Linear Extensions and the LYM Property

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Abstract

Let $\mathcal{P}$ be a finite ranked partially ordered set with whitney numbers $N_1, N_2, ..., N_n$ and let $e(\mathcal{P})$ denote the number of linear extensions of $\mathcal{P}$. It is, in general, very difficult to determine the size of $e(\mathcal{P})$ for all but the simplest classes of posets. This paper reviews some elementary probabilistic techniques for establishing an upper bound on $e(\mathcal{P})$. We discuss the LYM inequality and its generalization to the class of LYM posets. Finally, we prove an upper bound for the number of linear extensions of an LYM poset:

$$e(\mathcal{P}) \leq \prod_{i=0}^{n} N_i^{N_i}.$$

This bound was introduced for the boolean lattice by Jichang Sha and D. J. Kleitman in 1987 and generalized by Graham Brightwell in 2003.

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Chapter 1

Introduction

This paper explores certain combinatorial properties of finite sets. Some of the ideas discussed can be extended to infinite sets but we will not pursue such generalizations here. Our basic objects of study will be finite ranked partially ordered sets with the LYM property. The aim of this paper is to derive an upper bound for the number of linear extensions of any partially ordered set with these properties. A particularly well known example is the partial order of subsets of a given finite set S. If the set S contains n elements, then a linear extension of the partial order of subsets is an ordering of the entire power set $\mathbf{2}^S$, say $A_1, A_2, ..., A_{2^n}$, so that $i \leq j$ whenever $A_i \subseteq A_j$. Even for this well known partial order, the problem of counting the number of linear extensions for an arbitrary n is open. A few simple upper bounds were given by Sha and Kleitman in [14]. Of these, the most appealing said that this number could be no larger than $\prod_{i=0}^{n} \binom{n}{i}^{\binom{n}{i}}$.

My original intention was to carefully reconstruct the argument for this bound given in [14] and its generalization by Shastri in [16]. I encountered a number of difficulties, due at least in part to a paucity of details in both papers. Some of these difficulties, I could not resolve. I eventually found a much cleaner proof of a more general result given by Brightwell in [4] and decided that this would be preferable to the original.

In the course of working through [14], a few interesting conjectures arose that I felt should be included in the final product. Indeed, Brightwell's argument is similar enough to the Sha and Kleitman argument that I wanted to compare the two. My aim was to understand how Brightwell's argument managed to do so much more with so much less. In fact, Brightwell's success here reflects the value in generalizing the study of ranked partial orders to the more general study of weighted partial orders.[1]

It has been my intention in writing this paper to build the results from "the ground up" to the greatest extent possible. The reader is only assumed to be familiar with the usual basic set theory and algebra terminology, along with a few elementary counting techniques. The terminology of partially ordered sets and lattices is developed briskly, and the reader is directed to [8] for a more leisurely presentation.

To make the paper more readable, I have opted for the following conventions. Real numbers and maps to the real numbers will generally be given in lowercase, sometimes greek, script (e.g., m, n, x, μ). Elements of a poset will always be capital letters (e.g., P, Q, X, Y). Posets and subsets of posets will always be given the "math calligraphy" font (e.g., $\mathcal{P}, \mathcal{Q}, \mathcal{X}, \mathcal{Y}$). In addition, we will use gothic letters and capital greek letters to denote maps between posets. Let us also adopt the convention that $\leq$ will always refer to the standard ordering of $\mathbb{R}$. Additionally, we will use $\subseteq$ to refer to standard set theoretic containment. Finally, the symbol $\preceq$ will, in general, refer to any other ordering whose identity will be (hopefully) clear from context.

[1] See, in particular, [9].

Chapter 2

Partially Ordered Sets and Lattices

2.1 Basic Terminology

In this section, we define our basic object of study and introduce some terminology. The section closes by deriving a few basic properties.

Recall that a binary relation $\preceq$ on a set $\mathcal{P}$ is defined to be a **partial order** on $\mathcal{P}$ when $\preceq$ is reflexive, transitive, and antisymmetric. Similarly, $\preceq$ is a **total order** on $\mathcal{P}$ when it is a partial order on $\mathcal{P}$ such that if both A and $B \in \mathcal{P}$, then either $A \preceq B$ or $B \preceq A$. By a **partially ordered set**, or **poset**, we mean an ordered pair $(\mathcal{P}, \preceq)$ so that $\preceq$ is a partial order on $\mathcal{P}$. If the relation $\preceq$ is a total order on $\mathcal{P}$, we call $(\mathcal{P}, \preceq)$ a **totally ordered set**, or **toset**.

Let $(\mathcal{P}, \preceq)$ be a poset. An element $X \in \mathcal{P}$ is called the **maximum** of $\mathcal{P}$ if $Y \preceq X$ for each $Y \in \mathcal{P}$. An element $X \in \mathcal{P}$ is **maximal** if it satisfies the weaker condition that for each $Y \in \mathcal{P}$, $X \preceq Y$ implies $X = Y$. Similarly, X is called the **minimum** of $\mathcal{P}$ if $X \preceq Y$ for each $Y \in \mathcal{P}$ and **minimal** if $Y \preceq X$ implies that $X = Y$. If $\mathcal{X} \subseteq \mathcal{P}$, and $\preceq'$ is the restriction of $\preceq$ to $\mathcal{X} \times \mathcal{X}$, then $(\mathcal{X}, \preceq')$ is a poset. We will abuse this notation slightly and refer just to $\mathcal{P}$ and its subsets as posets when no confusion is likely to arise. This allows us to naturally extend terminology describing posets to their subsets.

Example 2.1.1. The set $[\mathbf{4}] = \{1, 2, 3, 4\}$ under the usual ordering of the integers is a poset with

$\preceq_{[\mathbf{4}]}= \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$. The set $\mathcal{X} = \{1,3\}$ is a subset of $\{1,2,3,4\}$. The restriction of $\preceq_{[\mathbf{4}]}$ to $\{1,3\}$ is given by $\preceq_{\mathcal{X}}= \{1,1),(1,3),(3,3)\}$. It is easy to check that $\mathcal{X}$ is a poset under this relation. Then, the element 3 is a maximum of $\mathcal{X}$ but is not a maximum of $[\mathbf{4}]$. On the other hand, the element 1 is a minimum of both $\mathcal{X}$ and $[\mathbf{4}]$.

If n is any positive integer, then we define $[\mathbf{n}]$ to be the set $\{1,2,3,...,n\}$ ordered by $\leq$.

Let σ be any map from a poset $\mathcal{P}$ to a poset $\mathcal{Q}$. If, for each $X, Y \in \mathcal{P}$, $X \preceq_{\mathcal{P}} Y$ implies that $\sigma(X) \preceq_{\mathcal{Q}} \sigma(Y)$, then we call σ **order preserving**. Similarly, we call a map $\sigma : \mathcal{P} \longrightarrow \mathcal{Q}$ **order reversing** if for each $X, Y \in \mathcal{P}$, $X \preceq_{\mathcal{P}} Y$ implies that $\sigma(Y) \preceq_{\mathcal{Q}} \sigma(X)$. An injective order preserving (respectively order reversing) map is said to be **invertible** if its inverse is also order preserving (respectively order reversing). Posets $\mathcal{P}$ and $\mathcal{Q}$ are called **order isomorphic**, denoted by $\mathcal{P} \cong \mathcal{Q}$, when there is an invertible order preserving bijection between them. An order preserving injection σ from $\mathcal{P}$ to $\mathcal{Q}$ is called an **order embedding** of $\mathcal{P}$ in $\mathcal{Q}$ if $\sigma(X) \preceq_{\mathcal{Q}} \sigma(Y)$ implies that $X \preceq_{\mathcal{P}} Y$. Note that if σ is an order embedding of $\mathcal{P}$ in $\mathcal{Q}$, then $\mathcal{P} \cong \sigma[\mathcal{P}] \subseteq \mathcal{Q}$. The **dual** of a poset $\mathcal{P}$, denoted $\mathcal{P}^{\downarrow}$ is the poset $(\mathcal{P}, \preceq_{\downarrow})$ where $X \preceq_{\downarrow} Y$ if and only if $Y \preceq X$. Note that there is an invertible order reversing bijection between $\mathcal{P}$ and $\mathcal{Q}$ just in case $\mathcal{P} \cong \mathcal{Q}^{\downarrow}$. We call a poset $\mathcal{P}$ **symmetric** if $\mathcal{P} \cong \mathcal{P}^{\downarrow}$ or equivalently if there exists an order reversing bijection from $\mathcal{P}$ onto itself.

Given posets $\mathcal{P}$ and $\mathcal{Q}$, the disjoint union $\mathcal{P} + \mathcal{Q}$ is a poset whose order relation is defined by $\preceq_{\mathcal{P}+\mathcal{Q}}=\preceq_{\mathcal{P}} + \preceq_{\mathcal{Q}}$. It is important to note that the expression "$A + B$", when A and B are sets will always carry with it the explicit assumption that A and B are disjoint. If this assumption is not intended, then we will use "$A \bigcup B$" instead. The disjointness requirement is important in the context of ordered sets because it ensures that the resulting relation remains a partial order. If the sets are not disjoint, then their order relations may not be. This can cause trouble. For example, if $\preceq_{\mathcal{P}}$ has at least one pair of distinct elements in it, then $\preceq_{\mathcal{P} \cup \mathcal{P}^{\downarrow}}$ fails to be antisymmetric.[1] The cartesian product $\mathcal{P} \times \mathcal{Q}$ is a poset whose order relation is defined by $(X, Y) \preceq_{\mathcal{P} \times \mathcal{Q}} (Z, W)$ if and only if $X \preceq_{\mathcal{P}} Z$ and $Y \preceq_{\mathcal{Q}} W$. The operations $+$ and $\times$ are associative and commutative, so that $\sum$ and $\prod$ extend in a straight forward way to disjoint unions and cartesian products.

Let X and Y be distinct elements of $\mathcal{P}$. We say that Y **covers** X when $X \preceq Y$ and the set

[1] Of course, it is easy to just tag the elements of $\mathcal{P}^{\downarrow}$ to make them distinct from those of $\mathcal{P}$.

$\{Z \in \mathcal{P} \mid X \preceq Z \preceq Y\}$ contains only X and Y. A function $r : \mathcal{P} \longrightarrow \mathbb{N}$ is a **rank function** of $\mathcal{P}$ if

(i) there is a minimal element $X_0 \in \mathcal{P}$ so that $r(X_0) = 0$ and

(ii) $r(X) = r(Y) + 1$ whenever X covers Y.

The value $r(X)$ is called the **rank** of X. A **ranked poset** is a triple $(\mathcal{P}, \preceq, r)$ where $(\mathcal{P}, \preceq)$ is a poset and r is a rank function.[2] The **rank** of a ranked poset $\mathcal{P}$ is the number $r(\mathcal{P}) = \max\{r(X) \mid X \in \mathcal{P}\}$.

More generally, a function from $\mathcal{P}$ to $\mathbb{R}^+$ is called a **weight function** w. The function is extended to subsets of $\mathcal{P}$ by taking $w(\mathcal{S}) = \sum_{X \in \mathcal{S}} w(X)$ for $\mathcal{S} \subseteq \mathcal{P}$. A simple example is the rank function r itself. Another is the "size" function s whose value on any element of $\mathcal{P}$ is 1 and whose value on any subset $\mathcal{S}$ is $s(\mathcal{S}) = \sum_{X \in \mathcal{S}} w(X) = |\mathcal{S}|$.

A ranked poset is called **graded** if

(i) $r(X) = 0$ if X is minimal and

(ii) if X and Y are maximal in $\mathcal{P}$, then $r(X) = r(Y)$.

If $\mathcal{X}$ is a set of elements of a ranked poset, then we let $\mathcal{X}_k$ denote the set $\{X \in \mathcal{X} \mid r(X) = k\}$. The sets $\mathcal{P}_k$ are of particular interest and the number $N_k := |\mathcal{P}_k|$ is called the kth **whitney number** of $\mathcal{P}$.

One of the best tools for visualizing a poset $\mathcal{P}$ is the **hasse diagram** of $\mathcal{P}$, a directed graph whose vertex set is the set $\mathcal{P}$ and whose edge set consists of those ordered pairs (X, Y) so that Y covers X in $\mathcal{P}$. Note that we always draw a ranked poset $\mathcal{P}$ by giving each element of rank i along the same vertical position and arranging each $\mathcal{P}_i$ just above $\mathcal{P}_{i-1}$ and just below $\mathcal{P}_{i+1}$. Figures 2.1-2.6 are all examples of hasse diagrams.

We call a subposet $\mathcal{C}$ of $\mathcal{P}$ a **chain** if either $X \preceq Y$ or $Y \preceq X$ for each pair of elements X and $Y \in \mathcal{C}$. Note that this is equivalent to $\mathcal{C}$ being a toset. Proposition 2.1.2 below guarantees that for each finite chain $\mathcal{C}$, there is an isomorphism $\sigma : C \longrightarrow [\mathbf{k}]$ where $k = |\mathcal{C}|$ so that the elements of $\mathcal{C}$ may be represented by a sequence $c_1, c_2, \ldots, c_k$ where $c_i = \sigma^{-1}(i)$. A chain $\mathcal{C}$ is called **maximal** in $\mathcal{P}$ if for any chain $\mathcal{C}'$ of $\mathcal{P}$, $\mathcal{C} \subseteq \mathcal{C}'$ implies that $\mathcal{C} = \mathcal{C}'$. The **length** of $\mathcal{C}$ is given by $|\mathcal{C}| - 1$.

A subset $\mathcal{X}$ of $\mathcal{P}$ is an **antichain** if neither $X \preceq Y$ nor $Y \preceq X$ for each X and $Y \in \mathcal{X}$. Any antichain $\mathcal{A}$ is called **maximal** if for any element $X \in \mathcal{P} - \mathcal{A}$, the set $\mathcal{A} \cup \{X\}$ is not an antichain.

[2] As above, when appropriate, we will simply speak of a ranked poset $\mathcal{P}$, instead of $(\mathcal{P}, \preceq, r)$

An antichain $\mathcal{A}$ of a ranked poset is **k-saturated** if for each $X \in \mathcal{P}$ so that $r(X) = k$, the set $\{X\} \cup \mathcal{A}$ is not an antichain.

The following unary operations on the power set of a given a poset $\mathcal{P}$ are useful in characterizing its subsets. Given a subset $\mathcal{X}$ of $\mathcal{P}$, an element X is **maximal in $\mathcal{X}$** if, for all $Y \in \mathcal{X}$, $X \preceq Y$ implies that $X = Y$ and **minimal in $\mathcal{X}$** if, for all $Y \in \mathcal{X}$, $Y \preceq X$ implies that $X = Y$. We follow the usual practice of letting $\max \mathcal{X}$ denote the set of maximal elements of $\mathcal{X}$ and $\min \mathcal{X}$ denote the set of minimal elements of $\mathcal{X}$. The **shadow** of $\mathcal{X}$ is the set of elements of $\mathcal{P}$ covered by some element of $\mathcal{X}$. Similarly, the **cover** of $\mathcal{X}$ is the set of elements of $\mathcal{P}$ covering some element of $\mathcal{X}$. The shadow and cover of $\mathcal{X}$ are respectively denoted by $\vartriangle(\mathcal{X})$ and $\triangledown(\mathcal{X})$. We call $\mathcal{X}$ an **ideal** of $\mathcal{P}$ just in case for each $X \in \mathcal{X}$, if $Y \preceq X$, then $Y \in \mathcal{X}$. In light of the above definition, $\mathcal{X}$ is an ideal just in case $\vartriangle(\mathcal{X}) = \emptyset$. Similarly, we call $\mathcal{X}$ a **filter** if, for each $X \in \mathcal{X}$, $X \preceq Y$ implies that $Y \in \mathcal{X}$. Again, note that this is equivalent to $\triangledown(\mathcal{X}) = \emptyset$. For any $\mathcal{X}$, the **ideal** (respectively **filter**) **generated by** $\mathcal{X}$ is the smallest ideal (respectively filter) containing $\mathcal{X}$. Stanley's notation,[3] $\langle \mathcal{X} \rangle$, for the ideal generated by $\mathcal{X}$ will be useful here. We will follow the general terminology of ideals in abstract algebra here and refer, if $\mathcal{X} = \{X\}$ for some $X \in P$, to the ideal (respectively filter) generated by $\mathcal{X}$ as a **principal** ideal (respectively filter). In such cases, we abbreviate for example $\vartriangle(\{X\})$ by omitting the brackets and writing $\vartriangle(X)$.

It is convenient to introduce an additional pair of operators acting on a $\mathcal{X}$. The **total shadow** of $\mathcal{X}$, denoted $\blacktriangle(\mathcal{X})$, is defined to be the ideal generated by $\vartriangle(\mathcal{X})$. Similarly, the **total cover** of $\mathcal{X}$, denoted $\blacktriangledown(\mathcal{X})$, is defined to be the filter generated by $\triangledown(\mathcal{X})$.

The following propositions recount some useful facts about posets.

Proposition 2.1.2. *If $\mathcal{Q}$ is a toset and $|\mathcal{Q}| = n$, then $Q \cong [\mathbf{n}]$.*

Proof. If $\mathcal{Q}$ is a toset of size n, then the map $X \longrightarrow |\langle X \rangle|$ is an invertible order preserving bijection from $\mathcal{Q}$ to $[\mathbf{n}]$. □

An implication of this proposition is that all chains of a given length i are isomorphic. Following this,

[3] See [17].

we sometimes use $\mathcal{C}_i$ to designate an arbitrary context appropriate representative of this isomorphism class.

Proposition 2.1.3. *If* $\Psi : \mathcal{P} \longrightarrow \mathcal{Q}$ *is an isomorphism between posets and* $\mathcal{X} \subseteq \mathcal{P}$*, then*

$$\Psi\left[\triangle(\mathcal{X})\right] = \triangle(\Psi\left[\mathcal{X}\right]) \text{ and } \Psi\left[\triangledown(\mathcal{X})\right] = \triangledown(\Psi\left[\mathcal{X}\right]).$$

If Φ *is an invertible order reversing bijection, then*

$$\Phi\left[\triangle(\mathcal{X})\right] = \triangledown(\Phi\left[\mathcal{X}\right]) \text{ and } \Phi\left[\triangledown(\mathcal{X})\right] = \triangle(\Phi\left[\mathcal{X}\right]).$$

If we replace $\triangle$ *with* $\blacktriangle$ *and* $\triangledown$ *with* $\blacktriangledown$*, these statements remain true.*

Proof. Let $Y \in \Psi\left[\triangle(\mathcal{X})\right]$. Then $\Psi^{-1}(Y) \in \triangle(\mathcal{X})$ so that there is an $X \in \mathcal{X}$ such that $\Psi^{-1}(Y)$ is covered by X. Since Ψ is order preserving, Y is covered by $\Psi(X)$ and therefore, $Y \in \triangle(\Psi\left[\mathcal{X}\right])$. Alternately, let $Y \in \triangle(\Psi\left[\mathcal{X}\right])$. Then there is an $X \in \Psi[\mathcal{X}]$ so that X covers Y. Since Ψ^{-1} is also order preserving, $\Psi^{-1}(X) \in \mathcal{X}$ and covers $\Psi^{-1}(Y)$. It follows that $\Psi^{-1}(Y) \in \triangle(\mathcal{X})$ and therefore $Y \in \Psi\left[\triangledown(\mathcal{X})\right]$.
The arguments for all other cases are nearly identical. Replacing $\triangle$ with $\blacktriangle$, for example, requires that we replace "covered by" and "covers" with $\preceq$. $\square$

This proposition also provides us with an important property of symmetric posets.

Corollary 2.1.4. *If* $\mathcal{P}$ *is symmetric, then for each* $X \in \mathcal{P}$*,* $(\triangledown(X))^{\downarrow} = \triangle(X^{\downarrow})$*.*

Of course, this corollary can be similarly extended to $\triangle$, $\blacktriangledown$, and $\blacktriangle$.

Proposition 2.1.5. *If* $\mathcal{A}$ *is an antichain in* $\mathcal{P}$*, then we have that* $\blacktriangle(\mathcal{A})$*,* $\mathcal{A}$*, and* $\blacktriangledown(\mathcal{A})$ *are disjoint. The antichain* $\mathcal{A}$ *is maximal if and only if* $\mathcal{P} = \blacktriangle(\mathcal{A}) + \mathcal{A} + \blacktriangledown(\mathcal{A})$*.*

Proof. Consider, for example $\mathcal{A} \bigcap \blacktriangle(\mathcal{A})$. If $X \in \mathcal{A} \bigcap \blacktriangle(\mathcal{A})$, then $X \in \mathcal{A}$ and there is a $Y \in \mathcal{A} - \{X\}$ so that $X \preceq Y$. It follows that $\mathcal{A}$ is not an antichain. The argument for $\mathcal{A} \bigcap \blacktriangledown(\mathcal{A})$ is identical. If $X \in \blacktriangle(\mathcal{A}) \bigcap \blacktriangledown(\mathcal{A})$, then there exist $Y$and $Z \in \mathcal{A}$ so that $Y \preceq X \preceq Z$. If either of $\mathcal{A} \bigcap \blacktriangle(\mathcal{A})$ or $\mathcal{A} \bigcap \blacktriangledown(\mathcal{A})$ are non-empty, then we already know that $\mathcal{A}$ is not an antichain. If both are empty, then it follows that X, Y, and Z are all distinct so that, in particular, $Y \neq Z$. By the transitivity of the order relation, $Y \preceq Z$ so that $\mathcal{A}$ is not an antichain. Any element of $\mathcal{P} - \blacktriangle(\mathcal{A}) + \mathcal{A} + \blacktriangledown(\mathcal{A})$ is not comparable to any element of $\mathcal{A}$. It follows that this set is empty if and only if $\mathcal{A}$ is maximal. $\square$

2.2 Examples

The toset $[\mathbf{n}]$ is a simple, but important poset. In this section we will introduce a few other important posets and try to make some of the definitions from the previous section more concrete.

First, we develop a simple example to illustrate these definitions.

Example 2.2.1. Consider the set

$$\mathcal{I}^2(k) := \left\{(i, j) \in \mathbb{Z}^2 \mid 0 \leq i, j \leq k\right\}$$

with the order relation $\preceq$ defined by $(i, j) \preceq (m, l)$ if and only if $i \leq m$ and $j \leq l$. This poset has a maximum element, (k, k) and a minimum element $(0, 0)$. The dual $\mathcal{I}^2(k)^{\downarrow}$ has the same elements but with the rule that $(i, j) \preceq^{\downarrow} (m, l)$ if and only if $m \leq i$ and $l \leq j$. Note that $\mathcal{I}^2(k)$ and $\mathcal{I}^2(k)^{\downarrow}$ are isomorphic since the map $\sigma : \mathcal{I}^2(k) \longrightarrow \mathcal{I}^2(k)^{\downarrow}$ defined by

$$\sigma(i, j) = (n - i, n - j)$$

is an order preserving bijection. With the rank function $r : \mathcal{I}^2(k) \longrightarrow \mathbb{N}$ defined by

$$r(i, j) = i + j$$

$\mathcal{I}^2(k)$ becomes graded with rank $2n$. Note that (i, j) is covered by (m, l) just in case either $i = m+1$ and $j = l$ or $j = l + 1$ and $i = m$. Figure 2.1 shows the Hasse diagram of $\mathcal{I}^2(2)$. In this poset,

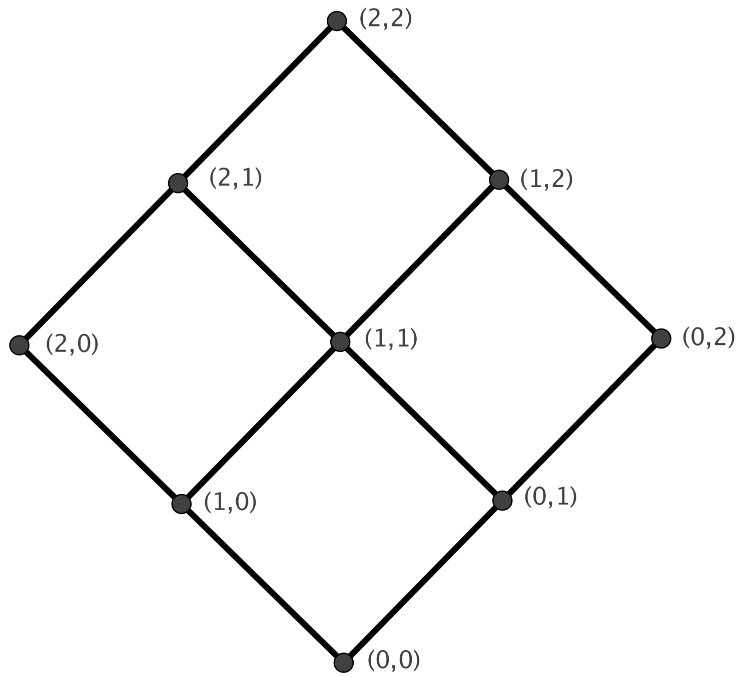

Figure 2.1: The poset $\mathcal{I}^2(k)$

consider the sets $\mathcal{C} = \{(1, 0), (2, 0)\}$ and $\mathcal{A} = \{(1, 1), (2, 0)\}$. Note that $\mathcal{C}$ is a chain and $\mathcal{A}$ is an antichain. Finally, we list the different shadow and cover sets of $\mathcal{C}$ and $\mathcal{A}$:

$$\triangle\, (\mathcal{C}) = \blacktriangle(\mathcal{C}) = \{(0, 0)\}, \qquad \triangledown(\mathcal{C}) = \{(1, 1), (2, 1)\},$$

and

$$\blacktriangledown(\mathcal{C}) = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

while

$$\blacktriangle(\mathcal{A}) = \{(0, 0), (0, 1), (1, 0)\}, \qquad \triangledown(\mathcal{A}) = \{(1, 2), (2, 1)\},$$

and

$$\blacktriangledown(\mathcal{A}) = \{(1, 2), (2, 1), (2, 2)\}.$$

Note that $\mathcal{I}^2(2) - (\blacktriangle(\mathcal{A}) + \mathcal{A} + \blacktriangledown(\mathcal{A})) = \{(0,2)\}$ and indeed $\mathcal{A} + \{(0,2)\}$ is an anti chain properly containing $\mathcal{A}$.

Posets (even finite posets!) are ubiquitous across all areas of mathematics and our arguments will apply to a large class of these, but we will restrict attention to a few important examples.

Example 2.2.2. The **unordered poset** with n elements, $\{n\}$, is the poset $\mathcal{P}$ whose size is n and whose order relation $\preceq_{\mathcal{P}}$ is the indentity relation, $\{(X,X)\}_{X\in\mathcal{P}}$, on $\mathcal{P}$. It is easy to check that this defines a partial order.

Example 2.2.3. This poset is called the **boolean lattice**,[4] denoted $\boldsymbol{\mathcal{B}}^S$. Of course, $\boldsymbol{\mathcal{B}}^S$ is isomorphic to $\boldsymbol{\mathcal{B}}^T$ just in case S and T have the same cardinality. It is therefore typical to denote this isomorphism class by $\boldsymbol{\mathcal{B}}^n$ and take the underlying set to be $[\mathbf{n}]$. Defining the rank function $r(X) = |X|$ for each $X \in \boldsymbol{\mathcal{B}}^n$ turns $\boldsymbol{\mathcal{B}}^n$ into a graded poset.

Example 2.2.4. Recall that a **multiset** $\mathcal{M} = (\mathcal{S}, m)$ is a finite set $\mathcal{S}$ together with a function m, mapping $\mathcal{S}$ to $\mathbb{Z}^{\geq 0}$. For each $X \in \mathcal{S}$ the number $m(X)$ is called the **multiplicity** of X in $\mathcal{M}$. A multiset $\mathcal{M}' = (\mathcal{S}', l)$ is a **submultiset** of $\mathcal{M}$ when $\mathcal{S}' = \mathcal{S}$ and for each $X \in S'$, $l(X) \leq m(X)$.

The set of all submultisets of a multiset $\mathcal{M}$ can thus be made into a poset where $\mathcal{N}' \preceq \mathcal{N}$ exactly when $\mathcal{N}'$ is a sub multiset of $\mathcal{N}$. This poset is known as the **multiset lattice** of $\mathcal{M}$. Certainly, $\boldsymbol{\mathcal{B}}^n$ can be viewed as a multiset lattice for which $|\mathcal{S}| = n$ and $m(X) = 1$ for each $X \in \mathcal{S}$. It should be clear that the multiset lattice of any two multisets $\mathcal{M} = (\mathcal{S}, m)$ and $\mathcal{M}' = (\mathcal{S}', l)$ with $|\mathcal{S}| = |\mathcal{S}'| = k$ are isomorphic just in case their underlying sets can be indexed $\mathcal{S} = \{X_1, X_{2,\ldots,}X_k\}$ and $\mathcal{S}' = \{Y_1, Y_{2,\ldots,}Y_k\}$ so that $m(X_i) = l(Y_i)$ for each i. We shall therefore denote this poset by $\boldsymbol{\mathcal{M}}^{n_1,n_2,\ldots,n_k}$ where $n_i = m(X_i)$ for each i. This poset is graded by the rank function defined by $r(\mathcal{M}') = \sum\limits_{X\in\mathcal{M}'} l(X)$ for each submultiset $\mathcal{M}'$ of $\mathcal{M}$.

Multisets capture the idea of a set whose members may include multiple copies of the same object. Similarly, the multiset lattice $\boldsymbol{\mathcal{M}}^{n_1,n_2,\ldots,n_k}$ can be thought of as a sublattice of $\boldsymbol{\mathcal{B}}^n$ where

[4] Note that the term "lattice" will be defined in the next section. For now, it is just a name.

$n = \sum n_i$. For each submultiset $\mathcal{N} = (\mathcal{S}, l)$ of $\mathcal{M}$ with $\mathcal{S} = \{X_1, X_{2,...,}X_k\}$ and $n_i = m(x_i)$, define the subset

$$\mathcal{N}_i = \left\{ \sum_{j=1}^{i-1} n_j + 1, \sum_{j=1}^{i-1} n_j + 2, ..., \sum_{j=1}^{i} n_j + l_i \right\}$$

of $[\mathbf{n}]$ for each i. Then the map $\Phi : \boldsymbol{\mathcal{M}}^{n_1,n_2,...,n_k} \longrightarrow \boldsymbol{\mathcal{B}}^n$ defined by $\Phi(\mathcal{N}) = \bigcup_{i=1}^{k} \mathcal{N}_i$ is an order embedding of $\boldsymbol{\mathcal{M}}^{n_1,n_2,...,n_k}$ into $\boldsymbol{\mathcal{B}}^n$. For example, if $\mathcal{M}$ is the multiset $(\{A, B, C\}, m)$ with $m(A) = m(B) = m(C) = 3$, then $n = 3 + 3 + 3 = 9$. For the submultiset $\mathcal{N} =(\{A, B, C\}, m')$ with $m'(A) = m'(B) = m'(C) = 2$, we have $\Phi(\mathcal{N}) = \{1, 2\} \cup \{4, 5\} \cup \{7, 8\}$, a subset of $[\mathbf{9}]$. In this way, we can think of a multiset and its submultisets as subposets of an appropriate subset lattice.

Example 2.2.5. $\mathcal{I}^2(k)$ may be generalized. Let $k, n_1, n_2, ...,$ and n_k be positive integers. Then, there is a poset $\mathcal{I}^k(n_1, n_2, ..., n_k) = \left\{(i_1, i_2, ..., i_k) \in \mathbb{Z}^k \mid 0 \leq i_j \leq n_j\right\}$ with the obvious analogous order relation and rank function called the **integer lattice**[5].

The next proposition gives us an important and useful way to label the elements of a multiset lattice.

Proposition 2.2.6. $\boldsymbol{\mathcal{M}}^{n_1,n_2,...,n_k} \cong \mathcal{I}^k(n_1, n_2, ..., n_k)$.

Proof. Consider the map taking the submultiset $(\mathcal{S}, l)$ to the ordered k-tuple $(l(X_1), l(X_2), ..., l(X_k))$. This map is certainly order preserving and injective. In fact, given any element $(a_1, a_2, ..., a_k) \in \mathcal{I}^k(n_1, n_2, ..., n_k)$, the multiset $(\mathcal{S}, l)$ with $l(X_i) = a_i$ for each i is guaranteed to be a submultiset of $\mathcal{M}$. The map taking $(a_1, a_2, ..., a_k)$ to $(\mathcal{S}, l)$ is also order preserving and injective. In fact, the two maps are obviously inverses of each other and our proposition follows. $\square$

It is often convenient to replace $(a_1, a_2, ..., a_k)$ with the concatenation $a_1a_2...a_k$. This gives us a useful representation of multiset lattices. Often such a representation makes it easy to see connections with other poset structures. Here is a well known example.

[5]We will follow that standard convention of writing $\mathcal{I}^k(n)$ whenever $n_i = n_j$ for each i, j pair. Similarly, we write $\mathcal{I}(n)$ whenever $k = 1$.

Example 2.2.7. The **lattice of divisors of an integer** n, denoted $D(n)$, defined by introducing, to the set $\{k \in \mathbb{N} | dk = n \text{ for some } d \in \mathbb{N}\}$, the ordering given by $s \preceq t$ exactly when s divides t. Using the prime factorization $n = \prod_{i=1}^{k} p_i^{n_i}$, we get an order isomorphism with $\boldsymbol{\mathcal{M}}^{n_1,n_2,\ldots,n_k}$ by mapping $s = \prod_{i=1}^{k} p_i^{a_i}$ to the multiset $a_1 a_2 \ldots a_k$.

The divisor lattice $D(2^3 3^2 5)$ is given below in fig 2.2. Note that, for example, divisor $2^2 3$ corresponds to the submultiset $(\{2,3,5\}, l)$ where $l(2) = 2$, $l(3) = 1$, and $l(5) = 0$. The divisor lattice $D(2^3 3^2 5)$ is isomorphic to $\boldsymbol{\mathcal{M}}^{1,2,3}$.

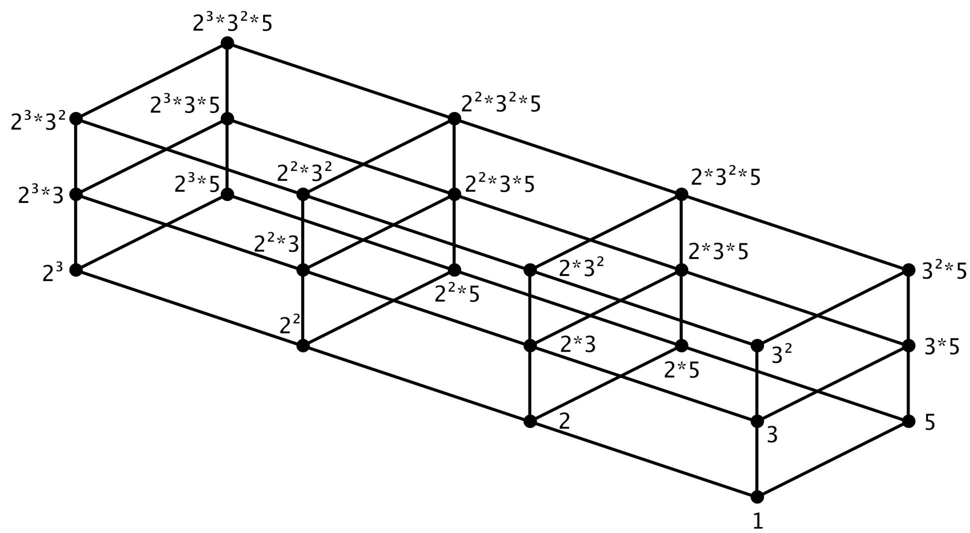

Figure 2.2: The divisor lattice $D(2^3 3^2 5)$

Proposition 2.2.8. $\boldsymbol{\mathcal{M}}^{n_1,n_2,\ldots,n_k} \cong \prod_{i=1}^{k} \mathcal{C}_{n_i}$

Proof. Note that, from the definition, it is obvious that $\mathcal{I}^k(n_1, n_2, \ldots, n_k) \cong \prod_{i=1}^{k} \mathcal{I}(n_i)$. Since $\mathcal{I}(n) \cong \mathcal{C}_n$, the proposition follows from Proposition 2.2.1. $\square$

Proposition 2.2.9. $\boldsymbol{\mathcal{M}}^{n_1,n_2,\ldots,n_k}$ *is symmetric.*

Proof. The map $\boldsymbol{\mathcal{M}}^{n_1,n_2,...,n_k} \xrightarrow{\Psi} (\boldsymbol{\mathcal{M}}^{n_1,n_2,...,n_k})^{\downarrow}$ that sends $a_1a_2...a_k \longrightarrow (n_1-a_1)(n_2-a_2)...(n_k-a_k)$ provides an invertible order preserving bijection from $\boldsymbol{\mathcal{M}}^{n_1,n_2,...,n_k}$ to its dual. $\square$

Of course, as a special case of these two propositions, we get that $\boldsymbol{\mathcal{B}}^n \cong \prod_{i=1}^{k} \mathcal{C}_1$ and also symmetric. Note that since $\mathcal{C}_1 \cong [\mathbf{2}]$, some authors denote the boolean lattice with $\mathbf{2}^n$.

A few other important examples follow, although the list is far from complete. [6] Given an algebraic structure, e.g. a group, we may define a poset of substructures ordered by set inclusion. This will always result in a subposet of a boolean lattice. In this paper, we will (briefly) discuss subgroup and vector subspace lattices.

Example 2.2.10. Given a group G, we denote the poset of subgroups of G by $\text{sub}(G)$, called the **subgroup lattice of** G. Since the poset $\text{sub}(G)$ always has $\{e\}$ as a unique minimal element and G itself as a unique maximal element, it becomes graded under the rank function $r(H)$ defined to be the minimum size of any subset of H that generates H as a group. Note that our assumption of finiteness of $\text{sub}(G)$ does not require that G itself be finite, but only that G have a finite number of subgroups.

Similarly, so long as V is a finite dimensional vector space over a finite field $\mathbb{F}_q$, its subspace poset will also be finite.

Example 2.2.11. Since vector spaces over the same field, of the same finite dimension, are isomorphic, we denote the poset of subspaces of an n dimensional vector space over $\mathbb{F}_q$ by $L[n,q]$, called the **subspace lattice** of $\mathbb{F}_q^n$. Since $L[n,q]$ poset always has $\{0\}$ as a unique minimal element and V itself as a unique maximal element, it becomes graded under the rank function $r(W) = \dim(W)$.

Example 2.2.12. Given a set $\mathcal{S}$ with n elements, a partition of $\mathcal{S}$ is a collection, $\{\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_k\}$, of nonempty disjoint subsets of $\mathcal{S}$, such that for each $X \in \mathcal{S}$, $X \in \mathcal{S}_i$ for some i. The sets $\mathcal{S}_i$ are called the **blocks** of the partition. A partition $\{\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_l\}$ of $\mathcal{S}$ is a **refinement** of $\{\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_k\}$ if $k \leq l$ and for each i, there is a j so that $\mathcal{S}_i \subseteq \mathcal{T}_j$. The **partition lattice** of a set $\mathcal{S}$ is the set

[6] Some nice lists may be found in [3] Section 1.3, [8] Sections 1.4-1.7, and [2] Section I.2.

of partitions of $\mathcal{S}$ ordered so that $\{\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_k\} \preceq \{\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_l\}$ exactly when $\{\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_k\}$ is a refinement of $\{\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_l\}$. Since the structure of this poset only depends on the size of $\mathcal{S}$, it is denoted Π_n.

Under this ordering, the partition of $\mathcal{S}$ into n blocks is a unique minimal element and the set $\mathcal{S}$ itself, the partition into one block, is a unique maximal element. It follows that Π_n becomes a graded poset under the rank function $r\left(\{\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_k\}\right) = n - k$.

Our final example is the only example considered here that does not happen to be a lattice.

Example 2.2.13. Let D and C be finite sets and consider the set of all functions f mapping some subset of D into C. If g is another such function, we introduce an ordering of this set by requiring that $f \preceq g$ just in case $\text{dom}(f) \subseteq \text{dom}(g)$ and $f(a) = g(a)$ for all $a \in \text{dom}(f)$. This poset is called the **function poset**. Since the structure of this poset is determined entirely by $n = |D|$ and $k = |C|$ it is denoted $\mathcal{F}_k^n$. Of course, the function whose domain is the empty set is a unique minimal element of this set, however, note that so long as $|C| > 1$, there is no maximal element. Still we may introduce the rank function $r(f) = |\text{dom}(f)|$. Under this rank function, f is maximal if and only if $r(f) = |D|$ and it follows that $\mathcal{F}_k^n$ is graded.

Suppose that $D = d_1, d_2, ..., d_n$ and $C = c_1, c_2, ..., c_k$. A function f partially defined from D to C can be uniquely represented as an ordered n-tuple of integers $(a_1, a_2, ..., a_n)$ as follows. First, let ν be the map from C to $\{k\}$ given by $\nu(c_i) = i$. Now set $a_i = \begin{cases} \nu(f(d_i)) & \text{if } d_i \in \text{dom} f \\ 0 & \text{otherwise} \end{cases}$. This defines an injective, order preserving map from $\mathcal{F}_k^n$ to the integer lattice $\mathcal{I}^n(k)$ and allows us to conveniently restate the order relation on $\mathcal{F}_k^n$ as $(a_1, a_2, ..., a_n) \preceq (b_1, b_2, ..., b_n)$ whenever, for each i, either $a_i = b_i$ or $a_i = 0$.

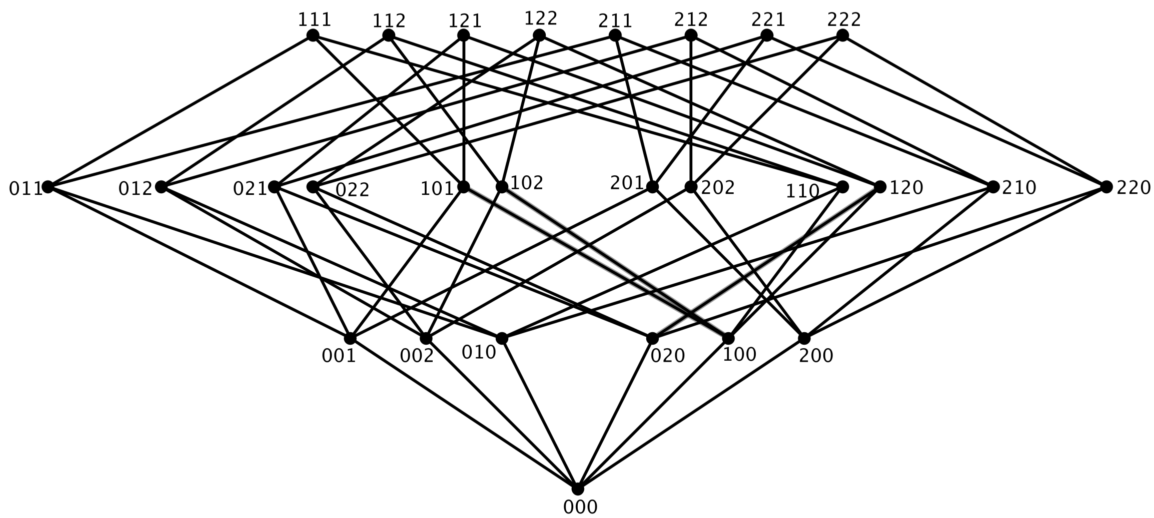

Figure 2.3: The function poset $\mathcal{F}_k^n$

In Figure 2.3 we have the function poset $\mathcal{F}_2^3$ with elements labeled according to this map. It is worth pointing out that since, for example, $(1, 0, 0, ..., 0)$ and $(2, 0, 0, ..., 0)$ are incomparable in $\mathcal{F}_k^n$ but certainly comparable in $\mathcal{I}^n(k)$, this map is not an order embedding whenever $k > 1$.

2.3 Lattices

Lattice theory is a huge subject that we will only briefly touch here. Although not technically necessary for our arguments, lattice theory provides a useful lens through which to view our problem. The goal in this section is to prove an interesting correspondence between posets and a certain class of lattices. First we introduce the basics of lattice theory. In what follows we continue to assume that all structures in sight are finite. [7]

Given a poset $(\mathcal{P}, \preceq)$, let $\wedge$ and $\vee$ be associative and commutative binary operations defined on $\mathcal{P}$. We call $(\mathcal{P}, \preceq, \wedge, \vee)$ a **lattice** if the following statements hold true for each X and $Y \in \mathcal{P}$:

(I) $X \wedge (X \vee Y) = X \vee (X \wedge Y) = X$

and

[7] The main result of this section, and the argument used to prove it, has a natural extension to the so called **finitary** posets, that is, infinite posets $\mathcal{P}$ with the property that the set $\{Y \in \mathcal{P} \,|\, 0 \preceq Y \preceq X\}$ is finite for each $X \in \mathcal{P}$.

(II) $X \wedge Y = X \iff X \vee Y = Y \iff X \preceq Y$.

Intuitively, $\wedge$ and $\vee$ are intended to capture the properties of greatest lower bound and least upper bound respectively. For example, it follows from (II) that $X \preceq Z$ and $Y \preceq Z$ if and only if

$$Z \vee (X \vee Y) = (Z \vee X) \vee Y = Z \vee Y = Z$$

or, equivalently, $X \vee Y \preceq Z$. A similar argument shows that $Z \preceq X$ and $Z \preceq Y$ if and only if $Z \preceq X \wedge Y$. In fact, lattices can be equivalently defined as posets for which each pair of elements has a unique greatest lower bound and least upper bound. We follow the standard practice of calling $X \wedge Y$ the **meet** of X and Y and $X \vee Y$ the **join** of X and Y. Because each operation is commutative and associative, the abbreviations $\bigvee_{i=1}^{n} a_i = a_1 \vee a_2 \vee ... \vee a_n$ and $\bigwedge_{i=1}^{n} a_i = a_1 \wedge a_2 \wedge ... \wedge a_n$ are common.

Proposition 2.3.1. *In any finite lattice L, there is a maximum element and a minimum element.*

Proof. Suppose that X and Y are both minimal in L. In this case, $X \wedge Y \in L$. since $X \wedge Y \preceq X$ and $X \wedge Y \preceq Y$, we have immediately from (II) that $X = X \wedge Y = Y$. Replacing $\wedge$ with $\vee$ shows additionally that if X and Y are both maximal in L then $X = Y$. $\square$

It is standard to denote the minimum and maximum element of a lattice by $\mathbf{0}$ and $\mathbf{1}$ respectively. This makes $\mathcal{P}$ into a ring like structure with the join operation $\vee$ acting as addition and the meet operation $\wedge$ acting as multiplication. Note that under these operations, an order ideal $\mathcal{O}$ is an ideal in the ring theoretic sense that for each $X \in \mathcal{O}$, $X \wedge Y \in \mathcal{O}$ for each $Y \in \mathcal{O}$. However lattices are not, in general, rings because their elements can fail to have unique inverses.

A lattice $(\mathcal{P}, \preceq, \wedge, \vee)$ becomes a **distributive lattice** if for each X, Y, and $Z \in \mathcal{P}$, we have:

(III) $X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z)$ or equivalently $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$.

The most important finite distributive lattices are the boolean lattices, whose poset structure we defined above.

Proposition 2.3.2. $(\boldsymbol{\mathcal{B}}^n, \subseteq, \bigcap, \bigcup)$ *is a distributive lattice.*

Proof. Note that associativity, commutativity, (II), and (III) are all well known properties of $\bigcap$ and $\bigcup$.[8] To see that (I) holds, note that from (II) and (III),

$$X = X \bigcap (X \bigcup Y) = (X \bigcap X) \bigcup (X \bigcap Y) = X \bigcup (X \bigcap Y).$$

$\square$

In fact, this last argument is perfectly general so that (II) and (III) always imply (I). The converse is, however, not true as the following example, usually known as M_5, shows. In this lattice, $A \vee (B \wedge C) = A \vee 0 = A$ but $(A \vee B) \wedge (A \vee C) = 1 \wedge 1 = 1$.

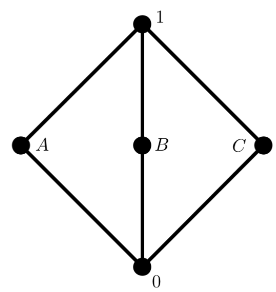

Figure 2.4: The lattice M_5

Proposition 2.3.2 can be generalized to multisets. Here, we define $(\mathcal{M}, l) \vee (\mathcal{M}, s) = (\mathcal{M}, t)$ where $t(X) = \max\{l(X), s(X)\}$ for each $X \in \mathcal{M}$ and $(\mathcal{M}, l) \wedge (\mathcal{M}, s) = (\mathcal{M}, t)$ where $t(X) = \min\{l(X), s(X)\}$ for each $X \in \mathcal{M}$.

Proposition 2.3.3. *Under the* min *and* max *operations defined above,* $\boldsymbol{\mathcal{M}}^{n_1, n_2, \ldots, n_k}$ is a distributive lattice.

Proof. It is straightforward to verify associativity, commutativity, and lattice properties (I) and (II) hold for these operations. To verify (III), note that if $X \in \mathcal{M}$ and p, l, and s are multiplicity functions of $\mathcal{M}$, then $\max\{p(X), \min\{l(X), s(X)\}\} = \min\{\max\{p(X), l(X)\}, \max\{p(X), s(X)\}\}$. $\square$

[8] See e.g. Halmos(CIT) pages 13-15.

Subgroup and vector subspace lattices also employ $\bigcap$ as a meet operation. However, because the union of, for example, two groups is not necessarily a group, we have to modify this operation so that the join of H and K is defined to be the subgroup generated by $H \bigcup K$. The join operation for $\mathbb{F}_q^n$ is defined analogously. Although it is straightforward to verify that these operations satisfy associativity, commutativity, and lattice properties (I) and (II), they are not, in general, distributive. Although they will not play a role in our paper, non-distributive join and meet operations can be given to the partition lattice as well.[9]

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are lattices, an order preserving map $\mathcal{L}_1 \xrightarrow{\phi} \mathcal{L}_2$ is a **lattice homomorphism** if it preserves both meets and joins. A bijective lattice homomorphism is a **lattice isomorphism**. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are lattices with the same ordering, the same join and meet operations, and if $\mathcal{L}_1 \subseteq \mathcal{L}_2$, then we call $\mathcal{L}_1$ a **sublattice of $\mathcal{L}_2$.**

Proposition 2.3.4. *A subposet $\mathcal{P}$ of a lattice $\mathcal{L}$ is a sublattice of $\mathcal{L}$ if and only if $\mathcal{P}$ is closed under the join and meet operations of $\mathcal{L}$. A sublattice of a distributive lattice is distributive.*

Proof. If it is closed under the join and meet operations, then $\mathcal{P}$ certainly satisfies (I), (II), and, if need be, (III) as all of these properties are inherited from the lattice structure of $\mathcal{L}$. If $\mathcal{P}$ is not closed under either operation, then it is not a lattice under those operations. $\square$

Two important order structures related to a poset $\mathcal{P}$ will play a central role in our characterization of linear extensions below. First, we (optimistically) define $\mathfrak{L}(\mathcal{P})$, **the lattice of ideals of $\mathcal{P}$,** to be the set of ideals of $\mathcal{P}$ under the subset ordering with the operations of union and intersection.[10]

Proposition 2.3.5. *The lattice of order ideals $(\mathfrak{L}(\mathcal{P}), \subseteq, \bigcup, \bigcap)$ of a finite poset $\mathcal{P}$ is indeed a distributive lattice.*

Proof. Note that $\mathfrak{L}(\mathcal{P})$ is contained in the subset lattice $\mathcal{B}^{\mathcal{P}}$ so that the distributive lattice operations $\bigcup$ and $\bigcap$ are inherited. It remains to check that $\mathfrak{L}(\mathcal{P})$ is closed under these operations. Let $\mathcal{O}_1$ and

[9] See e.g. [2] section I.2.

[10] In what follows, we are indebted to the excellent account in[3].

$\mathcal{O}_2$ be order ideals of $\mathcal{P}$. If $Y \in \mathcal{O}_1 \bigcup \mathcal{O}_2$, then either $Y \in \mathcal{O}_1$ or $Y \in \mathcal{O}_2$. In either case, we have that $X \preceq_{\mathcal{P}} Y$ implies that $X \in \mathcal{O}_1 \bigcup \mathcal{O}_2$. Similarly, if $Y \in \mathcal{O}_1 \bigcap \mathcal{O}_2$, then $Y \in \mathcal{O}_1$ and $Y \in \mathcal{O}_2$. Again, we have $X \preceq_{\mathcal{P}} Y$ implies that $X \in \mathcal{O}_1$ and $X \in \mathcal{O}_2$. Therefore $\mathcal{O}_1 \bigcup \mathcal{O}_2$ and $\mathcal{O}_1 \bigcap \mathcal{O}_2$ are both order ideals of $\mathcal{P}$. Since both $\emptyset$ and $\mathcal{P}$ are order ideals of $\mathcal{P}$, the theorem follows. □

Our main result for this section, Theorem 2.3.9, can be found in [17] where it is called The Fundamental Theorem of Finite Distributive Lattices; it is also known as Birkhoff's Representation Theorem.[11] To prove this result, we will need to prove a few lemmas. Let us call an element X of a distributive lattice $\mathcal{L}$ **join irreducible** if for every Y and $Z \in \mathcal{L}$, $X = Y \vee Z$ if and only if $X = Y$ or $X = Z$. We will write $\mathfrak{J}(\mathcal{L})$ for the set of join irreducible elements of $\mathcal{L}$. The following lemmas will be necessary to prove our main result.

Lemma 2.3.6. *If $\mathcal{L}$ is a distributive lattice, then for each $X \in \mathcal{L}$, there is a unique antichain $\mathcal{A} \subseteq \mathfrak{J}(\mathcal{L})$ so that $X = \bigvee_{Y_i \in \mathcal{A}} Y_i$.*

Proof. If X is itself join irreducible, then clearly $\mathcal{A} = \{X\}$ is an antichain in $\mathfrak{J}(\mathcal{L})$ and $X = X$. Otherwise, we may suppose that $X = Y \vee Z$ for some $Y, Z \in \mathcal{L}$. If either or both of Y and Z fails to be join irreducible, then they may be likewise replaced. With only a finite number of elements to work with, this process must end with $X = \bigvee_{Y_i \in \mathcal{S}} Y_i$ for some subset $\mathcal{S}$ of $\mathfrak{J}(\mathcal{L})$. We can then use lattice property (II) to remove any elements of $\mathcal{S}$ that are not join irreducible. This process ends with an antichain $\mathcal{A}$ of join irreducible elements of $\mathcal{L}$ so that $X = \bigvee_{Y_i \in A} Y_i$.

To see that $\mathcal{A}$ is unique, if $X = \bigvee_{Y_i \in \mathcal{A}} Y_i = \bigvee_{Z_j \in \mathcal{B}} Z_j$, and both $\mathcal{A}$ and $\mathcal{B}$ are antichains in $\mathfrak{J}(\mathcal{L})$, then for each Y_i, we have that $Y_i \preceq \bigvee_{Z_j \in \mathcal{B}} Z_j$ so that (II) guarantees that there is a $Z_j \in \mathcal{B}$ so that $Y_i \preceq Z_j$. This argument can also show that for each Z_j, there is a Y_i so that $Z_j \preceq Y_i$. Recalling that $\preceq$ must be transitive and that $\mathcal{A}$ and $\mathcal{B}$ are both antichains we see that the sets $\mathcal{A}$ and $\mathcal{B}$ must be equal. □

The next lemma shows that the join irreducible elements of $\mathfrak{L}(\mathcal{P})$ are particularly simple.

[11] See, for example, [2]

Lemma 2.3.7. *If $\mathcal{P}$ is a poset, then $\mathcal{O} \in \mathfrak{L}(\mathcal{P})$ is join irreducible if and only if $\mathcal{O}$ is a principal order ideal of $\mathcal{P}$.*

Proof. To prove this lemma, first note that from the definition of order ideal, we have $\mathcal{O} = \bigcup\limits_{X \in \mathcal{O}^+} \langle X \rangle$ where $\mathcal{O}^+$ is the set of maximal elements in $\mathcal{O}$. Note that each $\langle X \rangle$ is itself an element of $\mathfrak{L}(\mathcal{P})$. Supposing that there are $k > 1$ maximal elements of $\mathcal{O}$, let us index the members of $\mathcal{O}^+$ as $X_1, X_2, ..., X_k$. By the associativity of the join operation, we can rewrite the above identity as $\mathcal{O} = \left(\bigcup\limits_{i=1}^{k-1} \langle X_i \rangle \right) \bigcup \langle X_k \rangle$ thus non-trivially reducing $\mathcal{O}$ to the join of two other elements of $\mathfrak{L}(\mathcal{P})$. The result is that $\mathcal{O}$ has more than one maximal element if and only if $\mathcal{O}$ can be written as the join of two strictly smaller ideals. $\square$

This lemma suggests a natural map Φ from $\mathcal{P}$ to $\mathfrak{J}(\mathfrak{L}(\mathcal{P}))$ taking each $X \in \mathcal{P}$ to $\langle X \rangle$. This map is clearly a bijection. Noting that $\langle X \rangle \subseteq \langle Y \rangle$ if and only if $X \subseteq Y$ this map is actually order preserving and invertible. This leads to the following corollary.

Corollary 2.3.8. $\mathcal{P} \cong \mathfrak{J}(\mathfrak{L}(\mathcal{P}))$

Corollary 2.3.8 provides us with a clue to proving the fundamental theorem.

Theorem 2.3.9. *(The fundamental theorem of finite distributive lattices) $\mathcal{L}$ is a distributive lattice if and only if there is a poset $\mathcal{P}$ so that $\mathcal{L} \cong \mathfrak{L}(\mathcal{P})$.*

Proof. Given a poset $\mathcal{P}$, Proposition 2.3.5 above ensures that $\mathfrak{L}(\mathcal{P})$ is always a distributive lattice. Given a distributive lattice $\mathcal{L}$, we must construct a set $\mathcal{P}$ so that $\mathcal{L} \cong \mathfrak{L}(\mathcal{P})$. Lemma 2.3.7 and Corollary 2.3.8 suggest that $\mathfrak{J}(\mathcal{L})$ is a good candidate. We need only show that $\mathcal{L} \cong \mathfrak{L}(\mathfrak{J}(\mathcal{L}))$. Let $X \in \mathcal{L}$ and define

$$I(X) = \{Y \in \mathfrak{J}(\mathcal{L}) \,|Y \preceq_{\mathcal{L}} X\}.$$

Then, $I(X)$ is an order ideal of $\mathfrak{J}(\mathcal{L})$, that is, an element of $\mathfrak{L}(\mathfrak{J}(\mathcal{L}))$. The map Ψ sending $X \in \mathcal{L}$ to $I(X)$ is order preserving since $X \preceq_{\mathcal{L}} Y$ implies that $I(X) \subseteq I(Y)$. In Lemma 2.3.6, we saw that each X could be written uniquely as a product of incomparable elements of $\mathfrak{J}(\mathcal{L})$. Since $X = \bigvee\limits_{Y_i \in A} Y_i$ implies, for each $Y_i \in \mathcal{A}$, that $Y_i \preceq X$, we have that $\mathcal{A} \subseteq I(X)$. In fact, the proof of Lemma

2.3.6 shows that $\mathcal{A}$ is exactly the set of maximal elements of $I(X)$. It follows that $I(X)$ is an order preserving bijection from $\mathcal{L}$ to $\mathfrak{L}(\mathfrak{J}(\mathcal{L}))$. To see that $I(X)$ is also a lattice isomorphism, note that

$$\begin{aligned} I(X \vee Y) &= \{Z \in \mathfrak{J}(\mathcal{L}) \,|\, Z \preceq_{\mathcal{L}} X \vee Y\} \\ &= \{Z \in \mathfrak{J}(\mathcal{L}) \,|\, Z \preceq_{\mathcal{L}} X \text{ or } Z \preceq Y\} \\ &= I(X) \bigcup I(Y). \end{aligned}$$

The argument for $\wedge$ is made dually, replacing $\vee$ with $\wedge$ and "or" with "and". See the figure below for an illustration.[12] $\square$

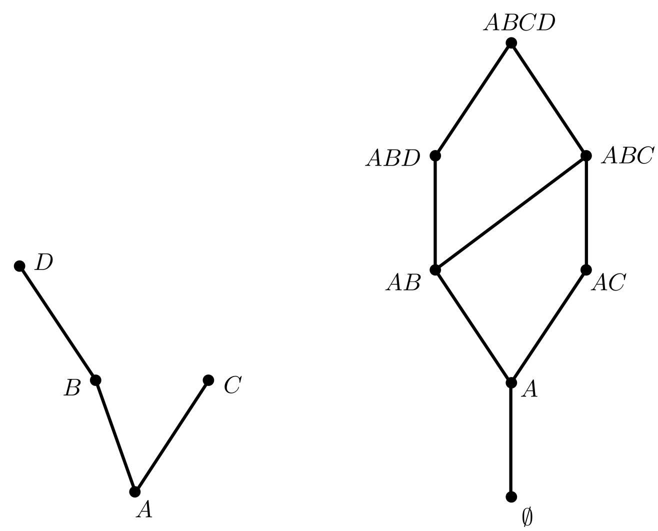

Figure 2.5: A small poset and its lattice of ideals

The join irreducible elements of $\mathfrak{L}(\mathcal{P})$, namely A, AB, AC, and ABD, are exactly the principal ideals of $\mathcal{P}$. It is easy to see that these elements form a subposet of $\mathfrak{L}(\mathcal{P})$ isomorphic to $\mathcal{P}$.

A consequence of this theorem is that finite distributive lattices are uniquely generated by their join irreducible elements. A consequence of this is that any order preserving map $\mathcal{P} \xrightarrow{\phi} \mathcal{Q}$ can be

[12] In this image, we adopt the useful convention of labeling the order ideals of $\mathcal{P}$ with a concatenation of its elements. So, for example, "ABD" refers to the order ideal $\{A, B, D\}$.

extended to a lattice homomorphism $\mathfrak{L}(\mathcal{P}) \xrightarrow{\overline{\phi}} \mathfrak{L}(\mathcal{Q})$ by letting, for example, $\overline{\phi}(a \wedge b) = \phi(a) \wedge \phi(b)$. Similarly, it follows from (II) that any lattice homomorphism must be order preserving and map join irreducible elements to join irreducible elements so that the map $\phi \longrightarrow \overline{\phi}$ has an obvious inverse in the restriction of $\overline{\phi}$ to $\mathcal{P}$. In the language of category theory, the fundamental theorem provides us with covariant functors between the category of finite posets with order preserving maps to the category of finite distributive lattices with lattice homomorphisms that act as the identity functors under composition.

For exceptionally simple classes of posets, the lattice of order ideals can sometimes be described easily.

Proposition 2.3.10. $\mathfrak{L}([\mathbf{n}]) \cong [\mathbf{n+1}]$

Proof. Any order ideal of $[\mathbf{n}]$ is a chain $\mathcal{C}$. The map $\mathcal{C} \longrightarrow |\mathcal{C}| + 1$ is an invertible order preserving bijection from $\mathfrak{L}([\mathbf{n}])$ to $[\mathbf{n+1}]$. □

Proposition 2.3.11. $\boldsymbol{\mathcal{B}}^n \cong \mathfrak{L}(\{n\})$ *and more generally,* $\boldsymbol{\mathcal{M}}^{n_1,n_2,\ldots,n_k} \cong \mathfrak{L}([\mathbf{n_1}] + [\mathbf{n_2}] + \ldots + [\mathbf{n_k}])$.

Proof. Both $\boldsymbol{\mathcal{B}}^n$ and $\boldsymbol{\mathcal{M}}^{n_1,n_2,\ldots,n_k}$ are finite distributive lattices. An element of $\boldsymbol{\mathcal{B}}^n$ is join irreducible if and only if it has exactly one element.

An element $(\mathcal{S}, l)$ of $\boldsymbol{\mathcal{M}}^{n_1,n_2,\ldots,n_k}$ is join irreducible if and only if $\mathcal{S}$ has exactly one element. For each $X \in \mathcal{S}$, the set of multisets $\{(\{X\}, m_i)\}_{1 \leq i \leq l(X)}$ forms a chain of join irreducible elements of length l. □

The map $\mathfrak{L}$ provides an interesting example of an injective map from an infinite set to a proper subset of itself. Note also that it can be iterated. Of particular combinatorial interest is $\mathfrak{L}(\boldsymbol{\mathcal{B}}^n) \cong \mathfrak{L}(\mathfrak{L}(\{n\}))$.[13] In figure 2.6, below, we show $\mathfrak{L}\left(\boldsymbol{\mathcal{B}}^3\right)$. The set $\mathfrak{J}(\mathfrak{L}(\boldsymbol{\mathcal{B}}^3))$ appears in the lattice as open circles. The reader may readily verify that this subposet is isomorphic to $\boldsymbol{\mathcal{B}}^3$.

In the arguments below, we will refer to the set of antichains of a poset $\mathcal{P}$ which can be made into a poset $\mathfrak{A}(\mathcal{P})$ by defining $\mathcal{A} \preceq_{\mathfrak{A}} \mathcal{B}$ if and only if for each $X \in \mathcal{A}$, there is a $Y \in \mathcal{B}$ so that

[13] See for example [2] and [8].

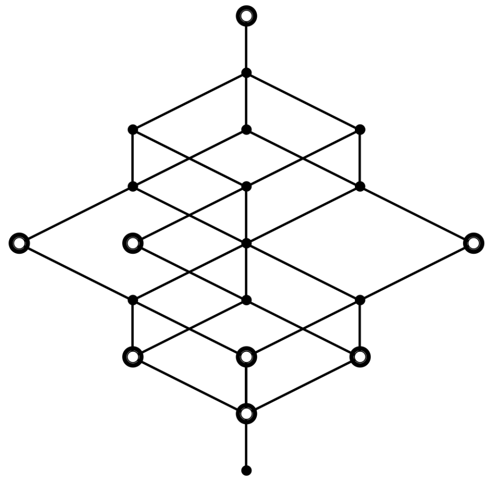

Figure 2.6: The lattice $\mathfrak{L}(\boldsymbol{\mathcal{B}}^3)$

$X \preceq_{\mathcal{P}} Y$ or equivalently, $\langle \mathcal{A} \rangle \subseteq \langle \mathcal{B} \rangle$. As a map from $\mathfrak{A}(\mathcal{P})$ to $\mathfrak{L}(\mathcal{P})$, note that $\langle \_ \rangle$ provides a nice correspondence between order ideals and antichains that we have already made implicit use of in our proof of Theorem 2.3.8.

Proposition 2.3.12. *As posets,* $\mathfrak{L}(\mathcal{P}) \cong \mathfrak{A}(\mathcal{P})$.

Proof. Consider the map from $\mathfrak{A}(\mathcal{P})$ to $\mathfrak{L}(\mathcal{P})$ sending $\mathcal{A}$ to $\langle \mathcal{A} \rangle$ and the map from $\mathfrak{L}(\mathcal{P})$ to $\mathfrak{A}(\mathcal{P})$ sending $\mathcal{O}$ to $\max \mathcal{O}$. Both maps are clearly order preserving bijections. If $\mathcal{A} \in \mathfrak{A}(\mathcal{P})$, then for each $X \in \mathcal{P}$, $X \in \mathcal{A}$ if and only if X is maximal in $\langle \mathcal{A} \rangle$. It follows that $\max\{\langle \mathcal{A} \rangle\} = \mathcal{A}$ and $\langle \max\{\mathcal{O}\} \rangle = \mathcal{O}$ so that $\langle \rangle$ is an order isomorphism. $\square$

Of course, the lattice structure of $\mathfrak{L}(\mathcal{P})$ can now be applied to $\mathfrak{A}(\mathcal{P})$.

Chapter 3

Linear Extensions

3.1 Linear Extensions

Given two partial orders $\mathcal{P}$ and $\mathcal{Q}$ on the same set, we say that $\mathcal{Q}$ is an **extension** of $\mathcal{P}$ if, for each $x, y \in \mathcal{P}$, $x \preceq_{\mathcal{Q}} y$ whenever $x \preceq_{\mathcal{P}} y$. It is natural, in this context, to view $\mathcal{P}$ and $\mathcal{Q}$ in terms of their order relations. In this light, $\mathcal{Q}$ is an extension of $\mathcal{P}$ whenever $\preceq_{\mathcal{P}}$ is a subset of $\preceq_{\mathcal{Q}}$ or, equivalently, whenever the identity map $\mathcal{P} \xrightarrow{\iota} \mathcal{Q}$ is order preserving. If $\mathcal{Q}$ is totally ordered then it is a **linear extension** of $\mathcal{P}$. Since, in this case, $\mathcal{Q}$ is isomorphic to $[n]$ for $n = |\mathcal{Q}|$, such a $\mathcal{Q}$ is a linear extension of $\mathcal{P}$ if and only if there is an order preserving bijection $\mathcal{P} \xrightarrow{\varepsilon} [n]$. In this case, we often call the map ε a linear extension of $\mathcal{P}$. Both perspectives are useful. On the one hand, it is helpful to see that we obtain a linear extension by adding appropriate elements of $\mathcal{P} \times \mathcal{P}$ to $\preceq_{\mathcal{P}}$ until the result of any further addition would not be a partial order. On the other hand, the map ε allows us index the elements of the total order $\mathcal{Q}$ by their place in the total order by writing $X_{\varepsilon(X)}$. This allows us to represent the total order explicitly as a chain $X_1 \preceq X_2 \preceq ... \preceq X_{|\mathcal{P}|}$ where the ith term, X_i, is the unique $X \in \mathcal{P}$ so that $\varepsilon(X) = i$. In what follows, we generally prefer this latter characterization, if for no better reason than that it provides us, by referring to the map ε, with a compact name for a linear extension.

Let $E(\mathcal{P})$ denote the set of all linear extensions of $\mathcal{P}$. The following propositions can be found in [18].

Proposition 3.1.1. *If $\mathcal{P}$ is a poset, then $E(\mathcal{P})$ is nonempty. If $\mathcal{P}$ is finite, then $E(\mathcal{P})$ is finite.*

Proof. If $\mathcal{P}$ is a toset, then $E(\mathcal{P}) = \{\mathcal{P}\}$. Otherwise, there is pair $(X, Y) \in \mathcal{P} \times \mathcal{P}$ with X and Y not comparable in $\preceq_{\mathcal{P}}$. Let $(X, Y)^{tr} = \{(X, Z) \,|\, Z \in \langle Y \rangle\}$, also called the **transitive closure** of (X, Y). Then, we claim that $\preceq_{\mathcal{P}} \bigcup (X, Y)^{tr}$ is always a partial order on the set $\mathcal{P}$. To see this, note that the relation inherits reflexivity from $\mathcal{P}$. Taking the transitive closure of (X, Y) ensures that the new relation is transitive. Finally, since X and Y are not comparable, every element of $\langle Y \rangle$ does not cover X. This ensures that the new relation is also antisymmetric.

Of course, this construction works whenever $\mathcal{P}$ is not a total order. This allows us to recursively define a sequence of order relations on the set $\mathcal{P}$. Let $\preceq_{\mathcal{P}_0} = \preceq_{\mathcal{P}}$, and $\preceq_{\mathcal{P}_i} = \preceq_{\mathcal{P}_{i-1}} \bigcup (X, Y)^{tr}$ for some incomparable pair $X, Y \in \mathcal{P}_{i-1}$. Since $\mathcal{P}$ is supposed to be finite, $\preceq_{\mathcal{P}} \subseteq \mathcal{P} \times \mathcal{P}$ must also be finite so that this construction must terminate for some i. In this case, we have a partial order $\mathcal{P}_j$ such that there is no pair of incomparable elements in $\preceq_{\mathcal{P}_j}$. This is equivalent to $\mathcal{P}_j$ being a total order. Since by our construction $\mathcal{P}_j$ is clearly an extension of $\mathcal{P}$, we have constructed our desired linear extension. □

Proposition 3.1.2. *If X and Y are incomparable in $\mathcal{P}$, then there is an $\varepsilon \in E(\mathcal{P})$ so that $X \preceq_{\varepsilon} Y$.*

Proof. In the proof of Proposition 2.4.1, we presented an algorithm for constructing a linear extension ε. To ensure that $X \preceq_{\varepsilon} Y$, it is sufficient to begin by adding $(X, Y)^{tr}$ to $\preceq_{\mathcal{P}}$. The result will be a poset to which we can further apply the algorithm. The result is a linear extension of $\mathcal{P}$ in which $X \preceq_{\varepsilon} Y$. □

Proposition 3.1.3. *$\mathcal{P}$ is the intersection of all of its linear extensions.*

Proof. This proposition is equivalent to the claim that $X \preceq_{\mathcal{P}} Y$ if and only if $X \preceq_{\varepsilon} Y$ for each $\varepsilon \in E(\mathcal{P})$. Of course by definition, $X \preceq_{\mathcal{P}} Y$ implies that $X \preceq_{\varepsilon} Y$ for each $\varepsilon \in E(\mathcal{P})$. Alternately, if $X \preceq_{\varepsilon} Y$ for each $\varepsilon \in E(\mathcal{P})$, then Proposition 2.4.2 guarantees that $X \preceq_{\mathcal{P}} Y$. □

Intuitively, we can imagine the following process for building a linear extension of $\mathcal{P}$. Beginning by choosing a minimal element, X_1, of $\mathcal{P}$, we then select a minimal element, X_2, of $\mathcal{P} - \{X_1\}$ and so on. Continuing on in this way, until the elements of $\mathcal{P}$ are exhausted, will result in a linear extension of $\mathcal{P}$. At the ith stage of construction, we have ordered some subset $\mathcal{O}_i = \{X_1, X_2, ..., X_i\}$ of $\mathcal{P}$. It is easy to see that the set $\mathcal{O}_i$ must be an ideal of $\mathcal{P}$ and that the map from $\mathcal{O}_i$ to $[\mathbf{i}]$ sending X_j to j is a linear extension of this set. The set of available choices for the next element in our ordering of $\mathcal{P}$ is then the set of minimal elements of $\mathcal{P} - \mathcal{O}_i$. If $X_1 \preceq X_2 \preceq ... \preceq X_{|\mathcal{P}|}$ is a linear extension of $\mathcal{P}$, then setting $\mathcal{O}_i = \{X_1, ..., X_i\}$ we obtain a unique sequence of order ideals $\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_{|\mathcal{P}|}$ so that $\mathcal{O}_i - \mathcal{O}_{i-1} = X_i$ for each i. In this way, the linear extensions of a poset $\mathcal{P}$ are in one to one correspondence with the maximal chains of $\mathfrak{L}(\mathcal{P})$.

As is clear from this argument, if ε and ε' are linear extensions of the same ideal $\mathcal{O}$ of $\mathcal{P}$, then they will have the same set of available choices given by $\min\{\mathcal{P} - \mathcal{O}\}$. It will be convenient in our arguments to use the $\mathfrak{a}$ from the lattice of order ideals in $\mathcal{P}$ to the lattice of antichains in $\mathcal{P}$ defined by $\mathfrak{a}\left(\mathcal{O}\right) = \min\left\{\mathcal{P} - \mathcal{O}\right\}$. Given this map, we define, for each ideal $\mathcal{O}$ of $\mathcal{P}$, the **choice antichain** of $\mathcal{O}$ to be the set $\mathfrak{a}(\mathcal{O})$. In the figure below, the open circles form an order ideal of $\boldsymbol{\mathcal{B}}^3$ while the solid diamonds make up its choice antichain.

We can thus associate a linear extension of $\mathcal{P}$ with a maximal ascending chain of order ideals $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_{|\mathcal{P}|}$ in $\mathfrak{L}(\mathcal{P})$ and a corresponding sequence of choice antichains $\mathfrak{a}(\mathcal{O}_1), \mathfrak{a}(\mathcal{O}_2), \ldots, \mathfrak{a}(\mathcal{O}_{|\mathcal{P}|})$ in $\mathfrak{A}(\mathcal{P})$. Note that $|\triangledown(\mathcal{O})|$ in $\mathfrak{L}(\mathcal{P})$ is equal to $|\mathfrak{a}(\mathcal{O})|$ in $\mathcal{P}$. Since each element of $\triangledown(\mathcal{O})$ is equal to $\mathcal{O} + \{X\}$ for some $X \in \mathcal{P}$, we can alternately define the choice antichain without reference to the poset $\mathcal{P}$ by $\mathfrak{a}(\mathcal{O}) = \left(\bigcup_{\mathcal{O}' \in \triangledown(\mathcal{O})} \mathcal{O}'\right) - \mathcal{O}$. It is clear from either definition that $\mathfrak{a}$ is an injective map. The following proposition shows that it is also surjective.

Proposition 3.1.4. *For each antichain $\mathcal{A}$, there is a unique order ideal $\mathcal{O}$ and filter $\mathcal{F}$ so that $\mathcal{A} = \mathfrak{a}(\mathcal{O})$, $\blacktriangledown(\mathcal{A}) = \mathcal{F}$ and $\mathcal{P} = \mathcal{A} + \mathcal{O} + \mathcal{F}$.*

Proof. Let $\mathcal{A}$ be an antichain in $\mathcal{P}$. Let $\mathcal{F} = \blacktriangledown(\mathcal{A})$. We claim that the order ideal $\mathcal{O} = \mathcal{P} - \mathcal{A} - \mathcal{F}$, has $\mathfrak{a}(\mathcal{O}) = \mathcal{A}$. To see this, note that $\mathcal{P} - \mathcal{O} = \mathcal{A} + \mathcal{F}$ and $\mathcal{A} = \min\{\mathcal{A} + \mathcal{F}\}$. $\square$

This is a property that the choice antichain map shares with the ideal-antichain map above.

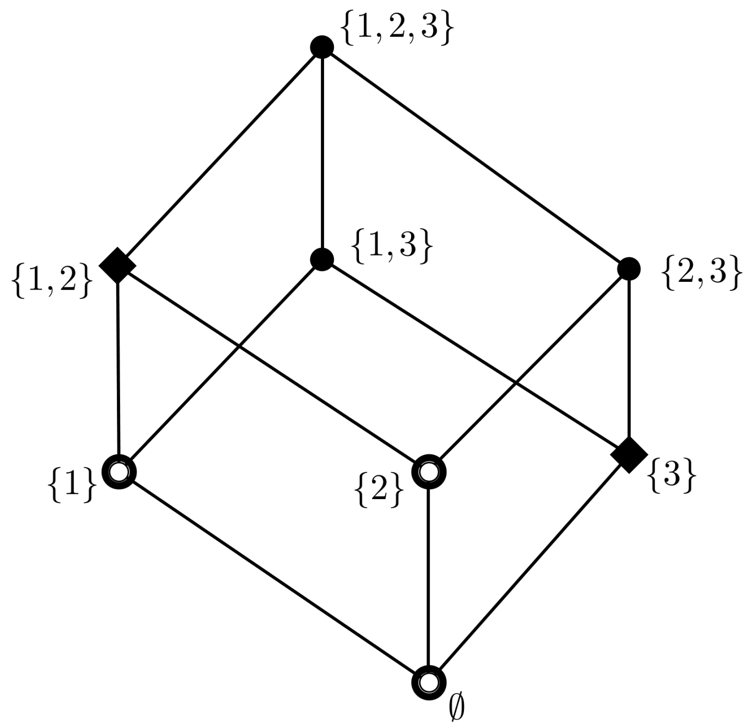

Figure 3.1: An ideal and its choice antichain in $\boldsymbol{\mathcal{B}}^3$

However, the two are also quite different. Indeed, the map $\mathfrak{a}$ is neither order preserving nor order reversing. Compare the order ideal, $\mathcal{O}_2$, of $\boldsymbol{\mathcal{B}}^3$ and its choice antichain given in Figure 3.2 below to the order ideal, $\mathcal{O}_1$, of $\boldsymbol{\mathcal{B}}^3$ given in Figure 3.1.

Note that $\mathcal{O}_1 \subseteq \mathcal{O}_2$ but $\mathfrak{a}(\mathcal{O}_1)$ is not contained in $\langle \mathfrak{a}(\mathcal{O}_2) \rangle$ and vice versa.

The following proposition shows that the choice antichain map is slightly better behaved when acting on order ideals with maximal choice antichains.

Proposition 3.1.5. *Let $\mathcal{A}$ be an antichain and $\mathcal{O}$ be an ideal such that $\mathcal{A} = \mathfrak{a}(\mathcal{O})$. Then, $\mathcal{O} = \blacktriangle(\mathcal{A})$ if and only if $\mathcal{A}$ is maximal in $\mathcal{P}$.*

Proof. If $\mathcal{O} = \blacktriangle(\mathcal{A})$, then $\mathcal{A} = \mathfrak{a}(\mathcal{O}) = \mathfrak{a}(\blacktriangle(\mathcal{A})) = \min(\mathcal{P} - \blacktriangle(\mathcal{A}))$. For any $X \in \mathcal{P} - \mathcal{A}$, either

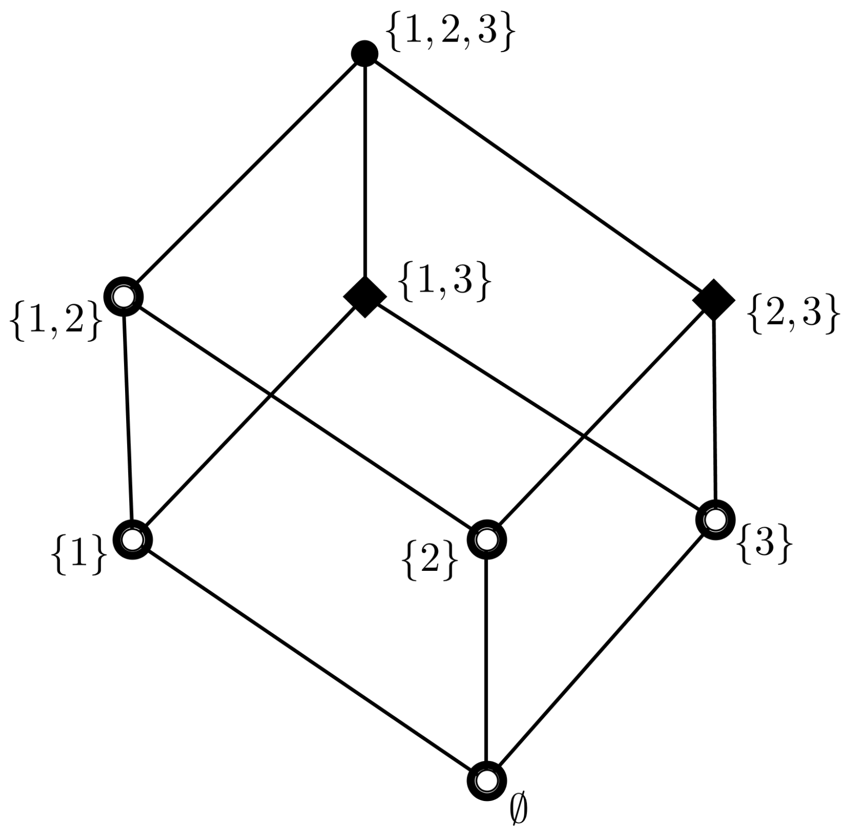

Figure 3.2: Another ideal and its choice antichain in $\boldsymbol{\mathcal{B}}^3$

$X \in \blacktriangle(\mathcal{A})$ or $X \in \mathcal{P} - \blacktriangle(\mathcal{A})$. In the latter case, there is some $Y \in \mathcal{A}$ so that $Y \preceq X$, while in the former, there is some $Y \in \mathcal{A}$ so that $X \preceq Y$. In either case, $\mathcal{A} \bigcup \{X\}$ is not an antichain.

If $\mathcal{A}$ is maximal in $\mathcal{P}$ then Proposition 3.1.4 implies that $\mathcal{P} = \blacktriangle(\mathcal{A}) + \mathcal{A} + \blacktriangledown(\mathcal{A})$. In this case, $\mathfrak{a}(\blacktriangle(\mathcal{A})) = \min(\mathcal{P} - \blacktriangle(\mathcal{A})) = \min(\mathcal{A} + \blacktriangledown(\mathcal{A})) = \mathcal{A}$. Since $\mathfrak{a}$ is injective, it follows that $\mathcal{O} = \blacktriangle(\mathcal{A})$. $\square$

Corollary 3.1.6. *If $\mathcal{O}$ is an ideal, then $\blacktriangle(\mathfrak{a}(\mathcal{O})) \subseteq \mathcal{O}$. Equality holds if and only if $\mathfrak{a}(\mathcal{O})$ is maximal.*

Proof. An element $X \in \blacktriangle(\mathfrak{a}(\mathcal{O}))$ exactly when there is a $Y \in \mathfrak{a}(\mathcal{O}) = \min\{\mathcal{P} - \mathcal{O}\}$ so that $X \preceq Y$. This immediately implies that $X \in \mathcal{O}$. The antichain $\mathfrak{a}(\mathcal{O})$ is not maximal, just in case there is an $X \in \mathcal{P} - \langle \mathfrak{a}(\mathcal{O}) \rangle$ so that $\mathfrak{a}(\mathcal{O}) + \{X\}$ is an antichain. Supposing, for proof by contradiction, that $X \notin \mathcal{O}$, we have that $X \in \blacktriangledown(\mathfrak{a}(\mathcal{O}))$ by proposition 3.1.4. Since this contradicts our choice of X it follows that $X \in \mathcal{O} - \blacktriangle(\mathfrak{a}(\mathcal{O}))$. □

Corollary 3.1.7. When $\mathfrak{a}$ is restricted to ideals with maximal choice antichains, it is order preserving.

Proof. If $\mathcal{O}_1 \subseteq \mathcal{O}_2$ and both ideals have maximal choice antichains, then $\blacktriangle(\mathfrak{a}(\mathcal{O}_1)) \subseteq \blacktriangle(\mathfrak{a}(\mathcal{O}_2))$. If $X \in \mathfrak{a}(\mathcal{O}_1) - \blacktriangle(\mathfrak{a}(\mathcal{O}_2))$, then since $\mathfrak{a}(\mathcal{O}_2)$ is maximal, either $X \in \mathfrak{a}(\mathcal{O}_2)$ or $X \in \blacktriangledown(\mathfrak{a}(\mathcal{O}_2))$. The latter case is not possible since $\{X\} \bigcup \mathcal{O}_2$ is not an order ideal whenever there is some $Y \in \mathcal{P} - (\mathcal{O}_2 \bigcup \{X\})$ so that $Y \preceq X$. If such a Y existed, it couldn't be in $\mathfrak{a}(\mathcal{O}_1)$ since it is covered by X and it couldn't be in $\mathcal{O}_1$ itself since $\mathcal{O}_1 \subseteq \mathcal{O}_2 = \blacktriangle(\mathfrak{a}(\mathcal{O}_2))$. It follows that $X \in \mathfrak{a}(\mathcal{O}_2)$ and therefore that $\mathfrak{a}(\mathcal{O}_1) \subseteq \langle \mathfrak{a}(\mathcal{O}_2) \rangle$. □

3.2 Counting Linear Extensions

Proposition 3.1.1 allows us to state our general problem: Given a finite poset $\mathcal{P}$ how many different linear extensions of $\mathcal{P}$ are there? This is, in general, a very difficult question to answer. [1]In light of this, we ask a more tractable question: What kinds of upper bounds can we place on the total number of linear extensions of any finite poset or on some special class of finite posets? To this end, let us define $e(\mathcal{P}) = |E(\mathcal{P})|$. Since any linear extension of $\mathcal{P}$ will correspond to some total ordering of the elements of $\mathcal{P}$, a trivial upper bound is given by $e(\mathcal{P}) \leq |\mathcal{P}|!$.

Sometimes it is possible to determine $e(\mathcal{P})$ exactly by a direct counting argument. For example, the unordered set on n elements has exactly $n!$ linear extensions corresponding to any possible

[1]In fact it has been shown in [6] that the general problem is $\#P$-complete. This means that the problem of constructing a polynomial-time algorithm that, given any poset $\mathcal{P}$, determines the number of linear extensions of $\mathcal{P}$ is equivalent to proving that $P = NP$.

ordering of its elements. Less trivial examples may be found in examples 3.5.3 through 3.5.5 of [17] including the following proposition.

Proposition 3.2.1. *If $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2 + ... + \mathcal{P}_n$ is the disjoint union of partially ordered sets and, for each i, $|\mathcal{P}_i| = m_i$, then $e(\mathcal{P}) = \binom{m_1+m_2+...+m_n}{m_1,m_2,...,m_n} e(\mathcal{P}_1)e(\mathcal{P}_2)...e(\mathcal{P}_n)$*

Proof. The proof is by induction on n. If $n = 1$, the claim is trivial. Suppose now that the claim holds for all $k < n$. Given $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2 + ... + \mathcal{P}_n$ with $|\mathcal{P}_i| = m_i$ for each i, let $\overline{\mathcal{P}} = \mathcal{P}_1 + \mathcal{P}_2 + ... + \mathcal{P}_{n-1}$. Then $\mathcal{P} = \overline{\mathcal{P}} + \mathcal{P}_n$ and $\left|\overline{\mathcal{P}}\right| = \sum_{i=1}^{n-1} m_i$ so that by our inductive hypothesis with $k = 2$, $e(\mathcal{P}) = \binom{m_1+m_2+...+m_n}{m_n} e(\overline{\mathcal{P}})e(\mathcal{P}_n)$. By our inductive hypothesis with $k = n-1$, we also have that

$$e(\overline{\mathcal{P}}) = \binom{m_1 + m_2 + ... + m_{n-1}}{m_1, m_2, ..., m_{n-1}} e(\mathcal{P}_1)e(\mathcal{P}_2)...e(\mathcal{P}_{n-1}).$$

Recalling that, in general,

$$\binom{m_1 + m_2 + ... + m_n}{m_1, m_2, ..., m_n} = \binom{m_1}{m_1}\binom{m_1 + m_2}{m_2}...\binom{m_1 + m_2 + ... + m_n}{m_n},$$

it follows that

$$\begin{aligned} e(\mathcal{P}) &= \binom{m_1 + m_2 + ... + m_n}{m_n}\left(\binom{m_1 + m_2 + ... + m_{n-1}}{m_1, m_2, ..., m_{n-1}} e(\mathcal{P}_1)e(\mathcal{P}_2)...e(\mathcal{P}_{n-1})\right) e(\mathcal{P}_n) \\ &= \binom{m_1 + m_2 + ... + m_n}{m_n}\left(\binom{m_1}{m_1}\binom{m_1 + m_2}{m_2}...\binom{m_1 + m_2 + ... + m_{n-1}}{m_{n-1}}\right) e(\mathcal{P}_1)e(\mathcal{P}_2)...e(\mathcal{P}_n) \\ &= \binom{m_1 + m_2 + ... + m_n}{m_1, m_2, ..., m_n} e(\mathcal{P}_1)e(\mathcal{P}_2)...e(\mathcal{P}_n). \end{aligned}$$

□

A purely combinatorial proof can also be given: Having independently linearly ordered each $\mathcal{P}_i$ with the linear extension ε_i, a linear extension δ of $\mathcal{P}$ may be constructed by choosing any m_1 elements of $[\mathbf{m_1} + \mathbf{m_2} + ... + \mathbf{m_n}]$, say $a_{1,1}, a_{1,2}, ..., a_{1,m_1}$, and then choosing m_2 elements of $[\mathbf{m_1} + \mathbf{m_2} + ... + \mathbf{m_n}] - \{a_1, a_2, ..., a_{m_1}\}$, say $a_{2,1}, a_{2,2}, ..., a_{2,m_2}$ and so forth. Setting $\delta(X) = a_{i,\varepsilon_i(X)}$

for each $X \in \mathcal{P}_i$ yields a linear extension δ of $\mathcal{P}$. Since, for each linear extension δ of $\mathcal{P}$, the restriction of δ to $\mathcal{P}_i$ yields a unique linear extension of $\mathcal{P}_i$, for each i, the theorem follows.

Since $e(\mathcal{C}) = 1$ for any chain $\mathcal{C}$, we have the following immediate consequence:

Corollary 3.2.2. *Let $dk = n$. If $\mathcal{P} = \sum_{i=1}^{d} \mathcal{C}_{k-1}$ is the disjoint union of d chains, each with k elements, then*

$$e(\mathcal{P}) = \binom{n}{k, k, \ldots, k} = \frac{n!}{(k!)^d}.$$

It is worth noting that no comparable formula is known for the cartesian product of posets, or even cartesian products of chains. This is unfortunate as we have seen that both the boolean lattice and the multiset lattice can be simply described as cartesian products of chains.

In what follows, we will make use of probabilistic arguments. Such arguments, in general, make use of elementary probability theory. In our case, we will only need the barest of additional definition. Let us call a function ρ from a finite set E to the interval $[0, 1]$ is a **probability distribution** over E if

$$\sum_{x \in E} \rho(x) = 1.$$

The following propositions show that we can bound the size of $e(\mathcal{P})$ by defining a suitable probability distribution over $e(\mathcal{P})$.

Proposition 3.2.3. *Let ρ be a probability distribution over $E(\mathcal{P})$. If there is a $k \in \mathbb{R}^+$ so that for each $\varepsilon \in E(\mathcal{P})$, $\rho(\varepsilon) \geq \frac{1}{k}$, then*

$$e(\mathcal{P}) \leq k.$$

Proof. The theorem immediately follows from the following inequality:

$$1 = \sum_{\varepsilon \in E(\mathcal{P})} \rho(\varepsilon) \geq \sum_{\varepsilon \in E(\mathcal{P})} \frac{1}{k} = \frac{e(\mathcal{P})}{k}.$$

□

The argument in [14] makes use of Proposition 3.2.3 by exploiting the special correspondence between linear extensions, sequences of order ideals, and sequences of antichains discussed in the previous section.

Suppose that we are given, for each $\mathcal{O} \in \mathfrak{L}(\mathcal{P})$, an upper bound on $|\mathfrak{a}(\mathcal{O})|$ that depends only on $|\mathcal{O}|$. The following theorem shows that Proposition 3.2.3 allows us to translate this collection of bounds into an upper bound on the total number of linear extensions of $\mathcal{P}$.

Theorem 3.2.4. *Let $\{a_i\}_{0\leq i\leq|\mathcal{P}|}$ be a collection of numbers with the property that for each order ideal $\mathcal{O}$ of $\mathcal{P}$, such that $|\mathcal{O}| = i$, we have that $|\mathfrak{a}(\mathcal{O})| \leq a_i$. Then*

$$e(\mathcal{P}) \leq \prod_{i=1}^{|P|} a_{i-1}.$$

If $|\mathfrak{a}(\mathcal{O}_i)| = a_i$, for each integer i, then

$$e(\mathcal{P}) = \prod_{i=1}^{|P|} a_{i-1}.$$

Proof. Note that the assumption of this theorem guarantees that, for each i, any ideal of $\mathcal{P}$ with i members will have a choice antichain of size at most a_i. Now imagine recursively constructing a linear extension of $\mathcal{P}$ as follows. Let $\mathcal{O}_0 = \emptyset$ and for each $i \geq 0$, let $\mathcal{O}_{i+1} = \mathcal{O}_i + \{X_i\}$ where X_i is chosen from $\mathfrak{a}(\mathcal{O}_i)$ with uniform probability. Since $\mathcal{P}$ is finite and $|\mathcal{O}_{i+1}| = |\mathcal{O}_i|+1$, this construction always terminates after $|\mathcal{P}|$ steps. The result is a linear extension of $\mathcal{P}$ given by $X_1, X_2, ..., X_{|\mathcal{P}|}$. In fact, since each distinct linear extension has a unique sequence of order ideals, we have that, given a linear extension ε with corresponding sequence of order ideals $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_{|\mathcal{P}|}$, our procedure results in ε if and only if we take X_i to be the unique element of $\mathcal{O}_{i+1} - \mathcal{O}_i$ for each $i > 0$.[2]

At the $i+1$st stage of this procedure, we have so far produced a partial extension of order ideal $\mathcal{O}_i$ of size i. By assumption there are at most a_i different candidates for X_i independently of our choices for the previous entries in the total order. Since we always chose from $\mathfrak{a}(\mathcal{O}_i)$ uniformly, the probability that any particular order ideal $\mathcal{O}_{i+1}$ of $\mathcal{P}$ results at the ith stage is always exactly $\frac{1}{|\mathfrak{a}(\mathcal{O}_i)|}$.

[2]Note that $\mathcal{O}_{i+1} - \mathcal{O}_i$ is always contained in $\mathfrak{a}(\mathcal{O}_i)$.

This allows us to define a function μ over the set $E(\mathcal{P})$ by assigning to ε the product $\prod_{i=1}^{|\mathcal{P}|} \frac{1}{|\mathfrak{a}(\mathcal{O}_i)|}$. Note that $\mu(\varepsilon)$ is exactly the probability that our construction results in ε. Since the construction is guaranteed to produce a linear extension of $\mathcal{P}$ we have that $\sum_{\varepsilon \in E(\mathcal{P})} \mu(\varepsilon) = 1$ so that μ is indeed a probability distribution. By our assumptions,

$$\mu(\varepsilon) = \prod_{i=1}^{|\mathcal{P}|} \frac{1}{|\mathfrak{a}(\mathcal{O}_i)|} \geq \prod_{i=1}^{|\mathcal{P}|} \frac{1}{a_i}$$

and applying Proposition 3.2.3, the theorem follows. $\square$

Let use apply Proposition 3.2.4 to an extended example to clarify these ideas.

Example 3.2.5. Consider $\boldsymbol{\mathcal{B}}^3$ the lattice of subsets of $\{A, B, C\}$. Two linear extensions of $\boldsymbol{\mathcal{B}}^3$ appear in Figure 2.9 below as dotted lines directed by arrows.

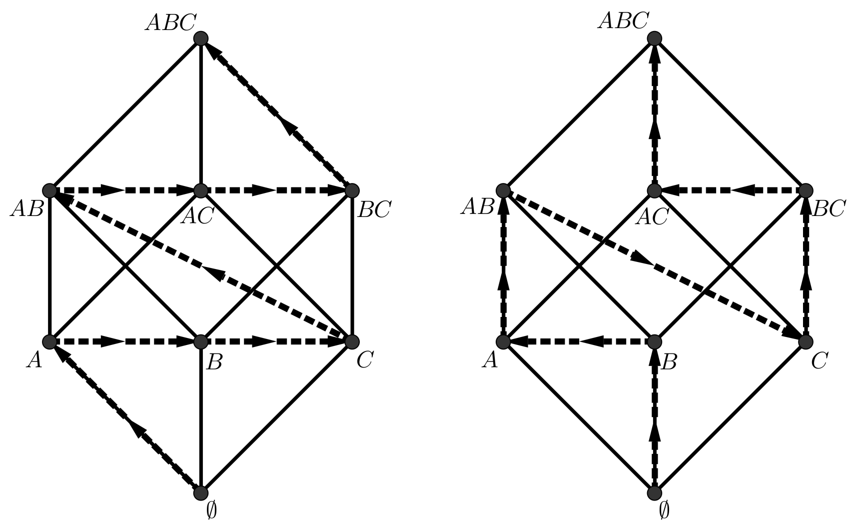

Figure 3.3: Two linear extensions of $\boldsymbol{\mathcal{B}}^3$

In the diagram on the left, let us call the given linear extension ε_1. The ordering is $\emptyset \preceq A \preceq B \preceq C \preceq AB \preceq AC \preceq BC \preceq ABC$. Following the construction above, we choose at each stage i, an element from $\mathfrak{a}(\mathcal{O}_i)$ with uniform probability.

$\mathcal{O}_i$	$\mathfrak{a}(\mathcal{O}_i)$	X_i	$\rho_i(X_i)$
$\emptyset$	$\{\emptyset\}$	$\emptyset$	1
$\{\emptyset\}$	$\{A, B, C\}$	A	1/3
$\{\emptyset, A\}$	$\{B, C\}$	B	1/2
$\{\emptyset, A, B\}$	$\{C, AB\}$	C	1/2
$\{\emptyset, A, B, C\}$	$\{AB, AC, BC\}$	AB	1/3
$\{\emptyset, A, B, C, AB\}$	$\{AC, BC\}$	AC	1/2
$\{\emptyset, A, B, C, AB, AC\}$	$\{BC\}$	BC	1
$\{\emptyset, A, B, C, AB, AC, BC\}$	$\{ABC\}$	ABC	1
$\mathcal{B}^3$	-	-	-

Table 3.1: The Linear Extension ε_1

From this table we can compute that the probability that our construction results in ε_1 to be the product $1 \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{72}$.

In the diagram on the right, let us call extension ε_2. The ordering is $\emptyset \preceq B \preceq A \preceq AB \preceq C \preceq BC \preceq AC \preceq ABC$.

$\mathcal{O}_i$	$\mathfrak{a}(\mathcal{O}_i)$	X_i	$\rho_i(X_i)$
$\emptyset$	$\{\emptyset\}$	$\emptyset$	1
$\{\emptyset\}$	$\{A, B, C\}$	B	1/3
$\{\emptyset, A\}$	$\{A, C\}$	A	1/2
$\{\emptyset, A, B\}$	$\{C, AB\}$	AB	1/2
$\{\emptyset, A, B, C\}$	$\{C\}$	C	1
$\{\emptyset, A, B, C, AB\}$	$\{AC, BC\}$	BC	1/2
$\{\emptyset, A, B, C, AB, AC\}$	$\{AC\}$	AC	1
$\{\emptyset, A, B, C, AB, AC, BC\}$	$\{ABC\}$	ABC	1
$\mathcal{B}^3$	-	-	-

Table 3.2: The Linear Extension ε_2

From this table we can compute that the probability that our construction results in ε_2 to be the product $1 \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 1 \cdot \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{24}$.

Chapter 4

The LYM Property

As noted in the introduction, our main result is a bound on $e(\mathcal{P})$ that applies to a special class of posets known as LYM posets. The name "LYM" pays homage to the early use of this property by Lubell, Yamamoto, and Meshalkin[1] in studying antichains in the boolean lattice.

4.1 The Boolean Lattice and the LYM Inequality

Here we follow [11] p.g. 138 which in turn follows Lubell's classic proof of the result commonly known as the LYM inequality.[2]

Theorem 4.1.1. *(The LYM Inequality) Let $\mathcal{A}$ be an antichain in the Boolean Lattice $\boldsymbol{\mathcal{B}}^n$ and let $\mathcal{A}_k$ be the be the set of all rank k nodes in $\mathcal{A}$. Then*

$$\sum_{k=0}^{n} \frac{|\mathcal{A}_k|}{\binom{n}{k}} \leq 1.$$

Proof. Recall that, from proposition 2.3.11, $\boldsymbol{\mathcal{B}}^n \cong \mathfrak{L}(\{1, 2..., n\})$. As observed in the beginning of section 3.2, since any total ordering of $\{1, 2..., n\}$ is a linear extension, there are exactly $n!$ linear

[1]Apparently, Ballobos should also be included in this list, but for some reason he is not.
[2]Originally published in [15], another nice exposition of this argument can be found in[1].

extensions of $\{1, 2..., n\}$. As discussed in section 2.4, this also counts the number of maximal chains in $\boldsymbol{\mathcal{B}}^n$.

On the other hand, given an element $X \in \boldsymbol{\mathcal{B}}^n$ with rank k, we might construct a maximal chain in $\boldsymbol{\mathcal{B}}^n$ by ordering the k elements of X and independently ordering the $n-k$ remaining elements, then concatenating the results. This can be done in $k!(n-k)!$ ways. Note that a maximal chain so constructed will always contain X. It follows that we can construct $k!(n-k)!$ maximal chains in $\boldsymbol{\mathcal{B}}^n$ containing the node X.

Consider an antichain $\mathcal{A}$ in $\boldsymbol{\mathcal{B}}^n$ and let $\mathcal{A}_k$ denote the set of elements of $\mathcal{A}$ of rank k. For each $X \in \mathcal{A}_k$, we count, as above, $k!(n-k)!$ maximal chains in $\boldsymbol{\mathcal{B}}^n$ containing X. Note that for any chain $\mathcal{C}$ and any antichain $\mathcal{A}$ in $\boldsymbol{\mathcal{B}}^n$, $|\mathcal{C} \cap \mathcal{A}| \leq 1$. This means that no chain intersects an antichain more than once. It follows that the sum

$$\sum_{k=0}^{n} |\mathcal{A}_k| \, k!(n-k)!$$

counts the number of maximal chains in $\boldsymbol{\mathcal{B}}^n$ whose intersection with $\mathcal{A}$ is non-empty. Since this number must be smaller than the total the number of maximal chains in $\boldsymbol{\mathcal{B}}^n$, we have established that

$$\sum_{k=0}^{n} |\mathcal{A}_k| \, k!(n-k)! \leq n!.$$

Dividing both sides by $n!$ yields the desired inequality. $\square$

We note one nearly immediate consequence.

Corollary 4.1.2. *(Sperner's Theorem): Let $\mathcal{A}$ be an antichain in $\boldsymbol{\mathcal{B}}^n$. Then*

$$|\mathcal{A}| \leq \binom{n}{\left[\frac{n}{2}\right]}.$$

Proof. First, recall that for any k so that $0 \leq k \leq n$,

$$\binom{n}{k} \leq \binom{n}{\left[\frac{n}{2}\right]}$$

so that

$$\sum_{k=0}^{n} \frac{|\mathcal{A}_k|}{\binom{n}{[\frac{n}{2}]}} \leq \sum_{k=0}^{n} \frac{|\mathcal{A}_k|}{\binom{n}{k}} \leq 1.$$

It follows that

$$\sum_{k=0}^{n} |\mathcal{A}_k| \leq \binom{n}{[\frac{n}{2}]}.$$

□

4.2 The LYM property and Linear Extensions

Since, in the Boolean lattice $\mathcal{B}^n$, the kth whitney number N_k is given by the binomial coefficient $\binom{n}{k}$, the LYM inequality can be naturally extended to other posets. If $\mathcal{P}$ is a rank n poset with whitney numbers $N_0, N_1, ..., N_n$, then $\mathcal{P}$ has the **LYM property** if for each antichain $\mathcal{A} \in \mathcal{P}$,

$$\sum_{i=0}^{n} \frac{|\mathcal{A}_i|}{N_i} \leq 1.$$

Such posets are often called **LYM posets** or are said to be **LYM**. Proposition 4.1.2 may be easily extended to any LYM poset in an obvious way.

Proposition 4.2.1. *Let $\mathcal{P}$ be an LYM poset with rank k and whitney numbers $N_0, N_1, \ldots, N_k$. If $M = \max_{0 \leq i \leq k} N_i$, then for any antichain $\mathcal{A}$ in $\mathcal{P}$,*

$$|\mathcal{A}| \leq M.$$

Proof. The argument is identical to that for Corollary 4.1.2 with N_k used in place of $\binom{n}{k}$ and M used in place of $\binom{n}{[\frac{n}{2}]}$. □

As is clear from the proof, the Sperner property is weaker than the LYM property. The previous theorem shows us that every LYM poset is Sperner. However, the converse does not in general hold as Figure 4.1 below illustrates. In this poset, we think of the rank function as determined by levels in the hasse diagram.

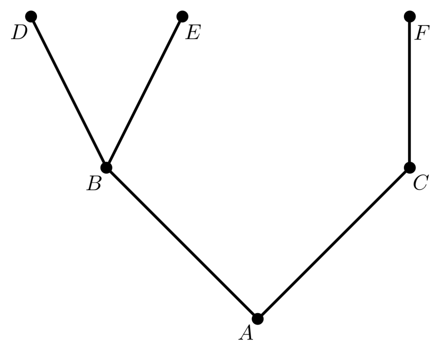

Figure 4.1: A poset that is not LYM

The antichain $\mathcal{A} = \{C, D, E\}$ then has $\mathcal{A}_0 = \emptyset$, $\mathcal{A}_1 = \{C\}$, and $\mathcal{A}_2 = \{D, E\}$. Since $N_0 = 1$, $N_1 = 2$, and $N_2 = 3$ we have

$$\sum_{i=0}^{n} \frac{|\mathcal{A}_i|}{N_i} = 0 + \frac{1}{2} + \frac{2}{3} > 1.$$

Proposition 4.2.1 gives us an upper bound on the size of any antichain in an LYM poset. In fact, since each rank of any ranked poset must comprise an antichain, the largest such rank will always be an antichain that attains this bound. It follows that this is the best possible global restriction on the size of any antichain in an LYM poset.

Theorem 4.2.2. *Let $\mathcal{P}$ be any poset with the sperner property and let M be defined as in Proposition 4.2.1. Then*

$$e(\mathcal{P}) \leq M^{|\mathcal{P}|}.$$

Proof. The sperner property guarantees that for each order ideal $\mathcal{O} \in \mathfrak{L}(\mathcal{P})$, $|\mathfrak{a}(\mathcal{O})| \leq M$. The theorem follows from Proposition 3.2.4, with $a_i = M$ for each i. $\square$

4.3 Kleitman's Theorem on LYM-Posets

We now prove a theorem of Kleitman's from [13] that is a standard result in the field.[3] We follow the proof in [3] for which we require a famous combinatorial theorem: Hall's Marriage Theorem. This theorem makes use of the idea of a **system of distinct representatives** or **s.d.r.** A collection of elements $a_1, a_2, \ldots, a_n$ is an s.d.r. for a collection of sets $A_1, A_2, \ldots, A_r$ if and only if $a_i \in A_i$ for each i and $a_i \neq a_j$ whenever $i \neq j$. For example, if $A_1 = \{1, 2, a\}$, $A_2 = \{3, 5, a, b\}$, $A_3 = \{1, 2, 3, b\}$, and $A_4 = \{4, 5, a, b\}$ is a collection of sets, then $a_1 = a$, $a_2 = 3$, $a_3 = b$, and $a_4 = 4$ is an s.d.r. for it. Alternately, if $A_1 = \{1\}$, $A_2 = \{2\}$, and $A_3 = \{1, 2\}$ then any choice of a representative for each of A_1, A_2, and A_3 will fail to be distinct. Therefore, the collection does not possess an s.d.r.

Hall's theorem provides a condition that is both necessary and sufficient for a collection of sets to possess an s.d.r that is commonly known as *Hall's condition.* Note that we do not require that each member of a collection be distinct. In this sense a collection is a multiset.

Theorem 4.3.1. *(Hall's Marriage Theorem) The collection of sets $A_1, A_2, \ldots, A_r$ has an s.d.r if and only if for each m with $0 < m \leq r$, the union of any m of the sets A_i contains at least m elements.*

Proof. First note that if any of the sets A_i are empty, then the theorem holds; no s.d.r is possible and the collection fails to satisfy Hall's condition. Let us then suppose that A_i is nonempty for each i. In this case, one implication is obvious. If there were, for some appropriate m, a collection of m of the sets $A_1, A_2, \ldots, A_r$ whose union had size smaller than m, then any choice of representatives from each of the m sets would contain less than m distinct elements. Since an s.d.r for $A_1, A_2, \ldots, A_r$ will include an s.d.r for any subcollection, we conclude that $A_1, A_2, \ldots, A_r$ cannot have an s.d.r. From this it follows that the existence of an s.d.r implies Hall's condition.

To establish the other direction, we follow the proof of Halmos and Vaughn in [10] and argue by induction on r. If $r = 1$, then any element of A_1 is an s.d.r and the collection always satisfies Hall's condition so that the theorem holds. Let r be greater than 1 and suppose for inductive hypothesis that the theorem holds for any collection of sets $B_1, B_2, \ldots, B_k$ with k less than r. Let $A_1, A_2, \ldots, A_r$

[3] See, for example, [3] pg. 15 and [11] pg. 145, exercise 3.3.6.

be a collection of sets satisfying Hall's condition. Our goal will be to construct an s.d.r $a_1, a_2, ..., a_r$ for this collection.

We begin with a special case. Suppose that for each m, with $0 < m < r$, the union of any m of the sets A_i contains at least $m+1$ elements. Choose any element x of A_1, set $a_1 = x$, and let $\overline{A_i} = A_i - \{x\}$ for each i. Then the collection $\overline{A_2}, \overline{A_3}, \ldots, \overline{A_r}$ has $r-1$ members and the union of any subcollection of size m will have at least m members. This new collection will therefore satisfy Hall's condition and so, by the inductive hypothesis, have an s.d.r, say $x_2, x_3, ..., x_r$. Setting $x_i = a_i$ for each i we obtain an s.d.r $a_1, a_2, ..., a_r$ for the original collection.

Next, let us suppose that there exists an $m < r$ so that there is at least one subcollection of m elements A_i of the collection $A_1, A_2, \ldots, A_r$ whose union has size exactly m. If necessary, relabel the original collection so that this subcollection is given by $A_1, A_2, ..., A_m$ and the remaining members are given by $A_{m+1}, A_{m+2}, ..., A_r$. By our inductive hypothesis the subcollection $A_1, A_2, ..., A_m$ has an s.d.r $a_1, a_2, ..., a_m$. Let $\widehat{A_i} = A_i - \{a_1, a_2, ..., a_m\}$ and note that, by Hall's Condition, $\widehat{A_i}$ must be nonempty for each $i > m$. Suppose for proof by contradiction that the collection $\widehat{A_{m+1}}, \widehat{A_{m+2}}, ..., \widehat{A_r}$ does not satisfy Hall's condition. In this case, for some h, there is a collection of h sets in $\widehat{A_{m+1}}, \widehat{A_{m+2}}, ..., \widehat{A_r}$ whose union has size less than h. If necessary, relabel this collection so that $\bigcup_{i=1}^{h} \widehat{A_{m+i}}$ has size less than h. Then

$$\bigcup_{i=1}^{m+h} A_i = \bigcup_{i=1}^{m} A_i + \bigcup_{i=1}^{h} \widehat{A_{m+i}} = \{a_1, a_2, ..., a_m\} + \bigcup_{i=1}^{h} \widehat{A_{m+i}}$$

and it follows that $\bigcup_{i=1}^{m+h} A_i$ has size less than $m+h$. Since this is a union of $m+h$ members of $A_1, A_2, \ldots, A_r$ with size smaller than $m+h$, we have contradicted our assumption that $A_1, A_2, \ldots, A_r$ satisfies Hall's Condition.

It follows that the collection $\widehat{A_{m+1}}, \widehat{A_{m+2}}, ..., \widehat{A_r}$ does satisfy Hall's condition and therefore, by our inductive hypothesis, has an s.d.r $\{a_{m+1}, ..., a_r\}$. Clearly, the set $\{a_1, a_2, ..., a_r\}$ is an s.d.r for $A_1, A_2, \ldots, A_r$. $\square$

This is a far-reaching theorem, and it is not immediately obvious how this theorem relates to our study of ranked posets. Recall that a graph G is **bipartite** if its vertex set can be partitioned into two disjoint sets A and B so that for any edge $\{X, Y\}$ of G, $X \in A$ if and only if $Y \in B$. For each $X \in A$, let $N(X)$ the set of vertices Y in B so that $\{X, Y\}$ is an edge of G. Then, indexing the elements of A as $X_1, X_2, ..., X_i$ we get a collection of sets $N(X_1), N(X_2), ..., N(X_i)$. An s.d.r of this collection, $Y_1, Y_2, ..., Y_i$ picks out a collection of non-adjacent edges, called a **matching**, $\{X_1, Y_1\}, \{X_2, Y_2\}, ..., \{X_i, Y_i\}$ that is said to **saturate** A, that is, pair each element of A with a unique element of B. Hall's Theorem then says that there is a matching that saturates A if and only if Hall's condition is satisfied on the collection $N(X_1), N(X_2), ..., N(X_i)$. A matching that saturates both A and B is a **perfect matching**. Note that an A saturating matching is a perfect matching if and only if $|A| = |B|$.

Let $\mathcal{P}$ be any rank n poset and consider the subposet $\mathcal{P}_i + \mathcal{P}_{i+1}$ for some i with $0 \leq i < n$. It is clear from the definition of the rank function that, for any i, no element of $\mathcal{P}_i$ covers and other element of $\mathcal{P}_i$. It follows that, for any i, the hasse diagram of the restriction of $\mathcal{P}$ to the set $\mathcal{P}_i + \mathcal{P}_{i+1}$ is a bipartite graph.[4]In this situation we usually let $\mathcal{P}_i$ play the role of A above so that $N(X)$ is given by $\nabla(X)$ for each $X \in \mathcal{P}_i$. We can now restate Hall's theorem to say that the collection $\{\nabla(X)\}_{X \in \mathcal{P}_i}$ has a $\mathcal{P}_i$ saturated matching if and only if for each $\mathcal{R} \subseteq \mathcal{P}$, $|\mathcal{R}| \leq |\nabla(\mathcal{R})|$.

With this in place, we are now able to prove Kleitman's theorem.

Theorem 4.3.2. *Let $\mathcal{P}$ be a ranked poset with rank k. Then the following conditions are equivalent:*

(i) $\mathcal{P}$ is an LYM poset

(ii) If $\mathcal{B} \subseteq \mathcal{P}_i$, for some i, then

$$\frac{|\mathcal{B}|}{N_i} \leq \frac{|\nabla(\mathcal{B})|}{N_{i+1}}.$$

*This condition is called the **Normalized Matching Property.***

(iii) There is a multiset $(\mathcal{C}, m)$ of maximal chains in $\mathcal{P}$ and for each pair $X, Y \in \mathcal{P}_j$, if $\mathcal{C}_X =$

[4] This observation actually allows us to show that the hasse diagram of $\mathcal{P}$ is itself a bipartite graph. Let $\mathcal{X} = \bigcup_{\substack{0 \leq i \leq k \\ i \text{ odd}}} \mathcal{P}_i$ and $\mathcal{Y} = \bigcup_{\substack{0 \leq i \leq k \\ i \text{ even}}} \mathcal{P}_i$. Then $\mathcal{P} = \mathcal{X} + \mathcal{Y}$ and if X_1 and X_2 are both in, say, $\mathcal{X}$, then either $r(X_1) = r(X_2)$, $r(X_1) < r(X_2) - 1$, or $r(X_2) < r(X_1) - 1$ so that X_1 cannot cover X_2, and vice versa. It follows that $\mathcal{X}$ and $\mathcal{Y}$ are disjoint and therefore that the hasse diagram of $\mathcal{P}$ is itself bipartite.

$\{C \in \mathcal{C} \,|\, X \in C\}$, then

$$\sum_{C \in C_X} m(C) = \sum_{C \in C_Y} m(C).$$

*Intuitively, this condition says that whenever X and Y have the same rank, they intersect the same number of chains in our collection. A poset satisfying this condition is said to have a **Regular Covering by Chains**.*

Proof. (i)⇒(ii) Let $\mathcal{B} \subseteq \mathcal{P}_i$. Then the set $\mathcal{B} + (\mathcal{P}_{i+1} - \nabla(\mathcal{B}))$ is an antichain in $\mathcal{P}$ with elements in $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$. By (i), we have that

$$\frac{|\mathcal{B}|}{N_i} + \frac{|\mathcal{P}_{i+1} - \nabla(\mathcal{B})|}{N_{i+1}} \leq 1.$$

Since $\nabla(\mathcal{B}) \subseteq \mathcal{P}_{i+1}$, it follows that $|\mathcal{P}_{i+1} - \nabla(\mathcal{B})| = N_{i+1} - |\nabla(\mathcal{B})|$ and our inequality becomes

$$\frac{|\mathcal{B}|}{N_i} + \frac{N_{i+1} - |\nabla(\mathcal{B})|}{N_{i+1}} \leq 1$$

which we rewrite as

$$\frac{|\mathcal{B}|}{N_i} \leq \frac{|\nabla(\mathcal{B})|}{N_{i+1}}.$$

(ii)⇒(iii) This proof is actually constructive. We are going to build a multiset of chains in $\mathcal{P}$ forming a regular covering. For each rank j of $\mathcal{P}$, let $\mu_j = \prod_{\substack{0 \leq i \leq k \\ i \neq j}} N_i$ and for each $X \in \mathcal{P}_j$, define the set

$$\mathcal{B}_X = \{X_t\}_{1 \leq t \leq \mu_j}.$$

consisting of μ_j indexed copies of X. Consider the poset $\overline{\mathcal{P}} = \bigcup_{x \in \mathcal{P}} \mathcal{B}_X$ where $X_l \preceq_{\overline{\mathcal{P}}} Y_m$ whenever $X \preceq_{\mathcal{P}} Y$. $\overline{\mathcal{P}}$ then inherits its rank function from $\mathcal{P}$ so that $X_l \in \overline{\mathcal{P}}$ has the same rank as $X \in \mathcal{P}$. Since each of the N_j distinct elements of $\mathcal{P}_j$ yields exactly μ_j distinct elements of $\overline{\mathcal{P}}_j$, we have that

$$\overline{N}_j = |\overline{\mathcal{P}}_j| = \prod_{0 \leq i \leq k} N_i.$$

This value is independent of j so that $\overline{\mathcal{P}}$ has the same whitney number for each of its ranks.

Consider the order preserving map Ψ sending X_l to X for each $X_l \in \overline{\mathcal{P}}$ let R be any set of r elements of $\overline{\mathcal{P}}_j$. Then, $\Psi[R]$ has at least $\frac{r}{\mu_j}$ distinct element of $\mathcal{P}_j$ in it. By the normalized matching property,

$$|\nabla(\Psi[R])| \geq |\Psi[R]| \frac{N_{j+1}}{N_j} \geq \frac{r}{\mu_j} \cdot \frac{N_{j+1}}{N_j} = \frac{r}{\mu_{j+1}}$$

so that there are at least $\frac{r}{\mu_{j+1}}$ elements of $\mathcal{P}_{j+1}$ covering elements of $\Psi[R]$. Taking $\Psi^{-1}[\nabla(\Psi[R])]$ we find at least r elements of $\overline{\mathcal{P}}_{j+1}$ covering R. Therefore, $\overline{\mathcal{P}}_j + \overline{\mathcal{P}}_{j+1}$ satisfies Hall's condition and by Theorem 4.3.1 we are guaranteed, for each rank j, a perfect matching M_j between $\overline{\mathcal{P}}_j + \overline{\mathcal{P}}_{j+1}$.

Now, consider $\bigcup_{i=0}^{k-1} M_i$ where M_i is any perfect matching between $\overline{\mathcal{P}}_i + \overline{\mathcal{P}}_{i+1}$. For any $X_t \in \overline{\mathcal{P}}_j$, there is a unique pair $(X_t, A_{j+1}) \in M_j$ and a unique pair $(A_{j-1}, X_t) \in M_{j-1}$. This determines a unique maximal chain, C_t, through X_t in $\bigcup_{i=0}^{k-1} M_i$. Note that, for any j,

$$|\mathcal{C}| = \left|\overline{\mathcal{P}}_j\right| = \prod_{0 \leq i \leq k} N_i.$$

Let $\mathcal{C}$ be the set of all maximal chains in $\bigcup_{i=0}^{k-1} M_i$ and consider the multiset $(\Psi[\mathcal{C}], m)$ where $m(C) = \left|\Psi^{-1}[C]\right|$ for each $C \in \Psi[\mathcal{C}]$. This is certainly a multiset of maximal chains in $\mathcal{P}$. To see that it is a regular covering, let $X \in \mathcal{P}_j$ and let $\mathcal{C}_X$ be the set of chains in $\mathcal{C}$ containing X_t for some t. Then, $|\mathcal{C}_X| = \mu_j$ and it follows that

$$\sum_{C \in \Psi[\mathcal{C}_X]} m(C) = \sum_{C \in \Psi[\mathcal{C}_X]} \left|\Psi^{-1}[C]\right| = \mu_j.$$

Since $\Psi[\mathcal{C}_X]$ is exactly the set of chains in $\Psi[\mathcal{C}]$ containing X, it follows that $(\Psi[\mathcal{C}], m)$ constitutes a regular covering of $\mathcal{P}$ by chains.

(iii)$\Rightarrow$(i) Let $\{\mathcal{C}_i\}_{i=1}^{t}$ be a regular covering of t (not necessarily distinct, but each with a distinct index) maximal chains in $\mathcal{P}$. Then, if $X \in \mathcal{P}_i$, X occurs in exactly $\frac{t}{N_i}$ of these chains. Let $\mathcal{A}$ be an antichain in $\mathcal{P}$. Then, exactly $|A_i| \cdot \frac{t}{N_i}$ chains in $\{\mathcal{C}_i\}_{i=1}^{t}$ intersect $\mathcal{A}$ at the ith level. Recall that

$\mathcal{C} \bigcap \mathcal{A} \leq 1$ for any chain and any antichain in $\mathcal{P}$. It follows that $\mathcal{A}$ intersects a total of

$$\sum_{i=0}^{k} |A_i| \cdot \frac{t}{N_i}$$

distinct chains in $\{\mathcal{C}_i\}_{i=1}^{t}$. Since this cannot be more than the total number of chains in $\{\mathcal{C}_i\}_{i=1}^{t}$, we have that

$$\sum_{i=0}^{k} |A_i| \cdot \frac{t}{N_i} \leq t$$

or

$$\sum_{i=0}^{k} \frac{|A_i|}{N_i} \leq 1.$$

$\square$

Certainly the most difficult part of this proof is the implication (ii)⇒(iii). Figure 4.2, below, illustrates the construction of $\overline{\mathcal{P}}$ and a possible choice of perfect matching for a small poset.

Theorem 4.3.2 is incredibly useful in determining what the class of LYM posets looks like as the next proposition from [9] illustrates.

Proposition 4.3.3. *If $\mathcal{P}$ is an LYM poset with rank n, then it is graded.*

Proof. Let $X \in \mathcal{P}$ be minimal and suppose that $r(X) = i > 0$. Then, note that $|\nabla(\mathcal{P}_{i-1})| < N_i$ so that

$$\frac{|\nabla(\mathcal{P}_{i-1})|}{N_i} < 1,$$

or equivalently,

$$1 = \frac{|\mathcal{P}_{i-1}|}{N_{i-1}} > \frac{|\nabla(\mathcal{P}_{i-1})|}{N_i}.$$

It follows that $\mathcal{P}$ is not LYM.

Let $X \in \mathcal{P}$ be maximal and suppose that $r(X) = j < n$. In this case, $\nabla X = \emptyset$ so that $\frac{|\nabla X|}{N_{j+1}} = 0$. It follows that

$$\frac{1}{N_j} = \frac{|\{X\}|}{N_j} > \frac{|\nabla X|}{N_{j+1}}.$$

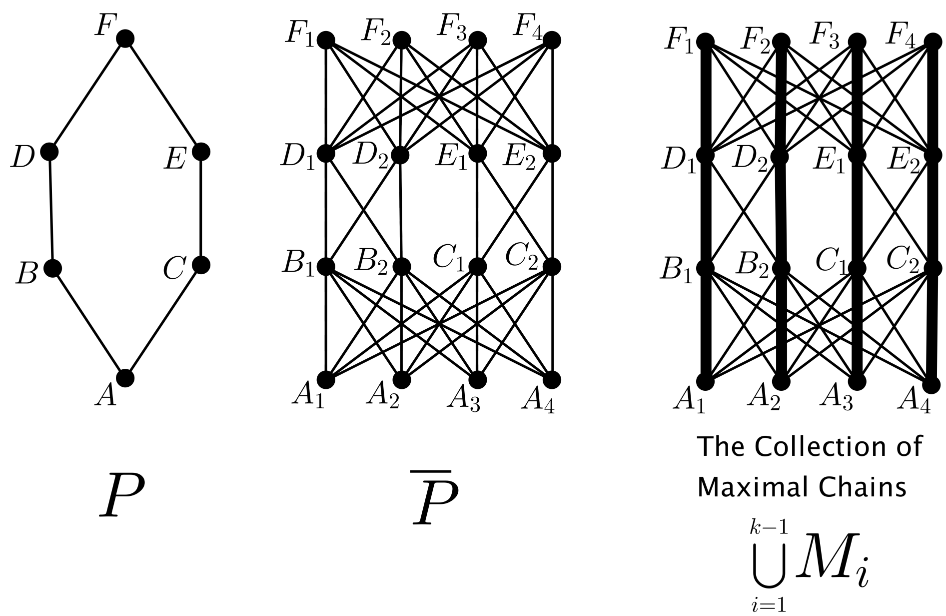

Figure 4.2: The construction of $\overline{\mathcal{P}}$

Again, it follows that $\mathcal{P}$ is not LYM. $\square$

The arguments in the following chapter do not actually make use of condition (iii), however this condition is useful in proving that certain posets are LYM as in the following characterization.

Proposition 4.3.4. *If $\mathcal{P}$ is a poset so that for each $X \in \mathcal{P}$, the sizes of $\triangledown X$ and $\vartriangle X$ are determined by the rank of X, then $\mathcal{P}$ has the LYM property.*

Proof. Suppose that $\mathcal{P}$ has rank n and that for each $X \in \mathcal{P}$ with rank i, $|\triangledown X| = a_i$ and $|\vartriangle X| = b_i$. Then if X has rank k, there are exactly $\prod_{i=1}^{k} a_{n-i} b_i$ maximal chains passing through X. Therefore, the set of all maximal chains in $\mathcal{P}$ is a regular covering by chains. It follows that $\mathcal{P}$ has the LYM property. $\square$

Corollary 4.3.5. *The lattice $L[n,q]$ of subspaces of the n-dimensional vector space $\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q$ with q elements is LYM.*

Proof. Recall that if X is a subspace of dimension i of $\mathbb{F}_q^n$ with basis $\{b_1, b_2, ..., b_i\}$, then $|X| = q^i$, corresponding to the q choices for each coefficient in the sum $a_1b_1 + a_2b_2 + ... + a_ib_i$. It follows that there are $q^n - q^i$ different elements in $\mathbb{F}_q^n - X$.[5] Let $b_{i+1} \in \mathbb{F}_q^n - X$, note that $\text{span}\{b_1, b_2, ..., b_i, b_{i+1}\} = \text{span}\{b_1, b_2, ..., b_i, a + \mu \cdot b_{i+1}\}$ for each $a \in X$ and $\mu \in \mathbb{F}_q - \{0\}$. In fact, if $c \in X$, $\lambda \in \mathbb{F}_q - \{0\}$ and $a + \mu \cdot b_{i+1} = c + \lambda \cdot b_{i+1}$, then $(a - c) = (\lambda - \mu) \cdot b_{i+1}$. Since $b_{i+1} \notin \text{span } \{b_1, b_2, ..., b_i\}$, it must be that $a = c$ and $\mu = \lambda$. Therefore, given b_{i+1}, each choice of $a \in X$ and $\mu \in \mathbb{F}_q - \{0\}$ yields a distinct element $a + \mu \cdot b_{i+1} \in \mathbb{F}_q^n - X$. Since there are $q - 1$ elements in $\mathbb{F}_q - \{0\}$ we have $(q-1)q^i$ different choices from $\mathbb{F}_q^n - X$ that result in the same subspace containing X. It follows that there are

$$\frac{q^n - q^i}{(q-1)q^i} = \frac{q^{n-i} - 1}{(q-1)} = 1 + q + ... + q^{n-i-1}$$

distinct subspaces of dimension $i + 1$ containing X. Equivalently,

$$|\triangledown(X)| = 1 + q + ... + q^{n-i-1}$$

in $L[n,q]$ whenever X has rank i.

To see that $\vartriangle X$ is similarly determined, we will show that $L[n,q]$ is symmetric and invoke Corollary 2.1.4. Let $\mathbb{F}_q^{n*}$ be the vector space of linear maps from $\mathbb{F}_q^n$ to $\mathbb{F}$ and let $L^*[n,q]$ be the lattice of subspaces of $\left(\mathbb{F}_q^n\right)^*$. Note that $L[n,q]$ and $L^*[n,q]$ are isomorphic as lattices via the dual map taking X to $X^* = \{u^* \,|u \in U\}$. Consider the bijective map ϕ from $L[n,q]$ to $L^*[n,q]$ taking X to $X^0 = \left\{f \in \left(\mathbb{F}_q^n\right)^* |f[U] = 0\right\}$. If $X \subseteq Y \subseteq V$, then $f[X] \subseteq f[Y]$ for each $f \in \left(\mathbb{F}_q^n\right)^*$ so that $Y^0 \subseteq X^0$. This map then provides us with an order reversing bijection from $L[n,q]$ to $L^*[n,q]$. Since these lattices are already order isomorphic, we have shown that $L[n,q]$ is symmetric. Since

[5] In the argument that follows, we make use of the well known fact that if X is a subspace of Y, then any basis B for X can be extended to a basis for Y containing B.

$\dim(X^0) = n - \dim(X)$, it follows that

$$|\triangle (X)| = \frac{q^n - q^{n-i}}{(q-1)q^{n-i}} = 1 + q + ... + q^{i-1}.$$

□

Corollary 4.3.6. *The function poset $\mathcal{F}_k^n$ is LYM.*

Proof. Employing the representation of $\mathcal{F}_k^n$ discussed in Example 2.2.13, it is obvious that a function has rank i just in case it has exactly i non-zero entries. If $(a_1, a_2, ..., a_n)$ has rank i, then we construct a rank $i + 1$ function covering $(a_1, a_2, ..., a_n)$ by choosing an a_i so that $a_i = 0$ and assigning any element of C to it. This can be done in exactly k^{n-i} ways. It follows that $|\triangledown f| = k^{n-i}$. Similarly, we construct a rank $i - 1$ function covered by $(a_1, a_2, ..., a_n)$ by replacing any non-zero a_i with a 0 entry. This can be done in exactly i ways so that $|\triangle f| = i$. □

Chapter 5

The Sha/Kleitman Bound

5.1 The Original Proof

The basic strategy of [14] is to find a bound on the size of $\mathfrak{a}(\mathcal{O})$ that depends only on the size of $\mathcal{O}$ and then employ Theorem 3.2.4.

Let $\mathcal{P}$ be an LYM poset. For each order ideal $\mathcal{O}$ of $\mathcal{P}$, let

$$u(\mathcal{O}) = \max\{r \in \mathbb{Z} \,|\, \mathcal{O}_r \neq \emptyset\}$$

and let

$$m(\mathcal{O}) = \min\{r \in \mathbb{Z} \,|\, \mathcal{P}_r - \mathcal{O}_r \neq \emptyset\}.$$

Let us call $\mathcal{O}$ **flat** if $u(\mathcal{O}) = m(\mathcal{O}) + 1$. In this case,

$$\mathcal{O} = \sum_{i=0}^{m(\mathcal{O})} \mathcal{P}_i + \mathcal{O}_{u(\mathcal{O})}$$

so that, in particular,

$$|\mathcal{O}| = \sum_{i=0}^{m(\mathcal{O})} N_i + \left|\mathcal{O}_{u(\mathcal{O})}\right|.$$

Note that if $\mathcal{O}$ is flat then $\mathfrak{a}(\mathcal{O}) \subseteq \mathcal{P}_{u(\mathcal{O})} + \mathcal{P}_{u(\mathcal{O})+1}$ and $(\mathcal{P} - \mathcal{O})_{u(\mathcal{O})} = \mathfrak{a}(\mathcal{O})_{u(\mathcal{O})}$.

Theorem 5.1.1. *Let $\mathcal{P}$ be an LYM poset. If $\mathcal{O}$ is any flat ideal of $\mathcal{P}$ and $x = \left|\mathcal{O}_{u(\mathcal{O})}\right|$, then*

$$|\mathfrak{a}(\mathcal{O})| \leq N_i + x\left(\frac{N_{i+1}}{N_i} - 1\right)$$

Proof. Let $\mathcal{O}$ be a flat ideal of $\mathcal{P}$, let $\mathcal{A} = \mathfrak{a}(\mathcal{O})$, and let $u(\mathcal{O}) = i$. Then, $\mathcal{A}$ is an antichain in $\mathcal{P}$ contained entirely in $\mathcal{P}_i + \mathcal{P}_{i+1}$. First, we write $\mathcal{A} = \mathcal{A}_i + \mathcal{A}_{i+1}$. Since $\mathcal{A}$ is an antichain, we have that $\nabla(\mathcal{A}_i) \subseteq P_{i+1} - A_{i+1}$. It follows that

$$\frac{|\nabla(\mathcal{A}_i)|}{N_{i+1}} \leq \frac{|\mathcal{P} - \mathcal{A}_{i+1}|}{N_{i+1}} = 1 - \frac{|\mathcal{A}_{i+1}|}{N_{i+1}} = 1 - \frac{|\mathcal{A}| - |\mathcal{A}_i|}{N_{i+1}}.$$

The normalized matching property, applied to $\mathcal{A}_i$, tells us that $\frac{|\mathcal{A}_i|}{N_i} \leq \frac{|\nabla(\mathcal{A}_i)|}{N_{i+1}}$ so that, from the above inequality, we have,

$$\frac{|\mathcal{A}_i|}{N_i} \leq 1 - \frac{|\mathcal{A}| - |\mathcal{A}_i|}{N_{i+1}}$$

which can be rewritten in the form

$$|\mathcal{A}| \leq N_{i+1} - |\mathcal{A}_i|\left(\frac{N_{i+1}}{N_i} - 1\right).$$

Since $(\mathcal{P} - \mathcal{O})_i = \mathcal{A}_i$, it follows that $|\mathcal{A}_i| = N_i - x$. Using this last identity to eliminate $|\mathcal{A}_i|$ from the above inequality yields our theorem. ☐

Since $\mathcal{O}$ is presumed to be flat, Theorem 5.1.1 tells us that

$$|\mathfrak{a}(\mathcal{O})| \leq N_i + x\left(\frac{N_{i+1}}{N_i} - 1\right)$$

whenever

$$|\mathcal{O}| = \sum_{i=0}^{m(\mathcal{O})} N_i + x.$$

Sha and Kleitman make their arguments entirely in the context of the boolean lattice where

$N_i = \binom{n}{i}$. Shastri restated these arguments in [16] in the more general setting. Sha and Kleitman's strategy is to extend this bound on the choice antichains of flat ideals to any ideal in the boolean lattice. They accomplish this by arguing that if $\mathcal{O}_1$ and $\mathcal{O}_2$ are order ideals of $\boldsymbol{\mathcal{B}}^n$ such that $|\mathcal{O}_1| = |\mathcal{O}_2|$, $\mathcal{O}_1$ is flat and $\mathcal{O}_2$ is not, then $|\mathfrak{a}(\mathcal{O}_1)| \geq |\mathfrak{a}(\mathcal{O}_2)|$. The argument is complex, sketchily presented, and makes use of an advanced result known in the literature as the Kruskal-Katona Theorem.[1] A full account of this argument is, unfortunately, out of our reach. Assuming that this argument succeeds, Corollary 3.2.4 produces the following rather unwieldy bound:

$$e(\boldsymbol{\mathcal{B}}^n) \leq \prod_{i=0}^{n} \left(\prod_{j=0}^{\binom{n}{i}-1} \left(\binom{n}{i} + j \left(\frac{\binom{n}{i+1}}{\binom{n}{i}} - 1 \right) \right) \right).$$

Sha and Kleitman then argue that the following inequality is easily obtained.

$$\prod_{i=0}^{n} \left(\prod_{j=0}^{\binom{n}{i}-1} \left(\binom{n}{i} + j \left(\frac{\binom{n}{i+1}}{\binom{n}{i}} - 1 \right) \right) \right) \leq \prod_{k=0}^{n} \binom{n}{k}^{\binom{n}{k}}.$$

(They offer no insight into how to do this. The word "easily" might be a misnomer here. See for example the so called "standard" derivation worked out by Cooper in [7].) Putting these last two inequalities together, we arrive at what we shall call the **Sha/Kleitman** bound:

$$e(\boldsymbol{\mathcal{B}}^n) \leq \prod_{k=0}^{n} \binom{n}{k}^{\binom{n}{k}}.$$

In his paper [16], Shastri presents a generalization of this result. Shastri's generalization attempts to extend the argument of Sha and Kleitman to any symmetric log concave unimodal LYM poset. Recall that a finite sequence of numbers $a_0, a_1, ..., a_n$ is **log concave** if $a_{i+1}a_{i-1} \leq a_i^2$ for each i and **unimodal** if, for some j, $a_0 \leq a_1 \leq ... \leq a_j$ and $a_n \leq a_{n-1} \leq ... \leq a_j$. The terms log concave and unimodal as applied to ranked posets refer to the sequence of whitney numbers $N_0, N_1, ..., N_n$. One can easily verify that $\boldsymbol{\mathcal{B}}^n$ satisfies this condition. We will not comment on this paper, beyond noting

[1] See for example chapter 7 in [3]or section 2.3 in [9].

that it is odd that Shastri adopts the argument to the more general setting wholesale while ignoring the use made by Sha and Kleitman of the Kruskal-Katona theorem, which applies specially to the Boolean Lattice. The general version of the above inequality given in [16] reads

$$\prod_{i=0}^{n}\left(\prod_{j=0}^{N_i-1}\left(N_i + j\left(\frac{N_{i+1}}{N_i}-1\right)\right)\right) \leq \prod_{k=0}^{n} N_i^{N_i}$$

where $N_0, N_1, ..., N_n$ are the rank numbers of any symmetric log concave unimodal LYM poset.

Note that for any finite sequence of numbers $a_1, a_2, ..., a_n$, a poset with whitney numbers $N_i = a_i$ for each rank i can easily be constructed. Let

$$\mathcal{P}_i = \{(i,j)\,|j \in \mathbb{Z} \text{ and } 1 \leq j \leq a_i\}\,.$$

Then, let $\mathcal{P} = \bigcup_{i=1}^{n} \mathcal{P}_i$ and define $\preceq_{\mathcal{P}}$ by $(i,j) \preceq_{\mathcal{P}} (l,m)$ if and only if $i \leq l$. Under this ordering, $\mathcal{P}$ is a graded LYM poset of rank n with Whitney numbers $N_i = a_i$. It follows that if Shastri's claim is true, the the above inequality holds for any finite symmetric, log concave sequence of numbers. This assertion seems incredible, but the author has not found a counterexample.

Presenting a complete account of these arguments is well beyond the scope of this thesis and so I will content myself with stating an interesting question that is left open. If a poset $\mathcal{P}$ satisfies the property that for all order ideals $\mathcal{O}_1$ and $\mathcal{O}_2$ of $\mathcal{P}$, if $\mathcal{O}_1$ is flat, $\mathcal{O}_2$ is not flat, and $|\mathcal{O}_1| = |\mathcal{O}_2|$ then $|\mathfrak{a}(\mathcal{O}_1)| \geq |\mathfrak{a}(\mathcal{O}_2)|$, let us call it **nice**. The natural question is then:

Problem 5.1.2. Which posets are nice?

The article [14] argues that $\boldsymbol{\mathcal{B}}^n$ is nice and [16] argues that this can be extended to any symmetric, log concave, unimodal, LYM poset. We will have to leave this problem unanswered.

5.2 Brightwell's Generalization

The right hand side of Sha/Kleitman bound may translated to any ranked poset $\mathcal{P}$ in a natural way as

$$\prod_{k=0}^{n} N_i^{N_i}$$

where $N_0, N_1, ..., N_n$ are the whitney numbers of $\mathcal{P}$. In this section we establish the surpising generalization that this number is an upper bound for $e(\mathcal{P})$ whenever $\mathcal{P}$ is LYM. We shall call this result, stated formally as Corollary 5.2.2, the **generalized Sha/Kleitman** bound.

The argument presented here appears in [4] and is discussed briefly in [5]. In some ways the proof comes from generalizing the construction used in the proof of proposition 3.2.4. There, we made use of a collection of uniform probability distributions over the antichains of $\mathcal{P}$. Brightwell alters this construction slightly and in doing so opens to door to a much simpler proof of the generalized Sha/Kleitman bound.

Theorem 5.2.1. *Let $\mathcal{P}$ be a ranked poset and let w be a weight function on $\mathcal{P}$. If $w(\mathcal{A}) \leq 1$ for each antichain $\mathcal{A}$ of $\mathcal{P}$ then*

$$e(\mathcal{P}) \leq \frac{1}{\prod_{X \in \mathcal{P}} w(X)}.$$

Proof. For each antichain $\mathcal{A}$ of $\mathcal{P}$, let $\rho_{\mathcal{A}}$ be the map from $\mathcal{A}$ to $\mathbb{R}$ defined by $\rho_{\mathcal{A}}(X) = \frac{w(X)}{w(\mathcal{A})}$. Recall that the choice antichain operation provided a one to one correspondence between antichains and order ideals so that for each antichain $\mathcal{A}$ in $\mathcal{P}$, we have some $\mathcal{O}$ so that $\mathfrak{a}(\mathcal{O}) = \mathcal{A}$. In what follows we will be interested the choice antichains of a sequence of order ideals corresponding to a linear extension. In this context, we will let ρ_i abbreviate $\rho_{\mathfrak{a}(\mathcal{O}_i)}$.

Consider the following slight variation on our recursive construction of a linear extension of $\mathcal{P}$ from Theorem 3.2.4. Let $\mathcal{O}_0 = \emptyset$ and for each $i \geq 0$, let $\mathcal{O}_{i+1} = \mathcal{O}_i + \{X_i\}$ where X_i is chosen from $\mathfrak{a}(\mathcal{O}_i)$ with probability $\rho_i(X_i)$ as defined above. Following the argument in Theorem 3.2.4, this procedure allows us to define a function μ over the set $E(\mathcal{P})$ by assigning, to each linear extension ε, the product $\prod_{i=1}^{|\mathcal{P}|} \rho_i(X_i)$ where X_i is the unique element of $\mathcal{O}_i - \mathcal{O}_{i-1}$. Note that $\rho_i(X_i)$ is exactly the probability that X_i is chosen at the ith stage of our construction given that $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_{i-1}$

have already been constructed. It follows that $\mu(\varepsilon)$ is exactly the probability that our construction results in ε. Since the construction is guaranteed to produce a linear extension of $\mathcal{P}$ we have that $\sum\limits_{\varepsilon \in E(\mathcal{P})} \mu(\varepsilon) = 1$.

Since for any linear extension ε, each $X \in \mathcal{P}$ appears in the corresponding sequence $X_1 \preceq X_2 \preceq ... \preceq X_{|\mathcal{P}|}$ exactly once, it follows from our assumptions that

$$\mu(\varepsilon) = \prod_{i=0}^{|\mathcal{P}|} \rho_i(X_i) = \prod_{i=0}^{|\mathcal{P}|} \frac{w(X_i)}{w(\mathfrak{a}(\mathcal{O}_i))} \geq \prod_{X \in \mathcal{P}} w(X).$$

Our theorem now follows from Proposition 3.2.3. □

To see how this relates to the LYM property, note that the condition

$$\sum_{i=0}^{n} \frac{|\mathcal{A}_i|}{N_i} \leq 1$$

can be rewritten as

$$\sum_{X \in \mathcal{A}} \frac{1}{N_{r(X)}} \leq 1.$$

Brightwell's argument makes use of Theorem 5.2.1 by defining a special weight function

$$w(X) = \frac{1}{N_{r(X)}}.$$

If $\mathcal{P}$ is an LYM Poset, then we have $w(\mathcal{A}) \leq 1$ for every antichain $\mathcal{A}$ in $\mathcal{P}$. In this way, Kleitman's bound can be easily derived.

Corollary 5.2.2. *(The generalized Sha/Kleitman bound) If $\mathcal{P}$ is an LYM poset with whitney numbers $N_0, N_1, N_2, ..., N_n$ then*

$$e(\mathcal{P}) \leq \prod_{i=0}^{n} N_i^{N_i}$$

Proof. Let w be defined as above. For any order ideal $\mathcal{O}$ of $\mathcal{P}$, let $\rho_{\mathcal{O}}$ be the map $\mathfrak{a}(\mathcal{O}) \longrightarrow \mathbb{R}^+$ defined by $\rho(X) = \frac{w(X)}{w(\mathfrak{a}(\mathcal{O}))}$, for each $X \in \mathfrak{a}(\mathcal{O})$. Since $w(\mathfrak{a}(\mathcal{O})) = \sum\limits_{X \in \mathfrak{a}(\mathcal{O})} w(X)$, it follows that

$\frac{w(X)}{w(\mathfrak{a}(O))} \leq 1$ for each $X \in \mathfrak{a}(\mathcal{O})$ and that $\sum\limits_{X \in \mathfrak{a}(\mathcal{O})} \frac{w(X)}{w(\mathfrak{a}(O))} = 1$. Therefore, $\rho_{\mathcal{O}}$ defines a probability distribution over $\mathfrak{a}(\mathcal{O})$.

If $\mathcal{P}$ is LYM, we have that $w(\mathfrak{a}(O)) \leq 1$ and therefore, $\rho(X) = \frac{w(X)}{w(\mathfrak{a}(O))} \geq w(X)$ for each $X \in \mathfrak{a}(\mathcal{O})$. By Theorem 5.2.1,

$$e(\mathcal{P}) \leq \frac{1}{\prod\limits_{X \in \mathcal{P}} w(X)} = \frac{1}{\prod\limits_{X \in \mathcal{P}} \frac{1}{N_{r(X)}}} = \prod_{X \in \mathcal{P}} N_{r(X)}$$

where r is the rank function of $\mathcal{P}$. Since for each i, there are exactly N_i elements of $\mathcal{P}$ with rank i, the theorem follows. $\square$

This proof is a substantial improvement on the original. It is always welcome when a generalization of a result admits a cleaner and simpler proof.

Recall the sum of disjoint chains, $\mathcal{P} = \sum\limits_{i=1}^{d} \mathcal{C}_{k-1}$, discussed in Corollary 3.2.2. There we deduced that the exact number of linear extensions of the poset $\mathcal{P}$ is

$$e(\mathcal{P}) = \frac{n!}{(k!)^d}$$

where $n = kd$. Since each of the k ranks of $\mathcal{P}$ has exactly d elements, the generalized Sha/Kleitman bound yields

$$e(\mathcal{P}) \leq \left(d^d\right)^k = d^n.$$

At the very end of [5], Brightwell and Tetali point out that, by using standard asymptotic arguments, one can show,

$$\tfrac{n!}{(k!)^d} = \left(d(1 - o(1))\right)^n.$$

Recall that a function f is equivalent to $o(1)$ exactly when $\lim\limits_{x \to \infty} f(x) = 0$. Therefore, as either d or k approaches to infinity, the Sha/Kleitman bound approaches the correct value.

This example can actually be greatly simplified. Note that a chain of size n is a rank n poset in which $N_i = 1$ for each i. Since a chain has exactly one linear extension, the Sha/Kleitman bound actually gives the correct number of linear extensions in this case. Since any chain is obviously LYM, the Sha/Kleitman bound must be the best possible general bound on the number of linear extension of an LYM poset.

This observation can also be turned around as the following proposition shows.

Proposition 5.2.3. *If $\mathcal{P}$ is not a chain, then Kleitman's bound is strict.*

Proof. In the argument above equality is attained just in case $\frac{w(X)}{w(\mathfrak{a}(O))} = w(X)$ for each $X \in \mathcal{P}$. This is equivalent to $w(\mathcal{A}) = 1$ for each antichain $\mathcal{A}$ of $\mathcal{P}$. If $|\mathcal{P}_i| > 1$ for some i, then let X and Y be distinct elements of $\mathcal{P}_i$. Since $\{X\}$, $\{Y\}$, and $\{X, Y\}$ are all antichains, it follows that $w(\{X, Y\}) = w(\{X\}) + w(\{Y\})$. □

Chapter 6

Conclusion

We have assembled a number of different bounds for LYM posets generally. It might be worthwhile, in closing, to look at the important example of the boolean lattice and see, first-hand, how these bounds compare.

The numbers $e(\boldsymbol{\mathcal{B}}^n)$ for $n = 1, 2, 3, 4, 5$ and 6 may be found in the OEIS under article A046873. They are given in Table 6.1 below, where they are contrasted with the corresponding Sha/Kleitman bound, the bound derived from Sperner's theorem in Section 4.2, the trivial bound discussed in the beginning of Section 3.2, and a well-known elementary lower bound. Brightwell and Tetali published improvements on the Sha/Kleitman bound for the boolean lattice in [5]. In this paper, they use the sophisticated "entropy" method attributed to Kahn and Kim[1] to derive the bound

$$e(\boldsymbol{\mathcal{B}}^n) \leq e^{6\cdot 2^n \cdot \frac{\ln n}{n}} \prod_{i=0}^{n} \binom{n}{i}!.$$

This bound is farther off for small values of n. For example, for $n = 3$,

$$\prod_{i=0}^{n} \binom{n}{i}^{\binom{n}{i}} = 739$$

[1]See [12]

while

$$e^{6\cdot 2^n \cdot \frac{\ln n}{n}} \prod_{i=0}^{n} \binom{n}{i}! \approx 1.55 \times 10^9.$$

For $n = 8$,

$$\prod_{i=0}^{n} \binom{n}{i}^{\binom{n}{i}} \approx 2.78 \times 10^{420}$$

while

$$e^{6\cdot 2^n \cdot \frac{\ln n}{n}} \prod_{i=0}^{n} \binom{n}{i}! \approx 2.26 \times 10^{491}.$$

(Note that Brightwell and Tetali's bound first surpasses the trivial bound here.) On the other hand it does eventually converge to the correct value. The first value of n for which this bound does better than the Sha/Kleitman bound is $n = 18$. In this case,

$$\prod_{i=0}^{n} \binom{n}{i}^{\binom{n}{i}} \approx 2.10 \times 10^{1173310}$$

and

$$e^{6\cdot 2^n \cdot \frac{\ln n}{n}} \prod_{i=0}^{n} \binom{n}{i}! \approx 1.58 \times 10^{1169187}.$$

It is also closely related to the lower bound for $e(\boldsymbol{\mathcal{B}}^n)$ given by

$$\prod_{i=0}^{n} \binom{n}{i}! \leq e\left(\boldsymbol{\mathcal{B}}^n\right).$$

The quantity on the left counts exactly the linear extensions ε of $\boldsymbol{\mathcal{B}}^n$ so that $r(X) \leq r(Y)$ implies that $X \preceq_{\varepsilon} Y$.

n	$\prod_{i=0}^{n} \binom{n}{i}!$	$e(\boldsymbol{\mathcal{B}}^n)$	$\prod_{i=0}^{n} \binom{n}{i}^{\binom{n}{i}}$	$\binom{n}{\lfloor \frac{n}{2} \rfloor}^{2^n}$	$2^n!$
1	1	1	1	1	2
2	2	2	4	16	24
3	36	48	729	6561	40320
4	4.15×10^5	1.680384×10^6	3.06×10^9	2.82×10^{12}	2.09×10^{13}
5	1.9×10^{17}	$1.4807804035657359360 \times 10^{19}$	9.77×10^{26}	10^{32}	2.63×10^{35}
6	2.16×10^{48}	$1.4137791169722788711719597031620079563020547695771648O \times 10^{53}$	4.38×10^{70}	1.84×10^{83}	1.72×10^{89}
7	7.08×10^{126}	?	2.81×10^{175}	4.37×10^{197}	3.86×10^{215}
8	9.15×10^{317}	?	2.78×10^{420}	2.21×10^{472}	8.58×10^{506}

Table 6.1: Estimates of our bounds for $e(\boldsymbol{\mathcal{B}}^n)$.
Exact values of $e(\boldsymbol{\mathcal{B}}^n)$ are given for $n = 1, 2, ..., 6$. The values $e(\boldsymbol{\mathcal{B}}^7)$ and $e(\boldsymbol{\mathcal{B}}^8)$ are, to date, not known.

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