Linear Extensions of LYM Posets

Ewan Kummel

Preliminaries

- A binary relation $\preceq$ on a set P is defined to be a **partial order** on P when $\preceq$ is reflexive, transitive, and antisymmetric.
- We will refer to the pair $(P, \preceq)$ as the **partially ordered set**, or **poset**, P.
- The relation is a **total order** if X and $Y \in P$ implies that $X \preceq Y$ or $Y \preceq X$.
- A map σ from a poset P to a poset Q is **order preserving** if, for each X and $Y \in P$, $X \preceq_P Y$ implies that $\sigma(X) \preceq_Q \sigma(Y)$.
- An order preserving bijection $\varepsilon : P \longrightarrow Q$ is a **linear extension** of P if Q is totally ordered.
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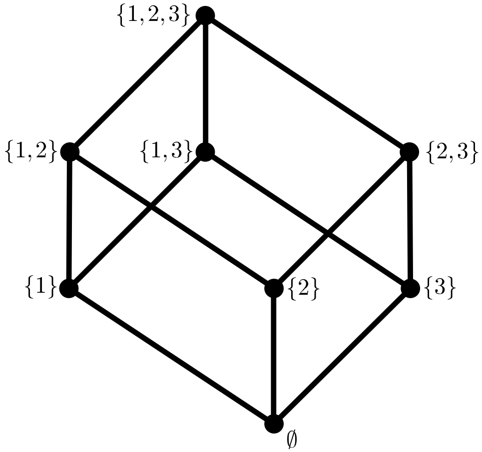
A Linear Extension

The poset $\mathscr{B}^3$



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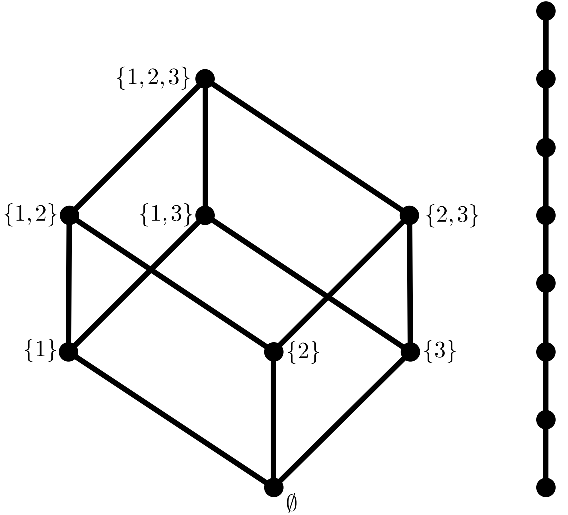

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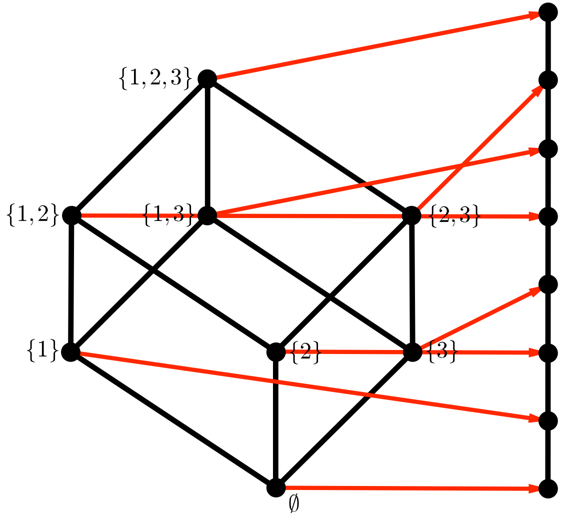

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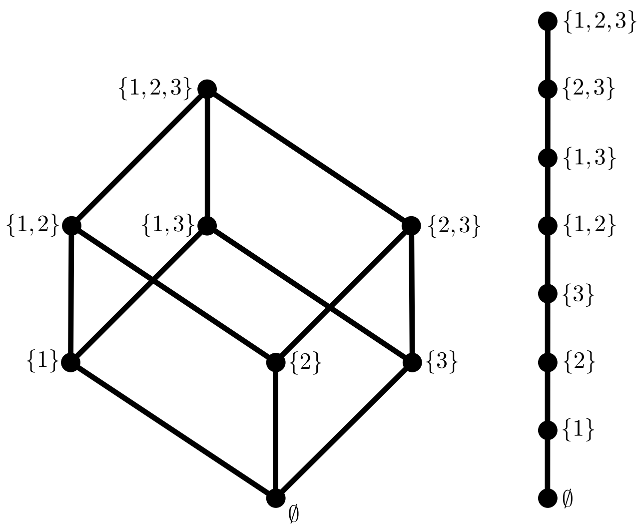

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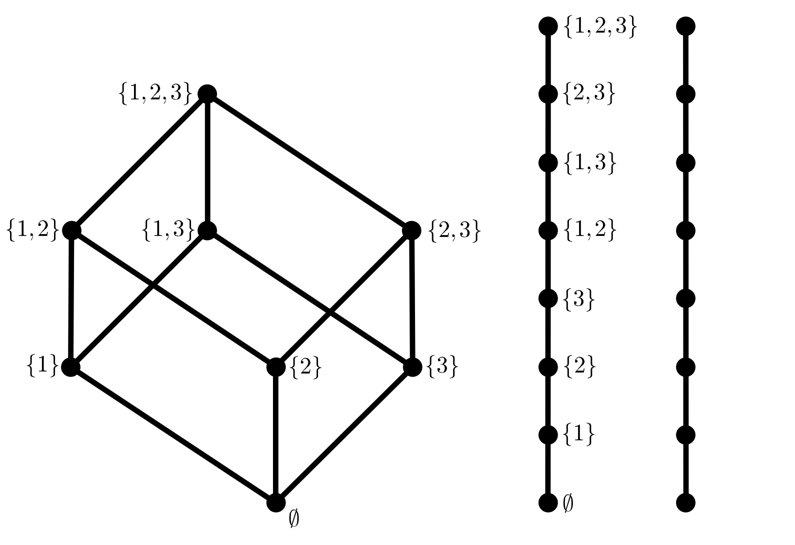

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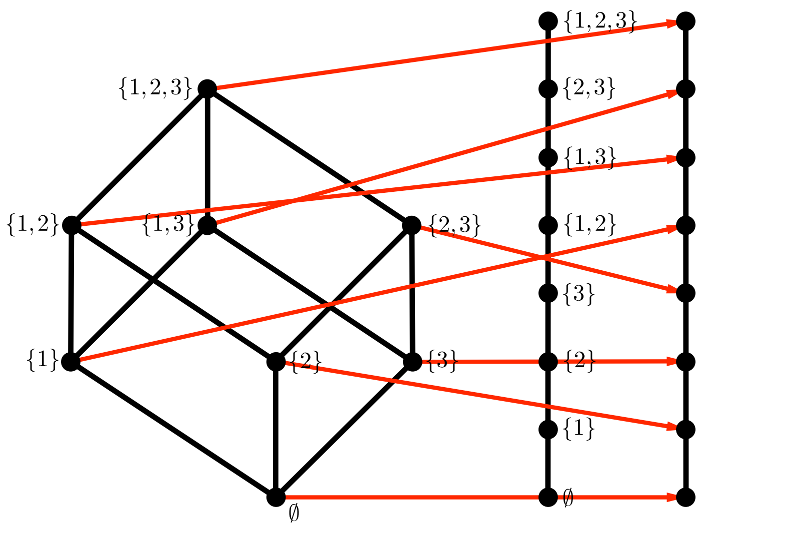

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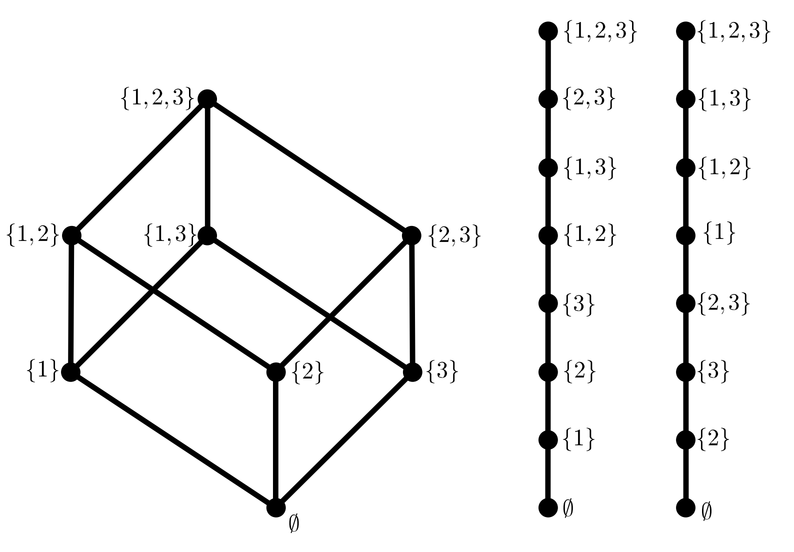

Counting The Linear Extensions of a Finite Poset

- Let $E(P)$ be the set of linear extensions of P. If P is finite then $E(P)$ is finite.
- We define $e(P)$ to the the size of $E(P)$.

A trivial upper bound is

$$e(P) \leq |P|!$$

(The right hand side counts the number of total orderings of the set P.)

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Subsets of Posets

Let Q be a subset of a partially ordered set P.

- Q is an **order ideal** if for each $X \in Q$, $Y \preceq X$ implies $Y \in Q$ for all $Y \in P$.
- Q is a **filter** if for each $X \in Q$, $X \preceq Y$ implies $Y \in Q$ for all $Y \in P$.
- Q is a **chain** if for each X and $Y \in Q$ either $X \preceq Y$ or $Y \preceq X$.
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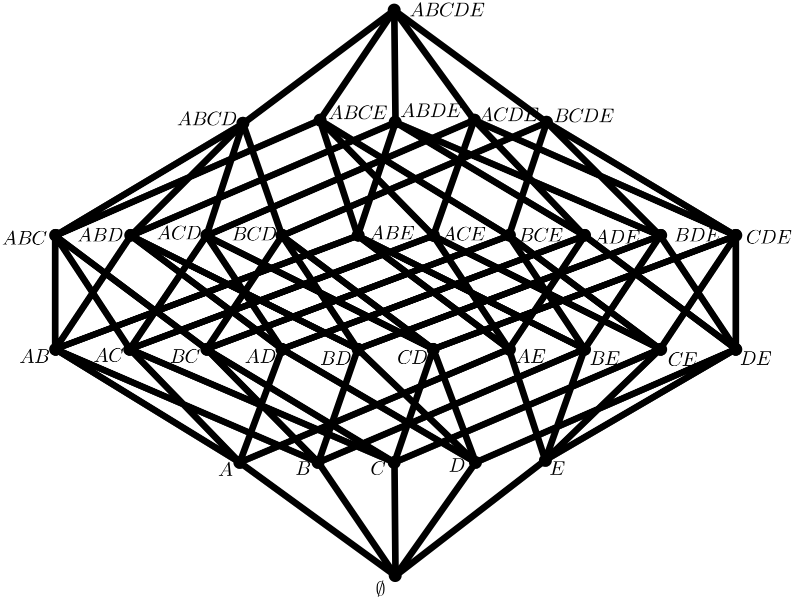

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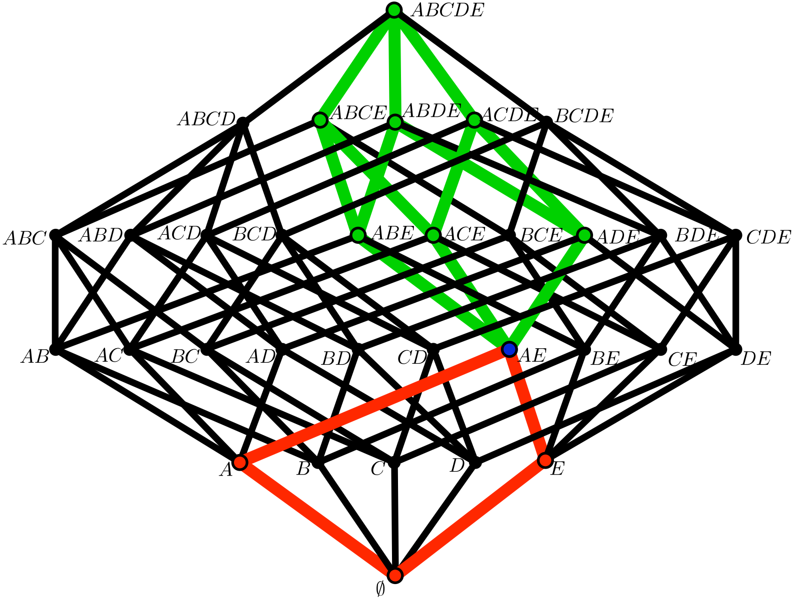

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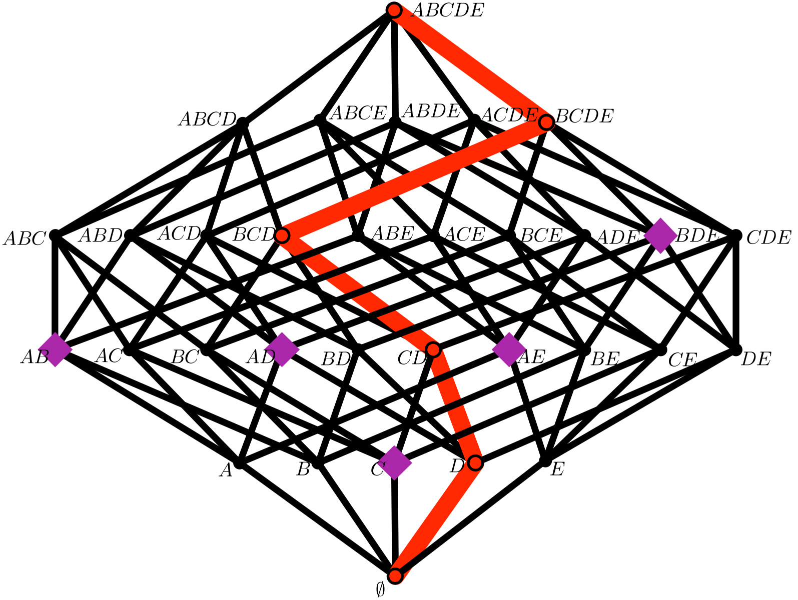

Linear Extensions, Order Ideals, and Antichains

- If ε is a linear extension of a poset P then the elements of P can be written $X_1, X_2, \ldots, X_{|P|}$ so that $X_i \preceq_\varepsilon X_j$ if and only if $i \leq j$. In fact, this sequence uniquely characterizes ε.
- Letting $O_i = \{X_1, X_2, \ldots, X_i\}$ we can construct a sequence of order ideals $O_1, O_2, \ldots, O_{|P|}$ of P. Again, this sequence uniquely characterizes ε.
- Given an ideal O of P, we define the map $\mathfrak{a}$ by

  $$\mathfrak{a}(O) = \min\{P - O\}.$$

  $\mathfrak{a}(O)$ is always an antichain, called the **choice antichain** of O. This map establishes a bijection between the order ideals of P and the antichains of P.
- This allows us to translate the the sequence of ideals $O_1, O_2, \ldots, O_{|P|}$ into a sequence of antichains $\mathfrak{a}(O_1), \mathfrak{a}(O_2), \ldots, \mathfrak{a}(O_{|P|})$. This sequence also uniquely characterizes ε.

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The Choice Antichain

- Intuitively, the choice antichain of O is the set of every element X of $P - O$ so that the set
$$O \bigcup \{X\}$$
is also an ideal of P.

For the first given linear extension of $\mathscr{B}^3$, we have the following sequences:

X_i	O_i	$\alpha(O_i)$
$\emptyset$	$\{\emptyset\}$	$\{\{1\},\{2\},\{3\}\}$
$\{1\}$	$\{\emptyset,\{1\}\}$	$\{\{2\},\{3\}\}$
$\{2\}$	$\{\emptyset,\{1\},\{2\}\}$	$\{\{3\},\{1,2\}\}$
$\{3\}$	$\{\emptyset,\{1\},\{2\},\{3\}\}$	$\{\{1,2\},\{1,3\},\{2,3\}\}$
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Ranked Posets

- A **rank function** on a poset P is a function $r : P \longrightarrow \mathbb{N}$ such that
  1. There is a minimal element $X_0 \in \mathscr{P}$ so that $r(X_0) = 0$

  and

  2. $r(X) = r(Y) + 1$ whenever X covers Y.

Given any ranked poset P,

- the number $\max\{r(X)\}_{X \in P}$ is the **rank** of P.
- For any subset Q of P, the set $\{X \in Q | r(X) = k\}$ is denoted by Q_k.
- The numbers $N_k = |P_k|$ are the **whitney numbers** of P.

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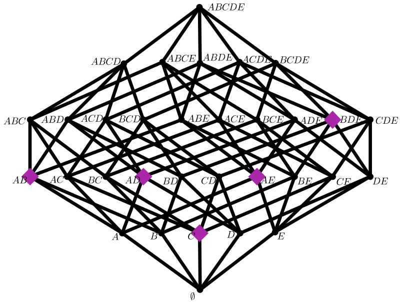
The LYM Property

Let P be a rank n poset, with whitney numbers $N_0, N_1, \ldots, N_n$.
P has the **LYM property** if for each antichain $A \in P$,

$$\sum_{k=0}^{n} \frac{|A_k|}{N_k} \leq 1.$$

The LYM Property



- The whitney number N_k of $\mathscr{B}^5$ is the binomial coefficient $\binom{5}{k}$.
- The antichain A has $|A_0| = |A_4| = |A_4| = 0$, $|A_1| = |A_3| = 1$, and $|A_2| = 3$.
- So,

$$\sum_{k=0}^{5} \frac{|A_k|}{\binom{5}{k}} = \frac{1}{5} + \frac{3}{10} + \frac{1}{10} = \frac{3}{5} < 1$$

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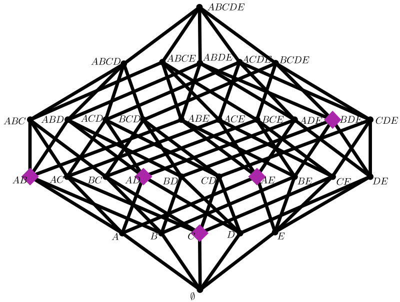

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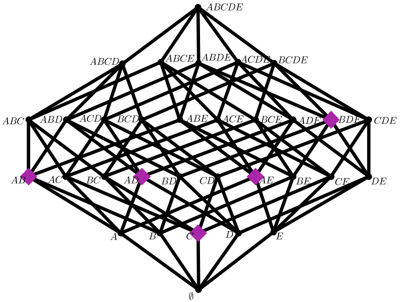

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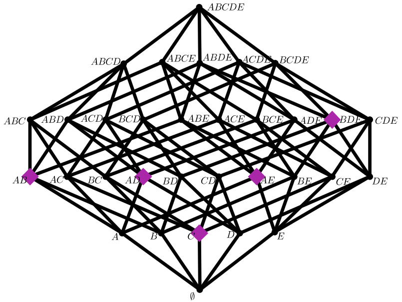

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The Boolean Lattice

Theorem

(The LYM Inequality) Let $\mathscr{A}$ *be an antichain in the Boolean Lattice* $\mathscr{B}^n$ *and let* $\mathscr{A}_k$ *be the be the set of all rank* k *nodes in* $\mathscr{A}$*. Then*

$$\sum_{k=0}^{n} \frac{|\mathscr{A}_k|}{\binom{n}{k}} \leq 1.$$

The Boolean Lattice

$\mathscr{B}^n$ contains exactly $n!$ maximal chains.

If $X \in \mathscr{B}^n$ and $r(X) = k$ then X generates an ideal of rank k isomorphic to $\mathscr{B}^k$ and a filter of rank $n-k$ isomorphic to $\mathscr{B}^{n-k}$. It follows that there are exactly $k!(n-k)!$ maximal chains in $\mathscr{B}^n$ containing X.

If $\mathscr{A}$ is an antichain in $\mathscr{B}^n$ and then for each $X \in \mathscr{A}_k$ there are exactly $k!(n-k)!$ maximal chains in $\mathscr{B}^n$ containing X.

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If $X \in \mathscr{B}^n$ and $r(X) = k$ then X generates an ideal of rank k isomorphic to $\mathscr{B}^k$and a filter of rank $n-k$ isomorphic to $\mathscr{B}^{n-k}$. It follows that there are exactly $k!(n-k)!$ maximal chains in $\mathscr{B}^n$ containing X.

If $\mathscr{A}$ is an antichain in $\mathscr{B}^n$ and then for each $X \in \mathscr{A}_k$ there are exactly $k!(n-k)!$ maximal chains in $\mathscr{B}^n$ containing X.

The Boolean Lattice

Given any antichain A and any chain C of any poset P, $A \bigcap C$ contains at most 1 element.

Therefore, there are exactly

$$\sum_{k=0}^{n} |\mathscr{A}_k| \, k!(n-k)!$$

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Since there are at most $n!$ maximal chains in $\mathscr{B}^n$ containing some member of $\mathscr{A}$,

$$\sum_{k=0}^{n} |\mathscr{A}_k|\, k!(n-k)! \leq n!.$$

Dividing through by $n!$ gives

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$\square$

Probabilistic Arguments

We will be using a discrete probability distribution over $E(P)$ to get an upper bound on its size, e(P).

- A function ρ from a finite set E to the interval $[0,1]$ is a **probability distribution** over E if

$$\sum_{x \in E} \rho(x) = 1.$$

- A **weight function** on P is a function $w : \mathscr{P}[P] \longrightarrow \mathbb{R}^+$ so that for every subset Q of P,

$$w(Q) = \sum_{X \in Q} w(X).$$

For each antichain A of P, the function $\rho_A : A \longrightarrow \mathbb{R}$ defined by

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The Generalized Sha/Kleitman Bound

Theorem

Let P be a ranked poset and let w be a weight function on P. If $w(A) \leq 1$ for each antichain A of P then

$$e(P) \leq \frac{1}{\prod_{X \in P} w(X)}.$$

Brightwell's Proof

Define a procedure for generating linear extensions of P as follows:

$$\begin{aligned} O_0 &= \emptyset \\ O_{i+1} &= O_i + \{X_i\} \end{aligned}$$

where X_i is chosen from $\mathfrak{a}(O_i)$ with probability $\rho_{O_i}(X_i)$.

The process terminates after the $|P|$th step when $O_{|P|} = P$ and $\mathfrak{a}(O_{|P|}) = \emptyset$. The generated sequence $O_1, O_2, \ldots, O_{|P|}$ determines a unique linear extension of P.

Alternately, given any sequence $O_1, O_2, \ldots, O_{|P|}$, characterizing a linear extension, the construction results in $O_1, O_2, \ldots, O_{|P|}$ only if the choice of X_i at the ith stage is exactly the single element of $O_{i+1} - O_i$.

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For each partial sequence $O_1, O_2, \dots, O_{i-1}$, the value $\rho_{O_i}(X_i)$ is exactly the probability that X_i is chosen at the ith stage of our construction given that $O_1, O_2, \dots, O_{i-1}$ have already been constructed.

It follows that, for any linear extension ε of P, the probability that our construction produces ε is exactly

$$\mu(\varepsilon) = \prod_{i=1}^{|P|} \rho_{O_i}(X_i).$$

where the sequences $X_1, \dots, X_{|P|}$ and $O_1, O_2, \dots, O_{|P|}$ are defined as above. Therefore, μ is a probability distribution over the set $E(P)$ assigning non-zero probability to each element $\varepsilon \in E(P)$.

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Brightwell's Proof

By our assumptions, for any order ideal O and any $X \in O$, we have

$$\rho_O(X) = \frac{w(X)}{w(\mathfrak{a}(O))} \geq w(X).$$

Since every element of P appears exactly once in the sequence $X_1, \ldots, X_{|P|}$,

$$\prod_{X \in P} w(X) \leq \prod_{i=1}^{|P|} \rho_{O_i}(X_i) = \mu(\varepsilon).$$

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Finally, since

$$\sum_{\varepsilon \in E(P)} \mu(\varepsilon) = 1$$

it follows that

$$e(P) \cdot \left(\prod_{X \in P} w(X) \right) = \sum_{\varepsilon \in E(P)} \left(\prod_{X \in P} w(X) \right) \leq \sum_{\varepsilon \in E(P)} \mu(\varepsilon) = 1.$$

□

Brightwell's Proof

Corollary

If P is an LYM poset with whitney numbers $N_0, N_1, N_2, ..., N_n$ then

$$e(P) \leq \prod_{i=0}^{n} N_i^{N_i}.$$

Brightwell's Proof

Let $w(X) = \frac{1}{N_{r(X)}}$, where r is the rank function on P. Note that w is a weight function on P.

If P is LYM, we have $w(A) \leq 1$ for every antichain A in P.

Therefore, by the previous theorem,

$$e(P) \leq \frac{1}{\prod_{X \in P} w(X)} = \frac{1}{\prod_{X \in P} \frac{1}{N_{r(X)}}} = \prod_{X \in P} N_{r(X)}.$$

Since for each i, there are exactly N_i elements of P with rank i, the corollary follows.

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Conclusion

- This bound is achieved by chains, but it is easy to see that it is not attained by any other poset.
- It is not asymptotic but for small values of n it is the best upper bound we have for $\mathscr{B}^n$.

n	$\prod_{i=0}^{n} \binom{n}{i}!$	$e(\boldsymbol{\mathcal{B}}^n)$	$\prod_{i=0}^{n} \binom{n}{i}^{\binom{n}{i}}$	$2^n!$
1	1	1	1	2
2	2	2	4	24
3	36	48	729	40320
4	4.15×10^5	1.680384×10^6	3.06×10^9	2.09×10^{13}
5	1.9×10^{17}	$1.4807804035657359 36 \times 10^{19}$	9.77×10^{26}	2.63×10^{35}
6	2.16×10^{48}	$1.4137791169722788711719597031620079563020547695771648 0 \times 10^{53}$	4.38×10^{70}	1.72×10^{89}
7	7.08×10^{126}	?	2.81×10^{175}	3.86×10^{215}
8	9.15×10^{317}	?	2.78×10^{420}	8.58×10^{506}

Conclusion

- Using a very sophisticated probabilistic approach Brightwell and Tetali have published an asymptotic bound on $e(\mathscr{B}^n)$ given by

$$e(\mathscr{B}^n) \leq e^{6 \cdot 2^n \cdot \frac{\ln n}{n}} \prod_{i=0}^{n} \binom{n}{i}!$$

- It first outdoes the Sha/Kleitman bound at $n = 18$ where

$$\prod_{i=0}^{n} \binom{n}{i}^{\binom{n}{i}} \approx 2.10 \times 10^{1173310}$$

and

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