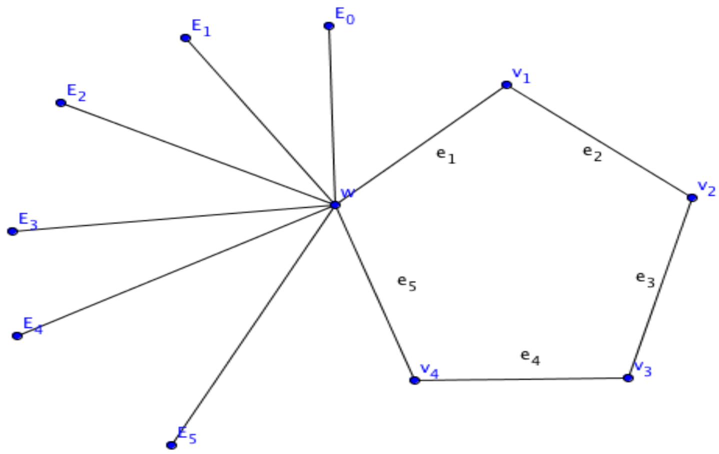
We are now ready to state and prove our main result. This proof builds on the intuition that we cannot get any better savings than 4 calls on a 4-cycle, or our scheme breaks down. The proof is by contradiction, where if we suppose there is a scheme with fewer than 2n-4 edges, we create a contradiction.

Gossip Theorem: For $n \geq 4$, the minimum number of edges in a gossip scheme on n vertices is $2n$-4.

Proof: We will prove by induction on n. Suppose 2n-4 is not the optimal scheme for n vertices. Then we can add calls to the optimal scheme until we end up with a scheme G with 2n-5 calls. For n greater than 4, assume that every gossip scheme with n-1 vertices has at minimum 2n-6 calls.

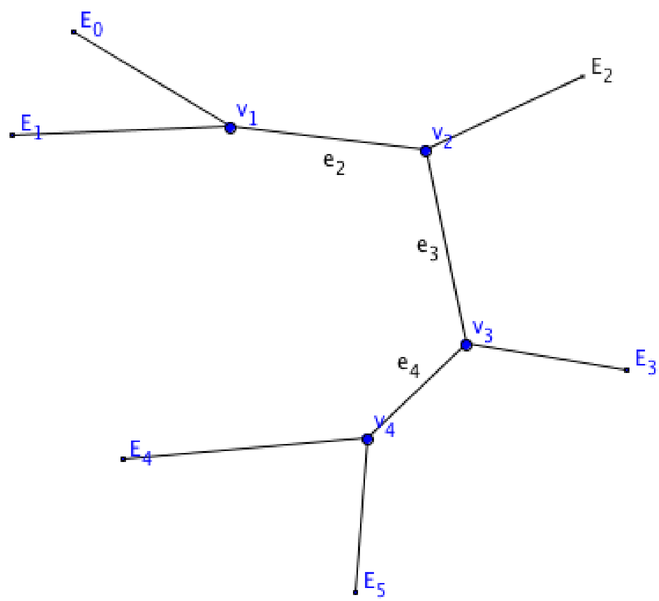
Claim 1. G satisfies the NOHO property:

Otherwise, G has some increasing path from w to v_k along edges $e_1,\ldots,e_k$ followed by an edge e_{k+1} that goes from v_k to w. The following example is for k = 4:



In this case, $E_0, \ldots, E_k$ are portioned edge sets that might happen in between calls on the wv_k-path. So E_0 contains all calls before e_1, E_1 all calls before e_2, and in general for $1 \le i \le k$ E_i is the set of calls before e_i and E_{k+1} is the call after e_k.

We will delete w, e1, and e5, moving all calls $E_0, \ldots, E_5$. Now not all calls might be connected to w, but they will only be affected if they are. So we will assume all calls in the sets $E_0, \ldots, E_5$ connect to w. When we remove w, e1, and e5, for each edge



e $\in$ Ei replace w with vi.

We now have a gossip scheme on $n-1$ vertices, with 2n-7 edges. $2n-7=2(n-1)-5$.

This contradicts our induction hypothesis. Therefore G must satisfy NOHO.

Claim 2: d(x) – 3 calls are useless to u, and hence $\delta(G)\geq 3$.

This is saying that each vertex in our graph has exactly three useful edges for our gossip scheme. So we must have at least 3 edges incident to each vertex.

Let $O(u)$ be the set of all useful calls coming from *u*. This means the set of calls where some vertex is reached for the first time by an increasing path from *u*. Let $I(u)$ be the set of all useful calls coming in to *u*. In other words, *I* is the set of all calls where u is reached for the first time coming from some other vertex. $I(u)$ is $O(u)$ for the reverse ordering on *E(G)*.

These are all the calls that are useful to *u*. So if we want to count up all the useless calls, we take the total calls, subtract all the calls in *O(u),* all the calls in *I(u),* and add back in all the calls in the intersection. Since *G* is a gossip scheme and there is an increasing path from any point to any other, *I(u)* and *O(u)* will each form a spanning tree, so will each have n-1 edges. All that remains is to find the number of edges in the intersection. We will show that this number equals *d(u)*.

$v \in N(u) \Rightarrow uv \in I(u) \cap O(u).$

If there is an uv adjacent to u that is not an element of O(u), then there must be some uv-path that is a part of O(u) that eventually reaches v, and then uv violates NOHO at u. A similar argument works for I(u). If there is an uv adjacent to u that is not an element of I(u), then there must be some vu-path that is a part of I(u) that eventually reaches u, and then uv violates NOHO at v.

$uv \in I(u) \cap O(u) \Rightarrow v \in N(u)$

In the opposite direction, suppose $e \in I(u) \cap O(u)$, but is not in *N(u).* Then an increasing path from x to e and an increasing path to x from e combine to form an increasing closed walk which violates NOHO. Therefore $e \in N(u).$

The calls useless to u are all the calls not in the union of I(u) and O(u), $\overline{O(u) \cup I(u)}$. Start with *2n-5*, the total calls. Take away *n-1*, all the calls in *I(u)*, since *G* is a gossip scheme, *I(u)* will be a spanning tree. Take away *n-1* ,all the calls in *O(u)*. Add back in *d(u)* which is all the calls in the intersection. We have the useless calls = *d(u) – 3*.

$$\left|\overline{O(u) \cup I(u)}\right| = 2n - 5 - (n-1) - (n-1) + d(u)$$

$$= 2n - n - n - 5 + 2 + d(u) = d(u) - 3$$

Claim 3: The subgraph obtained by deleting the first call and last call made by each vertex has a least five components and has no isolated vertex.

Let *uv* be the first call involving u. If the first call involving *v* is *vz* with $z \neq u$, then by definition it occurs before *uv*, and these two calls do not communicate from *u* to *z* (since that path would not be increasing). Since *vz* comes before *uv*, and *uv* is the first call from *u*, an increasing *uz*-path would violate NOHO once it got to *z*. Another way to see this is that a call between to vertices must be the first for both or neither, since if u calls v as its first call, but v's first call was to z, then in order for z to get u's information it must hear its own and violate NOHO. So no two first calls can be adjacent, hence the set of first calls is a matching, F, with $\frac{n}{2}$ elements. Likewise, a call is either the last call for both gossips or for neither. Since after getting a final call from v, u knows everything, so any further call would violate NOHO. So the set of last calls, *L*, is also a matching, also with $\frac{n}{2}$ elements.

Another way to say this is that a call is either the last call for both gossips or for neither. Since after getting a final call from v, u knows everything, so any further call would violate NOHO. Likewise,

Then $G-F-L$ has $2n-5-\frac{n}{2}-\frac{n}{2}=2n-5-n=n-5$ elements. Then $G-F-L$ has $n-(n-5)$ components, or 5 components. It has no isolated vertex, since $\delta(G)\geq 3$ and we removed at most 2 edges from each vertex.

Contradiction: We first show that some vertex has degree exactly three. From claim 2 we know that every vertex has degree at least 3. Now suppose $d(x) \geq 4, \forall x \in V(G)$. Then $|E(G)| = \frac{4 \cdot n}{2} = 2n$. But we know $|E(G)| = 2n - 5 < 2n$ therefore there must be some u such that d(u)=3.

Take the graph G-F-L from claim 3, and let C_1, C_2, and C_3 be the three components of G-F-L that contain u, u's first neighbor, and u's last neighbor, respectively. Information can propagate from u only into C1 and C2, since u's final call must also be the final call for his neighbor, so no further information can be sent from u to C3. So edges of G-F-L belong to O(x) only via paths that start in C1 and C2. Likewise, u gets information only from its middle and last neighbor, since its first call is also the first call for its first neighbor. So any edge in I(u) must start in C1 or C3.

The edges of the remaining components, which we showed numbered at least 2 in claim 3, are useless to u. We know they have edges since G-F-L has no isolated vertex. But from claim two we know that we can only have d(u)-3 = 0 useless edges to u.

QED

Further Questions

With the theorem proved, it is natural to wonder about further questions related to graphs and information spread. How would the solution be changed if instead of two-way information, we only had one-way transmissions? Or perhaps in the age of facebook, Google, and Skype, we think phone calls are an outdated model. How would conference calls or wall posts be modeled?

We can also look at structure. Given that I was originally planning on modeling a school, it seems silly to assume we have everyone talking to everyone. Some people will not know each other or not get along. How does the structure of a representative graph affect our solution? What sorts of structure, if any do we have to have in order to achieve the optimal scheme?

The NOHO property was extremely interesting when I was studying it for the proof. For example, it is impossible to create NOHO schemes on graphs where n is odd (n>3). Which leads to many questions of its own: Why is this? Are there other restrictions on what graphs can have a NOHO property? Why is the NOHO property important in the first place?

Clearly there are many further places to go for extensions. This is just the tip of the iceberg. Some of the questions seem to be extremely practical applications, while others seem to just be interesting mental puzzles. Regardless, there is lots of room for further exploration for anyone so inclined.

Relating the Curriculum to the Exploration

The curriculum I wanted to design was for the high school Algebra 2 level because my original idea was about using exponents to explore the rate of population growth in how rumors spread. However as I got further into a proof, I saw that the mathematics for a deep exploration required much more graph theory. To fully understand the gossip problem theorem students would need one to two classes of graph theory and a solid grasp of proof techniques. So this created the interesting challenge of identifying what key ideas from my exploration could be incorporated into a more basic math class.

I chose an exploration and inquiry approach with the goal of getting students to use graphs to represent gossip, and a general understanding of the optimal solution of a gossip scheme and why it is optimal. The activities I designed ended up being for Math 105. Students at this level can be expected to have a basic understanding of algebraic notation and be able to follow basic algorithms.

In the end I settled on 2 activities to walk students through the workings of gossip schemes, with a 3rd I added as an extension after teaching the lesson. They are outlined in the next section.

Intro to Curriculum

In the following section I will discuss the lessons I taught and outline the plan for a 3^{rd} follow up activity that I came up with after teaching the first two. This section will include an overall goal and rationale for each activity as a whole.

The activities are mostly broken up into tasks. I will show the tasks that the students are to work through, and give a rationale for each task. A part of the rationale will include an explanation of how the expected student work for each task builds toward the lesson goals.. Following this will be example of student work from selected tasks. Finally I will have a reflection on the activity as a whole.

It is worth mentioning here that I taught Activities 1 and 2 as a single lesson. The class was 1 hour and 50 minutes long and I taught for 1 hour and 30 of them. However I thought that the activities would be more useful broken into more manageable chunks. Activity 3 will not have any student work or reflection since it was not actually taught.

Introduction to Activity 1

Instructional Goal: Students will develop conceptual framework for fitting information spread (gossip) into graph notation.

Time needed: 45 min. Three total tasks, with time for class discussion in between.

Prerequisite knowledge: Students should have some familiarity with the basics of graph theory. They should know what edges and vertices are and have basic algebraic understanding that allows them to make sense of expressions like 2n-3.

Supplies: Students only need the task sheets handed out by teacher.

Class Organization: Tasks should be done in groups of 4.

Task 1:

Each of you has the piece of a murder mystery on a card. Your goal is to get everyone the entire story with the least number of "calls".

Rules:

> **Only 2 people can be in on a call.**
>
> **Anytime two people are on a call, they share all the information they have so far.**
>
> **You are done when everyone in your group has the whole story.**

Write down your group's method and draw a diagram that shows how you shared your information (maybe use a graph theory to help).

Task 2:
One way to solve this problem is by making one person in the group be the Busybody.

In this method everyone calls one person in the group. After they have all the information the Busybody calls everyone back.

Use this method to share your murder mystery information and record the process using a diagram.

Will the Busybody method always work as a way to get everyone all the information?

How do you know?

How many calls does the Busybody method take for your group of 4?

What about a group of 20 people?

Task 3:
For a group of n people, the Busybody method takes (n - 1) + (n - 1) - 1 calls.

Draw a diagram to convince yourself that it will always take this many calls.

Task 1 Resource:

The victim was Ms. Peacock.

The murderer was Colonel Mustard.

The weapon was a socket wrench.

The murder happened on the balcony.

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Write down your group’s method and draw a diagram that shows how you shared your information (maybe use a graph theory to help).

This gets students used to the idea of transferring information in a gossip scheme, and most groups should reach the ideal solution of 4 calls. We will build off of this ideal solution later to push students towards the ideal solution of 2n-4 calls.

Task 2:

One way to solve this problem is by making one person in the group be the Busybody.

In this method everyone calls one person in the group. After they have all the information the Busybody calls everyone back.

Use this method to share your murder mystery information and record the process using a diagram.

Students work in groups of 4 to use a specific method to solve a gossip scheme. If they have already found this method for a group of 4 in task 1, then have them use the method to find a number of calls in a group of 5 people. Introduce the task by stating that while having the best scheme is important, having a method that always works is equally important. Students should come up with 5 calls for a group of 4, or 7 calls for a group of 5. The second page of the task sheet asks students to think about generalizing their understanding. At this point they don't need any formal way to prove this method always works. The point is just to get them thinking about justification. Likewise n= 20 is mainly an extension activity for groups that get done quickly.

Will the Busybody method always work as a way to get everyone all the information?

How do you know?

How many calls does the Busybody method take for your group of 4?

What about a group of 20 people?

This gets students used to the idea of how we can transfer information in a gossip scheme, and most groups should reach the ideal solution of 4 calls. We will use this to prove a point later.

Task 3:
For a group of n people, the Busybody method takes (n - 1) + (n - 1) - 1 calls.

Draw a diagram to convince yourself that it will always take this many calls.

For this activity emphasize that students are not just to draw a diagram showing it works for a specific n. They should have a diagram that is convincing to others in their group that we always get 2n-3 calls from this method. They should be able to see n-1, n-1, and another -1 in their diagram.

Student Work for Activity 1

Each of you has the piece of a murder mystery on a card. Your goal is to get everyone the entire story with the least number of "calls".

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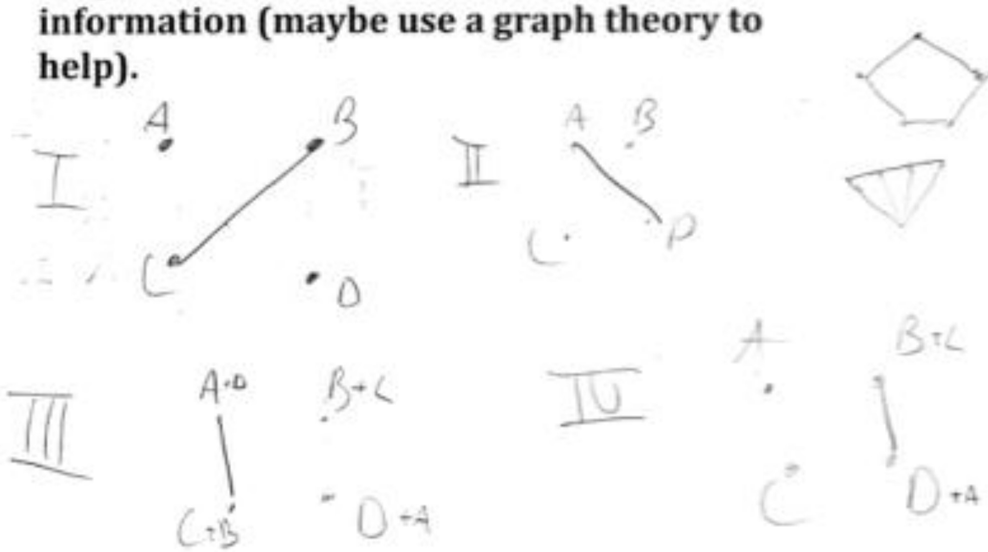
Write down your group's method and draw a diagram that shows how you shared your information (maybe use a graph theory to help).



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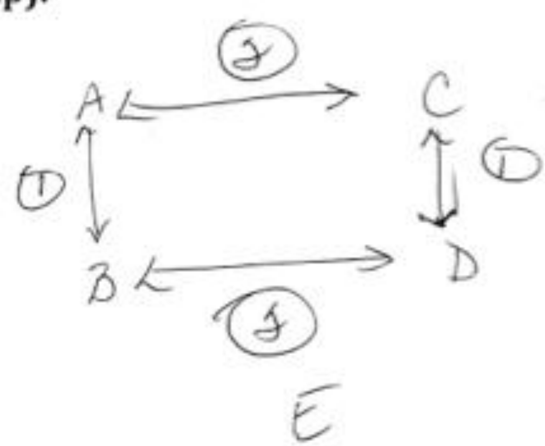

Student Work for Activity 1

Each of you has the piece of a murder mystery on a card. Your goal is to get everyone the entire story with the least number of "calls".

Rules:

> **Only 2 people can be in on a call.**
>
> **Anytime two people are on a call, they share all the information they have so far.**
>
> **You are done when everyone in your group has the whole story.**

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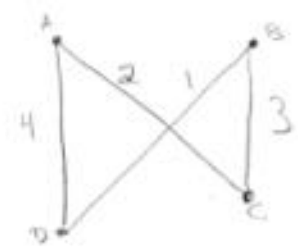

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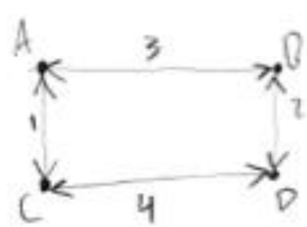


Student Work for Activity 1

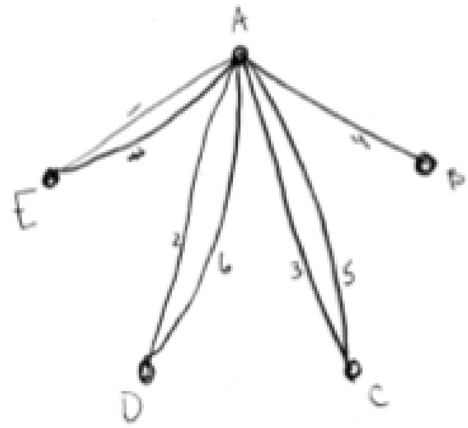
One way to solve this problem is by making one person in the group be the BusyBody.

In this method everyone calls one person in the group. After they have all the information the BusyBody calls everyone back.

Use this method to share your murder mystery information and record the process using a diagram.

5 people

7 calls



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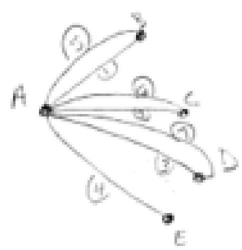
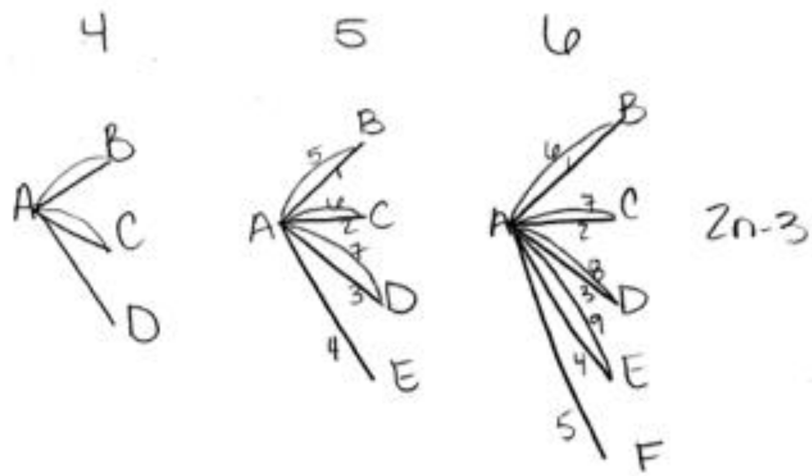
Busy Body



Student Work for Activity 1

For a group of n people, the BusyBody method takes (n - 1) + (n - 1) - 1 calls.

Draw a diagram to convince yourself that it will always take this many calls.

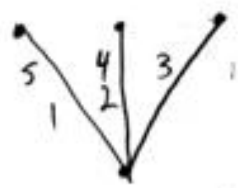


Student Work for Activity 1

For a group of n people, the BusyBody method takes (n - 1) + (n - 1) - 1 calls.

Draw a diagram to convince yourself that it will always take this many calls.

n−1, last pair knows everything



n − 1 , busybody doesn't call self

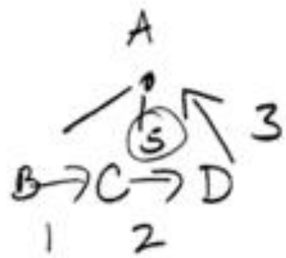
Student Work for Activity 1

For a group of n people, the BusyBody method takes (n - 1) + (n - 1) - 1 calls.

Draw a diagram to convince yourself that it will always take this many calls.

$$(5-1) + (5-1) - 1$$

$$4 + 4 - 1 = 7$$

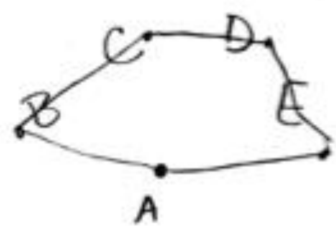


$$(4-1)+(4-1) - 1$$

$$3 + 3 - 1 = 5$$

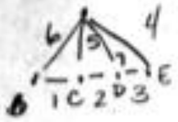
A

B C D E



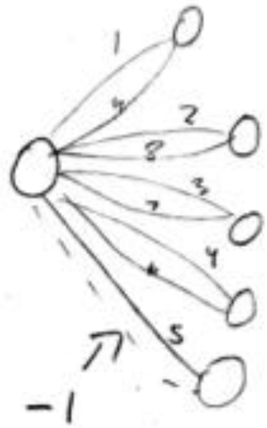
Student Work for Activity 1

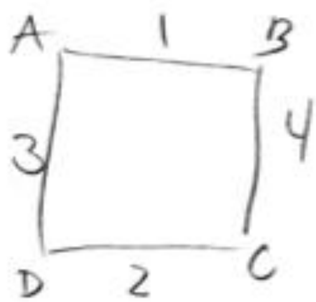
$(n-1) + (n-1) - 1$



5

$2n-3$





calls

		1	2	3 4
A	a	ab	ab	abcd
B	b	ab	ab	
C	c	c	cd	abcd
D	d	a	cd	

Reflection on Activity 1

I taught this lesson in Math 105, an exploratory math class for students who aren't planning on taking any more math in college. This lesson was trying to balance two tensions: the first was being accessible enough to be an interesting investigation. One of my primary motivations was that I wanted to bring something interesting like gossip into a mathematical realm for students. The second was trying to make it mathematically challenging.

I think that there was a lot of engagement for this activity. The students were all having lively discussions about possible solutions. They were eager to get a standard procedure like the Busy Body method that they could check against their own. I was able to have some very interesting discussions with students about how the graph pictures looked like networking diagrams, which was one of the broader exploratory connections I wanted them to make.

I also enjoyed in Task 3 watching students develop an understanding of the distinction of drawing a diagram that held true for my formula of 2n-3, and drawing one that showed why the Busy Body method will always give 2n-3 calls. I love showing students to draw convincing pictures. On the whole I'd say this activity was a rousing success.

Introduction to Activity 2

Instructional Goal: Students will continue to develop a conceptual framework for fitting information spread (gossip) into graph notation. They will gain a method to check if a given graph is a gossip scheme and learn what the optimal scheme is for any group of n people.

Time needed: 45 min. An introductory set of lecture notes and three total tasks, with time for class discussion in between.

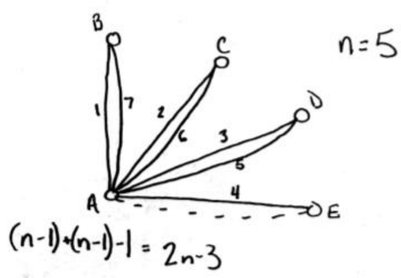
Prerequisite knowledge: Students should have some familiarity with the basics of graph theory. They should know what edges and vertices are and have basic algebraic understanding that allows them to make sense of expressions like 2n-3.

Supplies: Students only need the task sheets handed out by teacher.

Class Organization: Tasks should be done in groups of 4.

Lecture Notes:

We need a way to convince ourselves that a given graph we've drawn is a gossip scheme. We can use a table to make this check. So for example:



In the graph above, we want to be able to show that all of our callers have the whole story. Let the lower case letters in the table below be the information our callers know. Then we can show what information each person knows at any given call. In order to have a gossip scheme we need to have abcde in every cell in the final column.

Caller	Call 0	Call 1	Call 2	Call 3	Call 4	Call 5	Call 6	Call 7
A	a	ab	abc	abcd	abcde	abcde	abcde	abcde
B	b	ab	ab	ab	ab	ab	ab	abcde
C	c	c	abc	abc	abc	abc	abcde	abcde
D	d	d	d	abcd	abcd	abcde	abcde	abcde
E	e	e	e	e	abcde	abcde	abcde	abcde

We do, so we know we have a gossip scheme.

Task 4:
With 4 people, the Busybody method would take $(4 - 1) + (4 - 1) - 1 = 5$ calls to share all the information.

Can you beat that?

Task 5: With 5 people, the Busybody method would take (5 – 1) + (5 – 1) – 1 = 7 calls to share all the information.

Can you beat that? If so, show how. (Hint: Yes. Yes you can.)

Task 6:

Now that you know it is possible, try to share all information with 6 people using 2(6)-4= 8 calls.

Task 4:

With 4 people, the Busybody method would take (4 – 1) + (4 – 1) – 1 = 5 calls to share all the information.

Can you beat that?

This task is short. It is getting students to realize that in the intro activity they found a scheme with one less call that what they need for the Busy Body Method.

Task 5: With 5 people, the Busybody method would take (5 – 1) + (5 – 1) – 1 = 7 calls to share all the information.

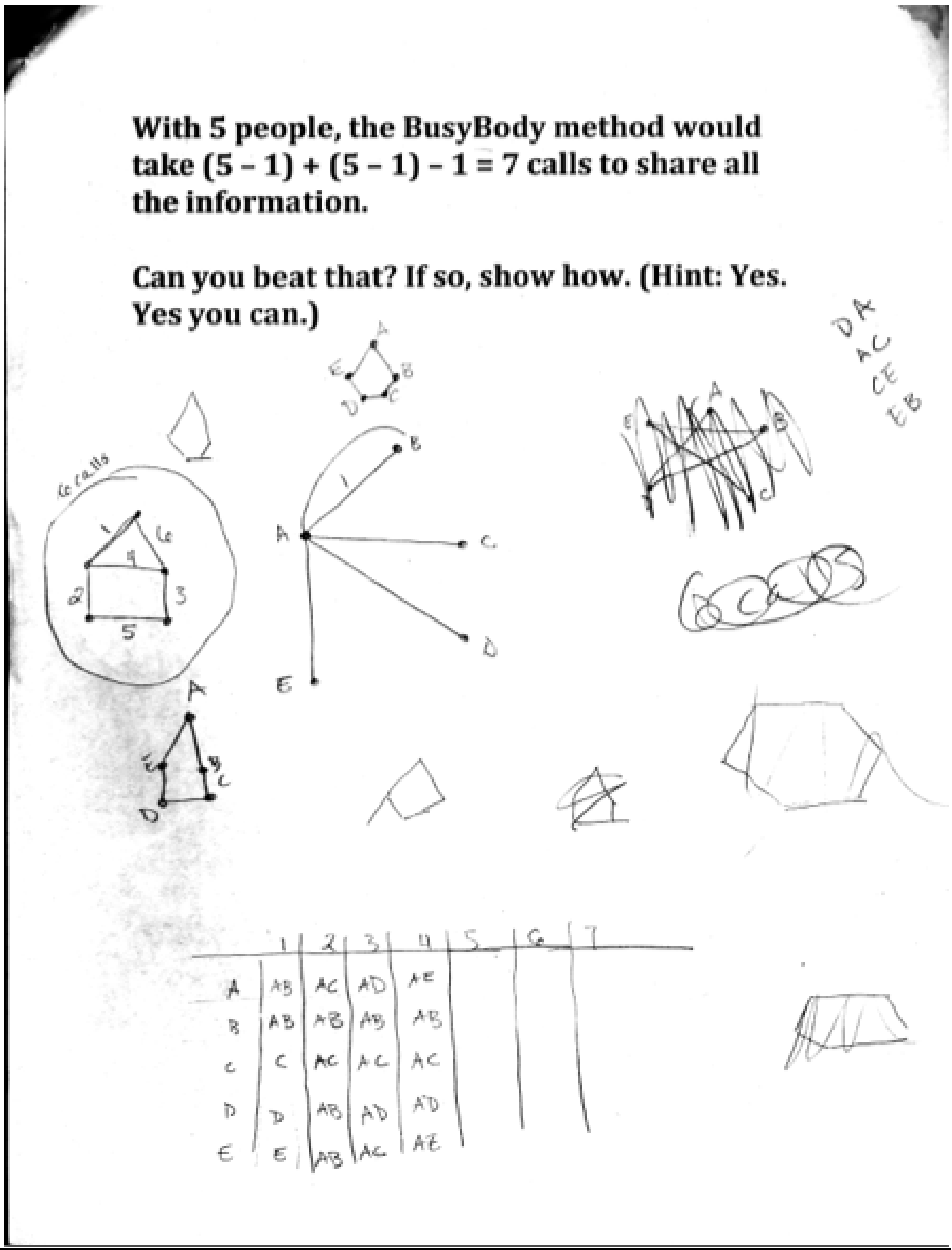
Can you beat that? If so, show how. (Hint: Yes. Yes you can.)

In this task students work with their groups to try to expand the gains given from their 4 call method. If students get stuck ask them if there is any way to use the method they just worked out in task 4. Another hint would be to ask them to see if they can get this 5th persons information into the group of 4 before they do their thing. This is building an intuitive grasp of our optimal scheme. The idea is that after working this out for themselves, they are much better at grasping an explanation of why the optimal scheme works. Likewise, they should be more convinced that there is no better scheme since they couldn't find a scheme with fewer than 6 calls for n=5.

Task 6:
Now that you know it is possible, try to share all information with 6 people using 2(6)-4= 8 calls.

The rationale for this activity is very similar to Task 5. Students are expanding and generalizing their strategy for finding optimum calls. After this we have our closing reveal of how the optimal method works.

Student Work for Activity 2



With 5 people, the BusyBody method would take (5 – 1) + (5 – 1) – 1 = 7 calls to share all the information.

Can you beat that? If so, show how. (Hint: Yes. Yes you can.)

	1	2	3	4	5	6	7
A	AB	AC	AD	AE			
B	AB	AB	AB	AB			
C	C	AC	AC	AC			
D	D	AB	AD	AD			
E	E	AB	AC	AE			

Student Work for Activity 2

With 5 people, the BusyBody method would take (5 – 1) + (5 – 1) – 1 = 7 calls to share all the information.

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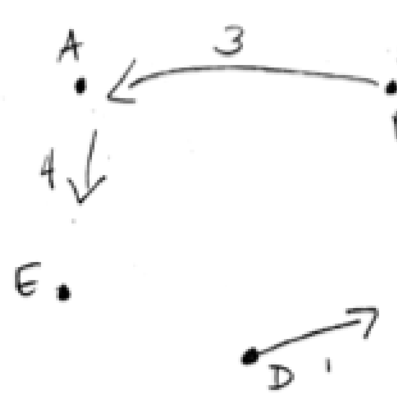
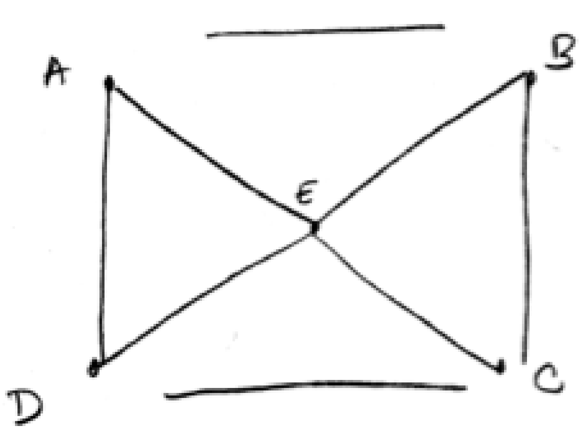
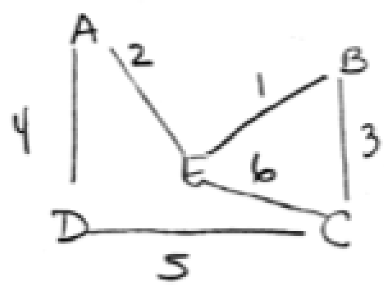


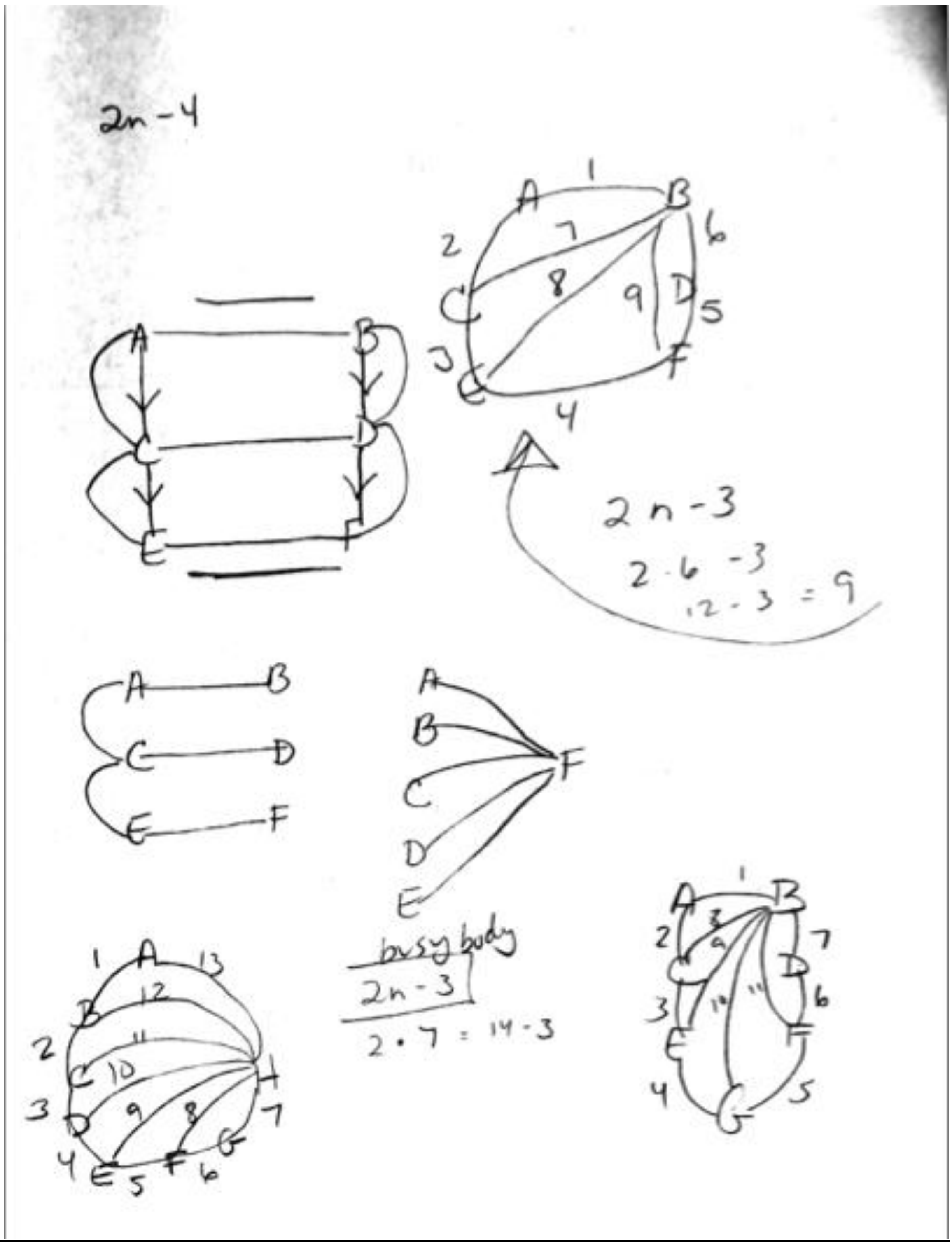
	1	2	3	4	5	6
A		ea				
B	eb	eb	abc bc			
C						
D						
E	eb	ebA				





EB 1
EA 2
BC 3
AD 4
DC 5
CE 6

Student Work for Activity 2



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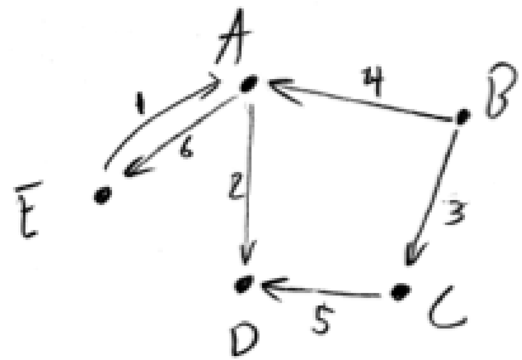


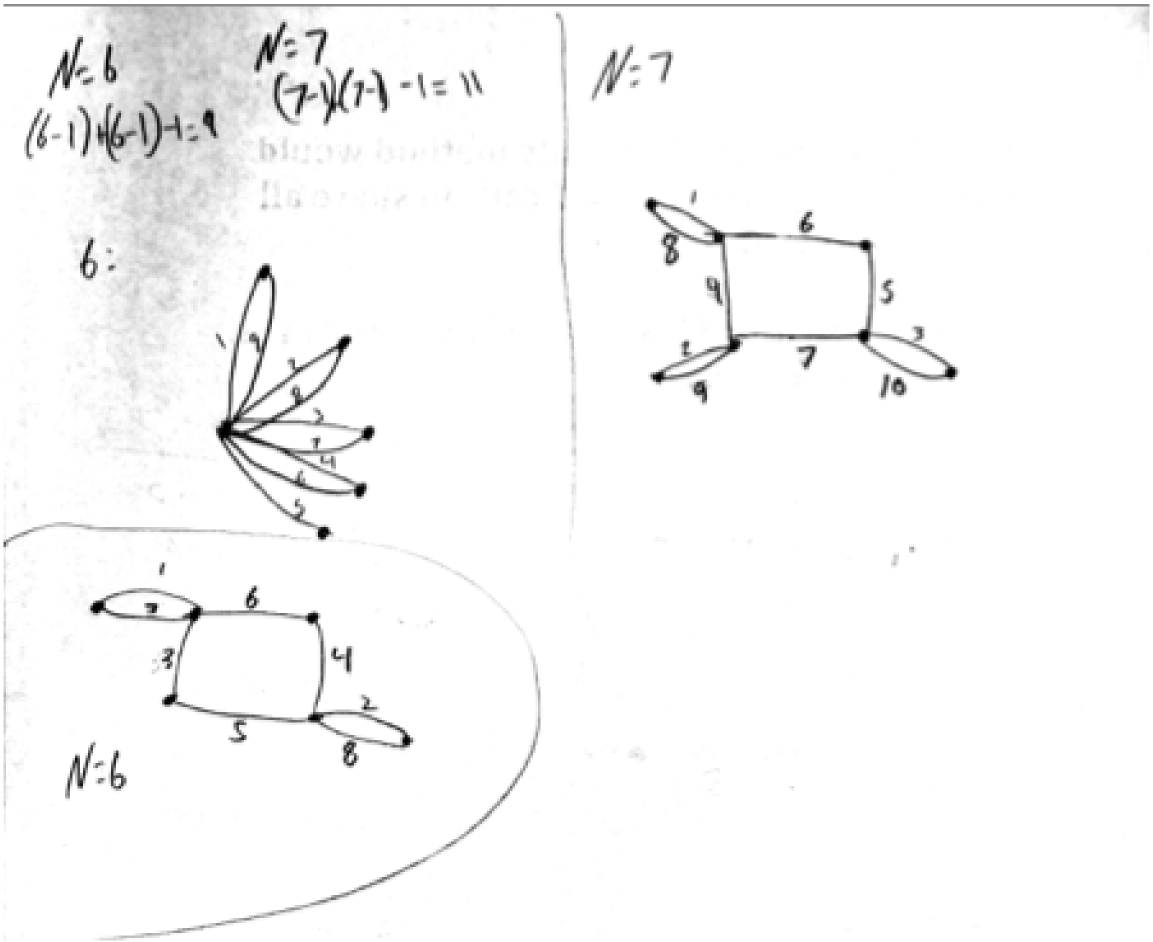
1. $E \to A$ 2. $A \to D$
3. $B \to C$ 4. $B \to A$
5. $A \to E$

1. E:EA A:EA
2. A:EAD D:EAD
3. B:BC C:BC
4. B:EADBC A:EADBC
5. D:EADBC C:EADBC
6. E: EADBC

Student Work for Activity 2

Now that you know it is possible, try to share all information with 6 people using $2(6)-4= 8$ calls.

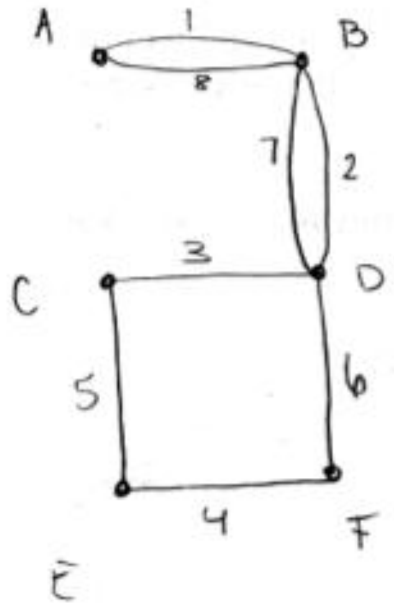
Student Work for Activity 2



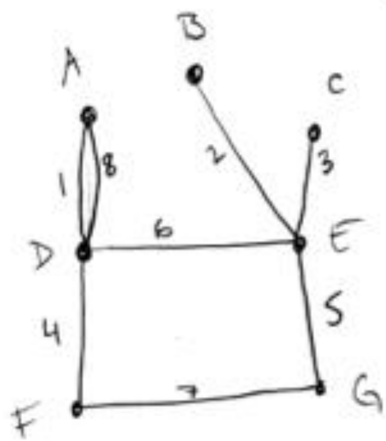
Student Work for Activity 2

$n = 6$

$\boxed{2n-4}$



$n = 7$



Reflection on Activity 2

This lesson was taught as the second half of a single lesson, however I think it fits as well or better as the second day of 50-minute classes. This lesson started off extremely well. We went over the lecture notes to give students a way of checking that they had in fact created a gossip scheme.

We then went on to explore the optimum solution of 2n-4. In the class I taught task 4 was a trivially short activity because students were only being asked to refer back 45 minutes to when they had done the intro activity task 1, and they still had the optimal solution written down. However I would expect it to take slightly longer when taught on a different day. Still I would treat this as a short refresher task.

Most of the lesson was spent on task 5, finding the optimal solution for n=5. Students here had an appropriately challenging time finding ways to find a solution with 6 calls. When students got stuck I would talk about thinking of our existing scheme for 4 people, then having the fifth caller get their information into that scheme. Many students didn't even need that much of a hint, and almost all got it after that much of a hint.

By the time we got to Task 6 enthusiasm flagged. Part of this may just have been a time factor. We had been working for about 80 minutes at that point. However I think that part of the problem was that while I was able to bring students around to an understanding of the optimal scheme, they didn't have enough mathematical background for the result to be impressive in and of itself.

In fact by the time I got to the punch line that 2n-4 was the optimal result, a couple of students were angry at the arbitrary importance of 4 people. They were angry that I didn't have a reason for why calling a group of 4 gave us the best possible solution. I tried to answer their frustration by pointing them to further investigations they could do that would convince them that four people gave optimal savings for a good reason.

On the whole I think the lesson went well. If was going to do something differently I would spend more time practicing the table check of a gossip scheme as I saw students continue to be unsure if they had created a gossip scheme well into the activity.

Introduction to Activity 3

Instructional Goal: Students will relate the visual structure of various graphs and gossip schemes to the optimal solution of 2n-4. Specifically, any graph that is an optimal gossip scheme must contain a 4-cycle. This lesson should hopefully address the anger the two students felt at the end of activity 2 by letting them make sense for themselves just why this structure always pops up.

Time needed: 45 min.

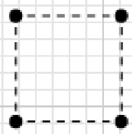
Prerequisite knowledge: Students should have gotten the basic goals from activities 1 and 2.

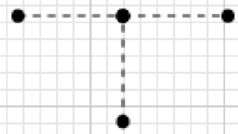
Supplies: Students only need the task sheets handed out by teacher.

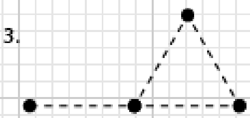
Class Organization: This worksheet has 4 problems and can be completed in class in small groups or given as a follow up homework assignment after activity 2.

Gossip Activity

Find the fewest number of calls for the given group of people.

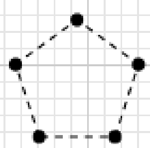
1. 

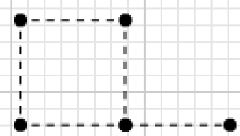
2. 

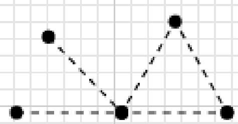
3. 

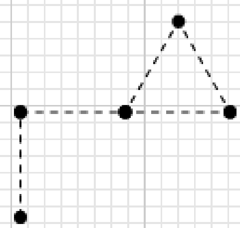
Which arrangement was easiest? Find a way to convince yourself and others that you have found the smallest possible number of calls. For example, if you think 5 is the fewest calls, how do you know?

2. Now find the fewest number of calls for 5 people in the given configurations.

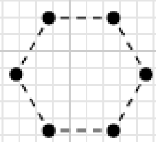


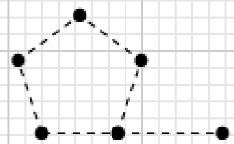


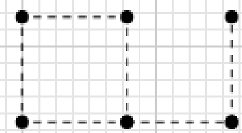


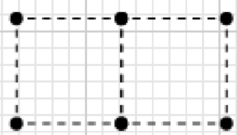


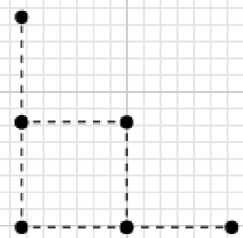
3. Now find the fewest number of calls for 6v people in the given configurations.











4. What shape is always needed in a graph to have the optimum 2n-4 calling scheme? Can you explain why?

Final Reflection

My intention with this project was two bring a relatable discrete math subject to students' attention who had previously mostly been exposed to only a typical continuous curriculum. I think that I met those goals. In my first activity I had students actively engaged in drawing graphs to represent information spread. There were several side conversations with students about how this sort of problem is expanded to cell phone networks or internet protocols. These conversations made me extremely happy as I was hoping to find a problem that was relatable but also brought to light that lots of important mathematics is puzzle solving and drawing and not just algebra.

I also feel that students bought in to the discovery and inquiry. Even in times when I got caught off guard, like with the two angry students at the end of activity 2, I think that this showed tremendous buy in. I thought that the fact that 4 people arbitrarily gives us the best savings is fascinating. We can't really say why. After my exploration the most I can do is prove in detail why it won't work for any larger group of people, but the fundamental problem that was frustrating these students is open to philosophical debate.

Student's left this activity with an ability to relate ordered graphs to information spreading via telephones, and evaluate whether all involved callers had all the information the group started with. This was my main concern as far as skills were concerned so I am satisfied here.

Conceptually I wanted to create a curriculum that mirrored actual mathematics. I know doing the exploration myself it was fascinating to look at this proof that had all these parts that seemed to come from no where and struggle to pry information out of them. Then, as I was trying to pull tasks for students to do and I thought about how we could use common sense to explain that we couldn't get more savings than 2n-4, and I tried to make that common sense more precise, the proof I had just worked through fell out. I don't think that my students saw all the formalism underlying their activities, but I do like to think that they were having fun not realizing how advanced the mathematics they were doing was. This is always my goal as a teacher, so all in all I think this lesson went great.

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