Rook ‘em, Danno.

The Combinatorics and Pedagogy of Matchings
Polynomials and Rook Polynomials

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Introduction

Counting is one of the earliest mathematical endeavors accessible to children. Indeed, it is widely held to be one of the most natural, fundamental, and directly mathematical processes that we, as humans, employ. Counting techniques can, however, lead to mathematical problems that are surprisingly sophisticated and complex. Combinatorics, the branch of mathematics most clearly concerned with the principles of counting, is becoming increasingly relevant in our computerized age, and yet as students get into high school and college, they tend to experience a great deal of difficulty as they encounter increasingly complex counting problems. This obscurity can be disorienting for them, precisely because counting is seen as such a basic procedure. Facility with counting principles is valued as a necessary part of students' mathematical educational experience; in fact the Number and Operations Standards of the National Council of Teachers of Mathematics recommends that "in grades 9-12 all students should develop an understanding of permutations and combinations as counting techniques." (NCTM, 2000). It is significant, then, to ask whether there are novel mathematical insights or pedagogical methods that could better motivate students, facilitate their understanding, and allow them to feel more comfortable with combinatorial ideas.

The topic of rook polynomials, which is the focus of this paper, can be readily associated with a wide variety of significant combinatorial principles. In addition, it proves to be quite effective as a didactic tool for the teaching of combinatorics. This paper consists of two major parts:

1. a mathematical investigation of rook polynomials (and the more general matchings polynomial),
2. a curriculum which draws upon some of the mathematical principles found in the investigation and explores their pedagogical potential.

Although the mathematics developed in the first part of the paper is fairly sophisticated, the curriculum is intended to be accessible to an advanced high school class.

Perhaps the most interesting aspect of rook polynomials is that they unite two major components of combinatorics: enumeration and graph theory. Such a perfect marriage of these two topics is exciting for any combinatorialist, and, from a pedagogical point of view, this connection offers students a clear view of the overarching span of the field. In this sense, rook polynomials are an ideal object of study, providing entry points to a dizzying array of combinatorial concepts from a convenient and contextualized perspective.

A commonly faced challenge in teaching combinatorics is the apparent disparate nature of the material, which includes a large number of topics that often appear, to the untrained eye, as unrelated and ill-motivated. Surprisingly, however, in the context of rooks, many of these central topics arise naturally – almost magically – in a way that is not only inextricably linked to other topics, but also easy to teach, natural to motivate, and ripe with potential for the classroom. The goal of the curriculum, then, is to illuminate some of these notorious topics through the use of rooks. In particular, the curriculum developed herein addresses the following three major topics in combinatorics: counting principles, generating functions, and matchings. The effortless unification of these topics will convey to the reader the pedagogical effectiveness of employing rooks in the classroom, and will convey to the student the strength and beauty of combinatorics itself.

Acknowledgements

The mathematics presented in this paper can be found in the first chapter of Chris Godsil's book, Algebraic Combinatorics. His mathematics is the backbone of this paper, and I have simply fleshed out his already-established mathematics with more detailed explanations and examples. I am indebted to him for his novel and fascinating treatment of the matchings polynomial as it relates to a number of different areas of math.

Equally important to the development of this paper was Dr. John Gorman and his class at Jesuit High School. Dr. Gorman allowed me to test the curriculum that follows in his classroom, and both he and his students provided invaluable insights and feedback that contributed greatly to the paper. This would not have been possible without their enthusiastic participation.

I also wish to thank my advisor, Dr. John Caughman. His support, advice, and encouragement were invaluable throughout this process, and the resulting paper would not be what it is without his dedicated involvement. Thank you.

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Part One:

The Mathematics of Matchings Polynomials

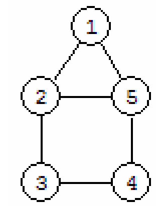
Overview of Matchings and Generating Functions

In this introductory section, we address two concepts that are fundamental to the mathematics in this paper: *matchings* and *generating functions*. Although a deep knowledge of these subjects is not required to understand the mathematics that follows, it *is* important that the reader become acquainted with certain ideas and terminology. We include several examples throughout this section to aid the reader and enhance their understanding of these basic ideas.

Matchings

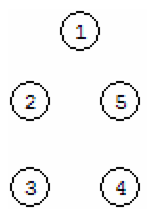
A **graph** G consists of a set of **vertices**, a set of **edges**, and a relation that associates each edge with exactly two vertices (which are called the **endpoints** of the edge). All the graphs discussed in this paper are **simple**, which means that they have no **loops** (edges whose endpoints are the same vertex) and no **multiple edges** (distinct edges with the same endpoints). A **matching** in a graph G is a set of edges such that no two edges share an endpoint. Or, said another way, a matching is a set of edges, no two of which have a vertex in common (West, 2001).

Since matchings are edge sets, a given graph can have matchings of various sizes; an ***r*-matching** in a graph G, then, is a set of exactly r edges, no two of which share a common vertex. The following graph H is often called the “house” graph; we will use it to exemplify the notion of r-matchings in graphs.



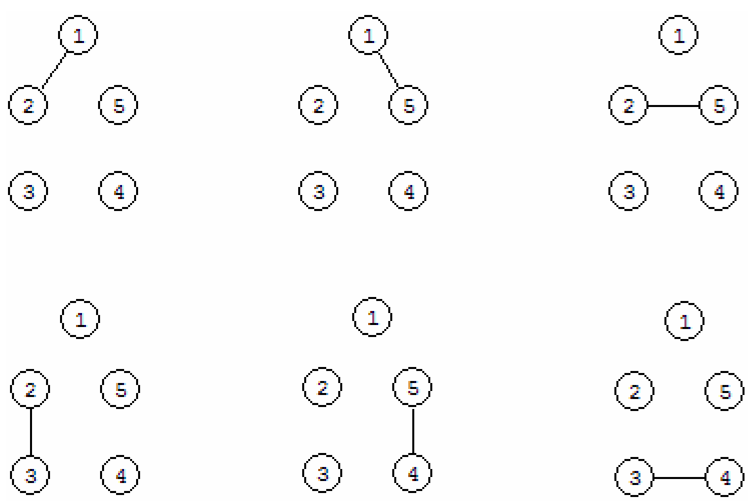
H

When $r = 0$, an *r*-matching is an edge set containing zero edges. The figure below depicts the empty edge set, the only possible *r*-matching when $r = 0$.



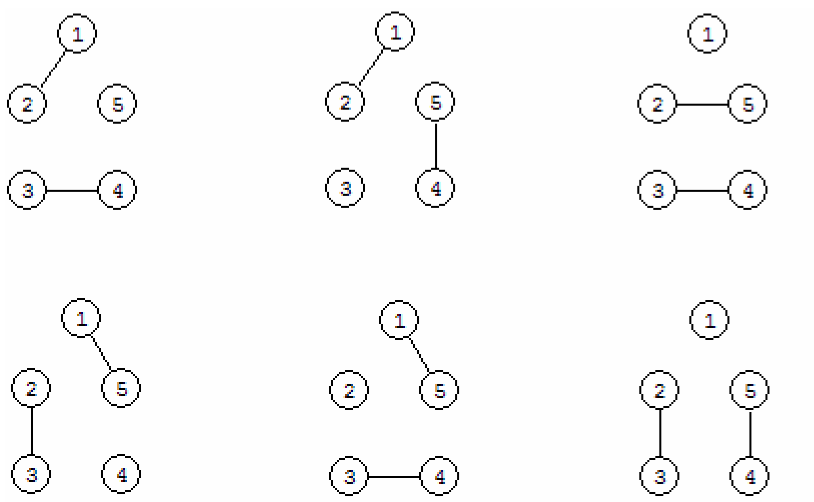
The unique 0-matching in *H*

When $r = 1$, an *r*-matching is an edge set containing one edge. The next figure displays all six of the possible 1-matchings in the house graph.



The 1-matchings in *H*

When $r = 2$, an r-matching is an edge set containing two vertex-disjoint edges. The figure below depicts all six of the possible 2-matchings in the house graph.

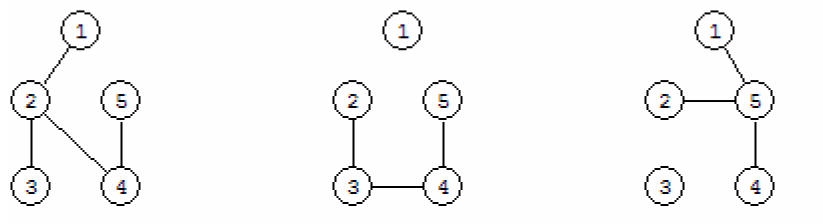


The 2-matchings in H

Although such diagrams are useful, notice that we could easily describe an r-matching in terms of its edges without resorting to a figure. Denoting by uv an edge that has endpoints u and v, we can express the first 2-matching listed above as $M = \{12,34\}$. This notation is more compact and also emphasizes the fact that matchings are indeed edge sets.

Note that the house graph does not have any matchings of size three or greater. This follows from the definition, which stipulates that no two edges in a matching can share an endpoint. Since the graph only has five vertices, it is not possible to form an r-matching where $r \geq 3$.

By way of illustrating some non-examples, the edge sets indicated below are not matchings; in each case, two or more of the edges share a common vertex.



Examples of edge sets in H that are not matchings

Generating Functions

A **generating function** of a given number sequence is a polynomial function (or power series) whose coefficients are the terms of that sequence. Associating the terms of a sequence with the coefficients of a polynomial is a process that has a number of surprising combinatorial benefits, as we will soon discover. In this paper, we will treat generating functions as purely formal objects, viewing the variable x in the function not so much as an unidentified element of some number field, but rather as little more than a placeholder in our presentation of the coefficients, which are the terms of our given number sequence. As Herbert Wilf, author of the *Generatingfunctionology*, describes it, "A generating function is a clothesline on which we hang up a sequence of numbers for display (Wilf, 1994)."

We could, for example, easily describe a generating function for the Fibonacci sequence, in which $f_0 = 1$, $f_1 = 1$, and the n^{th} term is defined by $f_n = f_{n-2} + f_{n-1}$. Recall that the first several terms of this sequence are 1, 1, 2, 3, 5, 8, 13, …. We can form the generating function for this sequence in a straightforward way – we simply need a power series in which the r^{th} term in the sequence, f_r, is the coefficient of the term x^r. The generating function for the Fibonacci sequence, then, is

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + ...$$

Note that in this case the generating function is an infinite series; if our number sequence were finite, however, then the generating function would have a finite number of terms as well. In such cases the generating function would simply be a polynomial.

Another familiar example of a generating function is found in the binomial theorem. This theorem states that for any real number *x* and any natural number *n*,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n = \sum_{r=0}^{n}\binom{n}{r}x^r$$

where the **binomial coefficient** $\binom{n}{r}$ is given by $\dfrac{n!}{r!(n-r)!}$. These binomial coefficients have intrinsic meaning; indeed, the coefficient $\binom{n}{r}$ equals the number of ways of choosing *r* items from *n* items. So the expression $(1+x)^n$ represents, in closed form, the generating function for the finite number sequence whose terms are $\binom{n}{r}$, where *r* ranges from 0 to *n*.

In the above example of the house graph, we can count the number of possible 0, 1, and 2-matchings in the graph. There is a single 0-matching, there are six 1-matchings, and there are six 2-matchings. This leaves us with a small sequence of numbers, 1,6,6, where the r^{th} number in the sequence represents the number of *r*-matchings in a graph. One way to form a generating function that would encode the number of different matchings in a graph *G* is to associate the r^{th} term with the number of *r*-matchings that *G* has. Such a generating function for the number of matchings in the house graph *H* would be: $1(x^0)+6(x^1)+6(x^2)=1+6x+6x^2$.

In this paper we will be working extensively with a different, but similarly-defined, object called the 'matchings polynomial' of a graph. This matchings polynomial is, fundamentally, a generating function, and although the precise definition for the matchings polynomial that we will introduce below varies slightly from the examples presented here, the concepts behind the generating function remain the same. This brief introduction, then, should prepare the reader for the mathematical discussion ahead.

Chapter 1 – The Matchings Polynomial

Section 1.1 – Matchings and Generating Functions

In this section we introduce and discuss the matchings polynomial of a graph. Let G be a graph with n vertices. Denote the number of r-matchings in G by $p(G,r)$. Agreeing to set $p(G,0)=1$, we define the **matchings polynomial** as follows:

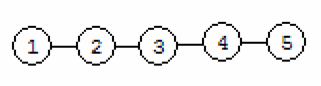
$$\mu(G,x) := \sum_{r\geq 0}(-1)^r\, p(G,r)x^{n-2r}\ .$$

Thus the matchings polynomial is a polynomial with alternating signs, in which the coefficients $p(G,r)$ represent the number of r-matchings in G. Unlike the generating functions discussed in the introduction, the coefficient $p(G,r)$ in the matchings polynomial corresponds to the term x^{n-2r}. There are several technical advantages to this convention that we need not dwell upon here. However, it is worth observing that our convention assures that the matchings polynomial will always be **monic**, meaning that the leading term is always 1. (This is true because when $r=0$, $p(G,0)=1$ and corresponds to the $x^{n-2(0)}$ term, or the x^n term). Furthermore, when n is even, the matchings polynomial is an even function, containing only even powers of x. Similarly, when n is odd, the polynomial is an odd function.

In order better to grasp the matchings polynomial, we determine the matchings polynomials for some simple classes of graphs: paths, cycles, complete graphs, and complete bipartite graphs. First, however, we note that if E_n denotes the empty graph with n vertices and no edges, then the matchings polynomial of E_n consists only of the term where $r=0$, since E_n can have at most a 0-matching. Hence $p(E_n,0)=1$, and $\mu(E_n,x)=x^n$.

Section 1.1 – Matchings in Paths

We now consider the matchings polynomials of the family of graphs known as paths. We begin with a definition. The **path** P_n has n vertices, two of which have degree one, and $n-2$ of which have degree two. For example, a path on five vertices, P_5, is pictured below. Vertices 1 and 5 have degree one, and vertices 2, 3, and 4 have degree two.



The number $p(P_n, r)$ of r-matchings in P_n is determined in the following way. If we view P_n as running from left to right, we can contract each edge in a given r-matching onto its left-hand endpoint. What results is a path with $n-r$ vertices, r of which are distinguished. Conversely, given a path on $n-r$ vertices with some subset of r of these vertices distinguished, we can reconstruct an r-matching in a path of n vertices. We do this by inserting an edge to the right of each distinguished vertex. This correspondence between r-matchings in P_n and selections of r distinguished vertices P_{n-r} allows us to count $p(P_n, r)$ easily. We simply choose any r of the $n-r$ vertices of P_{n-r} where we will insert an edge that will belong to our r-matching in P_n. Thus we have $\binom{n-r}{r}$ ways of determining an r-matching, so $p(P_n, r) = \binom{n-r}{r}$. Then, by definition of the matchings polynomial, we have the following result.

Proposition 1.1: For any natural number n, the matchings polynomial of the path P_n is given by

$$\mu(P_n, x) = \sum_{r \geq 0} (-1)^r \binom{n-r}{r} x^{n-2r} .$$

❑

<u>Example</u>: Let us work through an example of a path, utilizing the above proposition. Consider P_5, the path on 5 vertices, pictured in the text above. In order to compute $\mu(P_5, x)$, we must examine the r-matchings as r ranges from 0 to n. Clearly there is only one matching when $r = 0$. Using the above counting argument, we consider the case where $r = 1$. By examining a path on $5-1=4$ vertices, which is P_4, we note that we could insert the one edge of our 1-matching to the right of any of the 4 vertices. Similarly for $r = 2$, looking at a path on $5-2=3$ vertices, we could insert the edges of our 2-matching to the right of any of those 3 vertices. So we indeed see that $p(P_5, r) = \binom{5-r}{r}$, and we use this fact to construct the following table:

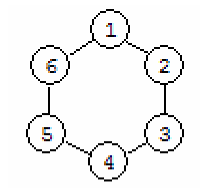
r-value	$p(P_5, x)$
0	$\binom{5-0}{0} = 1$
1	$\binom{5-1}{1} = \binom{4}{1} = 4$
2	$\binom{5-2}{2} = \binom{3}{2} = 3$

Thus there is one r-matching where $r = 0$, there are four r-matchings where $r = 1$, and there are three r-matchings where $r = 2$. So by using Proposition 1.1, we have arrived at the following:

$$\mu(P_5, x) = x^5 - 4x^3 + 3x.$$

<u>Section 1.2</u> – <u>Matchings in Cycles</u>

Next we consider the matchings polynomial of the family of cycles. A **cycle** C_n is a graph with *n* vertices, all of which have degree two and are joined in a circular fashion. For example, the cycle C_6 is shown.



We can determine the coefficients for the matchings polynomial of C_n as follows. Labeling the vertices of the cycle clockwise from 1 to *n*, we contract each edge in a given *r*-matching onto its endpoint in the clockwise direction. We look at the vertex labeled 1, and our search for *r*-matchings breaks down into two cases.

Let *e* be the edge that is directly left (counter-clockwise) of vertex 1. Then *e* either belongs or does not belong to any given *r*-matching *M*. Said another way, $e \in M$ or $e \notin M$. If $e \in M$, then by definition of a matching, the two edges adjacent to *e* cannot be included in *M*. The remainder of the graph, then, which must containing the remaining *r* – 1 edges of *M*, is a path on *n* – 2 vertices. Therefore, to count *r*-matchings containing *e*, we seek the number of $(r-1)$-matchings in P_{n-2}, which is denoted by $p(P_{n-2}, r-1)$. If $e \notin M$, then all edges of *M* are contained in the remainder of the graph, which forms a path on *n* vertices. We thus want the number of *r*-matchings in P_n, which is denoted by $p(P_n, r)$. In Section 1.1, we found that $p(P_n, r) = \binom{n-r}{r}$. By the addition principle, then, we express the number of *r*-matchings in the cycle C_n on *n* vertices in the following way:

$$
\begin{aligned}
p(C_n, r) &= p(P_n, r) + p(P_{n-2}, r-1) \\
&= \binom{n-r}{r} + \binom{n-r-1}{r-1}
\end{aligned}
$$

$$= \frac{(n-r)!}{r!(n-2r)!} + \frac{(n-r-1)!}{(r-1)!(n-2r)!}$$

$$= \frac{n(n-r)!}{r!(n-2r)!(n-r)}$$

$$= \frac{n}{n-r}\binom{n-r}{r}$$

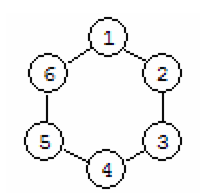
Thus, by the definition of the matchings polynomial, we arrive at the following result about the matchings polynomial of cycles.

<u>Proposition 1.2:</u> For any natural number *n*, the matchings polynomial of the cycle C_n is given by

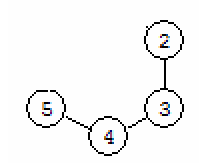
$$\mu(C_n, x) = \sum_{r \geq 0} (-1)^r \frac{n}{n-r}\binom{n-r}{r} x^{n-2r} \text{ .}$$

❑

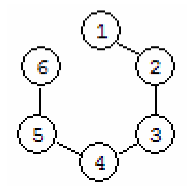
<u>Example</u>: Let us work through an example of the above proposition. Consider C_6, the cycle on 6 vertices pictured below. To compute $\mu(C_6, x)$ directly, we must examine the *r*-matchings as *r* ranges from 0 to *n*. Following the counting argument mentioned above, we consider the edge *e* with endpoints 1 and 6, the edge directly left of vertex 1. Any *r*-matching will either include this edge, or it won't.



To count the r-matchings that include e, we look for the number of $(r-1)$-matchings in the graph without the edge *e* or either of its endpoints. But this graph is simply a path on 4 vertices, P_4.



To count the r-matchings that do not include e, we look for the number of r-matchings in the graph without the edge e, seen below. But this graph is simply a path on 6 vertices, P_6.



Thus we see that $p(C_6, r) = p(P_6, r) + p(P_4, r-1)$. As our computation above indicates, this simplifies to $\frac{6}{6-r}\binom{6-r}{r}$. Using this fact we can construct the following table:

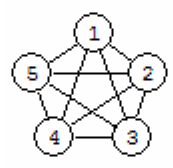
r-value	$p(C_6, x)$
0	$\frac{6}{6-0}\binom{6-0}{0} = \frac{6}{6}\binom{6}{0} = 1$
1	$\frac{6}{6-1}\binom{6-1}{1} = \frac{6}{5}\binom{5}{1} = 6$
2	$\frac{6}{6-2}\binom{6-2}{2} = \frac{6}{4}\binom{4}{2} = 9$
3	$\frac{6}{6-3}\binom{6-3}{3} = \frac{6}{3}\binom{3}{3} = 2$

Thus there is one r-matching where $r = 0$, there are six r-matchings where $r = 1$, there are nine r-matchings where $r = 2$, and there are two r-matchings where $r = 3$. So we have arrived at the following:

$$\mu(C_6, x) = x^6 - 6x^4 + 9x^2 - 2.$$

Section 1.3 – **Matchings in Complete Graphs**

The **complete graph** K_n on *n* vertices is the graph in which every vertex is adjacent to every other vertex; every possible edge is included. For example, the complete graph K_5 is shown below.



These complete graphs form another family of graphs with easily-computable matchings polynomials. Note that each edge in an r-matching covers exactly two vertices. Thus, to count r-matchings in K_n, we could first pick some subset of 2*r* vertices from the *n* vertices of the graph, and then find the number of r-matchings among those 2*r* vertices. This second step is the same as counting the number of r-matchings in K_{2r}, the complete graph on 2*r* vertices. So by the multiplication principle,

$$p(K_n, r) = \binom{n}{2r} p(K_{2r}, r).$$

To find the term $p(K_{2r}, r)$ in this expression, we first select an edge to cover the least-numbered of our chosen 2*r* vertices. There are $(2r-1)$ edges incident with this vertex in K_{2r}, and once any such edge has been chosen, we seek to complete our matching by choosing an (*r* – 1) matching in the complete graph on the remaining $(2r - 2)$ uncovered vertices. Thus

$$p(K_{2r}, r) = (2r-1)\, p(K_{2r-2}, r-1).$$

Proceeding inductively (noting that $p(K_2,1) = 1$), we arrive at the following result:

$$
\begin{aligned}
p(K_{2r}, r) \quad &= (2r-1)\,p(K_{2r-2}, r-1) \\
&= (2r-1)(2r-3)\,p(K_{2r-4}, r-2) \\
&= (2r-1)(2r-1)(2r-5)...5\cdot 3\cdot 1 \\
&= \frac{(2r)!}{(2r)(2r-2)(2r-4)...4\cdot 2} \\
&= \frac{(2r)!}{2^r r!}
\end{aligned}
$$

But since we determined that $p(K_n, r) = \binom{n}{2r} p(K_{2r}, r)$, we conclude that

$$
\begin{aligned}
p(K_n, r) \quad &= \binom{n}{2r} p(K_{2r}, r) \\
&= \binom{n}{2r} \frac{(2r)!}{2^r r!} \\
&= \frac{n!}{(n-2r)!2^r r!}
\end{aligned}
$$

By the above equation and the definition of the matchings polynomial, we have arrived at the following result about the matchings polynomial of complete graphs.

<u>Proposition 1.3</u>: For any natural number *n*, the matchings polynomial of the complete graph K_n is given by

$$\mu(K_n, x) = \sum_{r\geq 0} (-1)^r \frac{n!}{(n-2r)!2^r r!} x^{n-2r} \ .$$

❑

<u>Example</u>: Let us work through an example in which we compute the matchings polynomial of a complete graph. Consider K_5, the complete graph on 5 vertices, pictured in the beginning of this section. In order to compute $\mu(K_5, x)$ we must examine the *r*-matchings as *r* ranges from 0 to *n*. Using our result that $p(K_n, r) = \frac{n!}{(n-2r)!2^r r!}$, we can construct the following table:

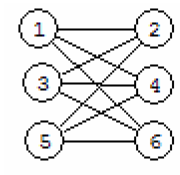
r-value	$p(K_5, x)$
0	$\frac{5!}{(5-2\cdot 0)!\cdot 2^0 \cdot 0!} = \frac{5!}{5!\cdot 1\cdot 1} = 1$
1	$\frac{5!}{(5-2\cdot 1)!\cdot 2^1 \cdot 1!} = \frac{5!}{3!\cdot 2\cdot 1} = 10$
2	$\frac{5!}{(5-2\cdot 2)!\cdot 2^2 \cdot 2!} = \frac{5!}{1!\cdot 4\cdot 2} = 15$

Thus we see that there is one *r*-matching where $r = 0$, there are ten *r*-matchings where $r = 1$, and there are fifteen *r*-matchings where $r = 2$. So by using Proposition 1.3, we have arrived at the following:

$$\mu(K_5, x) = x^5 - 10x^3 + 15x .$$

Section 1.4 – Matchings in Complete Bipartite Graphs

Finally, we consider the matchings polynomial of one more family of graphs. The **complete bipartite graph**, $K_{m,m}$, consists of two sets of *m* vertices, so it has a total of 2*m* vertices. Within each set, the vertices are mutually non-adjacent. However, every vertex in one set is adjacent to every vertex in the other set. The diagram below illustrates such a graph, $K_{3,3}$. Note that in this depiction, the vertices 1,3,5 form one set of three vertices, while 2,4,6 form the other. Each vertex in a given set is adjacent to every vertex in the other set but to none of the vertices in its own set.



To find the number of r-matchings in such a graph, we can first pick any r vertices from the first set, done in $\binom{m}{r}$ ways. Then we can pick any r vertices from the second set, also done in $\binom{m}{r}$ ways. Finally, there are $r!$ ways of assigning disjoint edges to pair up these two sets of vertices. Thus, using the multiplication principle, we find that $p(K_{m,m}, r) = \binom{m}{r}\binom{m}{r} r! = \binom{m}{r}^2 r!$. By the definition of the matchings polynomial, the result below follows.

<u>Proposition 1.4</u>: For any natural number m, the matchings polynomial of the complete bipartite graph $K_{m,m}$ is given by

$$\mu(K_{m,m}, x) = \sum_{r \geq 0} (-1)^r \binom{m}{r}^2 r! \, x^{2m-2r} .$$

❑

<u>Example</u>: Let us work through an example in which we compute the matchings polynomial of a complete bipartite graph. Consider $K_{3,3}$, the complete bipartite graph on 6 vertices (3 in each set) pictured above. In order to compute $\mu(K_{3,3}, x)$, we must determine the number of r-matchings, as r ranges from 0 to n. Using the fact given above that $p(K_{m,m}, r) = \binom{m}{r}^2 r!$, we can construct the following table:

r-value	$p(K_{3,3}, x)$
0	$\binom{3}{0}^2 0! = 1$
1	$\binom{3}{1}^2 1! = 9$
2	$\binom{3}{2}^2 2! = 18$
3	$\binom{3}{3}^2 3! = 6$

Thus we see that there is one r-matching where $r = 0$, there are nine r-matchings where $r = 1$, there are eighteen r-matchings where $r = 2$, and there are six r-matchings where $r = 3$. So by using Proposition 1.4, we have arrived at the following:

$$\mu(K_{3,3}, x) = x^6 - 9x^4 + 18x^2 - 6 .$$

Chapter 2 – Reduction Theorems for Matchings Polynomials

When graphs are large and complex, computing their matchings polynomial from the definition can be arduous. It is almost always more feasible first to compute the matchings polynomial of smaller, simpler graphs. Therefore, we seek ways to simplify graphs in order to ease the computation of their matchings polynomials. In this chapter, we describe a number of theorems that ultimately allow for such simplification; for clarification, we also provide examples of each theorem.

Section 2.1 – The Matchings Polynomial of a Disjoint Union

This first theorem states that the matchings polynomial of the union of two disjoint graphs is equal to the product of the matchings polynomials of the two graphs.

Theorem 2.1: For two graphs *G* and *H* with disjoint vertex sets,

$$\mu(G \cup H, x) = \mu(G, x)\mu(H, x).$$

Proof: First consider the coefficient of x^{n-2r} in $\mu(G \cup H, x)$. Each *r*-matching in the graph $G \cup H$ consists of an s-matching in *G*, combined with an ($r-s$) matching in *H*, for some *s*. Summing all such combinations (over s from 0 to *r*) will give us the total number of *r*-matchings in $G \cup H$. Thus $p(G \cup H) = \sum_{s=0}^{r} p(G,s)p(H,r-s)$. By the definition of the matchings polynomial, then, the coefficient of x^{n-2r} in $\mu(G \cup H, x)$ is $(-1)^r \sum_{s=0}^{r} p(G,s)p(H,r-s)$.

We now determine the coefficient of x^{n-2r} in $\mu(G,x)\mu(H,x)$. To do so we must consider pairs of terms, one each from $\mu(G,x)$ and $\mu(H,x)$, whose product contributes to the x^{n-2r} term. Such terms are expressed in the following table. In this table we have listed the terms starting with the "leading" term rather than starting with the constant term.

Note that $n_G + n_H = n$.

$\mu(G,x)$
$(-1)^0 p(G,0)x^{n_G-2(0)}$
$(-1)^1 p(G,1)x^{n_G-2(1)}$
$(-1)^2 p(G,2)x^{n_G-2(1)}$
$(-1)^3 p(G,3)x^{n_G-2(3)}$
$\dots$
$(-1)^{r-1} p(G,r-1)x^{n_G-2(r-1)}$
$(-1)^r p(G,r)x^{n_G-2(r)}$

$\mu(H,x)$
$(-1)^0 p(H,0)x^{n_H-2(0)}$
$(-1)^1 p(H,1)x^{n_H-2(1)}$
$(-1)^2 p(H,2)x^{n_H-2(2)}$
$(-1)^3 p(H,3)x^{n_H-2(3)}$
$\dots$
$(-1)^{r-1} p(H,r-1)x^{n_H-2(r-1)}$
$(-1)^r p(H,r)x^{n_H-2(r)}$

Products that contribute to the x^{n-2r} term

The terms that contribute to the coefficient of x^{n-2r} are formed by multiplying two terms from the above table (one from each column) in the following way: we multiply the first term in the left column by the last term in the right column, then we multiply the second term in the left column by the second-to-last term in the right column. We continue in this way until, ultimately, we multiply the last term in the left column by the first term in the right column.

Based on this table, then, we get the following possible combinations that contribute to the coefficient of x^{n-2r} in $\mu(G,x)\mu(H,x)$.

Product	Simplified Form
$(-1)^0 p(G,0)x^{n_G-2(0)} \times (-1)^r p(H,r)x^{n_H-2r}$	$= (-1)^{0+r} p(G,0)\,p(H,r)x^{n-2r}$
$(-1)^1 p(G,1)x^{n_G-2(1)} \times (-1)^{r-1} p(H,r-1)x^{n_H-2(r-1)}$	$= (-1)^{1+(r-1)} p(G,1)\,p(H,r-1)x^{n-2r}$
$(-1)^2 p(G,2)x^{n_G-2(2)} \times (-1)^{r-2} p(H,r-2)x^{n_H-2(r-2)}$	$= (-1)^{2+(r-2)} p(G,2)\,p(H,r)x^{n-2r}$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$(-1)^r p(G,r)x^{n_G-2(r)} \times (-1)^0 p(H,0)x^{n_H-2(0)} \qquad = (-1)^{r+0} p(G,r)p(H,0)x^{n-2r}$$

So to find the coefficient of the x^{n-2r} term, we must sum the coefficients of all these terms, as follows:

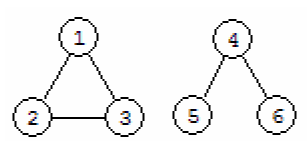
$$\sum_{s=0}^{r}(-1)^s p(G,s)\sum_{s=0}^{r}(-1)^{r-s} p(H,r-s) \qquad =\sum_{s=0}^{r}(-1)^s p(G,s)(-1)^{r-s} p(H,r-s)$$

$$=(-1)^r \sum_{s=0}^{r} p(G,s)p(H,r-s).$$

But we showed above that this same expression is the coefficient of x^{n-2r} in $\mu(G \cup H, x)$, so it follows that $\mu(G \cup H, x) = \mu(G,x)\mu(H,x)$.

❑

Example: Let G be the graph consisting of two components: C_3 and P_3, as shown below.



We use this graph to exemplify Theorem 2.1. If we count the number of r-matchings in G, this is the same as counting the number of r-matchings in $P_3 \cup C_3$. Counting this directly without Theorem 2.1 gives one 0-matching, five 1-matchings, and six 2-matchings. Thus by definition of the matchings polynomial,

$$\begin{aligned}\mu(P_3 \cup C_3) &= (-1)^0 1x^{6-2(0)} + (-1)^1 5x^{6-2(1)} + (-1)^2 6x^{6-2(2)} \\ &= x^6 - 5x^4 + 6x^2\end{aligned}$$

Let us now compare this with what Theorem 2.1 tells us. We can either count directly or use the identities defined in Section 1 in order to determine that $\mu(P_3) = x^3 - 2x$ and $\mu(C_3) = x^3 - 3x$. Thus

$$\mu(P_3)\mu(C_3) = (x^3 - 2x)(x^3 - 3x) = x^6 - 5x^4 + 6x^2 .$$

Comparing this with the polynomial above, we see that $\mu(P_3 \cup C_3) = \mu(P_3)\mu(C_3)$

<u>Section 2.2</u> – <u>Reduction by Deletion of an Edge and its Endpoints</u>

In this section, we obtain a theorem that allows us to reduce a given graph G by deleting an edge. We make use of the following notation: let $G \setminus e$ denote the graph G with the edge e removed, and let $G \setminus \{uv\}$ denote the graph G with vertices u and v removed. (Note that the removal of a vertex from a graph results in the removal of all edges adjacent to that vertex as well).

<u>Theorem 2.2:</u> For any edge $e \in G$ with endpoints u and v,

$$\mu(G,x) = \mu(G \setminus e, x) - \mu(G \setminus \{uv\}, x) .$$

<u>Proof:</u> The r-matchings in G consist of 2 kinds – those that use edge e and those that do not. As these are two disjoint cases, we will use the addition principle to count them separately and then add the results.

Any matching that uses e will determine a unique $(r-1)$-matching in the graph not including the endpoints of e. In other words, such a matching determines an $(r-1)$-matching in the graph $G \setminus \{uv\}$, as u and v are the endpoints of e. Thus the number of r-matchings in G which use e equals $p(G \setminus \{uv\}, r-1)$.

Any matching that does not use e will be an r-matching in the graph $G \setminus e$. Note, in this case, an r-matching may still use vertex u and/or vertex v. Thus the number of r-matchings not using e equals $p(G \setminus e, r)$.

So the addition principle gives

$$p(G,r) = p(G \setminus e, r) + p(G \setminus \{uv\}, r-1),$$

and this is true for $r \geq 1$.

Now let us consider $\mu(G,x) = \sum_{r\geq 0} (-1)^r p(G,r) x^{n-2r}$. In order to incorporate the above relation on the coefficients in this expression, we must be careful with our index of summation. We first pull out the $r = 0$ term, giving

$$\mu(G,x) = (-1)^0 p(G,0)x^n + \sum_{r\geq 1} (-1)^r p(G,r) x^{n-2r}.$$

Now, based on what we know $p(G,r)$ to be for $r \geq 1$, we substitute to get the following:

$$\begin{aligned}
\mu(G,x) &= (-1)^0 p(G,0)x^n + \sum_{r\geq 1} (-1)^r p(G,r) x^{n-2r} \\
&= (-1)^0 p(G,0)x^n + \sum_{r\geq 1} (-1)^r [p(G \setminus e, r) + p(G \setminus \{uv\}, r-1)] x^{n-2r}
\end{aligned}$$

But since $(-1)^0 p(G,0)x^n = 1 = (-1)^0 p(G \setminus e, 0)x^n$, we can adjust the first term.

$$\begin{aligned}
\mu(G,x) &= (-1)^0 p(G \setminus e,0)x^n + \sum_{r\geq 1} (-1)^r p(G \setminus e, r) x^{n-2r} + \sum_{r\geq 1} (-1)^r p(G \setminus \{uv\}, r-1) x^{n-2r} \\
&= \sum_{r\geq 0} (-1)^r p(G \setminus e, r) x^{n-2r} + \sum_{r\geq 1} (-1)^r p(G \setminus \{uv\}, r-1) x^{n-2r} \\
&= \mu(G \setminus e, x) + \sum_{r\geq 1} (-1)^r p(G \setminus \{uv\}, r-1) x^{n-2r}
\end{aligned}$$

We desire the summation in right hand side of the equation to begin at $r \geq 0$ instead of $r \geq 1$, so we momentarily re-index in order to address this problem. Let $r - 1 = t$, and thus $r = t + 1$. The equation above then becomes

$$= \mu(G \setminus e, x) + \sum_{t+1 \geq 1} (-1)^{t+1} p(G \setminus \{uv\}, t) x^{n-2(t+1)}$$

$$= \mu(G \setminus e, x) + (-1) \sum_{t \geq 0} (-1)^t p(G \setminus \{uv\}, t) x^{n-2t-2}.$$

Now since this right-hand expression is in a more familiar form, we will re-index back to r's, so let $t = r$. Hence

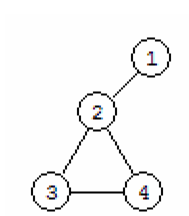
$$= \mu(G \setminus e, x) + (-1) \sum_{r \geq 0} (-1)^r p(G \setminus \{uv\}, r) x^{n-2r-2}.$$

Notice that the summation on the right is exactly $\mu(G \setminus \{uv\}, x)$. To see this, observe that because $G \setminus \{uv\}$ has two fewer vertices than G (due to the deletion of vertices u and v), its matchings polynomial gives the number of r-matchings in a graph with two fewer vertices, and so the powers of x take the form x^{n-2r-2} instead of x^{n-2r}. Thus,

$$\mu(G, x) = \mu(G \setminus e, x) - \mu(G \setminus \{uv\}, x).$$

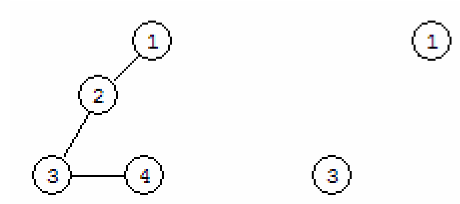
❑

<u>Example</u>: To illustrate the above result, let G be the following graph, where edge e has endpoints 2 and 4.



G

We consider two subgraphs, one where we delete e, and one where we delete the endpoints of e. $G \setminus e$ and $G \setminus \{24\}$ are the following respective subgraphs.



$G \setminus e$ $\qquad\qquad$ $G \setminus \{24\}$

Counting the matching polynomials of each subgraph gives:

$$\mu(G \setminus e, x) = x^4 - 3x^2 + 1 \qquad \text{and} \qquad \mu(G \setminus \{24\}, x) = x^2.$$

So

$$\mu(G \setminus e, x) - \mu(G \setminus \{24\}, x) \quad = \quad x^4 - 4x^2 + 1$$

Counting $\mu(G, x)$ directly results in the same polynomial, $x^4 - 4x^2 + 1$. Thus for the graph G we see that

$$\mu(G, x) = \mu(G \setminus e, x) - \mu(G \setminus \{24\}, x).$$

Section 2.3 – Reduction by Deletion of a Vertex and its Neighbors

In this section, we obtain a theorem that allows us to reduce a graph G by the deletion of a vertex u and its **neighbors**, which are defined to be all the vertices directly adjacent to vertex u. Recall that deleting a vertex from a graph results in the deletion of all edges adjacent to that vertex.

Theorem 2.3: For any vertex u of a graph G,

$$\mu(G, x) = x\mu(G \setminus u, x) - \sum_{i \sim u} \mu(G \setminus \{ui\}, x).$$

Proof: The r-matchings in G consist of two kinds – those that cover vertex u and those that do not. Again, as these are two disjoint cases, we will employ the addition principle.

Any matching that does use u will have to include an edge that has u as one of its endpoints. So for each vertex i adjacent to u, we must count $(r-1)$-matchings in $G \setminus \{ui\}$. To determine the total number of r-matchings in G that use u, we must sum these counts. This gives us a total of $\sum_{i \sim u} p(G \setminus \{ui\}, r-1)$ for the number of r-matchings that use vertex u.

Any matching that does not use u determines an r-matching in the graph G \ u. Thus the number of r-matchings that do not use u is $p(G \setminus u, r)$.

By the addition principle, then,

$$p(G,r) = p(G \setminus u, r) + \sum_{i \sim u} p(G \setminus \{ui\}, r-1),$$

and this is true for $r \geq 1$.

Now let's consider $\mu(G,x) = \sum_{r \geq 0} (-1)^r p(G,r) x^{n-2r}$. In order to incorporate the above relation on the coefficients in this expression, we must be careful with our index of summation. To take care of this we first pull out the $r = 0$ term, giving

$$\mu(G,x) = (-1)^0 p(G,0) x^n + \sum_{r \geq 1} (-1)^r p(G,r) x^{n-2r} .$$

Now, based on what we know $p(G,r)$ to be for $r \geq 1$, we substitute to get the following:

$$\begin{aligned} \mu(G,x) &= (-1)^0 p(G,0) x^n + \sum_{r \geq 1} (-1)^r [p(G \setminus u, r) + \sum_{i \sim u} p(G \setminus \{ui\}, r-1)] x^{n-2r} \\ &= (-1)^0 p(G,0) x^n + \sum_{r \geq 1} (-1)^r p(G \setminus u, r) x^{n-2r} + \sum_{r \geq 1} (-1)^r \sum_{i \sim u} p(G \setminus \{ui\}, r-1) x^{n-2r} \end{aligned}$$

But since $(-1)^0 p(G,0) x^n = 1 = (-1)^0 p(G \setminus u, 0) x^n$, we can adjust the first term.

$$\mu(G,x) = (-1)^0 p(G \setminus u, 0) x^n + \sum_{r \geq 1} (-1)^r p(G \setminus u, r) x^{n-2r} + \sum_{r \geq 1} (-1)^r \sum_{i \sim u} p(G \setminus \{ui\}, r-1) x^{n-2r}$$

$$= \sum_{r \geq 0} (-1)^r p(G \setminus u, r) x^{n-2r} + \sum_{r \geq 1} (-1)^r \sum_{i \sim u} p(G \setminus \{ui\}, r-1) x^{n-2r}$$

Consider the first sum on the right side. Since $G \setminus u$ has one less vertex than *G*, the definition of the matchings polynomial gives

$$\mu(G \setminus u, x) = \sum_{r \geq 0} (-1)^r p(G \setminus u, r) x^{(n-1)-2r} .$$

This expression differs from our sum only by a single factor of x, so

$$\mu(G, x) = x \sum_{r \geq 0} (-1)^r p(G \setminus u, r) x^{(n-1)-2r} + \sum_{r \geq 1} (-1)^r \sum_{i \sim u} p(G \setminus \{ui\}, r-1) x^{n-2r}$$

$$= x\mu(G \setminus u, x) + \sum_{r \geq 1} (-1)^r \sum_{i \sim u} p(G \setminus \{ui\}, r-1) x^{n-2r}$$

$$= x\mu(G \setminus u, x) + \sum_{r \geq 1} \sum_{i \sim u} (-1)^r p(G \setminus \{ui\}, r-1) x^{n-2r}$$

$$= x\mu(G \setminus u, x) + \sum_{i \sim u} \sum_{r \geq 1} (-1)^r p(G \setminus \{ui\}, r-1) x^{n-2r}$$

We desire the summation on the right to begin at $r \geq 0$ instead of $r \geq 1$, so we momentarily re-index in order to address this problem. Let $r - 1 = t$, and thus $r = t + 1$. Therefore, continuing from the equation above we have

$$= x\mu(G \setminus u, x) + \sum_{i \sim u} \sum_{t+1 \geq 1} (-1)^{t+1} p(G \setminus \{ui\}, t) x^{n-2(t+1)}$$

$$= x\mu(G \setminus u, x) + (-1) \sum_{i \sim u} \sum_{t \geq 0} (-1)^t p(G \setminus \{ui\}, t) x^{n-2t-2} .$$

Since this right-hand expression is now in a more familiar form, we will re-index back to *r*'s, so let $t = r$.

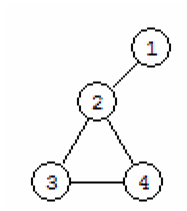
$$= x\mu(G \setminus u, x) + (-1) \sum_{i \sim u} \sum_{r \geq 0} (-1)^r p(G \setminus \{ui\}, r) x^{n-2r-2}$$

Note again that the sum $\sum_{r \geq 0} (-1)^r p(G \setminus \{ui\}, r) x^{n-2r-2}$ precisely equals $\mu(G \setminus \{ui\}, x)$, since $G \setminus \{ui\}$ has two fewer vertices than *G*. Hence

$$\mu(G,x) = x\mu(G \setminus u, x) - \sum_{i \sim u} \mu(G \setminus \{ui\}, x).$$

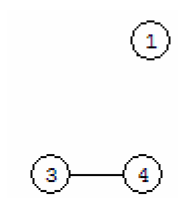
❑

<u>Example</u>: Using the graph *G* shown below, we now consider Theorem 2.3. Let vertex 2 be the vertex we delete.

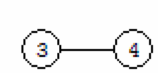


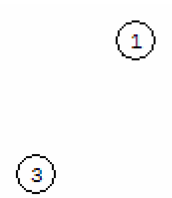
G

We consider two classes of subgraphs. The first consists of a single subgraph, one in which vertex 2 has been deleted. The second is a group of graphs, in each of which we have deleted vertex 2 and one vertex adjacent to it. The subgraphs are drawn below.



$G \setminus 2$





$G \setminus \{21\}$ $G \setminus \{23\}$ $G \setminus \{24\}$

Counting the matchings polynomials of these subgraphs gives us

$$\mu(G \setminus 2, x) = x\left(x^2 - 1\right) = x^3 - x,$$

by Theorem 2.1, and

$$\sum_{i\sim u}\mu(G\setminus\{2i\},x)=\left(x^2-1\right)+x^2+x^2=3x^2-1.$$

Thus, we get

$$x\mu(G\setminus 2,x)-\sum_{i\sim u}\mu(G\setminus\{2i\},x)=x\left(x^3-3\right)-\left(3x^2-1\right)=x^4-4x^2+1.$$

We know from above (or by direct counting) that $\mu(G,x)=x^4-4x^2+1$. So for the given graph *G* we have

$$\mu(G,x)=x\mu(G\setminus 2,x)-\sum_{i\sim u}\mu(G\setminus\{2i\},x).$$

Section 2.4 – The Derivative of the Matchings Polynomial

Since the matchings polynomial is a polynomial function, it is reasonable to inquire about its derivative. The following theorem provides an interesting result about the derivative of the matchings polynomial. We remark that our use of the derivative is purely formal. Since we are not interested in evaluating our expressions at any particular value of *x*, it is clear that we are not using the matchings polynomial as a function. In particular, then, we are not concerned with any particular rate of change when we are differentiating. Indeed, in this section we will employ the derivative merely as a formal operation which obeys the familiar 'power rule'.

Note here that $G\setminus i$ is a graph in which a vertex *i* has been deleted. Thus each graph $G\setminus i$ for some vertex $i\in G$ is called a **vertex-deleted subgraph** of *G*. So in the theorem below, the expression on the right represents the sum of the matchings polynomials of all the vertex-deleted subgraphs of *G*.

Theorem 2.4: For any graph *G*,

$$\frac{d}{dx}\mu(G,x)=\sum_{i\in V(G)}\mu(G\setminus i,x)$$

Proof: By the power rule for derivatives, the coefficient of the $x^{(n-1)-2r}$ term in $\frac{d}{dx}\mu(G,x)$ is equal to $(-1)^r(n-2r)p(G,r)$. Note that $(n-2r)p(G,r)$ represents the number of ways of first picking an r-matching in G (which is $p(G,r)$) and then choosing one vertex from G that is not covered by the r-matching (there are $n-2r$ of these, since each edge covers 2 vertices). We could instead, however, arrive at this same number if we first pick a vertex and then pick an r-matching that does not cover this vertex. This way of counting yields the expression $\sum_{i\in V(G)} p(G\setminus i, r)$.

Setting these equivalent expressions equal to one another, we obtain the equation

$$(n-2r)\,p(G,r) = \sum_{i\in V(G)} p(G\setminus i, r).$$

Note since $G\setminus i$ has one less vertex than G, its matchings polynomial is

$$\mu(G\setminus i, x) = \sum_{r\geq 0} (-1)^r\, p(G\setminus i, r)\, x^{(n-1)-2r}.$$

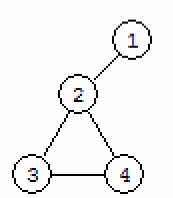
Therefore we can rewrite the derivative of $\mu(G,x)$ in the following way.

$$\begin{aligned}
\frac{d}{dx}\mu(G,x) &= \sum_{r\geq 0} (-1)^r (n-2r)\, p(G,r)\, x^{(n-1)-2r} \\
&= \sum_{r\geq 0} (-1)^r \sum_{i\in V(G)} p(G\setminus i, r)\, x^{(n-1)-2r} \\
&= \sum_{r\geq 0} \sum_{i\in V(G)} (-1)^r\, p(G\setminus i, r)\, x^{(n-1)-2r} \\
&= \sum_{i\in V(G)} \sum_{r\geq 0} (-1)^r\, p(G\setminus i, r)\, x^{(n-1)-2r} \\
&= \sum_{i\in V(G)} \mu(G\setminus i, x),
\end{aligned}$$

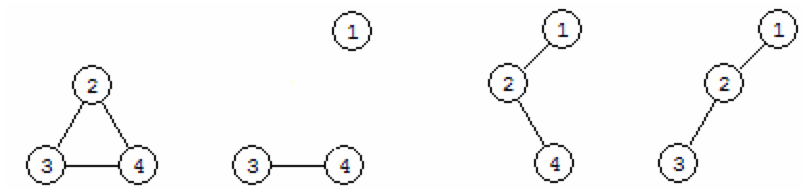
as desired.

❑

<u>Example</u>: Using the graph G pictured below, we consider Theorem 2.4.



Let us examine $\sum_{i \in V(G)} p(G \setminus i, r)$.



$\mu(G \setminus 1, x)$ $\mu(G \setminus 2, x)$ $\mu(G \setminus 3, x)$ $\mu(G \setminus 4, x)$

Counting matchings in each of these subgraphs gives us the following results:

$$\mu(G \setminus 1, x) = x^3 - 3x$$

$$\mu(G \setminus 2, x) = x(x^2 - 1) = x^3 - x$$

$$\mu(G \setminus 3, x) = x^3 - 2x$$

$$\mu(G \setminus 4, x) = x^3 - 2x$$

Then

$$\sum_{i \in V(G)} p(G \setminus i, r) = (x^3 - 3x) + (x^3 - x) + (x^3 - 2x) + (x^3 - 2x) = 4x^3 - 8x .$$

We know from the example in the previous section that $\mu(G, x) = x^4 - 4x^2 + 1$. So $\frac{d}{dx}\mu(G, x) = 4x^3 - 8x$. Thus, for the graph G we see that

$$\frac{d}{dx}\mu(G, x) = \sum_{i \in V(G)} p(G \setminus i, r) .$$

Chapter 3 – Three-Term Recurrences

Having now established a number of useful theorems related to the matchings polynomial, we turn our attention to recurrences for the matchings polynomials of familiar classes of graphs that we discussed above: paths, cycles, complete graphs and complete bipartite graphs. We use the above theorems to obtain three-term recurrence relations for these classes of graphs, and again we follow each discussion of a recurrence with an example.

Section 3.1 – Recurrences for Paths

Using Theorem 2.3 for any vertex *u* in a path P_{n+1} on $n+1$ vertices, we obtain the equation

$$\mu(P_{n+1},x) = x\mu(P_{n+1}\setminus u, x) - \sum_{i\sim u}\mu(P_{n+1}\setminus\{ui\}, x).$$

Choosing *u* to be an endpoint of P_{n+1}, we find that the graph $P_{n+1}\setminus u$ in our first term is simply a path on n vertices, P_n. In the second term, since *u* is only adjacent to one other vertex *i*, $P_{n+1}\setminus\{ui\}$ also represents a path P_{n-1}. Thus we arrive at the following result:

Proposition 3.1: For any natural number *n*,

$$\mu(P_{n+1},x) = x\mu(P_n,x) - \mu(P_{n-1},x).$$

❑

We can use this three-term recurrence to generate the first several matchings polynomials of the paths P_n.

$\mu(P_0,x) \quad = 1$

$\mu(P_1,x) \quad = x$

$$\mu(P_2, x) \qquad = x^2 - 1$$

$$\mu(P_3, x) \qquad = x(x^2 - 1) - x = x^3 - 2x$$

$$\mu(P_4, x) \qquad = x(x^3 - 2x) - (x^2 - 1) = x^4 - 3x^2 + 1$$

$$\mu(P_5, x) \qquad = x(x^4 - 3x^2 + 1) - (x^3 - 2x) = x^5 - 4x^3 + 3x$$

$$\mu(P_6, x) \qquad = x(x^5 - 4x^3 + 3x) - (x^4 - 3x^2 + 1) = x^6 - 5x^4 + 6x^2 - 1$$

<u>Example</u>: In Section 1.1 we developed a counting argument for computing the matchings polynomials of P_n. We found that

$$\mu(P_n, x) = \sum_{r \geq 0} (-1)^r \binom{n-r}{r} x^{n-2r} .$$

As an example, let us compare $\mu(P_5, x)$ using the counting method and the three-term recurrences.

Using the counting method we have

$$\mu(P_5, x) = \sum_{r \geq 0} (-1)^r \binom{5-r}{r} x^{5-2r} = x^5 - 4x^3 + 3x .$$

Note this agrees with the expression for $\mu(P_5, x)$ that we found above using the three-term recurrence.

<u>Section 3.2</u> – <u>Recurrences for Cycles</u>

In this section we derive a recurrence for cycles. If we apply Theorem 2.2 to the cycle C_n, we find that, for an edge e with endpoints u and v,

$$\mu(C_n, x) = \mu(C_n \setminus e, x) - \mu(C_n \setminus \{uv\}, x) .$$

But the graph $C_n \setminus e$ is just a path on n vertices, and the graph $C_n \setminus \{uv\}$ is a path on $n-2$ vertices. So if $n \geq 2$,

$$\mu(C_n, x) = \mu(P_n, x) - \mu(P_{n-2}, x) .$$

Now we use previous identities in order to develop a recurrence for the cycles. We ultimately seek to prove the following recurrence:

$$\mu(C_{n+1},x) = x\mu(C_n,x) - \mu(C_{n-1},x)$$

We begin by examining the left side of the recurrence.

It has been established that:

$$\mu(P_{n+1},x) = x\mu(P_n,x) - \mu(P_{n-1},x) \text{ and } \mu(P_{n-1},x) = x\mu(P_{n-2},x) - \mu(P_{n-3},x).$$

And it has been further found that:

$$\mu(C_{n-1},x) = \mu(P_{n-1},x) - \mu(P_{n-3},x).$$

By substitution, then, we obtain the following:

$$\begin{aligned} \mu(C_{n+1},x) &= \mu(P_{n+1},x) - \mu(P_{n-1},x) \\ &= [x\mu(P_n,x) - \mu(P_{n-1},x)] - \mu(P_{n-1},x) \qquad (1) \end{aligned}$$

Turning now to the right side of the desired recurrence, we use substitution to obtain

$$\begin{aligned} x\mu(C_n,x) - \mu(C_{n-1},x) &= x[\mu(P_n,x) - \mu(P_{n-2},x)] - \mu(C_{n-1},x) \\ &= x\mu(P_n,x) - x\mu(P_{n-2},x) - \mu(C_{n-1},x) \\ &= x\mu(P_n,x) - x\mu(P_{n-2},x) - [\mu(P_{n-1},x) - \mu(P_{n-3},x)] \\ &= x\mu(P_n,x) - \mu(P_{n-1},x) - [x\mu(P_{n-2},x) - \mu(P_{n-3},x)] \\ &= x\mu(P_n,x) - \mu(P_{n-1},x) - [\mu(P_{n-1},x)] \qquad (2) \end{aligned}$$

The expressions (1) and (2) above are equivalent. Keeping our initial assumption of $n \geq 2$ in mind, we find that for $n \geq 2$,

$$\mu(C_{n+1},x) = x\mu(C_n,x) - \mu(C_{n-1},x).$$

Thus we have obtained a three-term sequence for cycles.

<u>Proposition 3.2</u>: For any natural number $n \geq 2$,

$$\mu(C_{n+1},x) = x\mu(C_n,x) - \mu(C_{n-1},x).$$

❑

We can use this three-term recurrence to list out the first several matchings polynomials of the cycles C_n. We first note by direct counting that:

$\mu(C_0, x) \quad = 1$

$\mu(C_1, x) \quad = x$

$\mu(C_2, x) \quad = x^2 - 2$

By implementing our recurrence relation, we find the following:

$\mu(C_3, x) \quad = x(x^2 - 2) - x = x^3 - 3x$

$\mu(C_4, x) \quad = x(x^3 - 3x) - (x^2 - 2) = x^4 - 4x^2 + 2$

$\mu(C_5, x) \quad = x(x^4 - 4x^2 + 2) - (x^3 - 3x) = x^5 - 5x^3 + 5x$

$\mu(C_6, x) \quad = x(x^5 - 5x^3 + 5x) - (x^4 - 4x^2 + 2) = x^6 - 6x^4 + 9x^2 - 2$

<u>Example</u>: In Section 1.2 we developed a counting argument for computing the matchings polynomials of C_n. We found that

$$\mu(C_n, x) = \sum_{r \geq 0} (-1)^r \frac{n}{n-r} \binom{n-r}{r} x^{n-2r} .$$

As an example, let us compare $\mu(C_6, x)$ using the counting method and the three-term recurrences. Using the counting method we have

$$\mu(C_6, x) = \sum_{r \geq 0} (-1)^r \binom{6-r}{r} x^{6-2r} = x^6 - 6x^4 + 9x^2 - 2 .$$

Note this is agrees with the expression for $\mu(C_6, x)$ that we found above using the three-term recurrence.

<u>Section 3.3</u> – <u>Recurrences for Complete Graphs</u>

To derive a three-term recurrence for the complete graphs, we proceed in a similar manner as above. In particular, we begin by using Theorem 2.3 to obtain the following:

$$\mu(K_{n+1}, x) = x\mu(K_{n+1} \setminus u, x) - \sum_{i \sim u} \mu(K_{n+1} \setminus \{ui\}, x) ,$$

where u denotes any vertex in K_{n+1}. Observe that the graph $K_{n+1} \setminus u$ in our first term is simply a complete graph on n vertices, K_n. And in our second term, since a vertex u in K_{n+1} is adjacent to exactly n other vertices, and because we remove exactly two vertices from K_{n+1}, the term $\sum_{i \sim u} \mu(K_{n+1} \setminus \{ui\}, x)$ becomes n times $\mu(K_{n-1}, x)$. Thus we have the following three-term recurrence for complete graphs.

Proposition 3.3: For any natural number n,

$$\mu(K_{n+1}, x) = x \cdot \mu(K_n, x) - n \cdot \mu(K_{n-1}, x) .$$

❑

We can use this three-term recurrence to list out the first several matchings polynomials of the complete graphs K_n.

$\mu(K_0, x) \quad = 1$

$\mu(K_1, x) \quad = x$

$\mu(K_2, x) \quad = x^2 - 1$

$\mu(K_3, x) \quad = x(x^2 - 1) - 2x = x^3 - 3x$

$\mu(K_4, x) \quad = x(x^3 - 3x) - 3(x^2 - 1) = x^4 - 6x^2 + 3$

$\mu(K_5, x) \quad = x(x^4 - 6x^2 + 3) - 4(x^3 - 3x) = x^5 - 10x^3 + 15x$

$\mu(K_6, x) \quad = x(x^5 - 10x^3 + 15x) - 5(x^4 - 6x^2 + 3) = x^6 - 15x^4 + 45x^2 - 15$

Example: Recall that in Section 1.3, we developed a counting argument for computing the matchings polynomials of K_n. We found that

$$\mu(K_n, x) = \sum_{r \geq 0} (-1)^r \frac{n!}{(n-2r)! 2^r r!} x^{n-2r} .$$

We now compare the expression for $\mu(K_5, x)$ obtained using the counting method with the expression obtained from the three-term recurrence above. Using the counting method from Section 1.3 we have

$$\mu(K_5, x) = \sum_{r \geq 0} (-1)^r \frac{5!}{(5-2r)! 2^r r!} x^{5-2r} = x^5 - 10x^3 + 15x,$$

which agrees with the expression for $\mu(K_5, x)$ that we found above using the three-term recurrence.

Section 3.4 – Recurrences for Complete Bipartite Graphs

Obtaining a three-term recurrence for the family of complete bipartite graphs $K_{m,m}$ requires a bit more effort but proceeds along the same general lines. To begin with, we can count the number of r-matchings of $K_{m,m}$ as follows. Let X and Y be the two cells of the bipartite graph, and let us pick two vertices in opposite cells of $K_{m,m}$, say $u \in X$ and $v \in Y$. Then we identify the following cases. We consider the number of r-matchings that:

(1) Do not use u and do not use v. This is done in $p(K_{m-1,m-1}, r)$ ways, as we now have one fewer vertex in each set with which to form our r-matching.

(2) Use u. If we use u from set X, then we choose a vertex from set Y to be the other endpoint of the edge that covers u. Since the graph is complete bipartite, there are m vertices to choose from. No matter which vertex in Y we choose, we will need to complete our matching by selecting an (r – 1)-matching in the remainder of the graph, which can be done in $p(K_{m-1,m-1}, r-1)$ ways. So the number of r-matchings in $K_{m,m}$ that use vertex u must equal $m \cdot p(K_{m-1,m-1}, r-1)$.

(3) Use v. By the same argument as in case (2), we can form such an r-matching in $m \cdot p(K_{m-1,m-1}, r-1)$ ways.

Notice that if we were to sum these three cases, we would double count all of the r-matchings that use both u and v. Thus, from the sum of cases (1), (2), and (3), we must subtract the number of r-matchings that use both u and v. Such r-matchings fall into two disjoint categories, (4) and (5), which we count separately below. We consider the number of r-matchings that:

(4) Match u with v. One edge of each such r-matching will be the edge uv, so we seek to form an $(r-1)$-matching in the graph that remains, $K_{m-1,m-1}$. Hence there are $p(K_{m-1,m-1}, r-1)$ ways of forming an r-matching in $K_{m,m}$ that uses the edge uv.

(5) Use u and v but do not use edge uv. If we use u and v but do not use uv, then vertex u will have to form an edge with some vertex in Y other than v, and vertex v will have to form an edge with some vertex in X other than u. There are $m-1$ choices for u and $m-1$ choices for v, resulting in $(m-1)^2$ ways of forming the two edges that use u and v. What remains is to count the number of an $(r-2)$-matchings in the graph $K_{m-2,m-2}$. Thus the number of r-matchings that use u and v but do not use edge uv is equal to $(m-1)^2 \cdot p(K_{m-2,m-2}, r-2)$.

Putting it all together, we add cases (1), (2), and (3), and subtract cases (4) and (5) to learn that for $r \geq 2$,

$$\begin{aligned} p(K_{m,m}, r) \quad &= p(K_{m-1,m-1}, r) + m \cdot p(K_{m-1,m-1}, r-1) + m \cdot p(K_{m-1,m-1}, r-1) \\ &\qquad - p(K_{m-1,m-1}, r-1) - (m-1)^2 \cdot p(K_{m-2,m-2}, r-2) \\ &= p(K_{m-1,m-1}, r) + (2m-1)\, p(K_{m-1,m-1}, r-1) - (m-1)^2\, p(K_{m-2,m-2}, r-2). \end{aligned}$$

Now, by the definition of the matchings polynomial, we have the following:

$$\begin{aligned} \mu(K_{m,m}, x) \quad &= \sum_{r \geq 0} (-1)^r p(K_{m,m}, r) x^{n-2r} \\ &= x^n - m^2 x^{n-2} + \sum_{r \geq 2} (-1)^r p(K_{m,m}, r) x^{n-2r}, \end{aligned}$$

where $n = 2m$.

From above, we substitute based on what we know $p(K_{m,m}, r)$ to be.

$$\mu(K_{m,m},x) = x^n - m^2x^{n-2} + \sum_{r\geq 2}(-1)^r[p(K_{m-1,m-1},r) + (2m-1)p(K_{m-1,m-1},r-1) - (m-1)^2 p(K_{m-2,m-2},r-2)]x^{n-2r}$$

So now we have three summations to work with.

$$\begin{aligned}\mu(K_{m,m},x) &= x^n - m^2x^{n-2} + \sum_{r\geq 2}(-1)^r p(K_{m-1,m-1},r)x^{n-2r} \\ &\quad + (2m-1)\sum_{r\geq 2}(-1)^r p(K_{m-1,m-1},r-1)x^{n-2r} \\ &\quad - (m-1)^2\sum_{r\geq 2}(-1)^r p(K_{m-2,m-2},r-2)x^{n-2r}\end{aligned}$$

The first summation simplifies as follows:

$$x^2\sum_{r\geq 2}(-1)^r p(K_{m-1,m-1},r)x^{n-2-2r} = x^2[\mu(K_{m-1,m-1},x) - x^{n-2} + (m-1)^2x^{n-4}]$$

The second summation simplifies as well. We first re-index our summation, letting $t = r-1$, so that $r = t+1$. Once we have the form we desire, we re-index back to r's:

$$\begin{aligned}(2m-1)\sum_{t\geq 1}(-1)^{t+1} p(K_{m-1,m-1},t)x^{n-2(t+1)} &= (-1)(2m-1)\sum_{t\geq 1}(-1)^t p(K_{m-1,m-1},t)x^{n-2-2t} \\ &= -(2m-1)\sum_{r\geq 1}(-1)^r p(K_{m-1,m-1},r)x^{n-2-2r} \\ &= -(2m-1)[\mu(K_{m-1,m-1},x) - x^{n-2}]\end{aligned}$$

Finally we address the last summation. We will re-index again in a similar fashion. Here we let $t = r-2$, and so $r = t+2$.

$$\begin{aligned}-(m-1)^2\sum_{r\geq 2}(-1)^r p(K_{m-2,m-2},r-2)x^{n-2r} &= -(m-1)^2\sum_{t\geq 0}(-1)^{t+2} p(K_{m-2,m-2},t)x^{n-2(t+2)} \\ &= -(m-1)^2\sum_{t\geq 0}(-1)^t p(K_{m-2,m-2},t)x^{n-4-2t} \\ &= -(m-1)^2\sum_{r\geq 0}(-1)^r p(K_{m-2,m-2},r)x^{n-4-2r} \\ &= -(m-1)^2\mu(K_{m-2,m-2},x)\end{aligned}$$

Combining all these expressions and simplifying, we obtain

$$\mu(K_{m,m}, x) = (x^2 - 2m + 1)\mu(K_{m-1,m-1}, x) - (m-1)^2 \mu(K_{m-2,m-2}, x) .$$

Therefore we have written $\mu(K_{m,m}, x)$ in terms of its two previous values of m, and so we have derived the following three-term recurrence for complete bipartite graphs.

<u>Proposition 3.4</u>: For any natural number $m \geq 2$,

$$\mu(K_{m,m}, x) = (x^2 - 2m + 1)\mu(K_{m-1,m-1}, x) - (m-1)^2 \mu(K_{m-2,m-2}, x) .$$

❑

Based upon this recurrence we can list out the matchings polynomials of the first several complete bipartite graphs.

$$\begin{aligned}
\mu(K_{0,0}, x) &= 1 \\
\mu(K_{1,1}, x) &= x^2 - 1 \\
\mu(K_{2,2}, x) &= (x^2 - 3) \cdot (x^2 - 1) - 1 = x^4 - 4x^2 + 2 \\
\mu(K_{3,3}, x) &= (x^2 - 5) \cdot (x^4 - 4x^2 + 2) - 4(x^2 - 1) = x^6 - 9x^4 + 18x^2 - 6 \\
\mu(K_{4,4}, x) &= (x^2 - 7) \cdot (x^6 - 9x^4 + 18x^2 - 6) - 9(x^4 - 4x^2 + 2) \\
&= x^8 - 16x^6 + 72x^4 - 96x^2 + 24 \\
\mu(K_{5,5}, x) &= (x^2 - 9) \cdot (x^8 - 16x^6 + 72x^4 - 96x^2 + 24) - 16(x^6 - 9x^4 + 18x^2 - 6) \\
&= x^{10} - 25x^8 + 200x^6 - 600x^4 + 600x^2 - 120 \\
\mu(K_{6,6}, x) &= (x^2 - 11) \cdot (x^{10} - 25x^8 + 200x^6 - 600x^4 + 600x^2 - 120) \\
&\qquad - 25(x^8 - 16x^6 + 72x^4 - 96x^2 + 24) \\
&= x^{12} - 36x^{10} + 450x^8 - 2400x^6 + 5400x^4 - 4320x^2 + 720
\end{aligned}$$

<u>Example</u>: Recall that in Section 1.4, we developed a counting argument for the complete bipartite graph. For a graph $K_{m,m}$, where $n = 2m$,

$$\mu(K_{m,m}, x) = \sum_{r\geq 0} (-1)^r \binom{m}{r}^2 r! x^{n-2r} .$$

In fact, using the above expression, we found the matchings polynomial of $K_{3,3}$ to be

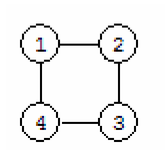
$$\mu(K_{3,3}) = x^6 - 9x^4 + 18x^2 - 6 .$$

Note this is the same matchings polynomial that we found above using the three-term recurrence.

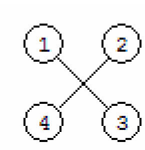
It is perhaps worth remarking that the significance of having a three-term recurrence on a family of polynomials is more than just computational. Indeed, the existence of such recurrence relations actually implies that each of these families of polynomials forms an orthogonal sequence with respect to an appropriate inner product. We will return to this curious fact in a later chapter.

<u>Chapter 4</u> – <u>Complements and Perfect Matchings</u>

Given a graph G, its **complement** $\overline{G}$ denotes a graph that has the same vertex set as G. However, the edge set of $\overline{G}$ is the opposite of the edge set of G. That is to say, if there is an edge between two vertices in G, then that edge is *not* in $\overline{G}$; if there is not an edge between two vertices in G, then there *is* such an edge in $\overline{G}$. The following two graphs exemplify this complementary relationship.



G

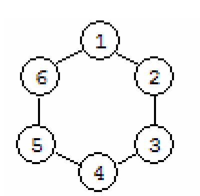


$\overline{G}$

In this chapter, we study some relationships between the matchings polynomials of a graph G and its complement $\overline{G}$. We will eventually express such relationships using integrals, but this requires the introduction of a particular kind of matching.

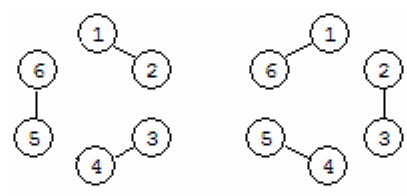
We define a **perfect matching** of a graph G to be a matching in which every vertex of G is an endpoint of an edge in the matching. Said another way, a matching is perfect if it covers every vertex in a graph. Let us denote the number of perfect matchings in a graph G by $pm(G)$.

<u>Examples</u>: Consider first the graph C_6, mentioned in Section 1.2.

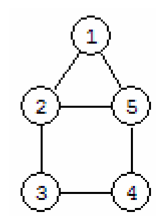


C_6

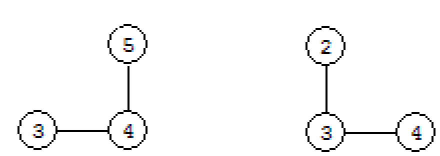
It has two perfect matchings, as shown below. Note that all of the vertices in the graph are included in both of these matchings, making the matchings perfect.



Contrast this with the house graph, which has 5 vertices.



Note in this graph, no set of edges will cover every vertex and still fulfill the definition of a matching. In order to include vertex 1 we would need to have {12} or {15} in our matching. But if we include either of these edges, we are left with the following graphs respectively. In neither case are we able to include the remaining three vertices in our matching. Thus the house graph has no perfect matching.



Indeed, since any matching must cover an even number of vertices, a graph with an odd number of vertices can never have a perfect matching.

Section 4.1 – **Perfect Matchings in the Complement of *G***

We begin this section by considering $pm(\overline{G})$, the number of perfect matchings in the complement of a graph *G*. We must first take note of two important identities. Let *e* be an edge in *G*. Since $e \in G$, *e* is not in $\overline{G}$. Since *e* is not in $G \setminus e$, it must be the case that *e* is in $\overline{G \setminus e}$. Thus $\overline{G \setminus e}$ will be exactly the same as $\overline{G}$, except it will also include the edge *e*. Therefore we have that

$$\overline{G \setminus e} = \overline{G} + e.$$

We turn now to the second identity. Arguing using the notion of set difference, we note that $\overline{G} \setminus \{uv\}$ can be written as $(K_n \setminus G) \setminus \{uv\}$ which, in turn, is equivalent to $(K_n \setminus \{uv\}) \setminus (G \setminus \{uv\})$. This last expression is the same as $\overline{G \setminus \{uv\}}$, and hence

$$\overline{G} \setminus \{uv\} = \overline{G \setminus \{uv\}}.$$

We will use these two simple observations to prove the following lemma.

Lemma 4.1: For any graph *G* and any edge *e* in *G* with endpoints *u* and *v*,

$$pm(\overline{G}) = pm(\overline{G \setminus e}) - pm(\overline{G \setminus \{uv\}}).$$

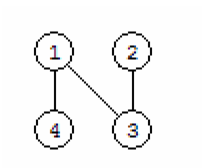
Proof: The edge *e* is clearly in $\overline{G}$ + *e*, so we conclude that *e* is in $\overline{G \setminus e}$ by the first identity above. The perfect matchings in $\overline{G \setminus e}$ consist of two kinds – those that use *e* and those that do not. Any perfect matching that *does not* use *e* is a perfect matching in $\overline{G \setminus e} \setminus e$, which is just the graph $\overline{G}$. Any perfect matching that *does* use *e* determines a perfect matching in $\overline{G} \setminus \{uv\}$. Thus

$$pm(\overline{G \setminus e}) = pm(\overline{G}) + pm(\overline{G} \setminus \{uv\}) .$$

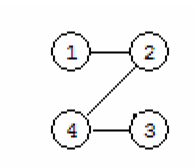
Rewriting this equation using our second identity above, we obtain the statement of the lemma.

❑

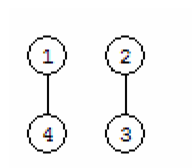
<u>Example</u>: We demonstrate Lemma 4.1 with the following example. Let G and $\overline{G}$ be the graphs below. Let e be the edge with the endpoints $\{13\}$. Listed below each graph is the number of perfect matchings in each graph, which can easily be found by direct counting.



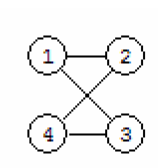
$pm(G \setminus e) = 1$



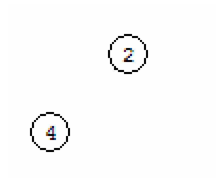
$pm(\overline{G \setminus e}) = 2$



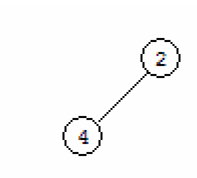
$pm(G) = 1$



$pm(\overline{G}) = 1$



$pm(G \setminus \{13\}) = 0$



$pm(\overline{G \setminus \{13\}}) = 1$

We see, then, that for this example it is true that

$$pm(\overline{G}) = pm(\overline{G \setminus e}) - pm(\overline{G \setminus \{13\}}) .$$

Section 4.2 – Perfect Matchings in the Complete Graph

In this section, we focus on perfect matchings in complete graphs. The lemma we obtain in this section will prepare us for the primary theorem in this chapter.

Lemma 3.2: $pm(K_n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} x^n \, dx$

Proof: We define $M(n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-x^2/2} dx$, which we evaluate using integration by parts. Letting $u = e^{-x^2/2}$ and $dv = x^n dx$, we find that $du = -xe^{-x^2/2}$ and $v = \frac{x^{n+1}}{n+1}$. So by the parts formula, $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$, leading us to calculate as follows:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} x^n dx = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{x^{n+1}}{n+1}\Bigg|_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x^{n+2}}{n+1} e^{-x^2/2} dx .$$

To compute the first term on the right hand side, we note that $\lim_{x\to\pm\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{x^{n+1}}{n+1}$ equals 0 by repeated applications of L'Hôpital's Rule. So we now have that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} x^n dx = 0 + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x^{n+2}}{n+1} e^{-x^2/2} dx ,$$

and this implies that

$$M(n) = \frac{1}{n+1} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{n+2} e^{-x^2/2} dx$$

$$= \frac{M(n+2)}{n+1} .$$

To exploit this recurrence, we now find values for $M(1)$ and $M(0)$. We have, by definition,

$$M(1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} x^1 dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} x dx .$$

We define the function $f(x) = xe^{-x^2/2}$, and observe that $f(-x) = -xe^{-x^2/2}$ and $-f(x) = -xe^{-x^2/2}$. Since $f(-x) = -f(x)$, this is an odd function, so it is symmetric about the origin. Integrating from $-\infty$ to ∞, then, will result in a net area of 0. Thus $M(1) = 0$. Note since $M(n) = \frac{M(n+2)}{n+1}$, this implies that $M(n) = 0$ whenever n is odd.

Next we find $M(0) = 1$, and we use multivariate integration to confirm this.

Let $I = \int_{-\infty}^{\infty} e^{-x^2} dx$. Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dydx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dydx = \left(\int_{-\infty}^{\infty} e^{-y^2} dy\right)\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right) = I^2 .$$

Switching to polar coordinates, we find that

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} rdrd\theta .$$

So now, substituting $u = -r^2$ and $du = -2rdr$, we have

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} rdrd\theta = -\frac{1}{2}\int_0^{2\pi}\int_0^{-\infty} e^u dud\theta = -\frac{1}{2}\int_0^{2\pi}(-1)d\theta = \pi .$$

Therefore $I = \sqrt{\pi}$. Note that I could not equal $-\sqrt{\pi}$, as our integral clearly evaluates a region with positive area. Now, with a simple change of variables, we can determine M(0). Letting $u = \frac{x}{\sqrt{2}}$ and $du = \frac{\sqrt{2}}{2}$, we see that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2} \sqrt{2} du = \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u} du = 1.$$

Hence $M(0) = 1$. Since $M(n) = \frac{M(n+2)}{n+1}$, we have the following relationship:

$$(n+1) \cdot M(n) = M(n+2)$$

Using this relationship, we use $M(0) = 1$ and $M(1) = 0$ to compute as follows:

$$\begin{aligned}
&M(2) = 1 \cdot M(0) = 1 \\
&M(3) = 2 \cdot M(1) = 0 \\
&M(4) = 3 \cdot M(2) = 3 \cdot 1 \\
&M(5) = 4 \cdot M(3) = 0 \\
&M(6) = 5 \cdot M(4) = 5 \cdot 3 \cdot 1 \\
&M(7) = 6 \cdot M(5) = 0 \\
&M(8) = 7 \cdot M(6) = 7 \cdot 5 \cdot 3 \cdot 1 \\
&M(9) = 8 \cdot M(7) = 0 \\
&\qquad\qquad \vdots
\end{aligned}$$

Thus, because $M(1) = 0$, it follows that $M(n) = 0$ when n is odd. And because $M(0) = 1$, it follows that for any even integer $n = 2r$, $M(2r) = (2r-1)(2r-3)...3 \cdot 1$.

But note that a perfect matching in K_n can only exist when there are an even number of vertices. Therefore $pm(K_n) = 0 = M(n)$ when n is odd, and

$$pm(K_n) = p(K_{2r}, r) \quad \text{when } n = 2r.$$

And recall from Section 1.3 that

$$p(K_{2r}, r) = \frac{(2r)!}{2^r r!} = (2r-1)(2r-3)...3 \cdot 1 = M(2r).$$

Therefore, for any integer n, we have the desired result:

$$pm(K_n) = M(n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-x^2/2} dx .$$

❑

Because x^n is actually the matchings polynomial of the complement of K_n, Lemma 4.2 suggests a possible relationship between graphs, their complements, and perfect matchings. In the next section, we present a theorem that describes such a relationship.

Section 4.3 – The Matchings Polynomial of *G* and Perfect Matchings in its Complement

In the previous section, we established that the number of perfect matchings in the complete graph is given by the formula

$$pm(K_n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-x^2/2} dx .$$

The integrand on the right hand contains a factor of the polynomial x^n. This polynomial is in fact the matchings polynomial of the complement of K_n, a fact which motivates the following result.

Theorem 4.3: For any graph *G*,

$$pm(\overline{G}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu(G,x) e^{-x^2/2} dx .$$

Proof: Denote $I(G) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu(G,x) e^{-x^2/2} dx$. We proceed by induction on the number of edges in graph *G*. Lemma 3.2 has given our base case, the graph $\overline{K_n}$ with 0 edges, as

$$pm(K_n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu(\overline{K_n}, x) e^{-x^2/2} dx .$$

For our induction hypothesis, we assume that G has at least one edge e with endpoints u and v, and we assume the theorem is true for any graph with fewer edges than G. The induction step proceeds as follows. By Theorem 1.6, we have

$$\begin{aligned} I(G) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu(G,x) e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\mu(G \setminus e, x) - \mu(G \setminus \{uv\}, x)\right] e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu(G \setminus e, x) e^{-x^2/2} dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu(G \setminus \{uv\}, x) e^{-x^2/2} dx \\ &= I(G \setminus e) - I(G \setminus \{uv\}) \end{aligned}$$

But both $G \setminus e$ and $G \setminus \{uv\}$ have fewer edges than G, so the induction hypothesis applies. It tells us that

$$I(G \setminus e) = pm(\overline{G \setminus e}) \text{ and } I(G \setminus \{uv\}) = pm(\overline{G \setminus \{uv\}}) .$$

Hence

$$I(G) = pm(\overline{G \setminus e}) - pm(\overline{G \setminus \{uv\}}) ,$$

which equals $pm(\overline{G})$ by Lemma 4.1, and so we have the desired equation

$$I(G) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu(G,x) e^{-x^2/2} dx = pm(\overline{G}) .$$

❑

<u>Chapter 5</u> – <u>Orthogonality</u>

In this chapter we discuss the orthogonality of several families of matchings polynomials with respect to various inner products. Let $R[x]$ denote the vector space of all polynomials with real coefficients. We can then view $\mu(G,x)$, the matchings polynomial of a graph *G*, as a vector in this vector space $R[x]$. Before we discuss orthogonality, let us first review the notion of an inner product.

If V is any vector space over the real numbers, then an **inner product** is any function $\langle\ ,\ \rangle : V$› R that satisfies the following properties:

(1) $\langle u,v\rangle = \langle v,u\rangle$ for every $u,v \in V$,

(2) $\langle u,v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$ for every $u,v,w \in V$,

(3) $\langle k\cdot u,v\rangle = k\cdot\langle u,v\rangle$ for every $u,v \in V$ and $k \in R$,

(4) $\langle u,u\rangle \geq 0$ for every $u \in V$, and $\langle u,u\rangle = 0$ if and only if $u = 0$.

An example of an inner product is the usual dot product encountered in a typical calculus course. Two vectors u,v are said to be **orthogonal** with respect to an inner product $\langle\ ,\ \rangle$ whenever $\langle u,v\rangle = 0$.

In the following sections, we will encounter a number of inner products on the vector space $R[x]$, and we will find that, curiously, each of the families of matchings polynomials we have studied is indeed an orthogonal set of vectors in $R[x]$ with respect to an appropriate inner product. Only the orthogonality of the matchings polynomials for *complete graphs* is immediately relevant for the main development of the results in this paper, although the other families have equally interesting results. Therefore, in the sections that follow, we will discuss the family of complete graphs in depth while offering a less detailed treatment of the other three families (paths, cycles, and complete bipartite graphs).

Section 5.1 – **Orthogonality and Complete Graphs**

In this section we return to our discussion of complements and perfect matchings. Observe that the complement of $K_m \cup K_n$ is the complete bipartite graph $K_{m,n}$, since the graph $\overline{K_m \cup K_n}$ consists of two independent sets of sizes *m* and *n*, with all possible edges between them. (Recall a set of vertices is **independent** if no two of them are joined by an edge.)

Since a perfect matching must cover all of the vertices, we can only have a perfect matching in $K_{m,n}$ if $m = n$. On the other hand, if $m = n$, then a perfect matching does indeed exist, because there are m choices for the first vertex to be paired with, $(m-1)$ choices for the next vertex to be paired with, and so on. So when $m = n$, we in fact find that there are a total of m! (which is the same as n!) perfect matchings. (In other words, the existence is guaranteed by the counting argument.) Thus we arrive at the following equation:

$$pm(K_{m,n}) = \begin{cases} m! & if \quad m = n \\ 0 & otherwise \end{cases}.$$

Note that by Theorem 2.1, $\mu(K_m \cup K_n) = \mu(K_m, x)\mu(K_n, x)$. Hence, using the fact that $\overline{K_m \cup K_n} = K_{m,n}$, and applying Theorem 4.3, we get

$$\begin{aligned} pm(K_{n,m}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu(K_m \cup K_n, x) e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu(K_m, x)\mu(K_n, x) e^{-x^2/2} dx \end{aligned}$$

Thus we have shown the following result.

<u>Theorem 5.1</u>: For any natural numbers *m* and *n*, we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mu(K_m,x)\mu(K_n,x)e^{-x^2/2}dx=\begin{cases} m! & if \quad m=n \\ 0 & otherwise\end{cases}$$

❑

Theorem 5.1 indicates that the matchings polynomials of the complete graphs form an orthogonal family of polynomials. They are orthogonal with respect to the inner product

$$\langle p(x),q(x)\rangle=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}p(x)q(x)e^{-x^2/2}dx\,.$$

In fact, the matchings polynomials of the complete graphs belong to a well-studied family of orthogonal polynomials, known as the **Hermite polynomials**. For more about these polynomials, the reader is referred to (Leon, 2006).

<u>Example</u>: Let us examine two complete graphs, K_3 and K_4. Their respective matchings polynomials are given by $\mu(K_3,x)=x^3-3x$ and $\mu(K_4,x)=x^4-6x^2+3$. Substituting into the inner product above, we find that

$$\langle\mu(K_3,x),\mu(K_4,x)\rangle=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(x^3-3x)(x^4-6x^2+3)e^{-x^2/2}dx=0\,,$$

as expected. If we evaluate the inner product of $\mu(K_3,x)$ with itself, we obtain

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(x^3-3x)^2e^{-x^2/2}dx=3!$$

These are the results we would expect, given the orthogonality of the Hermite polynomials.

Section 5.2 – Orthogonality and Paths

We can also consider whether the matchings polynomials of paths form a family of orthogonal polynomials with respect to some inner product. Indeed, upon substitution of $2x$ for x in the matchings polynomial of paths, we find a familiar family of polynomials. In fact,

$$\mu(P_n, 2x) = U_n(x),$$

where the $U_n(x)$ are the so-called **Chebyshev polynomials of the second kind** (Leon, 2006). The Chebyshev polynomials are known to be orthogonal (indeed, orthonormal) with respect to the following inner product

$$\langle p(x), q(x) \rangle = \frac{2}{\pi}\int_{-1}^{1} p(x)q(x)\sqrt{1-x^2}\,dx .$$

Example: Let us examine two paths, P_3 and P_4. Their respective matchings polynomials are given by $\mu(P_3, x) = x^3 - 2x$ and $\mu(P_4, x) = x^4 - 3x^2 + 1$. Using the above relationship $\mu(P_n, 2x) = U_n(x)$, we find that $U_3(x) = 8x^3 - 4x$ and $U_4(x) = 16x^4 - 12x^2 + 1$. Substituting $U_3(x)$ and $U_4(x)$ into the inner product above, we find that

$$\frac{2}{\pi}\int_{-1}^{1} (8x^3 - 4x)(16x^4 - 12x^2 + 1)\sqrt{1-x^2}\,dx = 0 ,$$

as expected. If we evaluate the inner product of $U_3(x)$ with itself, we find that

$$\frac{2}{\pi}\int_{-1}^{1} (8x^3 - 4x)^2\sqrt{1-x^2}\,dx = 1 .$$

These are the results we would expect, given the orthogonality of the Chebyshev polynomials of the second kind.

Section 5.3 – **Orthogonality and Cycles**

We can also consider whether the matchings polynomials of cycles form a family of orthogonal polynomials with respect to some inner product. These polynomials are also intimately related to a well known family of orthogonal polynomials. Indeed, for $n \geq 1$,

$$\mu(C_n, 2x) = 2T_n(x),$$

where the $T_n(x)$ are the **Chebyshev polynomials of the first kind** (Leon, 2006). These Chebyshev polynomials are known to be orthogonal (indeed, orthonormal) with respect to the following inner product

$$\langle p(x), q(x) \rangle = \frac{2}{\pi} \int_{-1}^{1} p(x)q(x) \frac{1}{\sqrt{1-x^2}} dx .$$

Example: Let us examine two cycles, C_3 and C_4. Their respective matchings polynomials are given by $\mu(C_3, x) = x^3 - 3x$ and $\mu(C_4, x) = x^4 - 4x^2 + 2$. We first note that $\mu(C_3, 2x) = 8x^3 - 6x$ and $\mu(C_4, 2x) = 16x^4 - 16x^2 + 2$. Then, using the above relationship $\mu(C_n, 2x) = 2T_n(x)$, we find that $T_3(x) = 4x^3 - 3x$ and $T_4(x) = 8x^4 - 8x^2 + 1$. Substituting $T_3(x)$ and $T_4(x)$ into the inner product above, we find that

$$\frac{2}{\pi} \int_{-1}^{1} (4x^3 - 3x)(8x^4 - 8x^2 + 1) \frac{1}{\sqrt{1-x^2}} dx = 0,$$

as expected. If we evaluate the inner product of $T_3(x)$ with itself, we find that

$$\frac{2}{\pi}\int_{-1}^{1}(4x^3-3x)^2\sqrt{1-x^2}\,dx = 1 .$$

These are the results we would expect, given the orthogonality of the Chebyshev polynomials of the first kind.

Section 5.4 – Orthogonality and Complete Bipartite Graphs

Finally, we can also consider whether the matchings polynomials of complete bipartite graphs form a family of orthogonal polynomials with respect to some inner product. These polynomials are also closely related to a well known family of orthogonal polynomials. In this case,

$$\mu(K_{m,m},x) = m!\,L_m(x^2)(-1)^m,$$

where the $L_m(x)$ are the **Laguerre polynomials** (Leon, 2006). These Laguerre polynomials are known to be orthogonal (indeed orthonormal) with respect to the following inner product

$$\langle p(x),q(x)\rangle = \int_{0}^{\infty} p(x)q(x)e^{-x}\,dx .$$

Example: Let us examine two complete bipartite graphs, $K_{2,2}$ and $K_{3,3}$. We have $\mu(K_{2,2},x) = x^4-4x^2+2$ and $\mu(K_{3,3},x) = x^6-9x^4+18x^2-6$. Using the above relationship $\mu(K_{m,m},x) = m!\,L_m(x^2)(-1)^m$, we find that $L_2 = \frac{1}{2}x^2-2x+1$ and

$L_3 = -\left(\frac{1}{6}x^3 - \frac{3}{2}x^2 + 3x - 1\right)$. Substituting $L_2(x)$ and $L_3(x)$ into the inner product above, we find that

$$\int_0^\infty (\frac{1}{2}x^2 - 2x + 1)(-\frac{1}{6}x^3 + \frac{3}{2}x^2 - 3x + 1)e^{-x}dx = 0 ,$$

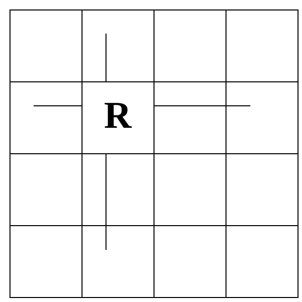
as expected. If we evaluate the inner product of $L_3(x)$ with itself, we find

$$\int_0^\infty (-\frac{1}{6}x^3 + \frac{3}{2}x^2 - 3x + 1)^2 e^{-x}dx = 1 .$$

These are the results we would expect, given the orthogonality of the Laguerre polynomials.

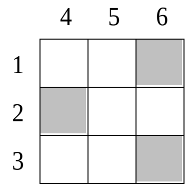
Chapter 6 – Rook Polynomials

We now turn our attention to a special case of the matchings polynomial, one that provides useful applications for counting problems and other combinatorial concepts. This special matchings polynomial, called the 'rook polynomial,' derives its name from the familiar game of chess. In this game, a rook is a piece which can move any number of squares horizontally or vertically; it moves exclusively along the rows and columns of a chessboard, as indicated below.

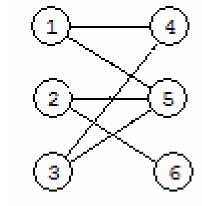


We can associate chessboard configurations with graphs in the following way. We define a **board** to be any subset of the squares of an $m \times m$ chessboard. Any such board *B* determines a bipartite graph G_B, a subgraph of $K_{m,m}$ as follows. The *m* vertices in one set of $K_{m,m}$ correspond to the rows of the containing $m \times m$ chessboard, and the *m* vertices of the other set correspond to the columns. We refer to these as the **row-vertices** and **column-vertices**, respectively. Any given row-vertex, together with a column-vertex, determines a unique square in the containing $m \times m$ chessboard. These vertices are joined by an edge in G_B if and only if the square they determine is in the board *B*.

For example, consider the diagrams that follow. In the chessboard below, the gray squares are **restricted positions**, or squares on which no rook can be placed. Thus the board *B* consists only of the white squares. Note we can see the relationship between the board *B* and the graph G_B: an edge exists between a row-vertex and a column-vertex in G_B exactly when the square defined by this row and column is in board *B*.

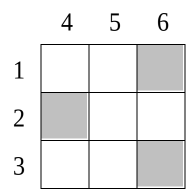


B

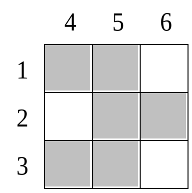


G_B

We also must define the complement of a board *B*, which we denote by $\overline{B}$. The **complement** $\overline{B}$ is a board that is made up of squares that were not in *B*; indeed the squares in $\overline{B}$ are exactly the restricted squares for *B*, and, conversely, the squares in *B* are exactly the restricted squares for $\overline{B}$. In fact, if *B* is any sub-board of an $m \times m$ chessboard, then $\overline{B}$ can be thought of as the board associated with the graph $K_{m,m} - G_B$. The board *B* and its complement $\overline{B}$ are given below.



B

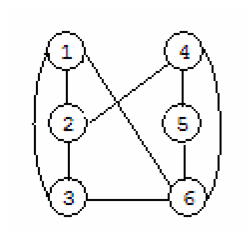


$\overline{B}$

Having now established a correspondence between boards and bipartite graphs, and having introduced the notion of the complement of a board, we can also consider the graph theoretic version of such board complementation. Observe that, under our definitions, $G_{\overline{B}}$ denotes the graph corresponding to the complement of board *B.* There is an important distinction to note here, however. The graph $G_{\overline{B}}$ is not the complement of the graph G_B.

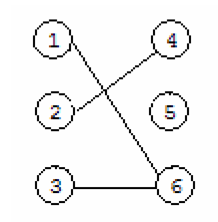
In order better to understand this distinction, consider the two graphs that follow. On the left below is $\overline{G_B}$, which is the *graph* complement of the graph G_B. On the right below is $G_{\overline{B}}$, which is the graph associated with the board $\overline{B}$. The graph on the left is obtained by first drawing G_B and then taking the graph complement of it. Note here that the vertices within the cells of the bipartition now must be adjacent, by definition of graph complementation. Hence this graph is no longer bipartite and thus no longer represents a board. The graph on the right, however, is clearly still bipartite and therefore is still associated with a board, namely $\overline{B}$. The graph $G_{\overline{B}}$ is often referred to as the **bipartite complement** of the graph G_B.

Complement of G_B



$\overline{G_B}$

Bipartite Complement of G_B



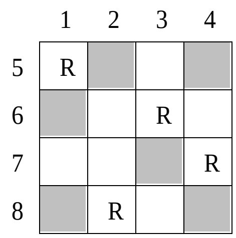
$G_{\overline{B}}$

In other words, in forming the bipartite complement, only the edges and non-edges *between* the two cells of the bipartition are interchanged – the non-edges within the cells are maintained. In general, we will use $\overline{\overline{G}}$ to denote the bipartite complement of a bipartite graph *G*.

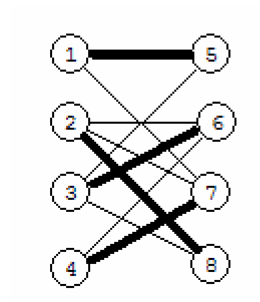
We have seen above that there is a natural interpretation relating any possible arrangement of rooks on *B* to a corresponding subset of edges in G_B. If, in an arrangement of rooks, there are no two rooks that in lie in the same row or same column, then the arrangement is said to be **non-attacking**. The edges in G_B corresponding to the locations of these rooks, then, will be disjoint. So in any non-attacking arrangement of *r*-rooks, the corresponding subset of edges in G_B is an *r*-matching. In particular, the number of non-attacking arrangements of r rooks is equal to $p(G_B, r)$.

Consider the following two diagrams which illustrate the notion of non-attacking arrangements of rooks. The board B' below consists of the white squares and displays such an arrangement of rooks; the R's represent rooks on the board. The graph $G_{B'}$

represents the graph associated with this board, and the bold edges represent the *r*-matching that corresponds with this particular arrangement of rooks.



B'



$G_{B'}$

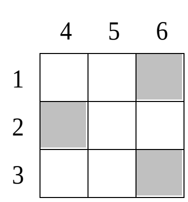
We now introduce a generating function for the number of non-attacking arrangements of rooks on a board. Given any sub-board *B* of an $m \times m$ chessboard, the **rook polynomial** of *B* is defined to be the following:

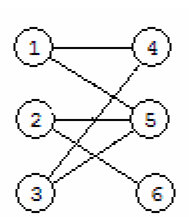
$$\rho(B,x) := \sum_{r \geq 0} (-1)^r \, p(G_B, r) x^{m-r} \ .$$

So, in particular, the rook polynomial of a board *B* is almost exactly the same as the matchings polynomial of the graph G_B associated with *B*. The only difference between the two polynomials concerns the exponents of the terms. Specifically, we have that

$$\rho(B, x^2) = \mu(G_B, x) \, .$$

<u>Example</u>: Let us study a simple example that compares the rook polynomial of a board *B* to the matchings polynomial of its graph G_B.





B $\qquad\qquad\qquad\qquad$ G_B

Notice that the rook polynomial of *B*,

$$\rho(B,x) = x^3 - 6x^2 + 8x - 2\,,$$

differs just slightly from the matchings polynomial of G_B,

$$\mu(G_B,x) = x^6 - 6x^4 + 8x^2 - 2\,.$$

The two only differ in the exponents of the terms, exemplifying the equality $\rho(B,x^2) = \mu(G_B,x)$ above.

The reader may well wonder why we need to introduce such a minor variation of the matchings polynomial. As we mentioned above, the bipartite complement is a different graph theoretic operation than the usual complement. In the following sections, we will see that the rook polynomial, unlike the matchings polynomial, behaves well under the operation of bipartite complementation. We will take advantage of this fact as we develop some analogous theorems to those in Chapter 4, counting perfect matchings in bipartite complements using rook polynomials of boards. Additionally, we will make use of these results to provide elegant solutions to some famous combinatorial problems.

Section 6.1 – Rook Analogs for Matchings Theorems

In this section, we prove the analogs of Theorem 2.1 and Theorem 2.2 for rook polynomials.

Theorem 6.1: For two disjoint boards B_1 and B_2,

$$\rho(B_1 \cup B_2, x) = \rho(B_1,x)\rho(B_2,x)\,.$$

Proof: We know from Theorem 2.1 that for two graphs G and H,

$$\mu(G \cup H, x) = \mu(G, x)\mu(H, x).$$

Then, by the fact that B_1 and B_2 correspond to the graphs G_{B_1} and G_{B_2} respectively, and by our above observation that $\rho(B, x^2) = \mu(G_B, x)$, it follows that

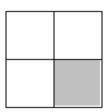
$$\rho(B_1 \cup B_2, x^2) = \rho(B_1, x^2)\rho(B_2, x^2),$$

which implies that

$$\rho(B_1 \cup B_2, x) = \rho(B_1, x)\rho(B_2, x).$$

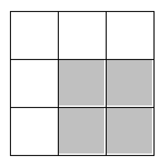
❑

Example: We examine two boards B_1 and B_2; their rook polynomials are given below. We can use these boards to exemplify Theorem 6.1.



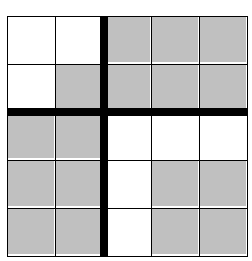
B_1

$$\rho(B_1, x) = x^2 - 3x + 1$$



B_2

$$\rho(B_2, x) = x^3 - 5x^2 + 4x$$



$$B_1 \cup B_2$$

$$\rho(B_1 \cup B_2, x) = x^5 - 8x^4 + 20x^3 - 17x^2 + 4x$$

Note that

$$\begin{aligned}\rho(B_1 \cup B_2, x) &= x^5 - 8x^4 + 20x^3 - 17x^2 + 4x \\ &= (x^2 - 3x + 1) \cdot (x^3 - 5x^2 + 4x) \\ &= \rho(B_1, x) \cdot \rho(B_2, x),\end{aligned}$$

which illustrates the theorem presented above.

We now obtain a reduction result.

Theorem 6.2: For any board *B*, and any square s in *B* located in row *u* and column *v*,

$$\rho(B, x) = \rho(B \setminus s, x) - \rho(B \setminus \{uv\}, x),$$

where $B \setminus s$ denotes the board *B* with the square s forbidden, and where $B \setminus \{uv\}$ denotes the board *B* with row *u* and column *v* forbidden.

Proof: We know from Theorem 2.2 that for any graph *G* and any edge $e \in G$ with endpoints *u* and *v*,

$$\mu(G, x) = \mu(G \setminus e, x) - \mu(G \setminus \{uv\}, x).$$

Note that *B* corresponds to a graph G_B, and square s corresponds to an edge *e* in G_B. Since $\rho(B, x^2) = \mu(G_B, x)$, it follows that

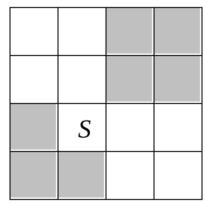
$$\rho(B, x^2) = \rho(B \setminus s, x^2) - \rho(B \setminus \{uv\}, x^2),$$

which implies that

$$\rho(B, x) = \rho(B \setminus s, x) - \rho(B \setminus \{uv\}, x).$$

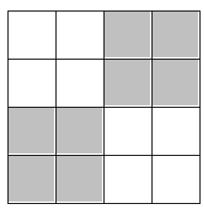
❑

Example: We examine the rook board B below and consider the square marked s to illustrate the use of Theorem 6.2. The rook polynomials of the boards B, $B \setminus s$ and $B \setminus \{uv\}$ are given below.



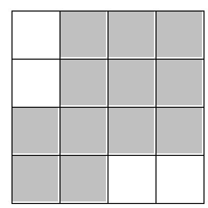
B

$$\rho(B,x) = x^4 - 8x^3 + 19x^2 - 12x$$



$B \setminus s$

$$\rho(B \setminus \{uv\}, x) = x^4 - 8x^3 + 20x^2 - 16x + 4$$



$B \setminus \{uv\}$

$$\rho(B \setminus s, x) = x^2 - 4x + 4$$

Note that

$$\begin{aligned} \rho(B,x) &= x^4 - 8x^3 + 19x^2 - 12x \\ &= (x^4 - 8x^3 + 20x^2 - 16x + 4) - (x^2 - 4x + 4) \\ &= \rho(B \setminus s, x) - \rho(B \setminus \{uv\}, x)\,, \end{aligned}$$

which illustrates the theorem presented above.

Although analogs of Theorem 2.3 and 2.4 could be formulated for rook polynomials, they are not directly pertinent to the focus of this paper. Therefore, we will not address them here.

Section 6.2 – Perfect Matchings in the Bipartite Complement of *G*

Throughout this section *G* will denote a bipartite graph. Recall that if *G* is a spanning subgraph of $K_{m,m}$, then its bipartite complement $\overline{\overline{G}}$ is the graph with the same vertex set as *G*, whose edge set is precisely those edges of $K_{m,m}$ not in *G*. Similar to our work in Chapter 4, we investigate an integral formula for $pm(\overline{\overline{G}})$, the number of perfect matchings in the bipartite complement of *G*.

We must first take note of two important identities. Let *e* be an edge in *G*. Certainly the graphs $\overline{\overline{G \setminus e}}$ and $\overline{\overline{G}}$ differ by at most the edge *e*. Since *e* is in *G*, it follows that *e* is not in $\overline{\overline{G}}$. Since *e* is also not in $G \setminus e$, we know that *e* is in $\overline{\overline{G \setminus e}}$, the bipartite complement of $G \setminus e$. Therefore, $\overline{\overline{G \setminus e}}$ is exactly the same graph as $\overline{\overline{G}}$, except that it includes the edge *e*. Thus we find that

$$\overline{\overline{G \setminus e}} = \overline{\overline{G}} + e .$$

Turning now to our second identity, we argue using the notion of set difference. Observe that $\overline{\overline{G}} \setminus \{uv\}$ can be written as $\left(K_{m,m} \setminus G\right) \setminus \{uv\}$ which, in turn, is equivalent to $\left(K_{m,m} \setminus \{uv\}\right) \setminus \left(G \setminus \{uv\}\right)$. This last expression is the same as $\overline{\overline{G \setminus \{uv\}}}$, and hence

$$\overline{\overline{G}} \setminus \{uv\} = \overline{\overline{G \setminus \{uv\}}} .$$

We will use these two simple observations to prove the following lemma.

Lemma 6.4: For any bipartite graph *G* and any edge *e* in *G* with endpoints *u* and *v*,

$$pm(\overline{\overline{G}}) = pm(\overline{\overline{G \setminus e}}) - pm(\overline{\overline{G \setminus \{uv\}}}).$$

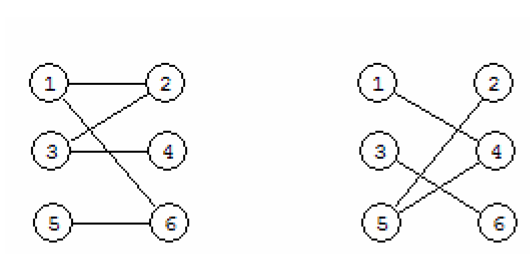
Proof: The edge *e* is clearly in $\overline{\overline{G}} + e$, so we conclude that $e \in \overline{\overline{G \setminus e}}$ by our first identity above. The perfect matchings in $\overline{\overline{G \setminus e}}$ consist of two kinds – those that use edge *e* and those that do not. Any perfect matching that does not use *e* is a perfect matching in $\overline{\overline{G \setminus e}} \setminus e$, which is just the graph $\overline{\overline{G}}$. Any perfect matching that does use *e* determines a unique perfect matching in $\overline{\overline{G}} \setminus \{uv\}$. From the second identity above, observe that $\overline{\overline{G \setminus \{uv\}}} = \overline{\overline{G}} \setminus \{uv\}$. Thus

$$pm(\overline{\overline{G \setminus e}}) = pm(\overline{\overline{G}}) + pm(\overline{\overline{G \setminus \{uv\}}}),$$

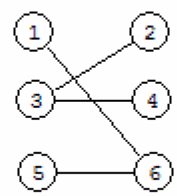
which can be written to obtain the statement of the lemma.

❑

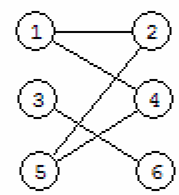
Example: We demonstrate Lemma 6.4 with the following example. Let G and $\overline{\overline{G}}$ be the graphs below. Let e be the edge with endpoints {12}. Listed below each graph is the number of perfect matchings of the graph, which we can count directly.



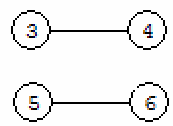
$pm(G) = 1$ $pm(\overline{\overline{G}}) = 1$



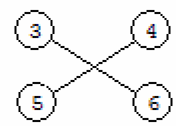
$pm(G \setminus e) = 0$



$pm(\overline{\overline{G \setminus e}}) = 2$



$pm(G \setminus \{12\}) = 1$



$pm(\overline{\overline{G \setminus \{12\}}}) = 1$

We see, then, that for the graph G in this example, $pm(\overline{\overline{G}}) = pm(\overline{\overline{G \setminus e}}) - pm(\overline{\overline{G \setminus \{12\}}})$.

<u>Section 6.3</u> – <u>Perfect Matchings in the Complete Bipartite Graph</u>

<u>Lemma 6.5</u>: For any natural number m,

$$pm(K_{m,m}) = \int_0^\infty x^m e^{-x} dx$$

<u>Proof</u>: We define $M(m) := \int_0^\infty x^m e^{-x} dx$, which we evaluate using integration by parts.

Letting $u = e^{-x}$ and $dv = x^m dx$, we find that $du = -e^{-x}$ and $v = \dfrac{x^{m+1}}{m+1}$. So using the parts formula $\int_a^b udv = uv - \int_a^b vdu$, we calculate as follows:

$$\int_0^\infty x^m e^{-x} dx = \left. \frac{e^{-x} x^{m+1}}{m+1} \right|_0^\infty + \int_0^\infty \frac{x^{m+1}}{m+1} e^{-x} dx$$

To compute the first term on the right hand side, we note that $\lim_{x\to\infty} \frac{x^{m+1}}{(m+1)e^x}$ must equal 0 by repeated applications of L'Hôpital's rule. Thus we have that

$$\int_0^\infty x^m e^{-x} dx \ = \ 0 + \frac{1}{m+1}\int_0^\infty x^{m+1} e^{-x} dx$$

which implies that

$$M(m) = \frac{1}{m+1}\int_0^\infty x^{m+1} e^{-x} dx$$

$$= \frac{M(m+1)}{m+1}.$$

This equation leads us to an important recurrence which we will utilize in the rest of this proof:

$$M(m+1) = (m+1)M(m).$$

Note it is easy to determine $M(0)$; we do so by letting $m = 0$ in the integral $M(m) = \int_0^\infty x^m e^{-x} dx$ we just obtained above. Thus

$$M(0) = \int_0^\infty x^0 e^{-x} dx = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = -\frac{1}{e^x} = 1.$$

Now by exploiting the above recurrence, we find that

$$M(m) = m!.$$

But we know that $pm(K_{m,m}) = p(K_{m,m}, m) = m!$ from Proposition 1.4, and hence

$$pm(K_{m,m}) = M(m) = \int_0^\infty x^m e^{-x} dx.$$

❑

As in Section 4.3, we note that x^m is the rook polynomial of the bipartite complement of the complete bipartite graph $K_{m,m}$. Therefore the above lemma suggests a possible relationship between bipartite graphs, their bipartite complements, and perfect matchings. In the following section we introduce a theorem that describes just such a relationship.

<u>Section 6.4</u> – <u>The Matchings Polynomial of *G* and Perfect Matchings in its Bipartite Complement</u>

We have just shown that the number of perfect matchings in the complete bipartite graph is given by the formula

$$pm(K_{m,m}) = \int_0^\infty x^m e^{-x} dx .$$

Note that x^m is in fact the rook polynomial of the bipartite complement of $K_{m,m}$. This observation motivates the following theorem.

<u>Theorem 6.6</u>: Let G be any bipartite graph and assume G is a spanning subgraph of $K_{m,m}$. Then the following holds:

$$pm(\overline{\overline{G}}) = \int_0^\infty \rho(G, x) e^{-x} dx .$$

<u>Proof</u>: We induct on the number of edges in G. Lemma 6.5 has given us our base case: when G has zero edges, then $\overline{\overline{G}}$ is the complete bipartite graph $K_{m,m}$. Then

$$pm(\overline{\overline{G}}) = pm(K_{m,m}) = \int_0^\infty x^m e^{-x} dx = \int_0^\infty \rho(G, x) e^{-x} dx ,$$

as desired.

For our induction hypothesis, we assume that G has at least one edge e. Let u and v denote the endpoints of e and assume that $pm(\overline{\overline{G}}) = \int_0^\infty \rho(G,x)e^{-x}dx$ holds for any subgraph of $K_{m,m}$ with fewer edges than G. By Lemma 6.4 above we have that

$$\begin{aligned} pm(\overline{\overline{G}}) &= pm(\overline{\overline{G \setminus e}}) - pm(\overline{\overline{G \setminus \{uv\}}}) \\ &= \int_0^\infty \rho(G \setminus e, x)e^{-x}dx - \int_0^\infty \rho(G \setminus \{uv\}, x)e^{-x}dx \\ &= \int_0^\infty [\rho(G \setminus e, x) - \rho(G \setminus \{uv\}, x)]e^{-x}dx\,, \end{aligned}$$

and by Theorem 6.2 we conclude

$$pm(\overline{\overline{G}}) = \int_0^\infty \rho(G,x)e^{-x}dx\ ,$$

as desired.

❑

<u>Section 6.5</u> – <u>Classic Counting Problems</u>

In this section, we will use our results about rook polynomials (Theorem 6.6 in particular) to solve two classic problems in enumeration. Although these problems are commonly found among advanced combinatorial textbooks, the methods we have developed in this paper are not. As we shall see, the results we have obtained provide novel and elegant solutions to these famous problems.

<u>Section 6.5a</u> – <u>Das Problem der Derangements</u>

The first classic counting problem can be described as follows. Let B denote an $m \times m$ chessboard with the diagonal squares forbidden. Then the number of perfect matchings in the associated graph G_B equals the number of permutations of $\{1,...,m\}$ with no fixed points. Such permutations are often called **derangements**, and it is precisely the number of these derangements that we wish to count.

Notice that the graph G_B is a subgraph of $K_{m,m}$, and its bipartite complement is the disjoint union of m copies of K_2. We denote this graph by mK_2. The rook polynomial of K_2 is $(x-1)$, and so by Theorem 2.1 the rook polynomial of $\overline{\overline{G_B}} = mK_2$ must equal $(x-1)^m$.

Let $D(m)$ denote the number of derangements of $\{1,\ldots,m\}$. By construction, $D(m) = pm(G_B)$, so we can use our results from this chapter to perform computations as follows.

$$D(m) = \int_0^\infty \rho(mK_2, x)e^{-x}dx = \int_0^\infty (x-1)^m e^{-x}dx$$

$$= \int_1^\infty (x-1)^m e^{-x}dx + \int_0^1 (x-1)^m e^{-x}dx$$

Upon substituting $y = x-1$, the first integral simplifies to $e^{-1}\int_1^\infty y^m e^{-y}dx$, which evaluates to $\frac{m!}{e}$ by earlier results in this chapter. We denote the value of the second integral – the remainder – simply by R_m. Thus we have

$$D(m) = \frac{m!}{e} + R_m$$

Notice, however, that

$$|R_m| \le \int_0^1 |(x-1)|^m e^{-x}dx < \int_0^1 |(x-1)|^m dx = \left.\frac{|x-1|^{m+1}}{m+1}\right|_0^1 = \frac{1}{m+1}.$$

So $|R_m|$ must be smaller than $\frac{1}{2}$ for every m. Therefore $D(m)$, the number of derangements of $\{1,\ldots,m\}$, is equal to the integer nearest to $\frac{m!}{e}$

Based upon this result, the following table provides us with an interesting sequence of approximations to the number *e*. Since there can be no derangement of a set with only one element, we begin our table with $m = 2$.

m	$\frac{m!}{D(m)}$
2	$\frac{2!}{1} = 2$
3	$\frac{3!}{2} = 3$
4	$\frac{4!}{9} = \frac{8}{3} \approx 2.\overline{6}$
5	$\frac{5!}{44} = \frac{30}{11} \approx 2.\overline{72}$
$\vdots$	$\vdots$
10	$\frac{10!}{1334961} = \frac{44800}{16481} \approx 2.718281658...$

We conclude this section with a simple example using derangements.

Example: Consider the following scenario. A professor hands back a quiz to his class of 23 students, and he would like them to be able to grade each other's papers. In how many ways can he pass the quizzes back such that no student receives their own paper to grade?

Solving this problem is simply a matter of counting the number of derangements of $\{1,...,23\}$. From above, we know that $D(23)$ is just the closest integer to $\frac{23!}{e}$ which equals 9.5104×10^{21}.

Section 6.5b – Le Problèm des Menages

The second classic counting problem that we consider using rook polynomials is called the **Ménages** problem. Here we are asked to find the number of ways of seating *n* married couples at a circular table, where we alternate between men and women, and where no one is seated next to their spouse. We can use our techniques to solve this problem as follows.

Note that we have *n* women and *n* men. We first seat the women, and since they are seated around a circle, we consider there to be exactly $(n-1)!$ distinct ways to accomplish this task. Next we label the women $1,2,3,..,n$ in a clockwise manner, and we assign the number of the i^{th} woman both to her spouse and to the seat directly counterclockwise of hers. Note that each possible seating arrangement for the men is determined by a permutation of $\{1,...,n\}$, as there are *n* places left for the men to sit at the table. But there is the specified restriction yet to consider. In particular, we are interested in counting only the permutations in which no number *i* gets mapped either to itself or to the number $i-1$. Said another way, we're concerned only with the permutations *f* such that $f(i) \notin \{i-1, i\}$. We note here that subtraction is understood to be taken modulo *n*, as we are considering positions at a circular table.

Counting such permutations, however, is equivalent (by an examination of the associated rook board) to counting the number of perfect matchings in the bipartite complement of the cycle C_{2n} on 2*n* vertices. From Section 2.2 , we have that

$$p(C_{2n}, r) = \frac{2n}{2n-r}\binom{2n-r}{r}$$

for each integer *r*. Therefore, we have

$$\begin{aligned} pm(\overline{\overline{C_{2n}}}) &= \int_0^\infty \rho(C_{2n}, x)e^{-x}dx \\ &= \int_0^\infty \sum_{r=0}^{n} (-1)^r\, p(C_{2n}, r)x^{n-r}e^{-x}dx \\ &= \sum_{r=0}^{n} (-1)^r\, p(C_{2n}, r)\int_0^\infty x^{n-r}e^{-x}dx \end{aligned}$$

Recall from Lemma 6.5 that $\int_0^\infty x^n e^{-x}\, dx = n!$, so by this fact, and Proposition 1.2, we have

$$pm(\overline{C_{2n}}) = \sum_{r=0}^{n} (-1)^r \frac{2n}{2n-r} \binom{2n-r}{r} (n-r)!$$

Hence the number of seatings at the circular table equals

$$(n-1)! \sum_{r=0}^{n} (-1)^r \frac{2n}{2n-r} \binom{2n-r}{r} (n-r)!$$

which solves our problem.

We conclude this section with an example of the Ménages problem.

<u>Example</u>: Consider the case when $n = 4$. Then we simply use the above expression to compute the number of ways in which 4 married couples can be seated at a circular table so that no one is seated next to their spouse. We find that there are

$$\begin{aligned}(n-1)! \sum_{r=0}^{n} (-1)^r \frac{2n}{2n-r} \binom{2n-r}{r} (n-r)! &= 3! \sum_{r=0}^{4} (-1)^r \frac{8}{8-r} \binom{8-r}{r} (4-r)! \\ &= 6\left[\frac{8}{8}\binom{8}{0}4! - \frac{8}{7}\binom{7}{1}3! + \frac{8}{6}\binom{6}{2}2! - \frac{8}{5}\binom{5}{3}1! + \frac{8}{4}\binom{4}{4}0!\right] \\ &= 6\,[24 - 48 + 40 - 16 + 2] \\ &= 12\end{aligned}$$

ways, which can be verified by direct enumeration.

Part Two:

Teaching Combinatorics Using Rook Polynomials and Matchings Polynomials

Overview of the Curriculum

Having investigated a variety of mathematical aspects of the matchings polynomial, the focus of the paper now shifts to the development of curricular materials related to these mathematical concepts. Although some of the mathematical details of the above investigation are inaccessible to high school students, the didactical implications are not as contrived as one might think. Indeed, if presented in a pedagogically appropriate form, the mathematics studied above has the potential to be both relevant and clearly understood, even within a high school classroom. To prepare the reader for the curriculum to come, we draw their attention to two important aspects of the design of the curriculum. First, we discuss the relationship between the mathematics in the previous part of the paper and the mathematics in the curriculum. Second, we discuss the specifics of the curriculum design and implementation.

The Mathematics of the Curriculum

The rook polynomial, which is a special case of the matchings polynomial, plays a much larger role in the curriculum than it did in the mathematical section of this paper. Indeed, it has proven to be an invaluable focal point of the curriculum that follows. Because students tend to be familiar with the game of chess, the problem of counting the number of ways of placing rooks on a chessboard is a natural point of entry. Not only is this type of problem intrinsically compelling for students, but also it is easy to visualize and conceptualize. Another benefit of studying counting problems in the context of rooks is that rook problems generalize quite naturally to a wide range of counting problems in a variety of contexts, including counting problems with restricted positions. Thus, the principles related to rook problems extend well beyond the original context, making such problems even more useful (and motivating) for students, providing a springboard to richer and more advanced mathematical concepts.

One minor technical adjustment that has been made in the transition from the mathematical investigation to the curricular implementation concerns a simplification of the form of the matchings (and rook) polynomials. Recall that the definition of the matchings polynomial given above is somewhat at odds with the typical definition of a generating function. In particular, the matchings polynomial was defined to be

$$\mu(G,x) := \sum_{r \geq 0} (-1)^r \, p(G,r) x^{n-2r} \, .$$

Observe the presence of the alternating sign and the association of $p(G,r)$, the number of r-matchings in a graph G, with the term x^{n-2r}. In the curriculum, however, the notation $m(G,r)$ is used to indicate the number of r-matchings in a graph G, and the matchings polynomial is defined by the simpler

$$\mu(G,x) := \sum_{r \geq 0} m(G,r) x^r \, .$$

Note that the rook polynomial has undergone the same modification throughout the curriculum as well.

It is worth remarking that this change is notationally convenient but has no essential bearing on the associated mathematics. Indeed, in so defining the matchings polynomial, many of the theorems still hold true while assuming a form that is substantially more accessible to the students. Although the development of the curriculum required careful attention to this issue, and although several results needed to be re-derived, the concordant elimination of distracting algebraic complexity is very satisfactory.

Curriculum Design and Implementation

Because of the appealing nature of rooks, the ultimate goal in developing the curriculum is to investigate the types of combinatorial principles that could effectively be taught using the basic rook setup (namely, counting the number of ways of placing non-attacking rooks on a chessboard). Combinatorially, the mathematics in this paper employs three major concepts which are important but are often difficult to convey in the classroom: counting principles, generating functions, and matchings. (These three concepts are rarely taught together, and certainly not from the standpoint of rook problems. In fact, only rarely do textbooks mention a relationship between rooks and matchings). The question emerges, then, whether students might be able to grasp all three of these concepts through investigating rooks on a chessboard. Could rooks be the entry point from which to introduce counting principles, generating functions, and even matchings? If so, would this new approach prove to be effective for students?

It was in this vein, then, that this curriculum was developed. The curriculum was tailored for and taught to an advanced group of high school seniors who were enrolled in a discrete math class. These students were undeniably bright and motivated, and they had been introduced to many concepts in discrete math prior to working with this curriculum. The reader should thus take note that these activities were specifically geared for a high-level class. The actual teaching of the lessons ranged from having the students work through and develop answers completely on their own to lecturing about the various topics. When lecturing did take place, every effort was made to have the instruction be as interactive as possible. More specifics related to this are discussed in reflections on the activities in the section that follows.

The curriculum consists of seven activities total, along with three assessments, all of which were designed to engage students in the three combinatorial concepts described above. The activities were primarily presented to the class during their math period on three consecutive Thursdays; some were given out between visits to the classroom. The schedule of the curriculum is outlined below. Because of the nature and timing of the classroom visits, the actual writing of the curriculum was somewhat spontaneous in nature. That is, rather than being pre-arranged and strictly implemented, the development of many of the activities was based upon how the prior activity had gone.

The presentation of the activities in this paper proceeds chronologically, according to how they were (and are meant to be) implemented in the classroom. Included with each activity is a brief description of the activity, a Teacher's Version (which includes comments for the teacher and an answer key), and a reflection on how the actual execution of the activity went.

Finally, it should be noted that despite the fact that this curriculum was presented to a high-level class, only minor modifications would be needed to alter the level of presentation. Furthermore, the activities and assessments need not be implemented in the particular timeframe outlined above. The curriculum that follows is not meant to be *the* definitive curriculum regarding rooks. Rather, the intention here is to provide an overarching structure for a unit on rooks and to illuminate the types of combinatorial ideas that could be taught using this context.

Schedule of Activities

Prior to Day 1:	Activity 1 – Rook Boards 101 Activity 2 – Rooks in the Real World
Day 1:	Activity 3 – Taboo Squares
Prior to Day 2:	Assessment 1 – Review Worksheet 1 Activity 4 – Bored with Boards Yet?
Day 2:	Activity 5 – Rook Kung Fu
Prior to Day 3:	Assessment 2 – Review Worksheet 2 Activity 6 – All Aboard for Matchings, Captain Rook!
Day 3:	Activity 7 – We're Gonna Rook Your World
After Day 3:	Assessment 3 – Rook Exam

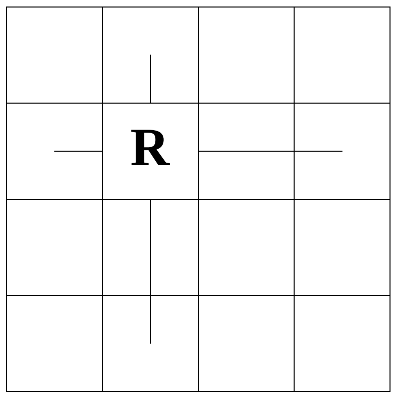
Introduction to Activity 1

This activity begins with an explanation of non-attacking configurations of rooks, with the intent of orienting students to the most basic ideas of rooks on a chessboard. The primary goal of this activity is to have students come up with a formula for the number of ways of placing *r* rooks on an $n \times n$ chessboard. In so doing, the students will encounter the addition and multiplication principles of counting. This activity focuses heavily on this particular context; the students can think about applying these principles solely to problems about rooks. At this early stage of investigation, they need not concern themselves with extrapolating these ideas to any other applications.

The use of groups will complement this activity nicely; the students should engage with this problem and discuss it with their peers before being given any hints or answers. It is a relatively short activity on its own, but it has the potential to lead to further discussion about binomial coefficient notation, representation of equations, and a variety of other topics (see Teacher's Version).

Activity 1 – Rook Boards 101

In chess, a rook can move any number of spaces in straight lines along the rows and columns of a board. A configuration of rooks on a board is called "non-attacking" if no two rooks occupy the same row or column.



1) With your group, find a formula for the number of ways of placing *r* non-attacking rooks on an $n \times n$ chessboard.

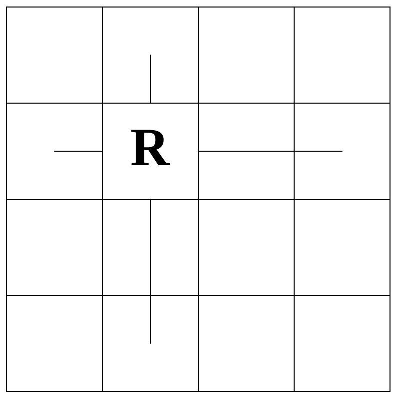
2) Use your formula above to complete the following table.

number of non-attacking rooks	0	1	2	3	4	5
number of configurations on a 5x5 board						

3) Make a similar table for a 14 x 7 chessboard.

Activity 1 – Rook Boards 101 (Teacher's Version)

In chess, a rook can move any number of spaces in straight lines along the rows and columns of a board. A configuration of rooks on a board is called "non-attacking" if no two rooks occupy the same row or column.



1) With your group, find a formula for the number of ways of placing r non-attacking rooks on an $n \times n$ chessboard.

Answer: There are several different ways to approach this problem. We proceed with a particular solution, but any method will do. First we choose r rows, and there are $\binom{n}{r}$ ways of doing this. In the first of these rows, we chose a column in which to place the first rook. There are n choices for this. In the second of the chosen rows, there are $n - 1$ columns available to place the 2nd rook. We continue in this fashion until we are left with $n - r + 1$ columns available to place the r^{th} rook in the last of the chosen rows. Thus, by the multiplication principle, we get the following formula for the number of ways of placing r non-attacking rooks on an $n \times n$ chessboard:

$$\binom{n}{r} n(n-1)(n-2)\ldots(n-r+1) = \binom{n}{r}\frac{n!}{(n-r)!}$$

Note that this expression could also be written as $\binom{n}{r}\binom{n}{r} r!$

We believe it is very important to make sure that all students see this last, most simplified, version of the formula and come to understand how it relates to the counting procedure, so suggest it if it doesn't arise from them. This formula will generalize nicely later.

2) Use your formula above to complete the following table.

number of non-attacking rooks	0	1	2	3	4	5
number of configurations on a 5x5 board	1	25	200	600	600	120

3) Make a similar table for a 14 x 7 chessboard.

number of non-attacking rooks	0	1	2	3	4	5	6	7
number of configurations on a 14x7 board	1	98	3822	76440	840840	5045040	15135120	17297280

- This is a rich problem for discussion, since any of the following issues may arise (if they don't, perhaps the teacher could raise them).
  - How many columns should there be in the table? 7? 14? Is it WRONG to have 14?
  - If we use more columns in our table, we could put zeros where appropriate. This relates to the custom of setting "m choose r" to be zero when r > m.

- In completing the table, students will (hopefully) come up with a formula similar to the one they found in answering #1. How well are they dealing with the variables *m*, *n*, and *r*?

- Get them to discuss different ways of writing a general formula for *r* rooks on an *m x n* board. Did they develop any alternative formulas? We'd like them to realize how a well-chosen version of the formula will, in fact, behave properly even in the troublesome cases. In particular, we'd like it to emerge that

$$\binom{m}{r}\binom{n}{r}r!$$

is a very nice expression that works in very general settings and also reflects a nice counting procedure that might, in fact, be different than the one they first derived. **Goal**: by the end of the day, have all students understand how we arrived at the above formula.

Reflection on Activity 1

Activities 1 and 2 had been given to the students prior to our first meeting with them. In Activity 1, they worked in groups of three or four to develop the formula for the number of ways of placing *r* rooks on an $n \times n$ chessboard. During our first visit to the class, we began by reviewing Activity 1 to see what the students had discovered. One student who had correctly derived the formula came to the board and explained how he had gotten it. The explanation was thorough and clear; it involved choosing columns then choosing rows, and it made explicit the constraint forbidding repetition in rows and columns. Furthermore, his explanation allowed for the extra *r*! as the number of ways of shuffling the *r* rows of a particular placement to get all the other placements. Using the students' counting formula, only a minor adjustment in notation was needed to arrive at the desired expression.

When asked about the maximum number of rooks they could place on an $m \times n$ board, one student explained that it wouldn't make sense to place more than the lesser of *m* and *n* rooks on a board. This led to an interesting conversation of the possibility of having more than min(*m*,*n*) rooks, and we were able to discuss conventions regarding the expression $\binom{m}{r}$ (where $r > m$). In the future, I'd like to talk about this more explicitly, possibly having them work through these ideas while they do the activity (particularly as they consider making the $m \times n$ table.) A potential issue to discuss more in depth (which we did mention, but only briefly) is the value of having different forms of equations – whether one form is more helpful than another depending upon the situation. Depending upon how they are written, some expressions tend to suggest different counting techniques. This would be a beneficial topic to have the students consider.

Overall, I was pleased with the activity. I was impressed with the students' counting abilities and with their capacity to articulate their arguments. This particular class was able to finish this activity without too much trouble in a single class period. However, the activity could be extended to span more than one class period if necessary, allowing the students to have more time to investigate these formulas and to consider some of the ideas mentioned above.

Introduction to Activity 2

This activity is designed to be a short warm-up to Activity 3. The goal is to have students understand and appreciate the applicability of rook problems to a wider variety of contexts. In particular, students are asked to relate rooks to a scenario of a high school dance. The students may answer the questions on their own, but they should discuss their answers in small groups at some point. While these questions are not overly complicated in and of themselves, this activity forces the students to articulate their thinking and describe the connections they make. By explicitly answering these questions, and by discussing them with their classmates, the students must demonstrate their understanding of the concepts introduced in Activity 2. The rest of the activities in this curriculum rely on students' abilities to generalize rook problems to various situations, and thus this solidifying activity is worthwhile.

<u>Activity 2</u> – <u>Rooks in the Real World</u>

In a very small school, there are fifteen boys and ten girls who want to go to the dance. These twenty-five people will not go with anyone outside of this group; every girl would be perfectly happy going with any of the boys, and vice versa. Obviously not all of the boys will get to go.

Assume that every girl attends the dance - each of them goes with one (and only one) of the boys. Recalling Activity 1, answer the following questions.

1) How does this relate to the problem of non-attacking rooks?

2) If a rook is placed on a given square, what does it mean in the context of the problem? In other words, what does the rook stand for?

3) Why rooks -- how does the movement of a rook relate to the problem?

Activity 2 – Rooks in the Real World (Teacher's Version)

In a very small school, there are fifteen boys and ten girls who want to go to the dance. These twenty-five people will not go with anyone outside of this group; every girl would be perfectly happy going with any of the boys, and vice versa. Obviously not all of the boys will get to go.

Assume that every girl attends the dance - each of them goes with one (and only one) of the boys. Recalling Activity 1, answer the following questions.

1) How does this problem relate to the problem of non-attacking rooks?
    - Rows and columns represent boys & girls, and rooks denote a pairing.

2) If a rook is placed on a given square, what does it mean in the context of the problem? In other words, what does the rook stand for?
    - It means the boy and the girl in whose column and row the rook is placed are going together to the dance. The rooks represent a date.

3) Why rooks -- how does the movement of a rook relate to the problem?
    - Once a boy and a girl go together, their 'rows and columns' are used up, as they can't then go with anyone else.

Note: The teacher should make sure they eventually model this as a rook problem, even if it has to be explicitly stated and demonstrated in front of the class. It is important for the students to understand that the rook boards can model these types of counting problems.

Reflection on Activity 2

As mentioned above, Activities 1 and 2 had been given to the students prior to our first meeting. During our first meeting, after we had discussed Activity 1, we briefly reviewed their work in Activity 2. It was immediately evident that they understood the relationship between the word problem and the rook board. One student did a great job explaining how a rook represents a date because "once a girl and a guy go together, they can’t go with anyone else." The generalization of rook problems to broader contexts seemed to come fairly naturally to the students.

Introduction to Activity 3

In this activity, the students handle a variation of the counting problem of Activity 2 which involved the high school dance. They should recognize that this problem is indeed equivalent to some rook problem, and thus the high school dance can be thought of in terms of rooks. In the first question of the activity, they are faced with the notion of a single restricted position on a chessboard. Counting configurations of rooks on a board with restricted positions brings up an important counting principle: the principle of inclusion/exclusion. This activity can serve as an introduction to, or a reminder of, this principle, depending upon how much exposure to counting principles the students have had previously. In either case, visualizing the counting problem in terms of a chessboard (regardless of the original context of the problem) potentially allows for a new way of thinking about the inclusion/exclusion principle.

In answering the first question, students should use inclusion/exclusion (or at least its most basic form: total-minus-bad) to develop a formula for handling one restricted position on a chessboard. In the second question, the students must utilize more complicated applications of this principle. In fact, they must develop two more formulas, each of which handles a special case of a board with exactly two restricted positions.

While the principle of inclusion/exclusion is not the crux of this curriculum, it is an important counting principle with which students should become familiar. It is not necessary that the students come up with this principle on their own (although some might be able to); rather, the goal is for them to gain a better understanding of the principle as they go through this activity. More than anything, this activity serves as a means of using rooks to discuss this valuable counting principle. In addition, because inclusion/exclusion increases in complexity with greater numbers of restricted positions, this activity can provide motivation for alternative counting techniques for more complicated boards.

<u>Activity 3</u> – <u>Taboo Squares</u>

Consider now a case of 25 boys and 25 girls who want to go to a dance. But now we say that one boy, Brian, and one girl, Ashley, are brother and sister, so they can't go to the dance together.

1) Develop a counting argument that counts the number of ways in which everyone can go to the dance (one boy and one girl in each date), but where Brian and Ashley are not a date.

2) Let us now consider a **<u>pair</u>** of restricted positions. In other words, develop a formula for counting the number of ways of placing r rooks on an $m \times n$ board that has 2 restricted positions.

Activity 3 – Taboo Squares (Teacher's Version)

Consider now a case of 25 boys and 25 girls who want to go to a dance. But now we say that one boy, Brian, and one girl, Ashley, are brother and sister, so they can't go to the dance together.

1) Develop a counting argument that counts the number of ways in which everyone can go to the dance (one boy and one girl in each date), but where Brian and Ashley are not a date.

Answer: In general, the counting argument is $\binom{m}{r}\binom{n}{r}r! - \binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)!$. Thus, the numerical answer should be $\binom{25}{25}\binom{25}{25}25! - \binom{24}{24}\binom{24}{24}24! = 25! - 24!$.

Notes

- The students should model this as a board with one restricted square
- This problem will just barely begin to introduce the inclusion/exclusion principle (total minus bad).

2) Let us now consider a **pair** of restricted positions. In other words, develop a formula for counting the number of ways of placing *r* rooks on an $m \times n$ board that has 2 restricted positions.

Answer: This breaks down into two cases:

When the forbidden positions were in distinct rows and columns, the general formula is
$$\binom{m}{r}\binom{n}{r}r! - 2\binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)! + \binom{m-2}{r-2}\binom{n-2}{r-2}(r-2)!.$$

When they were in the same row (or column), the formula is
$$\binom{m}{r}\binom{n}{r}r! - 2\binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)!.$$

Notes:

- This second problem will rely heavily on the principle of inclusion/exclusion.
- When the students are explaining their formulas on the board, the teacher should have the students explain each term in the formula with a diagram.
- In explaining the differences between the two cases, it's helpful to note that the final term in the formula for the first case is necessary because we had double counted cases in which rooks were in both restricted positions. But in the second case, we will *never* have rooks in both restricted positions, so this isn't even a possibility (and doesn't need to be accounted for).

Reflection on Activity 3

The students worked on this activity during the first day we visited the class. We had just gotten finished discussing Activities 1 and 2, and we simply handed out Activity 3 for them to work on. We gave no mention of the principle of inclusion/exclusion; we were curious about what they could come up with on their own. They first worked in groups of two or three on Question 1, and it didn't take very long for the groups to deal with the issue of a single restricted position. After some time we reconvened as an entire class to discuss the results. One student explained his work to the class, and he had used the notion of the "total minus the bad," which is precisely the idea behind inclusion/exclusion. Indeed, before we returned to the big group, there were at least four groups with solutions that were more or less equivalent to this "total minus the bad" result. Admittedly, some students had been explaining their solutions to others, but by the time we had the one student share out, most everyone had a grasp of how to count this. I wrote the general formula for this on the board, and students generally seemed to understand how we'd obtained it.

Next, the students worked in groups of two or three on Question 2. We had given them a concrete example in our original statement of Question 2, (using numbers in an effort to make things easier), but this proved to be unnecessary. Ultimately they arrived at the answer in a general form, and one group had even cleverly made the variables a little more colorful. They continued to work for a while, and again they realized the need for inclusion/exclusion on their own.

An interesting issue that arose was that the students only considered the case in which the restricted positions are in distinct rows and columns (this was likely due to the dance aspect of the problem). In reality, however, *two* cases of restricted position must be considered: one in which the restricted positions are in distinct rows and columns, and another in which they are in a common row or column. Not surprisingly, with a little bit of pushing they came up with the case breakdown themselves. In fact, when we pushed them to consider the *other* case of 2 restricted positions, a student quickly offered a case where Brian has 2 sisters – where the rooks are in the same row or column. I was pleased that this contextualization was suggested, relating the issue to the context of the dance. We then had them work on this second case, and it seemed like even the kids who typically got the answers right were not entirely sure how to count this. Eventually, though, they reasoned through it and presented it. We finished with a brief explanation of this which explained why the formulas were different (which was mentioned in Activity 3's Teacher's Version).

When two students presented their solution to Question 2, some of their classmates were not entirely satisfied with their explanation. We pushed them to draw diagrams to go along with the terms in their formula, and this proved to be helpful. Other students contributed to these diagrams as well, and we ultimately ended up with very satisfying diagrams that explained the formula. We ended the activity with a quick comment that they could naturally wonder what happens for three, four, or more restricted positions.

On the whole I was very happy about how things went. It should be noted that these students had seen the principle of inclusion/exclusion before, although I was somewhat surprised that they were so comfortable with using it. Their ability to think through and apply this principle in the context of rooks was impressive. At the very least, this activity served the purpose of reinforcing old principles (like counting principles, inclusion/exclusion) and setting the stage for some more exciting math (like generating functions and matchings)!

The activity would have gone quite differently had we presented these problems to students who had never encountered inclusion/exclusion before. It would not be better or worse; rather, I anticipate that it would require nothing more than a slight change in emphasis. In such a setting, I do believe that rooks could be used effectively to teach these principles to students for whom these ideas would be new.

Introduction to Review Worksheet 1

After students have worked through Activity 3, they are ready to handle Review Worksheet 1. The design of this first assessment is to provide fun problems in an effort to engage the students with their newly acquired tools, namely the three formulas they had developed in Activities 1-3. Specifically, in Part A of the worksheet, the students first revisit the basic formula that they derived for counting the number of ways of placing r rooks on an $m \times r$ chessboard; a couple of interesting contexts related to this formula are presented. In Part B, by studying a detailed situation involving a high school dance, the students work through problems that involve one restricted position. And in Part C, they consider the case of two restricted positions within this same context. Again, no new concepts are being taught here; this is designed to help students collect and unify their thoughts up to this point.

Assessment 1– Review Worksheet

Now that you have had some exposure to questions about rooks (and some real-life problems they model), let's review some of what we've learned.

A. Recall that the number of ways of placing r rooks on an $m \times n$ board (with no restrictions) is given by the beautiful expression:

$$\binom{m}{r}\binom{n}{r}r!$$

Use this formula to answer the following questions.

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1) Suppose there are 12 puppies at the pound, and 8 kids who want to adopt them. How many ways could 5 of the puppies get paired up with 5 of the kids?

2) Suppose you have 8 rooks and a chessboard with 8 rows. How many columns must your chessboard have, in order for the number of non-attacking configurations of your 8 rooks to exceed a trillion? (Note: 1 trillion = 10^{12}.)

3) Suppose you have a chessboard with 4 rows and 6 columns. What number of rooks gives the highest number of non-attacking configurations? Is it always true that "more rooks" means "more non-attacking configurations"? Either explain or give a counterexample.

B. Recall, too, that we found a formula for the number of ways of placing r rooks on an $m \times n$ board with **<u>one</u>** restricted position. The fabulous formula was:

$$\binom{m}{r}\binom{n}{r}r! - \binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)!$$

Use this formula to answer the following (very realistic) questions.

1) In a (small) senior class of 30 guys and 36 girls, we need to decide on a Prom King and Queen. The senior class must choose 5 couples, and then the entire school votes from among these final couples. But <u>Kyle and Bethany refuse to be paired up with each other</u>. How many different ways could the senior class come up with 5 "acceptable" final couples for the school to vote on, given this constraint?

2) The ever-popular math teacher, "Dr. G," told his class not to worry so much about Kyle and Bethany, because it wasn't very likely that they'd be paired up anyways. To see just how right he was, figure out what **percentage** of the total # of pairings of 5 couples are actually "acceptable," given the constraint.

3) After Dr. G's particularly difficult math exam, 1/2 of the guys and 1/2 of the girls suffered "severe" drops in their grades, making them ineligible for Prom King and Queen. Now that the eligible pool is down to 15 guys and 18 girls, what percentage of the total # of pairings of 5 couples are "acceptable"?

C. Finally, recall that we found 2 different formulas for counting the number of ways of placing r rooks on an $m \times n$ board with **<u>two</u>** restricted positions. Which formula to use depends on how the restricted positions are arranged.

- When the 2 forbidden positions are in distinct rows and columns, our formula is:

$$\binom{m}{r}\binom{n}{r}r! - 2\binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)! + \binom{m-2}{r-2}\binom{n-2}{r-2}(r-2)!$$

- When they are in the same row (or column), our formula is the simpler:

$$\binom{m}{r}\binom{n}{r}r! - 2\binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)!$$

Use these formulas to answer the following questions.

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1) Suppose that a pairing of Jeremy and Laura is impossible, **and** Kyle and Bethany still refuse to be paired up, **and** there are only 15 guys and 18 girls. How many matchings of 5 couples are we down to now?

2) If Kyle and Bethany resolve their issues, but if Jeremy somehow ruins his chances with both Laura AND Bethany, then how many matchings of 5 couples do we have?

<u>Assessment 1</u>– <u>Review Worksheet (Teacher's Version)</u>

Now that you have had some exposure to questions about rooks (and some real-life problems they model), let's review some of what we've learned.

A. Recall that the number of ways of placing *r* rooks on an $m \times n$ board (with no restrictions) is given by the beautiful expression:

$$\binom{m}{r}\binom{n}{r}r!$$

Use this formula to answer the following questions.

---

1) Suppose there are 12 puppies at the pound, and 8 kids who want to adopt them. How many ways could 5 of the puppies get paired up with 5 of the kids?

<u>Answer</u>: $\binom{12}{5}\binom{8}{5}5!$ = 5322240

2) Suppose you have 8 rooks and a chessboard with 8 rows. How many columns must your chessboard have, in order for the number of non-attacking configurations of your 8 rooks to exceed a trillion? (Note: 1 trillion = 10^{12}.)

<u>Answer</u>: Since $\binom{8}{8}$ is the 'rows term' in the formula, we're looking for n such that $\binom{n}{8} \cdot 8! > 1{,}000{,}000{,}000{,}000$. A little trial and error shows that the number of columns must be 36 to exceed a trillion. (Note: 18 columns would exceed a billion.)

3) Suppose you have a chessboard with 4 rows and 6 columns. What number of rooks gives the highest number of non-attacking configurations? Is it always true that "more rooks" means "more non-attacking configurations"? Either explain or give a counterexample.

<u>Answer</u>: This problem shows that placing 4 rooks (the maximum number) doesn't necessarily result in the most possibilities for configurations – in this case 3 rooks results in more possibilities (480 total) than 4 rooks (360 total).

B. Recall, too, that we found a formula for the number of ways of placing r rooks on an $m \times n$ board with **one** restricted position. The fabulous formula was:

$$\binom{m}{r}\binom{n}{r}r! - \binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)!$$

Use this formula to answer the following (very realistic) questions.

1) In a (small) senior class of 30 guys and 36 girls, we need to decide on a Prom King and Queen. The senior class must choose 5 couples, and then the entire school votes from among these final couples. But <u>Kyle and Bethany refuse to be paired up with each other</u>. How many different ways could the senior class come up with 5 "acceptable" final couples for the school to vote on, given this constraint?

<u>Answer</u>: $\binom{30}{5}\binom{36}{5}5! - \binom{29}{4}\binom{35}{4}4! = 6{,}416{,}988{,}177{,}600$

2) The ever-popular math teacher, "Dr. G," told his class not to worry so much about Kyle and Bethany, because it wasn't very likely that they'd be paired up anyways. To see just how right he was, figure out what **percentage** of the total # of pairings of 5 couples are actually "acceptable," given the constraint.

<u>Answer</u>: $$\frac{\binom{30}{5}\binom{36}{5}5! - \binom{29}{4}\binom{35}{4}4!}{\binom{30}{5}\binom{36}{5}5!} = \frac{6{,}416{,}988{,}177{,}600}{6{,}446{,}834{,}634{,}240} \approx 99.5\%$$

3) After Dr. G's particularly difficult math exam, 1/2 of the guys and 1/2 of the girls suffered "severe" drops in their grades, making them ineligible for Prom King and Queen. Now that the eligible pool is down to 15 guys and 18 girls, what percentage of the total # of pairings of 5 couples are "acceptable" (Kyle-and-Bethany free)?

<u>Answer</u>: $$\frac{\binom{15}{5}\binom{18}{5}5! - \binom{14}{4}\binom{17}{4}4!}{\binom{15}{5}\binom{18}{5}5!} = \frac{3{,}030{,}387{,}360}{3{,}087{,}564{,}480} \approx 98.1\%$$

C. Finally, recall that we found 2 different formulas for counting the number of ways of placing r rooks on an $m \times n$ board with **<u>two</u>** restricted positions. Which formula to use depends on how the restricted positions are arranged.

- When the 2 forbidden positions are in distinct rows and columns, our formula is:

$$\binom{m}{r}\binom{n}{r}r! - 2\binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)! + \binom{m-2}{r-2}\binom{n-2}{r-2}(r-2)!$$

- When they are in the same row (or column), our formula is the simpler:

$$\binom{m}{r}\binom{n}{r}r! - 2\binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)!$$

Use these formulas to answer the following questions.

1) Suppose that a pairing of Jeremy and Laura is impossible, **and** Kyle and Bethany still refuse to be paired up, **and** there are only 15 guys and 18 girls. How many matchings of 5 couples are we down to now?

<u>Answer</u>: $\binom{15}{5}\binom{18}{5}5! - 2\binom{14}{4}\binom{17}{4}4! + \binom{13}{3}\binom{16}{3}3! =$ 2,974,171,200

2) If Kyle and Bethany resolve their issues, but if Jeremy somehow ruins his chances with both Laura AND Bethany, then how many matchings of 5 couples do we have?

<u>Answer</u>: $\binom{15}{5}\binom{18}{5}5! - 2\binom{14}{4}\binom{17}{4}4! =$ 2,973,210,240

Introduction to Activity 4

In this activity, students get more practice applying the formulas they found for counting boards with restricted positions; certainly Part A consists solely of such computation. They should be comfortable with using these formulas by now, especially having completed Review Worksheet 1. The primary goal of this activity, however, rests in Part B. The large board they are asked to count really consists of the two sub-boards they computed in Part A. The students are thus given the tools they need to count this board, even though it is a larger board than they have previously dealt with and consists of many restricted positions.

The students should be given this activity without much advice or explanation; it is intended to be exploratory in nature. Again, they should work in small groups and discuss their processes and ideas with their classmates. While it is not particularly long or involved, the intent is that their investigation will lead them to discover a general rule for counting a large board that consists of two disjoint sub-boards. Ideally, they will recognize that counting the number of ways of placing r rooks on a large board involves several cases: taking 0 rooks from one sub-board and r rooks from the other, or 1 rook from the first sub-board and $r - 1$ from the other, etc. Even if they cannot express these concepts in such specific terms, they should begin to formulate thoughts that correspond to these ideas.

Activity 4 – Bored with Boards Yet?

Part A

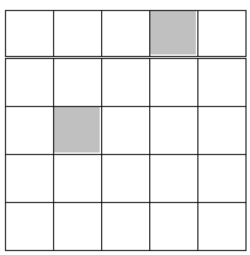
Putting the drama of high school Proms aside, we gladly return to the safe world of counting rooks on chessboards. As nice as our formulas are, we're still not quite to the big picture yet. So let us begin by considering the following two 5×5 boards, where a gray box represents a restricted position. Recall from Activity 3 the formulas we developed for computing the number of ways of placing r rooks on an $m \times n$ board where:

- The two restricted positions are in distinct rows and columns

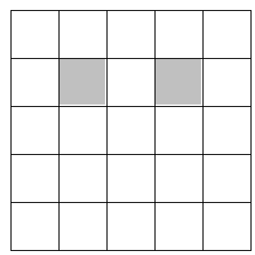
$$\binom{m}{r}\binom{n}{r}r! - 2\binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)! + \binom{m-2}{r-2}\binom{n-2}{r-2}(r-2)!$$

- The two restricted positions share a row or column

$$\binom{m}{r}\binom{n}{r}r! - 2\binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)!$$



B1



B2

For each board B1 and B2, use the above formulas to complete the following tables.

# of non-attacking rooks on B1	0	1	2	3	4	5
# of ways of placing these rooks on B1						

# of non-attacking rooks on B2	0	1	2	3	4	5
# of ways of placing these rooks on B2						

<u>Activity 4</u> – <u>How About Now?</u>

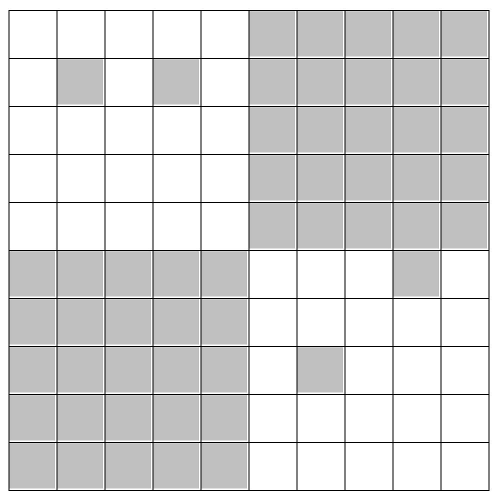
<u>Part B</u>

Now we present you with the following 10×10 board, where again gray boxes represent restricted positions.

1) How many ways are there to place 7 rooks on the given 10×10 board?

2) What process did you go through in order to solve this problem?

3) Can you extract any general principles from the way you worked through this problem?



Activity 4 – Bored with Boards Yet? (Teacher's Version)

Part A

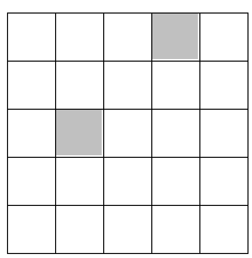
Putting the drama of high school Proms aside, we gladly return to the safe world of counting rooks on chessboards. As nice as our formulas are, we still don't see the big picture yet. So let us begin by considering the following two 5×5 boards, where a gray box represents a restricted position. Recall from Activity 3 the formulas we developed for computing the number of ways of placing r rooks on an $m \times n$ board where:

- The two restricted positions are in distinct rows and columns

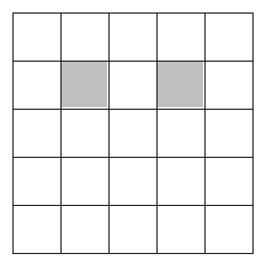
$$\binom{m}{r}\binom{n}{r}r! - 2\binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)! + \binom{m-2}{r-2}\binom{n-2}{r-2}(r-2)!$$

- The two restricted positions share a row or column

$$\binom{m}{r}\binom{n}{r}r! - 2\binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)!$$



B1



B2

For each board B1 and B2, use the above formulas to complete the following tables.

Answer: $\binom{m}{r}\binom{n}{r}r! - 2\binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)! + \binom{m-2}{r-2}\binom{n-2}{r-2}(r-2)!$

Let *m*,*n* = 5, and have *r* range from 0 to 5, and we arrive at the following completed table.

# of non-attacking rooks on B1	0	1	2	3	4	5
# of ways of placing these rooks on B1	1	23	169	465	426	78

Answer: $\binom{m}{r}\binom{n}{r}r! - 2\binom{m-1}{r-1}\binom{n-1}{r-1}(r-1)!$

Let *m*,*n* = 5, and have *r* range from 0 to 5, and we arrive at the following completed table.

# of non-attacking rooks on B2	0	1	2	3	4	5
# of ways of placing these rooks on B2	1	23	168	456	408	72

<u>Activity 4</u> – <u>How About Now? (Teacher's Version)</u>

<u>Part B</u>

Now we present you with the following 10×10 board, where again gray boxes represent restricted positions.

1) How many ways are there to place 7 rooks on the given 10×10 board?

<u>Answer</u>: $169 \cdot 72 + 465 \cdot 408 + 426 \cdot 456 + 78 \cdot 168 =$ 409248. The reflection on this activity gives further insight into methods for solving this problem.

2) What process did you go through in order to solve this problem?

<u>Answer</u>: They should have recognized that the large 10×10 board is exactly made up of the two smaller boards we computed above. Thus the way of placing 7 rooks on this board can be thought of as four distinct cases: putting 5 on the first board and 2 on the second, 4 and 3, 3 and 4, and 2 and 5. Each of these cases uses the multiplication principle, and we add the four cases together, which is how we got the result above. We hope they'll use the tables they just computed in order to find the answer.

3) Can you extract any general principles from the way this problem worked for you?

<u>Answer</u>: Hopefully they can begin to see that this sort of case breakdown would generalize to other boards with similarly disjoint sub-boards. Ultimately we're pushing a relationship between this and polynomial multiplication, but we don't expect them to make this connection yet. We don't want to push anything yet; it's good if they can just generally reflect upon how they found their result.

Reflection on Activity 4

The students were given Activity 4 to work on between our visits, almost as a pre-activity to our second visit. When we arrived for this second visit, we began the day by asking them to reflect upon Activity 4. They had recorded the number of ways of placing rooks on each of the two boards in the provided table, and, in turn, had proceeded to use these tables to count the number of ways of placing r rooks on the disjoint union of the two boards. One student very succinctly and eloquently explained how he'd approached the problem – which was exactly the type of answer I was hoping to hear. He had done what we'd aimed for, namely, used the two smaller boards to count the larger one; in fact his explanation included a nice description of the convolution of the two sequences associated with each board. Other students seemed to indicate that this was the approach they took as well. We asked whether anyone had tried counting the large board directly, and one student said it had been quite a bit harder than the convolution that the first student had described.

For the students, this activity might have seemed a bit trivial; many of them seemed to understand it fairly naturally. From our perspective, however, this was a hugely important step in the quest toward generating functions. This activity was a success because the students essentially recognized the idea of convolution of sequences, even if they weren't aware of the mathematical implications of what they were saying. Although sequence convolution is not yet being introduced explicitly, each subtle exposure to this idea is significant in the development of generating functions.

Introduction to Activity 5

This is a fairly long activity, among the most involved of anything the students will work on. Essentially, by the end of this activity, the students will have the tools necessary to solve any rook board they will ever encounter (it is remarkable that three relatively simple principles allow for this to be the case). However, in this activity the students will also be exposed to generating functions for the first time. This is one of the most important concepts in the entire curriculum, and it will take time and effort to convey this idea properly. Thus, there is an abundance of material for the students to engage with in this activity, and all of it contains extremely relevant combinatorial ideas.

The structure of the activity is as follows. The students will study the three principles (called "Rook Rules") that allow for any rook board to be reduced and counted: the Disjoint Board principle, the Use/Don't Use principle, and Switcheroo. In Part A, these rules are introduced solely on the basis of counting, while in Part B generating functions are introduced. Although the computations in Part A can be a bit unwieldy, the familiar ideas should come relatively naturally. It is unlikely, however, that the students will be able to develop each of these entirely on their own. A whole-class discussion of these ideas (an interactive lecture) might be a valuable approach in conveying these concepts. There are some practice problems throughout the activity for the students to work on (in the midst of such a whole class discussion) in order to confirm the ideas that are being taught. The students should appreciate the fact that these three Rook Rules enable them to reduce and count any rook board they may encounter.

In Part B, when generating functions are introduced, the analogs of these Rook Rules are presented – this time employing the idea of generating functions. This is a topic of enormous weight, and care should be taken in raising it. Again, an interactive lecture is a recommended means of communicating these ideas. Because of the subtle attention given to sequence convolution in Activity 4, the introduction to generating functions appears remarkably well-motivated. The students are primed for such a discovery. After having gone through the more cumbersome counting versions of the Rook Rules, the generating function versions should be a welcome relief.

This entire activity really serves to motivate an understanding of generating functions. There is a lot of material, but it is imperative that students understand it before moving on to subsequent activities. Because of the power that generating functions hold in other mathematical contexts, they are one of the most central topics of the entire curriculum, and they should be treated as such.

Activity 5 – **Rook Kung-Fu**

Part A

Goal: We want to establish a set of rules that will allow us to count ANY rook board we could ever encounter: the Kung-Fu of rook problems.

Notation: Given a board B, the number of ways of placing *r* non-attacking rooks on B will be denoted by $n_r(B)$. From now on, when we refer to rooks, we'll assume we are talking about non-attacking rooks.

Rook Rule #1: (**Disjoint Boards**) If a board C consists of two sub-boards A and B that do not overlap in any rows or columns, then

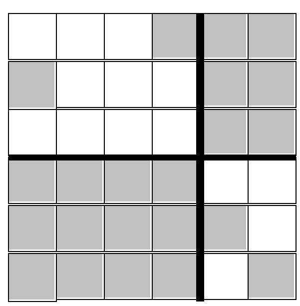
$$n_r(C) = n_r(A)n_0(B) + n_{r-1}(A)n_1(B) + \cdots + n_0(A)n_r(B)$$

---

1) The following tables describe the number of ways of placing *r* rooks on the two disjoint sub-boards (A, B) of the board C below. We used our restricted position formulas to construct these tables. (Note: let A be the sub-board on the 'upper left' and B be the sub-board on the 'lower right.')

R	0	1	2	3
$n_r(A)$	1	10	25	14

R	0	1	2	3
$n_r(B)$	1	4	3	0



C

Complete the following table using the Disjoint Board principle.

r	0	1	2	3	4	5	6
$n_r(C)$							

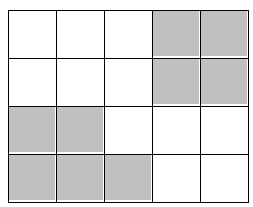
Rook Rule #2: (**Use / Don't Use**) If the *i*,*j*-square S of a board C is not a forbidden square, then

$$n_r(C) = n_{r-1}(C_1) + n_r(C_2),$$

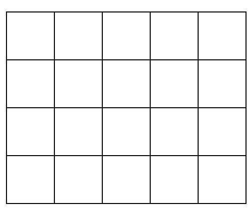
where C_1 is the board formed when we *use* S (and remove the i^{th} row and j^{th} column), and C_2 is the board formed when we *don't use* S (and S becomes a forbidden square).

(Note: it would make sense to define $n_{-1}(C) = 0$ for any board C.)

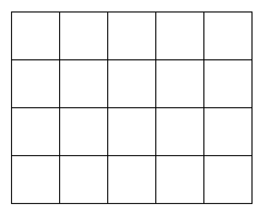
2) Select a square for S in the board B below which (after applying Rook Rule #2) will reduce the given board B to two simple rectangles. Shade in the restricted squares for your new boards B_1 and B_2, where B_1 is the board obtained after we *use* S, and B_2 is the board obtained after we *don't use* S. You do not need to reduce the board further or count the number of configurations.



B



B_1 - *use* S



B_2 - *don't use* S

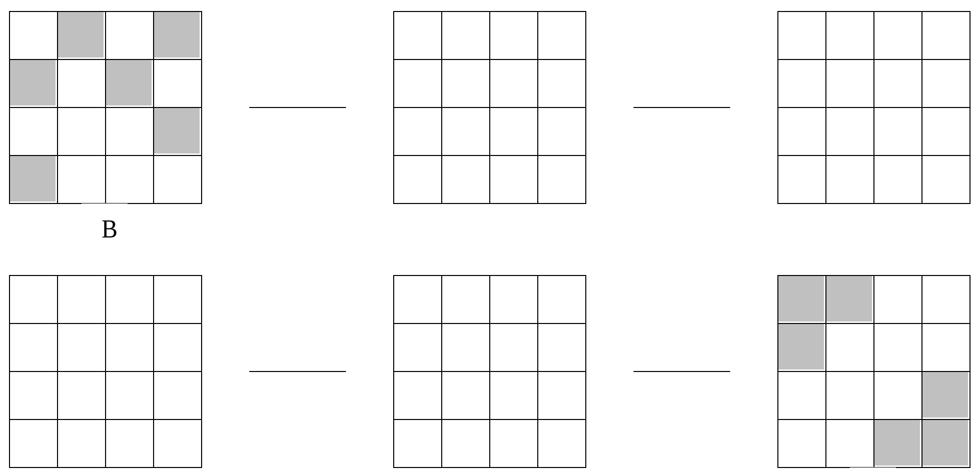
If we wanted to place *r* rooks on board B, how many rooks do we place on B_1 and B_2 respectively?

Rook Rule #3: (**Switcheroo**) Suppose a board B can be obtained from another board C simply by permuting rows and/or columns. Then for any integer r,

$$n_r(B) = n_r(C)$$

In other words, we can swap rows and columns without affecting the outcome.

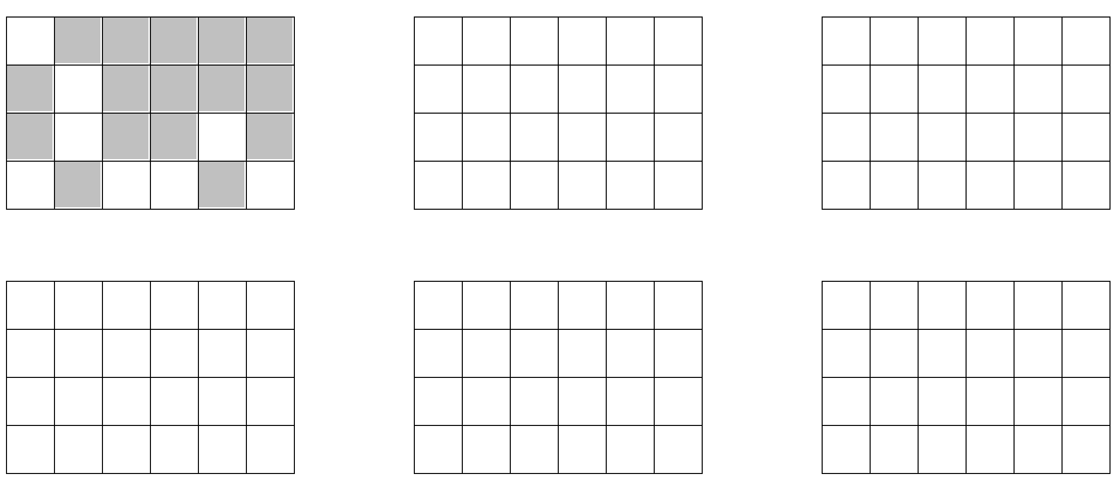
3) For the following boards B and C, perform a series of row and/or column switches to transform one into the other. You may use the empty boards to draw the intermediate stages.
Are you satisfied that we can switch rows and columns without changing the counting problem? Why or why not?



B

C

4) Since we know how to count disjoint sub-boards, transform the board below into a board containing two disjoint sub-boards.



Activity 5 – Rook Kung Fu

Part B

Notation: By the rook polynomial of a board B, we mean the polynomial

$$R(B,x) = \sum_{r\geq 0} n_r(B)x^r \ .$$

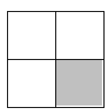
The rook polynomial of a board is a generating function where the coefficient of the r^{th} term is $n_r(B)$, the number of ways of placing *r* rooks on a chessboard B.

Rook Rule #1: **Disjoint Boards (polynomial version)** If a board C consists of two sub-boards A and B that do not overlap any rows or columns, then

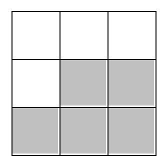
$$R(C,x) = R(A,x)R(B,x)\,.$$

---

5) Use rook polynomials Rook Rule #1 to complete the following problem. The rook polynomial of the boards *A* and *B* are given below.



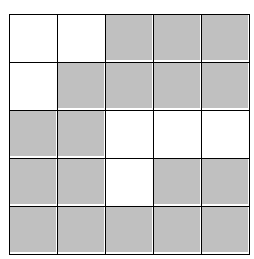
A



B

$R(A,x) = 1 + 3x + x^2$

$R(B,x) = 1 + 4x + 2x^2$



C

Then board C is simply the disjoint union of boards A and B.
According to Rook Rule #1, give the rook polynomial for board C:

$R(C,x) =$

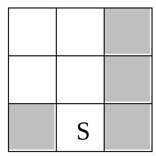
If we count the board C by hand, we arrive at the same result!

Rook Rule #2: **Use/Don't Use (polynomial version)** If the *i*,*j*-square S of a board C is not a forbidden square, then

$$R(C,x) = xR(C_1,x) + R(C_2,x),$$

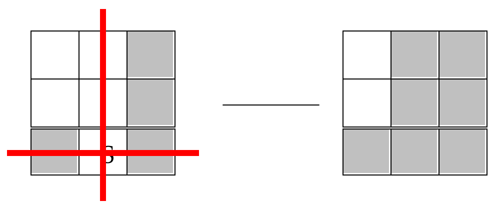
Where C_1 is the board formed when we *use* S (and forbid the i^{th} row and j^{th} column), and C_2 is the board formed when we *don't use* S (and S becomes a forbidden square).

6) Use rook polynomials and Rook Rule #2 to complete the following problem.



C

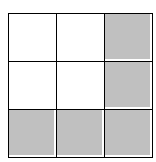
If we *use* square S, then we forbid the row and columns of S, and we get the following board C_1, whose rook polynomial is given below.



C_1

$R(C_1,x) = 1 + 2x$

If we *don't use* square S, then we get the following board C_2, whose rook polynomial is given below.



C_2

$R(C_2,x) = 1 + 4x + 2x^2$

Use Rook Rule #2, to write the rook polynomial of board C.

$R(C,x) = xR(C_1,x) + R(C_2,x) =$

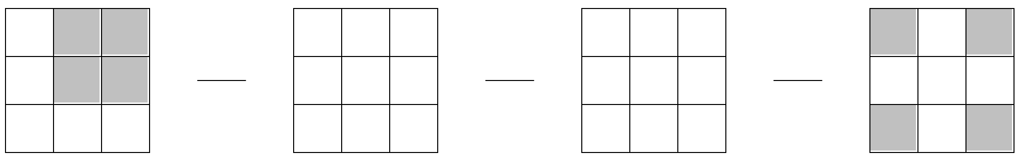
Again, counting board C directly yields the same result.

Rook Rule #3: **Switcheroo (polynomial version)** Suppose a board B can be obtained from another board C simply by permuting rows and/or columns. Then

$$R(B,x) = R(C,x)$$

In other words, we can swap rows and columns without affecting the outcome.

7) Transform board B into board C by switching rows and columns. The rook polynomial of board B is given.



B C

$R(B,x) = 1 + 5x + 4x^2$

According to Rook Rule #3, what is the rook polynomial of C?

$R(C,x) =$

Use counting methods to obtain $R(C,x)$ to confirm that you obtain the expected result.

Activity 5 – Rook Kung Fu (Teacher's Version)

Part A

Preliminary Notes: The teacher could begin the class by saying something like:

"Recall that we've developed formulas for counting the number of ways of placing r rooks on boards with 0, 1, and 2 restricted positions. We used some basic counting principles, (including the principle of inclusion/exclusion) to derive these formulas. It turns out, though, that applying the principle of inclusion/exclusion to increasingly complex boards gets *ridiculously* hard pretty quickly. Fortunately, we can continue to use some basic counting principles in order to simplify (and then count) more intricate boards.

In Activity 4 you made tables for two boards (using formulas we had derived), and then you figured out a way to place 7 rooks on the disjoint union of these two boards. In doing this problem, did you develop a conjecture about the general way to count such boards? What must be true of these boards in order to make your conjecture hold? How do you know? How do counting principles relate to this?"

Goal: We want to establish a set of rules that will allow us to count ANY rook board we could ever encounter: the Kung-Fu of rook problems.

Notation: Given a board B, the number of ways of placing r non-attacking rooks on B will be denoted by $n_r(B)$. From now on, when we refer to rooks, we'll assume we are talking about non-attacking rooks.

Rook Rule #1: (**Disjoint Boards**) If a board C consists of two sub-boards A and B that do not overlap in any rows or columns, then

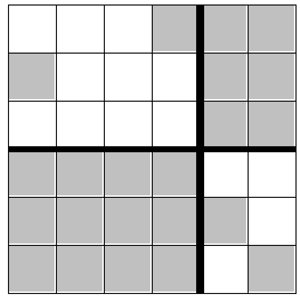
$$n_r(C) = n_r(A)n_0(B) + n_{r-1}(A)n_1(B) + \cdots + n_0(A)n_r(B)$$

---

1) The following tables describe the number of ways of placing r rooks on the two disjoint sub-boards (A, B) of the board C below. We used our restricted position formulas to construct these tables. (Note: let A be the sub-board on the 'upper left' and B be the sub-board on the 'lower right.')

r	0	1	2	3
$n_r(A)$	1	10	25	14

r	0	1	2	3
$n_r(B)$	1	4	3	0



C

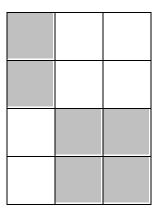
Complete the following table using the Disjoint Board principle.

Answer: By applying Rook Rule #1, the students should arrive at the following table.

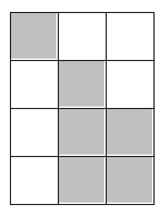
R	0	1	2	3	4	5	6
$n_r(C)$	1	14	68	144	131	42	0

Notes:

- Emphasize what "disjoint" means on the blackboard. For this Rook Rule #1 to hold, the boards must be completely disjoint; they cannot overlap in any row or column. For instance, below are two boards A and B. A consists of two disjoint sub-boards, but B does not. This is an important distinction for your students to recognize.



A



B

- Also, while Rook Rule #1 is stated for *two* disjoint sub-boards, it holds true for any number of disjoint sub-boards. This knowledge will benefit the students later. It might be worthwhile to ask them if they think the rule generalizes and then have them explain their reasoning.

Rook Rule #2: (**Use / Don't Use**) If the *i*,*j*-square S of a board C is not a forbidden square, then

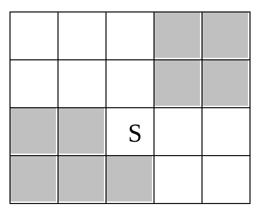
$$n_r(C) = n_{r-1}(C_1) + n_r(C_2)\,,$$

where C_1 is the board formed when we *use* S (and remove the i^{th} row and j^{th} column), and C_2 is the board formed when we *don't use* S (and S becomes a forbidden square).

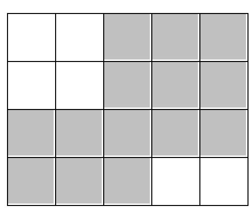
(Note: it would make sense to define $n_{-1}(C) = 0$ for any board C.)

2) Select a square for S in the board *B* below which (after applying Rook Rule #2) will reduce the given board B to two simple rectangles. Shade in the restricted squares for your new boards B_1 and B_2, where B_1 is the board obtained after we *use* S, and B_2 is the board obtained after we *don't use* S. You do not need to reduce the board further or count the number of configurations.

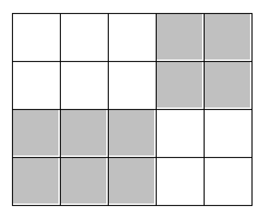
Answer: We show the desired square S and the resulting reduced boards below.



B



B_1 - *use* S



B_2 - *don't use* S

If we wanted to place *r* rooks on board *B*, how many rooks do we place on B_1 and B_2 respectively?

Answer: We seek to place $r-1$ rooks on B_1 and to place *r* rooks on B_2.

Notes: See transparency at the end of this activity for a complete example of this rule.

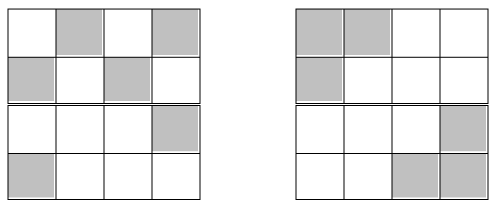
Rook Rule #3: (**Switcheroo**) Given any board B, and any board C that can be obtained from B merely by permuting rows and/or columns of C, we have

$$n_r(B) = n_r(C)$$

In other words, we can swap rows and columns without affecting the outcome.

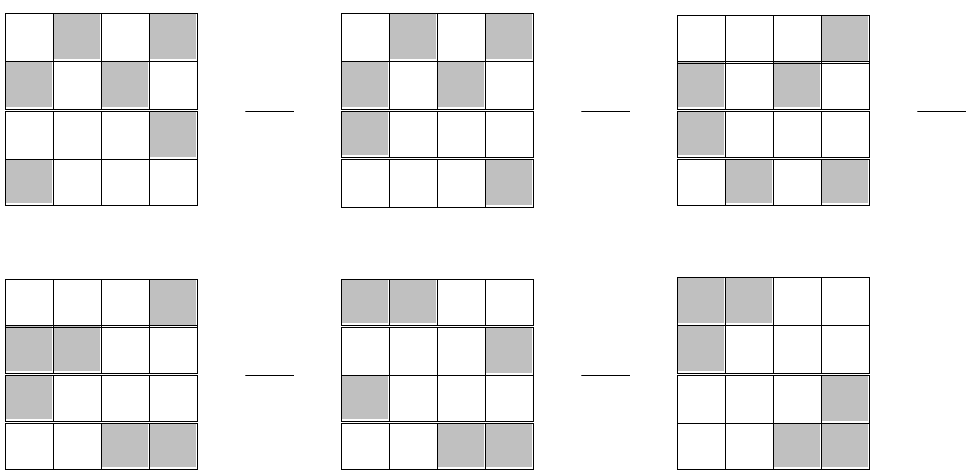
Notes: In these exercises, hopefully the students will recognize the fact that they can switch multiple rows and columns as well. For instance, they could interchange a set of two rows with a single row. This should make sense to them.

1) For the following boards B and C, perform a series of row and/or column switches to transform one into the other. Are you satisfied that we can switch rows and columns without changing the counting problem? Why or why not?



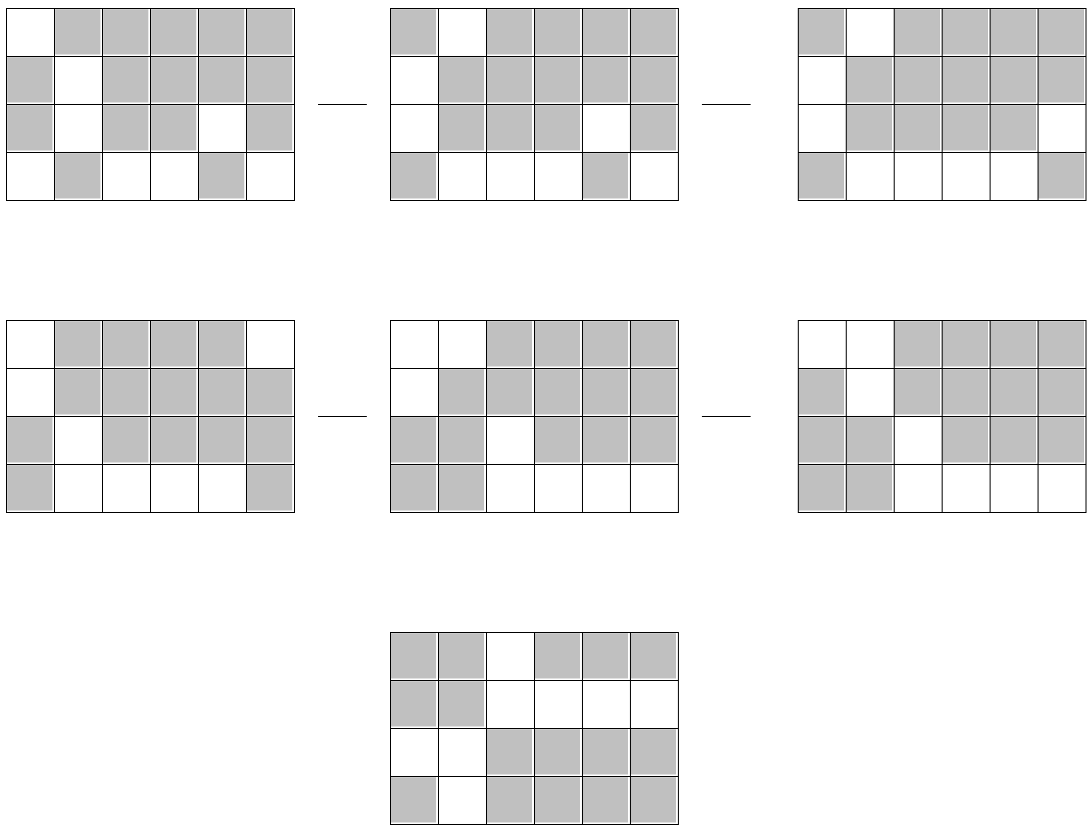
B C

Answer: This is one option of a scenario for transforming B into C.



2) Since we know how to count disjoint boards, deform the board B below into a board containing two disjoint sub-boards.

Answer: They should deform it to something like the board on the right, although there are other options as well.



Activity 5 – Rook Kung Fu (Teacher's Version)

Part B

Notation: By the rook polynomial of a board B, we mean the polynomial

$$R(B,x)=\sum_{r\geq 0} n_r(B)x^r$$

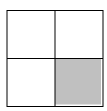
The rook polynomial of a board is a generating function where the coefficient of the r^{th} term is $n_r(B)$, the number of ways of placing r rooks on a chessboard B.

Rook Rule #1: **Disjoint Boards (polynomial version)** If a board C consists of two sub-boards A and B that do not overlap any rows or columns, then

$$R(C,x)=R(A,x)R(B,x)$$

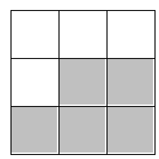
---

3) Use rook polynomials Rook Rule #1 to complete the following problem. The rook polynomial of the boards *A* and *B* are given below.



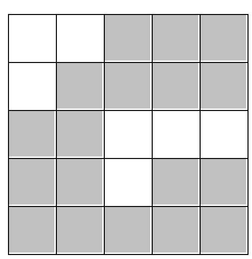
A

$R(A,x)=1+3x+x^2$



B

$R(B,x)=1+4x+2x^2$



C

Then board C is simply the disjoint union of boards A and B.
According to Rook Rule #1, give the rook polynomial for board C:
$R(C,x)=R(A,x)R(B,x)=(1+3x+x^2)\cdot(1+4x+2x^2)=1+7x+15x^2+10x^3+2x^4$

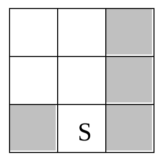
If we count the board C by hand, we arrive at the same result!

Rook Rule #2: **Use/Don't Use (polynomial version)** If the *i,j*-square S of a board C is not a forbidden square, then

$$R(C,x) = xR(C_1,x) + R(C_2,x),$$

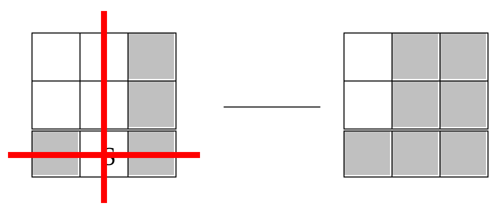
where C_1 is the board formed when we *use* S (and remove the i^{th} row and j^{th} column), and C_2 is the board formed when we *don't use* S (and S becomes a forbidden square).

4) Use rook polynomials and Rook Rule #2 to complete the following problem.



C

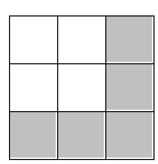
If we *use* square S, then we get the following board C_1, whose rook polynomial is given below.



C_1

$R(C_1,x) = 1 + 2x$

If we *don't use* square S, then we get the following board C_2, whose rook polynomial is given below.



C_2

$R(C_2,x) = 1 + 4x + 2x^2$

Use Rook Rule #2, to write the rook polynomial of board C.

$R(C,x) = xR(_1,x) + R(_2,x) = x(1+2x) + (1+4x+2x^2) = 1+5x+4x^2$

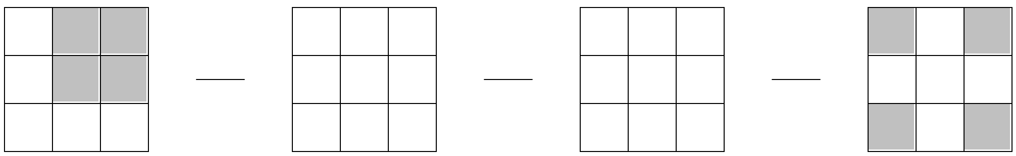
Again, counting board C directly yields the same result.

Rook Rule #3: **Switcheroo (polynomial version)** Suppose a board B can be obtained from another board C simply by permuting rows and/or columns. Then

$$R(B, x) = R(C, x)$$

In other words, we can swap rows and columns without affecting the outcome.

5) Transform board B into board C by switching rows and columns. The rook polynomial of board B is given.



B C

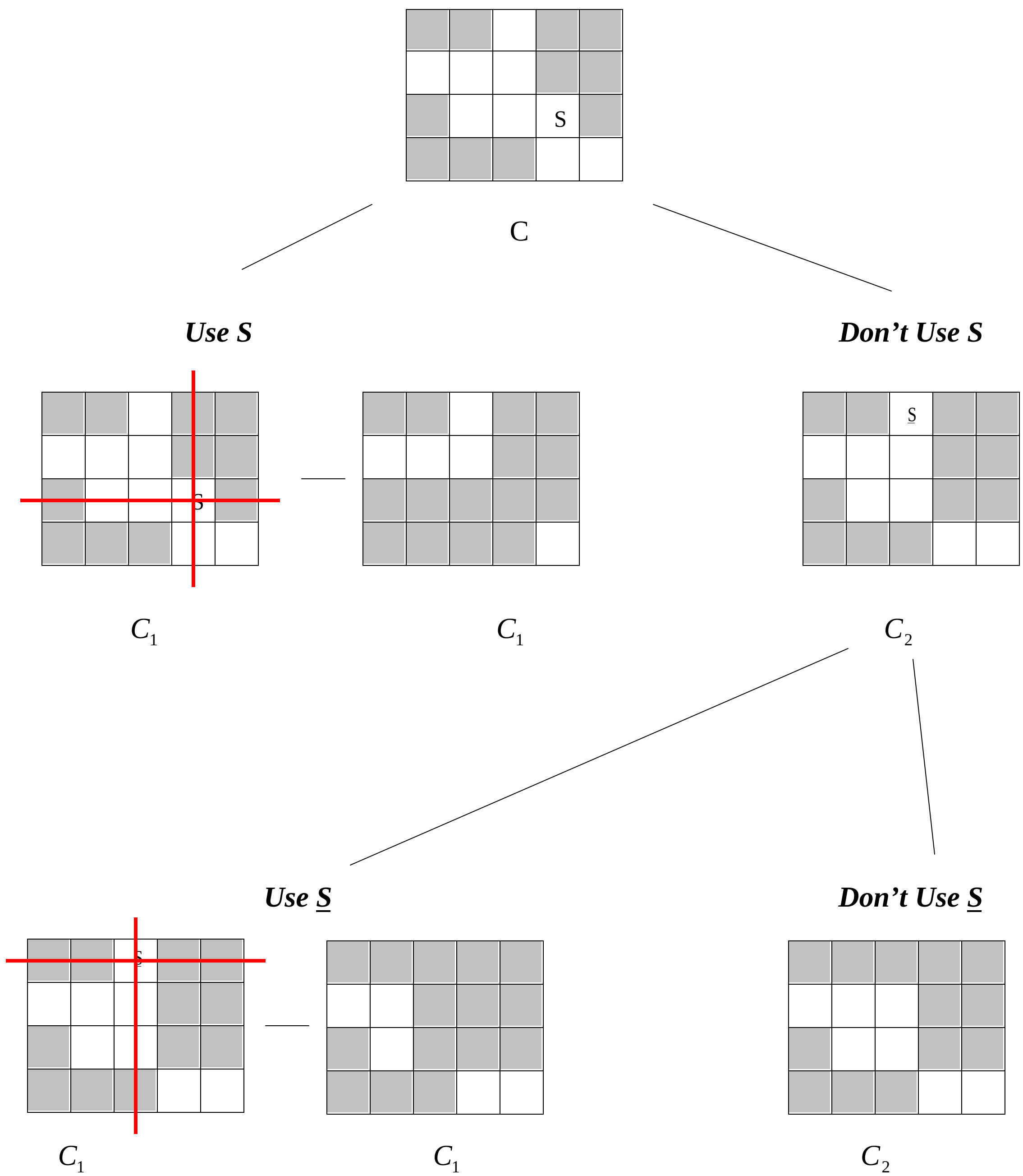
$R(B, x) = 1 + 5x + 4x^2$

According to Rook Rule #3, what is the rook polynomial of C?

$R(C, x) = 1 + 5x + 4x^2$

Use counting methods to obtain $R(C, x)$ to confirm that you obtain the expected result.

Given the board C below, use Rook Rule #2 to complete the given table for $n_r(C)$. We choose S strategically.



By our Rook Rule #2, we know that

$$
\begin{aligned}
n_r(C) &= n_{r-1}(C_1) + n_r(C_2) \\
&= n_{r-1}(C_1) + [n_{r-1}(\underline{C}_1) + n_r(\underline{C}_2)]
\end{aligned}
$$

Student Version

Now C_1, $\underline{C_1}$, and $\underline{C_2}$ are relatively easy to count.

r	0	1	2	3	4
$n_r(C_1)$					

r	0	1	2	3	4
$n_r(\underline{C_1})$					

r	0	1	2	3	4
$n_r(\underline{C_2})$					

Recall that by our Rook Rule #2, we know that $n_r(C) = n_{r-1}(C_1) + n_r(C_2) = n_{r-1}(C_1) + [n_{r-1}(\underline{C_1}) + n_r(\underline{C_2})]$

So by plugging the above table values into given equation, we can complete the desired table for $n_r(C)$. Keep in mind we define $n_{-1}(B) = 0$ for any board B.

r	0	1	2	3	4
$n_r(C)$					

Answer Key

Now C_1, $\underline{C_1}$, and $\underline{C_2}$ are relatively easy to count. Some of the counting may require formulas from activity one, but most of can be counted by hand.

r	0	1	2	3	4
$n_r(C_1)$	1	5	6	2	0

r	0	1	2	3	4
$n_r(\underline{C_1})$	1	5	7	2	0

r	0	1	2	3	4
$n_r(\underline{C_2})$	1	7	14	8	0

Recall that by our Rook Rule #2, we know that $n_r(C) = n_{r-1}(C_1) + n_r(C_2) =$

$n_{r-1}(C_1) + [n_{r-1}(\underline{C_1}) + n_r(\underline{C_2})]$

So by plugging the above table values into given equation, we can complete the desired table for $n_r(C)$.
Keep in mind we define $n_{-1}(B) = 0$ for any board B.

r	0	1	2	3	4
$n_r(C)$	1	9	24	21	4

Reflection on Activity 5

We went through this activity in a single class period. This involved more lecture than we had previously used, but because each key point in the discussion was followed immediately by a simple example and a short time of working in pairs, it was quite interactive in nature. Indeed, as we introduced concepts and raised questions, the students were very engaged and responsive. Furthermore, when I would query them as to what I should write on the board in various examples, they enthusiastically participated, offering valuable insights and suggestions. To introduce this activity we talked briefly about the fact that we needed to move away from inclusion/exclusion, as it can get complicated very quickly. We explained that we were giving them rules that will allow them to count *any* rook board *ever*.

We then had them work on the first problem of Activity 5 (Rook Rule #1), and the students seemed to understand the gist of it. I noticed that some students started to work out the formulas for counting the smaller boards, but their classmates pointed out that they could use the tables that were already provided. After some time, one student presented his solution on the white board. We'd written the tables for the smaller boards on the white board, and he was able just to point to the various combinations of r rooks from each board, which essentially demonstrated this idea of the convolution of sequences. Again, this was the exact connection I had been hoping they would make. It was nice to get a sense of how he reasoned through the problem, and the rest of the class seemed to understand his explanation. We asked them if this reminded them of anything, but no one recognized this as polynomial multiplication (we didn't push it and weren't expecting them to connect these ideas).

Next I lectured a bit about Rook Rule #2, doing a small example of the Use/Don't Use principle. The class was responsive and able to answer questions about how many rooks were to be placed on the reduced boards (r or $r - 1$, depending on the case). They seemed to be pretty convinced about why this rule would be true. We then had them work on Question 2. Admittedly, this was a little hard, and we have since refined it a bit for this activity. While some students were very careful about what to choose for S, others didn't make a very strategic choice. It was just a little much, and they got bogged down in the computation of it. Many of them were still able to arrive at the right answers, though, which was good. In fact, some of the girls said that they were enjoying reducing the boards in this way. If we had a little more time, we really could have hammered this home, particularly for some of the possibly weaker students. As I observed the small groups as they worked, not all of the students were clear about what was going on, and a couple more examples would have likely clarified this more completely. Ultimately this was a good lesson for me, however, as I learned to tone this example down a bit for the future.

Due to time, we barely touched on Rook Rule #3, just mentioning the fact that they could switch rows and columns if they so desired. Again, they seemed to understand this. With a little more time, as with the above activity, we could make sure that the entire class really understands this. We were anxious to get past this, though, and move on to rook polynomials. We also had the luxury of knowing that they could have a week to absorb the ideas we were introducing before we would return to continue with more.

Next, I briefly introduced the idea of a generating function. It was not clear at first that they completely followed the definition, but as I wrote the generating functions for the tables that were already up on the board, they obviously began to have a much better understanding of what I had said. We talked about the relationship between the counting we had done earlier and polynomial multiplication, and, to our surprise, the students grasped it immediately. In fact, I heard at least a couple of "that's awesome"s from the class, which was great. It was pretty natural for them to see the relationship between generating functions and our Rook Rule #1. Surrounded by vigorous nodding and nonverbal clues of encouragement, we proceeded to go through the other rules pretty quickly. In discussing Rook Rule #2, something important happened. We presented the rule, and a student asked what the extra "x" was doing there. Another student was able correctly to answer him in a manner that reflected an understanding of both the underlying counting principle as well as the algebraic significance of the generating function structure. So I think that, already, some of the students had a good sense of what was going on and were even excited about it.

At one point we had a couple of simple boards drawn on the white board, and I sort of talked through what the rook polynomials might be. I would say, "how many ways can we place 0 rooks?" and they would answer. Then, I would write that answer as the coefficient of the x^0 term. As I did this for a couple of boards, this helped the students get at the notion of what a rook polynomial was (and hopefully something about generating functions as well).

On the whole I was pretty pleased, especially because we successfully used rooks to introduce generating functions. The students seemed genuinely excited about the idea of generating functions, and I believe they realized the value of them, at least as computational time-savers. Everyone was more relaxed during this second visit, too, (including myself) which made the whole experience that much more enjoyable.

Introduction to Review Worksheet 2

Similar to the first review worksheet, this is designed to have students explore the new ideas to which the students have just been introduced. Intended to follow Activity 5 directly, this assessment specifically allows for more time spent exploring both the counting and polynomial versions of the Rook Rules. The purpose is to have interesting, somewhat fun problems to give them practice with these new concepts. This could easily be given as homework or as a group assignment in class.

Assessment 2 – Review Worksheet

We have established 3 pretty sweet counting principles related to rook problems, and these allow us to count ANY rook board we may encounter. Recall the three major principles we discussed…

Rook Rule #1: (**Disjoint Boards**) If a board C consists of two sub-boards A and B that do not overlap in any rows or columns, then

$$n_r(C) = n_r(A)n_0(B) + n_{r-1}(A)n_1(B) + \cdots + n_0(A)n_r(B)$$

Rook Rule #1**: Disjoint Boards (polynomial version)** If a board C consists of two sub-boards A and B that do not overlap in any rows or columns, then

$$R(C,x) = R(A,x)R(B,x)$$

Rook Rule #2: (**Use / Don't Use**) If the *i*,*j*-square S of a board C is not a forbidden square, then

$$n_r(C) = n_{r-1}(C_1) + n_r(C_2),$$

where C_1 is the board formed when we *use* S (and remove the i^{th} row and j^{th} column), and C_2 is the board formed when we *don't use* S (and S becomes a forbidden square).

Rook Rule #2: **Use/Don't Use (polynomial version)** If the *i*,*j*-square S of a board C is not a forbidden square, then

$$R(C,x) = xR(C_1,x) + R(C_2,x),$$

where C_1 is the board formed when we *use* S (and remove the i^{th} row and j^{th} column), and C_2 is the board formed when we *don't use* S (and S becomes a forbidden square).

Rook Rule #3: (**Switcheroo**) Suppose a board B can be obtained from another board C simply by permuting rows and/or columns. Then

$$n_r(B) = n_r(C)$$

In other words, we can swap rows and columns without affecting the outcome.

Rook Rule #3: **Switcheroo (polynomial version)** Suppose a board B can be obtained from another board C simply by permuting rows and/or columns. Then

$$R(B,x) = R(C,x)$$

In other words, we can swap rows and columns without affecting the outcome.

Now that we have these great principles about counting and rook polynomials, let's put them into practice! Use the principles we discussed in class (and the Rook Rules listed above) to answer the following problems.

1) Use rook polynomials to determine the number of ways of placing 4 rooks on the RED squares of a checkerboard (which has dimension 8×8).

2) Five kids are getting ready to buy the last five pets at a pet store. Their options are a hamster, a frog, a goldfish, a cockatiel, and a puppy. The only problem is some of the kids can't handle some of the pets:

- Carly only wants something with fur (feathers don't count).
- Sarah prefers amphibians.
- Brad would like anything that doesn't have claws or talons.
- Joanna only wants a puppy or a hamster.
- Derek wants a pet that can fly.

How many ways can we distribute the pets to these five kids? Keeping in mind our simplification methods, set up a rook board for this problem and solve.

3) Since Skating with Celebrities and Dancing with the Stars have been relatively big hits, the networks are already looking for some celebrities to appear on their newest show: Acting with the Stars. In this show, we take respected, Oscar-winning actors and pair them with B-list action heroes. They perform scenes in front of members of the Academy, and each week, one unlucky duo gets voted off. There are current negotiations attempting to give the winning team honorary Academy Awards.

Unbelievably, the networks have gotten 4 good actors and 5 less-good actors to agree to this. The networks have hired you to determine just how many ways we could get 4 couples from the given choices. Let's face it, though; some actors have egos. So the following restrictions apply:

- Sir Anthony Hopkins absolutely refuses to work with Steven Seagal, Carl Weathers and Jean-Claude Van Damme.
- Even though Al Pacino hasn't won in a few years, he just can't respect Carl Weathers or Steven Seagal.
- Meryl Streep *loves* accents, so she wants to work with Jean-Claude Van Damme or Lucy Lawless.
- Dame Judi Dench hates two things: Australian accents and facial hair, so she doesn't want to work with Lucy Lawless or Chuck Norris.

Use our theorems to count the number of ways of making 4 couples!

Assessment 2 – **Review Worksheet (Teacher's Version)**

We have established 3 pretty sweet counting principles related to rook problems, and these allow us to count ANY rook board we may encounter. Recall the three major principles we discussed…

Rook Rule #1: (**Disjoint Boards**) If a board C consists of two sub-boards A and B that do not overlap in any rows or columns, then

$$n_r(C) = n_r(A)n_0(B) + n_{r-1}(A)n_1(B) + \cdots + n_0(A)n_r(B)$$

Rook Rule #1**: Disjoint Boards (polynomial version)** If a board C consists of two sub-boards A and B that do not overlap in any rows or columns, then

$$R(C,x) = R(A,x)R(B,x)$$

Rook Rule #2: (**Use / Don't Use**) If the *i*,*j*-square S of a board C is not a forbidden square, then

$$n_r(C) = n_{r-1}(C_1) + n_r(C_2)\,,$$

where C_1 is the board formed when we *use* S (and remove the i^{th} row and j^{th} column), and C_2 is the board formed when we *don't use* S (and S becomes a forbidden square).

Rook Rule #2: **Use/Don't Use (polynomial version)** If the *i*,*j*-square S of a board C is not a forbidden square, then

$$R(C,x) = xR(C_1,x) + R(C_2,x)\,,$$

where C_1 is the board formed when we *use* S (and remove the i^{th} row and j^{th} column), and C_2 is the board formed when we *don't use* S (and S becomes a forbidden square).

Rook Rule #3: (**Switcheroo**) Suppose a board B can be obtained from another board C simply by permuting rows and/or columns. Then

$$n_r(B) = n_r(C)$$

In other words, we can swap rows and columns without affecting the outcome.

Rook Rule #3: **Switcheroo (polynomial version)** Suppose a board B can be obtained from another board C simply by permuting rows and/or columns. Then

$$R(B,x) = R(C,x)$$

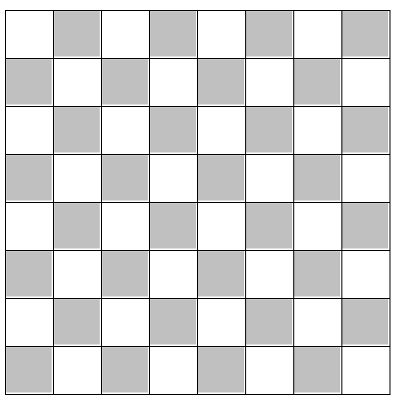
In other words, we can swap rows and columns without affecting the outcome.

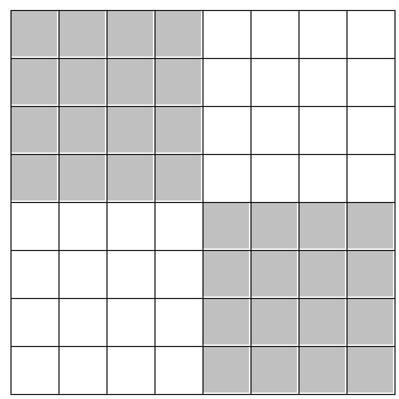
Now that we have these great principles about counting and rook polynomials, let's put them into practice! Use the principles we discussed in class (and listed above) to answer the following problems.

Notes: The purpose of these problems is to have the students become familiar with the principles we used. The hope is that they will recognize the various counting ideas, and that they will become familiar with how to implement them. They really could approach these either with the counting ideas or the rook polynomial versions of those ideas, but hopefully at some point they will realize that the rook polynomials (the generating functions) aid in computation.

1) How many ways can we place 4 rooks on the RED squares of a checkerboard?

Answer: This problem is utilizes Rook Rule #1 and #2. It's really just a matter of swapping rows and columns and then applying Rule #1.





We can use our previously-derived formulas to find that the rook polynomial for each sub-board SB (which also happens to be a 4×4 square).

$R(SB,x) = 1+16x+72x^2+96x^3+24x^4$

Thus the rook polynomial for the entire board B is

$$R(B,x) = (1+16x+72x^2+96x^3+24x^4)^2$$
$$= 1+32x+400x^2+2496x^3+8304x^4+14592x^5+12672x^6+4608x^7+576x^8$$

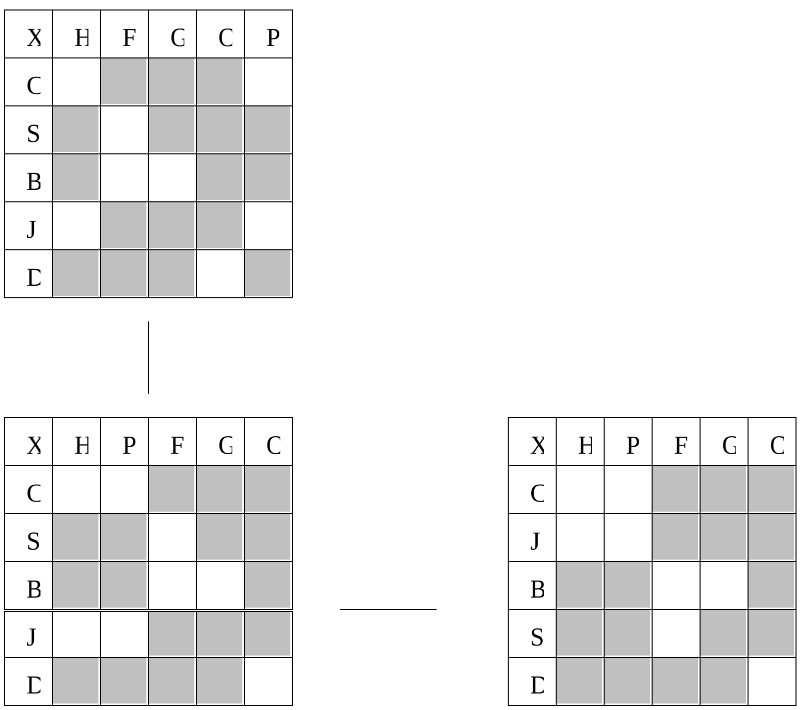
Thus the coefficient of the x^4 term, or the number of ways of placing 4 rooks, is 8304.

2) Five kids are getting ready to buy the last five pets at a pet store. Their options are a hamster, a frog, a goldfish, a cockatiel, and a puppy. The only problem is some of the kids can't handle some of the pets:

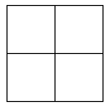
- Carly only wants something with fur (feathers don't count).
- Sarah prefers amphibians.
- Brad would like anything that doesn't have claws or talons.
- Joanna only wants a puppy or a hamster.
- Derek wants a pet that can fly.

How many ways can we distribute the pets to these five kids? Keeping in mind our simplification methods, set up a rook board for this problem and solve.

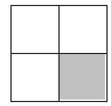
Answer: This answer begins with a given configuration of the board. Note that students may label their rows and columns differently to begin with. However, the actual rook polynomial should come out to be the same no matter what configuration they start with. Note the kids are listed as rows in the order they were mentioned, and the pets are listed as columns in the order they were mentioned. In this problem we labeled the rows and columns so the instructor can better understand this answer key. The rook board itself consists of the board without the row and column of labels.



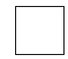
Note we can transform this into the board on the right above, which consists of three disjoint sub-boards that are easily countable. The rook polynomials for the boards are as follows:



$R(A,x) = 1 + 4x + 2x^2$



$R(B,x) = 1 + 3x + x^2$



$R(C,x) = 1 + x$

Because they are disjoint, the rook polynomial of the entire board is the product of the three,

$R(D,x) = (1 + 4x + 2x^2)\cdot(1 + 3x + x^2)\cdot(1 + x) = 1 + 8x + 22x^2 + 25x^3 + 12x^4 + 2x^5$

So there are only two ways to distribute these pets to these five kids!

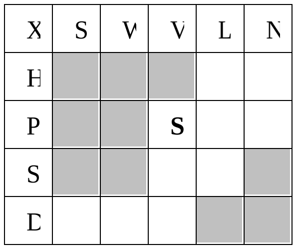
3) Since Skating with Celebrities and Dancing with the Stars have been relatively big hits, the networks are already looking for some celebrities to appear on their newest show: Acting with the Stars. In this show, we take respected, Oscar-winning actors and pair them with B-list action heroes. They perform scenes in front of members of the Academy, and each week, one unlucky duo gets voted off. There are current negotiations attempting to give the winning team honorary Academy Awards.

Unbelievably, the networks have gotten 4 good actors and 5 less-good actors to agree to this. The networks have hired you to determine just how many ways we could get 4 couples from the given choices. Let's face it, though; some actors have egos. So the following restrictions apply:

- Sir Anthony Hopkins absolutely refuses to work with Steven Seagal, Carl Weathers and Jean-Claude Van Damme.
- Even though Al Pacino hasn't won in a few years, he just can't respect Carl Weathers or Steven Seagal.
- Meryl Streep *loves* accents, so she wants to work with Jean-Claude Van Damme or Lucy Lawless.
- Dame Judi Dench hates two things: Australian accents and facial hair, so she doesn't want to work with Lucy Lawless or Chuck Norris.

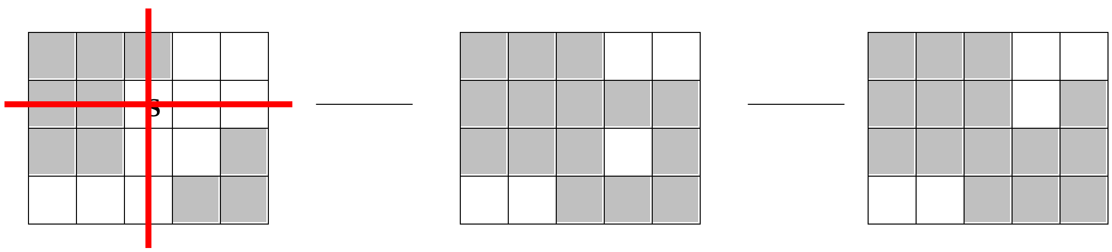
Use our theorems to count the number of ways of making 4 couples!

Answer: This problem really utilizes Rook Rule #2, and it could potentially be tricky. As above, this answer begins with a given configuration of the board. Note that students may label their rows and columns differently to begin with. However, the actual rook polynomial should come out to be the same no matter what configuration they start with. Note the "good actors" are listed as rows in the order they were mentioned; the "bad actors" are listed as columns in the order they were mentioned. The letters correspond to their last names. In this problem I labeled the rows and columns so the instructor can better understand this answer key. The rook board itself consists of the board without the row and column of labels. I eliminate them after the initial set-up of the board is clear.



D

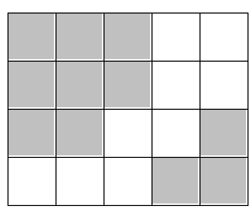
If we *use* S we get the following board, call it C_1. Then we use Rook Rule #3 to get the board below on the right.



C_1

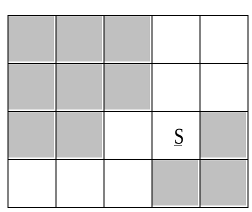
The rook polynomial of this board C_1 is the easily computable
$R(C_1, x) = (1+2x)\cdot(1+3x+x^2) = 1+5x+7x^2+2x^3$.

If we *don't use* S we get the following board, call it C_2.



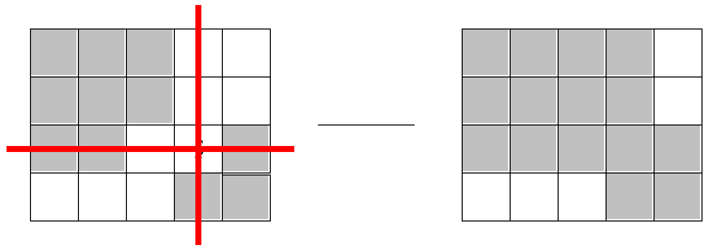
C_2

This board C_2, however, is still a little complicated to count, so we apply Use/Don't Use again to the C_2. We select square S̲.



C_2

If we *use* S, then we get the following board.

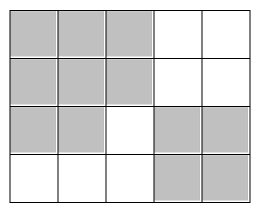


$\underline{C_1}$

The rook polynomial of this board $\underline{C_1}$ is the easily computable

$R(\underline{C_1}, x) = (1+3x)\cdot(1+2x) = 1+5x+6x^2$.

If we *don't use* S, then we get the following board.



$\underline{C_2}$

The rook polynomial of this board $\underline{C_2}$ is also easy to compute:

$R(\underline{C_2}, x) = (1+4x+2x^2)\cdot(1+4x+2x^2) = 1+8x+20x^2+16x^3+4x^4$

So now we recall Rook Rule #2, and we realize that before we add all of these together we need to multiply the rook polynomials of C_1 and $\underline{C_1}$ by x first.

So, in total, we use Rook Rule #2 to find the rook polynomial of our original board D. Since we applied Use/Don't use again to C_2 and found $\underline{C_1}$ and $\underline{C_2}$, we get

$R(D,x) = xR(C_1,x) + R(C_2,x) = xR(C_1,x) + [xR(\underline{C_1},x) + R(\underline{C_2},x)]$.

Just to recap, note that

$R(C_1,x) = (1+2x)\cdot(1+3x+x^2) = 1+5x+7x^2+2x^3$

$R(\underline{C_1},x) = (1+3x)\cdot(1+2x) = 1+5x+6x^2$

$R(\underline{C_2},x) = (1+4x+2x^2)\cdot(1+4x+2x^2) = 1+8x+20x^2+16x^3+4x^4$

Plugging in, we get

$$R(D,x) = x(1+5x+7x^2+2x^3) + x(1+5x+6x^2) + (1+8x+20x^2+16x^3+4x^4)$$
$$= 1+10x+30x^2+29x^3+6x^4$$

So there are 6 ways that we could make couples for the new TV show!

Introduction to Activity 6

This activity relates matchings, a topic from graph theory, to students' developing knowledge of rook boards. It asks specific questions that force them to consider the relationship between configurations of rooks and matchings in complete bipartite graphs. The activity, as it is written, assumes that the students have some prior exposure to matchings. If they have been introduced to matchings before they start this activity, it is reasonable to expect them to be able to work through it on their own or in small groups. If not, however, more time might be needed in order to explore the notion of matchings further. It is not necessary that students know any of the powerful theorems concerning matchings that are out there; rather, they need only a familiarity with the basic concepts and definitions involved.

This activity should be treated more as a pre-activity, just to get students to think about these ideas. The open-ended nature of the questions should force them to articulate their ideas clearly, ultimately making their learning more meaningful. Discussion of these problems among students is greatly encouraged.

Activity 6 – All Aboard For Matchings, Captain Rook!

Even though you guys are probably *pros* at matchings by now, take a moment to remind yourselves of the definition of a matching.

Matchings

Here we deal with simple graphs consisting of vertices and edges. A matching in a graph G is a set of edges such that no two edges share an endpoint. Or, said another way, a matching is a set of edges, no two of which have a vertex in common.

Since matchings are edge sets, we can find matchings of various sizes for a given graph; an r-matching in a graph G, then, is a set of r edges, no two of which share a common vertex.

Rooks and Matchings

Rook boards correspond to bipartite graphs in a natural way. Each row and each column is represented by a vertex, where the row vertices and the column vertices make up the two cells of the bipartition. An edge is drawn between a row vertex and a column vertex if the square in that row and column is not forbidden. For example, in an $m \times n$ board with no forbidden squares, the corresponding graph is the complete bipartite graph, $K_{m,n}$.

1) Draw a graph that corresponds to a 4×5 chessboard with no restricted positions.

2) Relate the rules of placing non-attacking rooks to the rules governing matchings.

3) Explain why rook boards always give rise to bipartite graphs with this construction. Can every bipartite graph be modeled by a rook board?

4) What does a restricted position in the rook setting correspond to in the setting of graphs? In other words, how might we describe a restricted position in a graph?

5) Using a formula we have about rooks, find the number of r-matchings in the complete bipartite graph $K_{m,n}$. Look at this formula and discuss how you would describe the counting process it reveals in terms of matchings.

Activity 6 – All Aboard For Matchings, Captain Rook! (Teacher's Version)

Even though you guys are probably *pros* at matchings by now, take a moment to remind yourselves of the definition of a matching.

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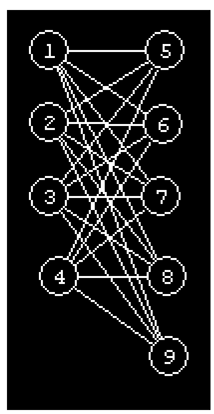
1) Draw a graph that corresponds to a 4×5 chessboard with no restricted positions.



2) Relate the rules of placing non-attacking rooks to the rules governing matchings.

Answer: In the rook board setting, once we place a non-attacking rook we cannot place any other rook in the same row or column as our given rook. Similarly, because of how matchings are defined, once we select an edge to be in a given matching, we cannot re-use either endpoint of that edge in that matching. Since the rows and columns each correspond to vertices in a cell of the bipartition, we see that just as a row or column is "used up" once a rook is placed there, so a vertex is "used up" once its edge is included in a matching. Said another way, a given row or column can contribute to at most *one* rook being placed on the board, and a given vertex can contribute to at most *one* edge in a matching.

3) Explain why rook boards always give rise to bipartite graphs with this construction. Can every bipartite graph be modeled by a rook board?

Answer: Because of the fact that the rows and columns of the rook board correspond to vertices in the two cells of the bipartition, these rook boards and bipartite graphs are inextricably linked. It works out ideally that matchings and non-attacking rooks have exactly the same restrictions, so the problems of rooks and matchings are perfectly analogous to one another. Note, however, that rook boards do not model other types of non-bipartite graphs.

4) What does a restricted position in the rook setting correspond to in the setting of graphs? In other words, how might we describe a restricted position in a graph?

Answer: A restricted position in the graph is a missing edge. Two vertices cannot be paired up (and an edge between them cannot be included in a matching) unless an edge exists between them. A missing edge essentially disallows a pairing of vertices, much like a restricted position in the rook setting.

5) Recalling a formula we have about rooks, find the number of r-matchings in the complete bipartite graph $K_{m,n}$. Look at this formula and discuss how you would describe the counting process it reveals in terms of matchings.

Answer: We realize that in our complete bipartite graph $K_{m,n}$, m and n are the number of vertices in each cell of the bipartition. In counting the r-matchings in such a graph, we are essentially looking for edge sets of size r. In order to do this, we must first pick r left endpoints from one cell (of size m) and then r right endpoints from the other cell (of size n). Once we've selected our endpoints, there are $r!$ ways to arrange edges among them. Therefore the number of r-matchings in the complete bipartite graph $K_{m,n}$ is given by the formula $\binom{m}{r}\binom{n}{r}r!$. Note this is really the same thing as picking rows and columns on a rook board.

Reflection on Activity 6

The students worked on this activity directly prior to our third visit with them. They had previously been exposed to matchings earlier in their coursework, and so the point of this activity was mainly to jog their memories. We began the third day by exploring rook polynomials and their relationship to bipartite graphs. The students were able to describe how a bipartite graph might model a rook board, and this was fairly natural for them. We then discussed briefly the fact that we can use graphs to model more general counting problems, and we explained the distinction between counting matchings in bipartite graphs versus counting matchings in any general graph. In the general, non-bipartite case, it is not necessary to match up members of two *distinct* sets – we can match up any vertices, provided they are joined by an edge. This, then, provided some motivation for learning about the matchings polynomial (which applies to graphs in general) instead of just the rook polynomial (which applies to bipartite graphs). We just briefly introduced the real-life wrestling problem as an example of this. Since this activity was just a warm-up for Activity 7, more reflections on how the students handled this material is included in the reflections for Activity 7.

Introduction to Activity 7

This final activity is fairly ambitious. It explores the matchings polynomial, which is the more general case of the rook polynomial. This matchings polynomial provides a nice bridge between generating functions and matchings, allowing students to make connections between two previous concepts. This activity motivates counting problems related to matchings, informing students that they might encounter problems in which a matchings polynomial would be more useful than the rook polynomial. Additionally, this activity investigates various properties of matchings, including four interesting theorems. Finally, the notion of perfect matchings is introduced, and a fascinating *integral* formula for computing the number of perfect matchings is discussed.

There are two primary goals for this activity. The first is to get students thoroughly comfortable with the idea of matchings. They should feel confident in identifying matchings in graphs, counting them, computing matchings polynomials, etc. Such familiarity with matchings will aid them in further mathematical studies. The second goal is for them to appreciate the breadth and variety of interesting results about matchings; the intent is to give students some insight into how remarkable these concepts are. The fact that there are interesting results when differentiating and integrating these polynomials is – let's face it – just plain cool, and this activity seeks to convey this.

This activity will likely be most effective in an interactive lecture setting, in which topics are explicitly explained, but where students feel free to ask questions and engage with the material. There are some questions throughout the activity where the students can take some time alone or in small groups in order to work through them. The Teacher's Version includes an additional example that could be used during the class as well. Because of the depth of the subject matter covered in this activity, the material included here could be spread over more than one class period.

Activity 7 – We're Gonna Rook Your World

We've talked some about the rook polynomial, which we defined to be a polynomial function whose coefficients represent the number of configurations of rooks on a chessboard. Specifically, the coefficient of the term x^r is the number of ways of placing *r* rooks on a board.

You have also recognized the very natural (and undeniably cool) relationship between rook boards and bipartite graphs, which includes the fact that a configuration of non-attacking rooks on a board represents a matching in a bipartite graph.

A Real-Life Example of Matchings

So why do we even *care* about matchings in graphs? Glad you asked!
Consider a counting problem like this one:

There are 10 kids in gym class who have to get matched up into 5 pairs of wrestling partners. Somehow, you (the cool-but-small math nerd) always seem to get matched up with Buzz (the guy with no neck). Suppose there are 4 kids who refuse to wrestle Buzz and 3 (different) kids who don't want to wrestle you. If a 5-matching is chosen at random, what is the probability that you'll have to wrestle Buzz? (Note: In this case, the 10 kids represent vertices in a graph, and a pairing represents an edge between them, but this graph doesn't have to be bipartite!)

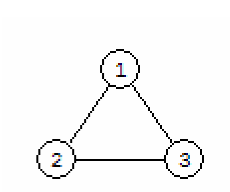
Since you're now well-versed in all of this, and since you're familiar with the notion of a generating function, we feel we're ready to unleash the big dog: the matchings polynomial.

The Matchings Polynomial

This is really just what it sounds like: it's a generating function where the coefficient of x^r represents the number of *r*-matchings in the graph. (Recall that an *r*-matching in a graph G is a set of *r* edges, no two of which have a vertex in common.) So if we denote the number of *r*-matchings by $m(G,r)$, then the matchings polynomial is defined as

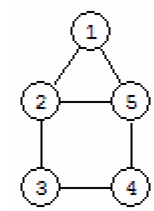
$$\mu(G,x) := \sum_{r\geq 0} m(G,r)x^r$$

For example, in a triangle graph (call it *G*), there is one 0-matching, there are three 1-matchings, and there are no 2-matchings. Thus the matchings polynomial of the triangle graph is $\mu(G,x) = 1 + 3x$



G

The following graph is often called the "house" graph (for obvious reasons); we will use it to familiarize ourselves with the matchings polynomial. We'll call the house graph *H*.



H

In order to find the matchings polynomial of the house graph, we must first determine the number of r-matchings in the graph, denoted $m(H,r)$. We find these by direct counting in this case; the table should make it easier to keep track of everything.

r	0	1	2	3
$m(H,r)$				

According to this table we have values for $m(H,r)$, and we can plug these into our definition of the matchings polynomial. Write the matchings polynomial for the house graph H below.

$\mu(H,x) =$

Cool Theorems about the Matchings Polynomial

Alright, so now that you have the matchings polynomial for the house graph, we're going to discuss some relevant theorems about this polynomial. Rather than proving these, we'll have you work these out with the house graph (and the triangle graph), whose matchings polynomials we already know.

Theorem 1: For any two disjoint graphs G and H, $\mu(G \cup H, x) = \mu(G,x)\mu(H,x)$

Example 1: Show that this is true for our two disjoint graphs: our triangle graph G and our house graph H.

Theorem 2: If e is an edge in G with endpoints u and v, and $G \setminus \{uv\}$ is the graph where we remove vertices u and v, then $\mu(G,x) = x\mu(G \setminus \{uv\}, x) + \mu(G \setminus e, x)$

Example 2: Pick any edge in our house graph H and verify that this works.

Theorem 3: If u is a vertex of a graph G, then $\mu(G,x) = x\sum_{v\sim u}\mu(G\setminus\{uv\},x) + \mu(G\setminus u,x)$

Example 3: Pick any vertex in our house graph H and verify that this works.

Theorem 4: For some edge e with endpoints u and v, $G\setminus\{uv\}$ is the graph where we remove vertices u and v, as well as any edge incident to either vertex. Then

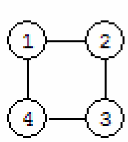
$$\frac{d}{dx}\mu(G,x) = \sum_{uv\in E(G)}\mu(G\setminus\{uv\},x)$$

Example 4: Show that this holds true for the house graph H.

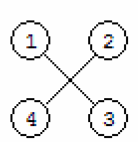
Now we switch gears a little bit and return to some good old-fashioned counting.

A Little Review

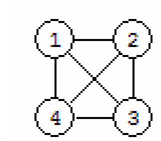
Recall that a complete graph on n vertices, K_n, contains edges between every pair of vertices; the vertices in K_n are all mutually adjacent. The complement of a graph on n vertices G, denoted $\overline{G}$, has the same vertex set as G, but it contains all edges in K_n not in G. Said another way, the edges in G, together with all edges in $\overline{G}$, make up the edges in K_n. The following two graphs exemplify this complementary relationship.



G



$\overline{G}$



K_4

Also, a <u>perfect matching</u> of a graph is a matching that includes every vertex. Note that a graph can only have a perfect matching if it has an even number of vertices. If the number of vertices is even, say $n = 2k$, then the number of perfect matchings will be the coefficient of the x^k term.

1) Develop a formula that counts $m(K_n, r)$, the number of r-matchings in the complete graph on n vertices.

2) Draw the complete graph on 4 vertices, K_4. Use the above formula you just found to write the matchings polynomials for this graph.

$\mu(K_4, x) =$

Now we introduce a random (but not as random as you'd think) formula that gives the number of perfect matchings in the complement of a graph $\overline{G}$. It's an *integral*, how cool is that?!

$$pm(\overline{G}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} x^n \mu(G, -\frac{1}{x^2}) dx$$

3) In order to convince yourselves of the verity of the above equation, find the matchings polynomial of the complement of K_n, and plug it in for $\mu(G, x$ in the formula above. Try it for a couple of values of n on your calculator. It turns out (check it sometime!) that if n is even, say $n = 2k$, then the integral equals $\binom{n}{k}\frac{k!}{2^k}$, which of course is the number of perfect matchings in K_n. (If n is odd, then this integral equals 0.)

4) Consider now the graph $K_4 \cup K_4$, the disjoint union of two copies of K_4. Since we have the matchings polynomial for K_4, we can use Rook Rule #1 (remember that?) about disjoint boards to compute $\mu(K_4 \cup K_4, x)$. Try it!

$\mu(K_4 \cup K_4, x) =$

5) Plug $\mu(K_4 \cup K_4, x)$ in to the integral formula above to give us the number of perfect matchings in the complement $\overline{K_4 \cup K_4}$.

6) Now, draw the graph for $\overline{K_4 \cup K_4}$, the complement of the disjoint union. Does this remind you of anything? (Hint: it should!!)

7) Just to hit our point home, use the very first formula we derived to find the number of ways of placing r non-attacking rooks on a 4×4 board with no restrictions. Now, compare this to the number of perfect matchings in the graph $K_4 \cup K_4$. Is that sweet or what?!

8) Remember our problem about pairing up the wrestlers? Given what you know now about polynomials, perfect matchings, integrals, life, can you come up with a solution?!

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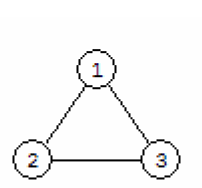

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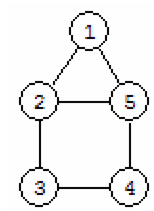

H

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r	0	1	2	3
$m(H,r)$	1	6	6	0

According to this table we have values for $m(H,r)$, and we can plug these into our definition of the matchings polynomial. Write the matchings polynomial for the house graph *H* below.

$\mu(H,x) = \ 1+6x+6x^2$

<u>Notes</u>: The students' ability to compute this matchings polynomial of *H* will rely on their understanding of matchings. It's important to emphasize that a given r-matching can only include a vertex at most once.

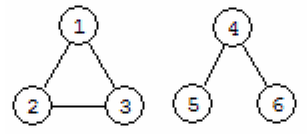
<u>Cool Theorems about the Matchings Polynomial</u>

Alright, so now that you have the matchings polynomial for the house graph, we're going to discuss some relevant theorems about this polynomial. Rather than proving these, we'll have you work these out with the house graph and the triangle graph, whose matchings polynomials we already know.

<u>Theorem 1</u>: For any two disjoint graphs *G* and *H*, $\mu(G \cup H, x) = \mu(G,x)\mu(H,x)$

<u>Note</u>: In this section that follows, the teacher ought to run through the following example provided in class. Then, the students can work through the same process with the house graph.

<u>Example in class</u>: Let *G* be the graph consisting of two components: a C_3 and a P_3, as shown below.



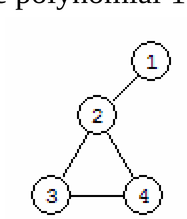
Counting the number of r-matchings in $P_3 \cup C_3$ gives us $\mu(P_3 \cup C_3, x) = 1 + 5x + 6x^2$.
We note that $\mu(P_3, x) = 1 + 2x$ and $\mu(C_3, x) = 1 + 3x$,
so $\mu(P_3, x)\mu(C_3, x) = (1 + 2x)(1 + 3x) = 1 + 5x + 6x^2$.
Thus, comparing the above results, we see that $\mu(P_3 \cup C_3, x) = \mu(P_3, x)\mu(C_3, x)$.

Example 1: Show that this is true for our two disjoint graphs: our triangle graph G and our house graph H.

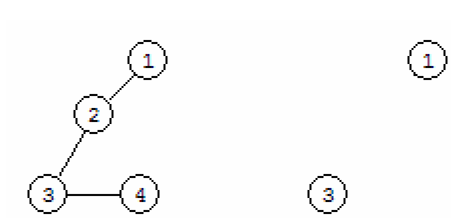
Answer: We found above that, for triangle graph G and house graph H,
$\mu(G, x) = 1 + 3x$ and $\mu(H, x) = 1 + 6x + 6x^2$.
We count directly and find that $\mu(G \cup H, x) = 1 + 9x + 24x^2 + 18x^3$.
Comparing this with $\mu(G, x)\mu(H, x) = (1 + 3x)(1 + 6x + 6x^2) = 1 + 9x + 24x^2 + 18x^3$, and we see that the theorem holds true.

Theorem 2: If e is an edge in G with endpoints u and v, and $G \setminus \{uv\}$ is the graph without the vertices u and v, then $\mu(G, x) = x\mu(G \setminus \{uv\}, x) + \mu(G \setminus e, x)$

Example in class: Let G be the following graph, where edge e has endpoints 2 and 4. Note, this theorem would still work regardless of the edge chosen. $\{12\}$ would be another interesting choice for edge e.
Counting $\mu(G, x)$ directly gives us the polynomial $1 + 4x + x^2$.



We consider two sub-graphs, one where we delete e, and one where we delete the endpoints of e. $G \setminus e$ and $G \setminus \{24\}$ are the following respective sub-graphs.



$G \setminus e$ $\qquad$ $G \setminus \{24\}$

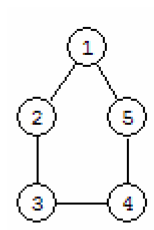
Counting the matching polynomials of each sub-graph gives us $\mu(G\setminus e,x)=(1+3x+x^2)$ and $\mu(G\setminus\{24\},x)=(1)$.

So $x\mu(G\setminus\{24\},x)+\mu(G\setminus e,x)=x+(1+3x+x^2)=1+4x+x^2$

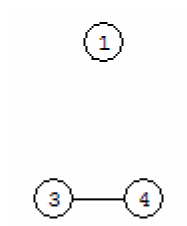
Thus for the graph G we see that $\mu(G,x)=x\mu(G\setminus\{24\},x)+\mu(G\setminus e,x)$.

Example 2: Pick any edge in our house graph H and verify that this works.

Answer: For our house graph H shown above, pick the edge e with endpoints 2 and 5.



$H\setminus e$



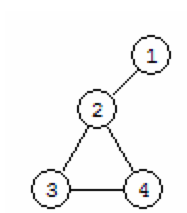
$H\setminus\{25\}$

$\mu(H\setminus e,x)=1+5x+5x^2$ and $\mu(H\setminus\{25\},x)=1+x$.

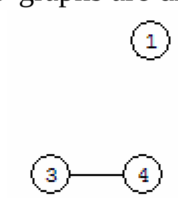
From the theorem above we find that $\mu(H,x)=x(1+x)+(1+5x+5x^2)=1+6x+6x^2$.
This checks out with what we know $\mu(H,x)$ to be.

Theorem 3: If u is a vertex of a graph G, then $\mu(G,x)=x\sum_{v\sim u}\mu(G\setminus\{uv\},x)+\mu(G\setminus u,x)$

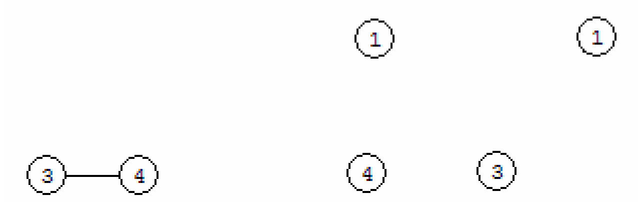
Example in class: We use the same graph G, shown below. Let vertex 2 be the vertex u we delete.



We consider two classes of sub-graphs. The first is a single sub-graph, one where we delete vertex 2. The second is a group of graphs, where in each graph we delete vertex 2 and one vertex adjacent to it. The sub-graphs are drawn below.



$G\setminus 2$



$G \setminus \{21\}$ $\quad$ $G \setminus \{23\}$ $\quad$ $G \setminus \{24\}$

Counting the matchings polynomials of these subgraphs gives us $\mu(G \setminus 2, x) = 1 + x$, and $\sum_{i \sim 2} \mu(G \setminus \{2i\}, x) = [(1+x)+1+1] = 3 + x$. Thus, we get

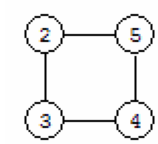
$\mu(G,x) = x\sum_{v \sim 2} \mu(G \setminus \{2v\}, x) + \mu(G \setminus 2, x) = x(3+x) + (1+x) = 1 + 4x + x^2$.

We know from above (or by direct counting) that $\mu(G,x) = x^4 + 4x^2 + 1$.

So for the given graph G and $u = 2$ we have $\mu(G,x) = x\sum_{v \sim u} \mu(G \setminus \{uv\}, x) + \mu(G \setminus u, x)$

Example 3: Pick any vertex in our house graph H and verify that this works.

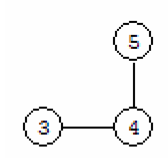
Answer: We pick vertex $u = 1$ to be the vertex we remove.



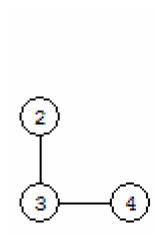
$H \setminus 1$

$$\mu(H \setminus 1, x) = 1 + 4x + 2x^2$$

We must also consider the graphs that remove the edges of which 1 is an endpoint, namely $H \setminus \{12\}$ and $H \setminus \{15\}$.



$H \setminus \{12\}$



$H \setminus \{15\}$

$\mu(H \setminus \{12\}, x) = 1 + 2x$ and $\mu(H \setminus \{15\}, x) = 1 + 2x$.

So $\sum_{v \sim u} \mu(H \setminus \{uv\}, x) = (1+2x) + (1+2x) = 2 + 4x$, and thus

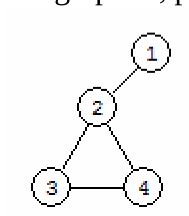
$x\sum_{v \sim u} \mu(H \setminus \{uv\}, x) = x(2+4x) = 2x + 4x^2$

Then $\mu(H,x) = (2x + 4x^2) + (1 + 4x + 2x^2) = 1 + 6x + 6x^2$, which we know is the matchings polynomial for the house graph.

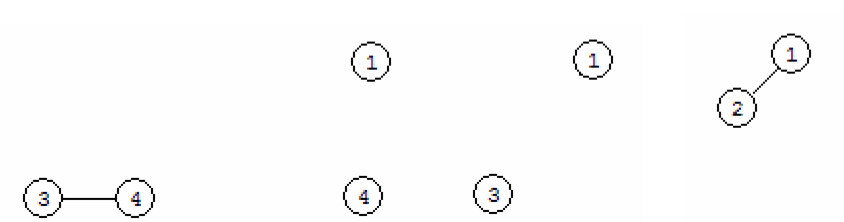
Theorem 4: For some edge e with endpoints u and v, $G \setminus \{uv\}$ is the graph where we remove vertices u and v, as well as any edge incident to either vertex. Then

$$\frac{d}{dx}\mu(G,x) = \sum_{uv \in E(G)} \mu(G \setminus \{uv\}, x).$$

Example in class: We consider the same graph G, pictured below.



Let us examine $\sum_{uv \in E(G)} \mu(G \setminus \{uv\}, x)$. Since there are four edges, there will be four subgraphs to consider.



$G \setminus \{12\}$ $\quad$ $G \setminus \{23\}$ $\quad$ $G \setminus \{24\}$ $\quad$ $G \setminus \{34\}$

We can compute the matchings polynomials for each of these subgraphs. Summing these polynomials will give the right hand side of the equation in the theorem.

$\mu(G \setminus \{12\} = 1 + x$
$\mu(G \setminus \{23\} = 1$
$\mu(G \setminus \{24\} = 1$
$\mu(G \setminus \{34\} = 1 + x$

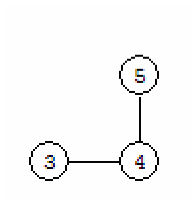
Thus, $\sum_{uv \in E(G)} \mu(G \setminus \{uv\}, x) = (1+x) + 1 + 1 + (1+x) = 4 + 2x$

We know from above that $\mu(G,x) = 1 + 4x + x^2$, and so $\frac{d}{dx}\mu(G,x) = 4 + 2x$.

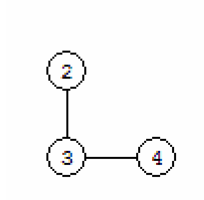
Therefore the theorem holds for this graph.

Example 4: Show that this holds true for the house graph *H*.

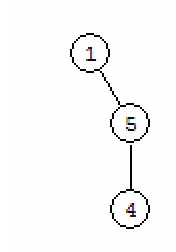
Answer: On the right hand side, we must consider $\sum_{uv \in E(G)} \mu(G \setminus \{uv\}, x)$, which is the matchings polynomials of all subgraphs of *H* where an edge (and its endpoints and their incident edges) is removed, $H \setminus 1$, $H \setminus 2$, $H \setminus 3$, $H \setminus 4$, and $H \setminus 5$.



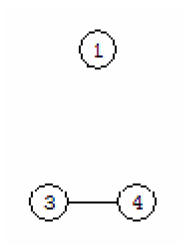
$H \setminus 12$



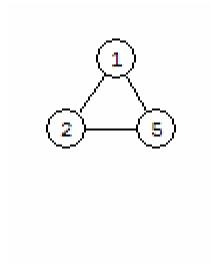
$H \setminus 15$



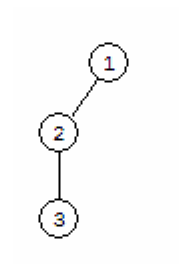
$H \setminus 23$



$H \setminus 25$



$H \setminus 34$



$H \setminus 45$

We compute the matchings polynomials for the above subgraphs and sum them.

$\mu(H \setminus \{12\}, x) = 1 + 2x$
$\mu(H \setminus \{15\}, x) = 1 + 2x$
$\mu(H \setminus \{23\}, x) = 1 + 2x$
$\mu(H \setminus \{25\}, x) = 1 + x$
$\mu(H \setminus \{34\}, x) = 1 + 3x$,
$\mu(H \setminus \{45\}, x) = 1 + 2x$

Then $\sum_{uv \in E(G)} \mu(G \setminus \{uv\}, x) = 4(1 + 2x) + (1 + x) + (1 + 3x) = 6 + 12x$

We compare this with $\frac{d}{dx}\mu(G, x)$.

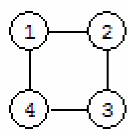
We know that $\mu(H, x) = 1 + 6x + 6x^2$, so $\frac{d}{dx}\mu(G, x) = 6 + 12x$.

Therefore, for the graph *H* it is true that $\frac{d}{dx}\mu(G, x) = \sum_{uv \in E(G)} \mu(G \setminus \{uv\}, x)$.

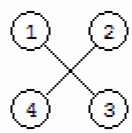
Now we switch gears a little bit and return to some good old-fashioned counting.

<u>A Little Review</u>

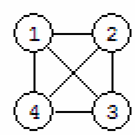
Recall that a complete graph on n vertices, K_n, contains edges between every pair of vertices; the vertices in K_n are all mutually adjacent. The <u>complement</u> of a graph on n vertices G, denoted $\overline{G}$, has the same vertex set as G, but it contains all edges in K_n not in G. Said another way, the edges in G, together with all edges in $\overline{G}$, make up the edges in K_n. The following two graphs exemplify this complementary relationship.



G



$\overline{G}$



K_4

Also, a <u>perfect matching</u> of a graph is a matching that includes every vertex. Note that a graph can only have a perfect matching if it has an even number of vertices. If the number of vertices is even, say $n = 2k$, then the number of perfect matchings will be the coefficient of the x^k term.

<u>Note</u>: Perfect matching is an important notion for the students to grasp. It would be worth running through a few examples of perfect matchings to hit this concept home.

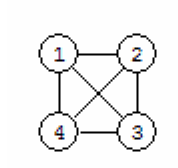
1) Develop a formula that counts $m(K_n, r)$, the number of r-matchings in the complete graph on n vertices.

<u>Answer</u>: $\dfrac{\binom{n}{2r}\binom{2r}{r}r!}{2^r} = \dfrac{n!}{(n-2r)!2^k r!}$. We get this in the following way. First we pick some $2r$ vertices from n, which represents picking a set of vertices for our r-matching (as every edge has two vertices). From those $2r$ vertices, we pick r of them as left endpoints, and we assign r right endpoints to these left endpoints in $r!$ ways. But since there are no left and right endpoints, we must divide by 2^r. Simplification gives us the second equation.

There are other ways to think of counting this as well, but they should result in this same formula.

2) Draw the complete graph on 4 vertices, K_4. Use the above formula you just found to write the matchings polynomials for this graph.

Answer:



K_4

$$\mu(K_4, x) = 1 + 6x + 3x^2$$

Now we introduce a random (but not as random as you'd think) formula that gives the number of perfect matchings in the complement of a graph $\overline{G}$. It's an *integral*, how cool is that?!

$$pm(\overline{G}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} x^n \mu(G, -\frac{1}{x^2}) dx$$

3) In order to convince yourselves of the verity of the above equation, find the matchings polynomial of the complement of K_n, and plug it in for $\mu(G, x$ in the formula above. Try it for a couple of values of n on your calculator. It turns out (check it sometime!) that if n is even, say $n = 2k$, then the integral equals $\binom{n}{k}\frac{k!}{2^k}$, which of course is the number of perfect matchings in K_n. (If n is odd, then this integral equals 0.)

Answer: The complement of K_n is just the empty graph on n vertices. The matchings polynomial of an empty graph will always just be 1 (as there is just one 0-matching in any empty graph). So we end up integrating $pm(\overline{K_n}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} (x^n \cdot 1) dx$,

and testing this for some values of n confirms this. Perhaps students could try different values for n on their calculators and then share their results with the class. Or, alternatively, the teacher could do this in Maple or some such program.

4) Consider now the graph $K_4 \cup K_4$, the disjoint union of two copies of K_4. Since we have the matchings polynomial for K_4, we can use Rook Rule #1 (remember that?) about disjoint boards to compute $\mu(K_4 \cup K_4, x)$. Try it!

$\mu(K_4 \cup K_4, x) =$

Answer: Because $\mu(K_4, x) = 1 + 6x + 3x^2$ and $\mu(K_4, x) = 1 + 6x + 3x^2$,

$$\begin{aligned}\mu(K_4 \cup K_4, x) &= (1+6x+3x^2)(1+6x+3x^2) \\ &= 1 + 12x + 42x^2 + 36x^3 + 9x^4\end{aligned}$$

5) Plug $\mu(K_4 \cup K_4, x)$ in to the formula above to give us the number of perfect matchings in the complement $\overline{K_4 \cup K_4}$.

Answer:

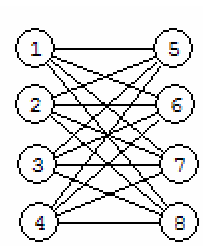
We know that $\mu(K_4 \cup K_4, -\frac{1}{x^2}) = 1 - \frac{12}{x^2} + \frac{42}{x^4} - \frac{36}{x^6} + \frac{9}{x^8}$, and we note that $n = 8$.

Then $x^8 \mu(K_4 \cup K_4, -\frac{1}{x^2}) = x^8 - 12x^6 + 42x^4 - 36x^2 + 9$.

Therefore, $pm(\overline{K_4 \cup K_4}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} (x^8 - 12x^6 + 42x^4 - 36x^2 + 9)dx = 24$

6) Now, draw the graph for $\overline{K_4 \cup K_4}$, the complement of the disjoint union. Does this remind you of anything? (Hint: it should!!)

Answer: It should! This is really a graph that models a 4×4 rook board with no restricted positions!!



7) Just to hit our point home, use the very first formula we derived to find the number of ways of placing *r* non-attacking rooks on a 4×4 board with no restrictions. Now, compare this to the number of perfect matchings in the graph $K_4 \cup K_4$. Is that sweet or what?!

Answer: $\binom{m}{r}\binom{n}{r}r! = \binom{4}{4}\binom{4}{4}4! = 24$ is the same as the number of perfect matchings in $K_4 \cup K_4$ that we just found above!

8) Remember our problem about pairing up the wrestlers? Given what you know now about polynomials, perfect matchings, integrals, life, can you come up with a solution?!

Answer: In our wrestling scenario, we're looking for a perfect matching (since we want a 5-matching in a 10-vertex graph). Let S_3 be a star with 3 leaves and S_4 be a star with 4 leaves, as seen below. These represent the restrictions related to you (the S_3) and to Buzz (the S_4). The wrestling graph is $G = K_{10} - S_3 - S_4$ (as the S_3 and S_4 are disjoint from each other).

The numerator we need is the number of 5-matchings that do pair you and Buzz up, so we look for a 4-matching in the graph not including you and Buzz. (Since the restrictions only affect you and Buzz, once you and Buzz are paired up, anyone else can be paired up together.) Deleting you and Buzz gives the graph $G' = K_8$. This has $\frac{\binom{8}{4}4!}{2^4} = 105$ matchings of size 4. Note, this numerator could also be found by plugging the matchings polynomial of the graph $\overline{K_8}$ (which happens to be 1) into the integral formula above. This would give the same result.

The denominator is the total number of 5-matchings in the graph G, which we count as the number of perfect matchings of the graph that is the complement of [$S_3 \cup S_4 \cup$ (3 isolated vertices)]. Since $\mu(S_3, x) = 1 + 3x$ and $\mu(S_4, x) = 1 + 4x$ (and the isolated vertices have a matchings polynomial = 1), the graph that *does* consist of [$S_3 \cup S_4 \cup$ (3 isolated vertices)] has matchings polynomial $(1+4x)(1+3x) = 1 + 7x + 12x^2$ by the disjoint union formula.

We replace x with $-\frac{1}{x^2}$ and multiply by x^{10} in our integral formula as above to give us the polynomial $x^{10} - 7x^8 + 12x^6$. Then using this in our integral formula, we need to integrate $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}}(x^{10} - 7x^8 + 12x^6)dx$. This gives us 390 matchings for our denominator. Therefore 105/390 is the probability that you and Buzz are going to get paired up.

Reflection on Activity 7

The students worked on this activity during our third and final visit to the classroom. After discussing Activity 6, we introduced the matchings polynomial as being analogous to the rook polynomial. I gave a quick example of a matchings polynomial but then realized that they hadn't had a prior introduction to matchings in non-bipartite graphs (which I had unwittingly assumed). Had I known this sooner, I might have been more explicit in describing matchings to them. However, despite this, they seemed to catch on pretty quickly to the idea of matchings, and, throughout the rest of the time, counting matchings came easily for them. When asked to compute the matchings polynomial in the house graph, they did it correctly and very quickly; I was impressed and even a little surprised.

We then delved into the four theorems about the matchings polynomial. I ran through a pretty simple example on the board, and even as I did so they were able to recognize what the various subgraphs and matchings polynomials should be. So, it became clear that counting matchings and finding matchings polynomials was a very natural thing for them to do. The first three theorems went pretty well. We explained how the first two theorems were analogous to our first two counting principles. And, although we did not explicitly connect Theorem 3 for them, one student volunteered the observation that this theorem represents the Use/Don't Use principle applied to a vertex.

Theorem 4 provides a startling connection to derivatives, which we included for its coolness factor. There was a technical difficulty surrounding the incorporation of this theorem in the curriculum. In particular, the change in form of the matchings polynomial that we employed in the curriculum – although it simplified every other result – actually complicated this one substantially. Specifically, the formula below appeared in the original version of the activity given to students

$$\frac{d}{dx}\left[x^n \mu(G,\frac{1}{x^2})\right] = \sum_{v\in V(G)} x^{n-1}\mu(G\setminus v,\frac{1}{x^2}) .$$

Notice the algebraic complexity of this expression. Although the students had previously dealt with sigma notation, set theoretic notation, and derivatives, the substitution of $\frac{1}{x^2}$ for x in the generating function was disorienting for them. Having to perform this substitution seemed arbitrary and unmotivated, and as a result the theorem appeared less impressive to them. In reflecting upon this activity, we discovered a substantial simplification of this theorem that retains all of the mathematical content while avoiding the substitution. We have incorporated the newer, cleaner version in the curriculum presented here.

In retrospect, I would have liked to have been able to take a little more time to hit home the idea of matchings and the matchings polynomial. In doing this, I would have emphasized some applications of these four theorems. I wonder if, without this motivation, working through these theorems (and finding matchings polynomials) felt too computational to them. I think that with a little more time I might have better made this connection.

We just barely got to discussing complements and complete graphs. While I think they understood this discussion, I don't think that they'd had much exposure to either of these terms. This makes me think that the rook problem (and the matchings polynomial) has the potential to introduce and allow students to learn about complements and complete graphs as well. Because of the powerful results that the matchings polynomial has about complete graphs and complements (which we didn't get to because of time), I think this exercise would provide an effective means for discussing these two important graph theory concepts.

We had to rush pretty quickly to count the number of matchings in the complete graph. But again, because these kids were good counters, they ultimately followed the formula that we, in the interest of time, presented to them. They might have even been able to come up with this formula on their own. We just started to have them draw the complete graphs when the class period abruptly ended prematurely, and we had to leave.

I think it went okay overall. We were purposefully ambitious about what we could get through given that this was our last day with them, and I wonder if a little more motivation wouldn't have been beneficial for them, just to give a little more purpose behind what we were doing. Even without this motivation, though, they *did* get a solid introduction to matchings and matchings polynomials, which is important, I think. I'm not sure that I had a good idea of what the purpose of the activity was going to be, but I see now that at the very least it was useful to discuss matchings and the matchings polynomial. In fact, by the end they exhibited a good understanding of both of these concepts. In that sense, then, as our ultimate goal involves trying to give them exposure to combinatorial principles, this activity was successful.

Introduction to Rook Exam

This exam is a final assessment tool, designed to gauge what the students came away with from this time. The hope is to have students unify their thoughts related to all of this material. This ‘exam’ does not necessarily need to be taken as an in-class exam; it could even be a group worksheet that they work through. The intent is simply to provide teachers with a means of comprehensively testing the student’s knowledge of the entirety of the rook materials. At the very least, this exam gives teachers some more problems and question types which they can pass along to their students.

<u>Assessment 3</u> – <u>Here's Rookin' at You, Kid</u>

1. Given a 10×49 chessboard with no restricted positions, what is the maximum number of rooks you could place on the board? Explain your answer.

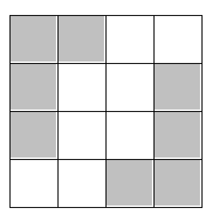
2. In class we found that the number of configurations of r non-attacking rooks on an $m \times n$ board is given by the formula $\binom{n}{r}\binom{n}{r}r!$. Considering the above question (1), use the language of rooks to relate this formula to the convention of letting $\binom{n}{r} = 0$ if $n < r$.

3. Consider the following scenario. Four college students want to go on an exchange program, and their school can send one student each to Spain, New Zealand, and China, and Honduras. There are, of course, some restrictions:

   Ander and Becky don't want to go to a Spanish-speaking country, but John only wants to go to one. Nick, on the other hand, will only be happy if he gets to go to Europe.

   Use rook rule #1 and rook rule #3 to count the number of ways that these students could be sent to the countries of their choice.

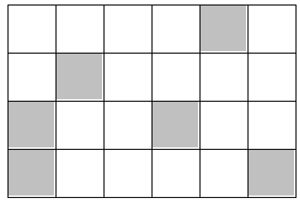
4. For the given board *B*, apply the **use/don't use** principle (rook rule #2) exactly twice in order to simplify the board. In other words, use this principle to obtain simpler boards (with disjoint sub-boards) that you can easily count using the disjoint board principle (rook rule #1). Then find the board *B*'s rook polynomial. Try to pick a strategic square for S.



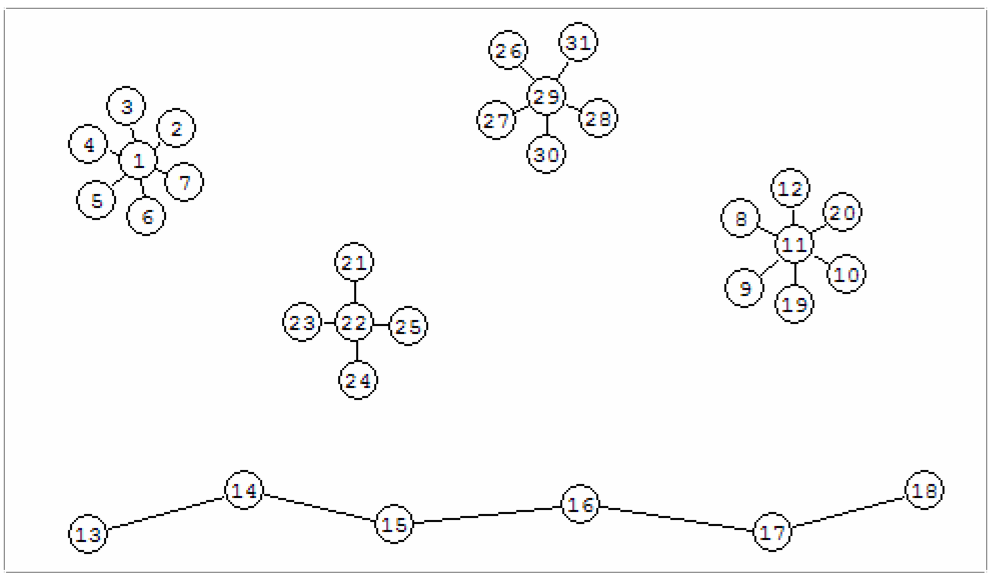
B

5. Could the following polynomials be rook polynomials for some board? If so, draw a board that represents it. If not, then why not?
   a. $1 + 2x^2$
   b. $2 + 4x + 4x^2$
   c. $1 + 6x + 5x^2$
   d. $1 + 3x + 3x^2 + 2x^3$

6. Come up with a story problem that could model the following board.



7. Find the matchings polynomial for the following graphs.
   a. A star with *n* vertices
   b. A path with *n* vertices
   c. Use these facts to compute the matchings polynomial of the graph given below (yes, those are supposed to be snowflakes).



G

8. Recall that we had a theorem (Theorem 3 from Activity 7) that states "If u is a vertex of a graph G, then $\mu(G,x) = x\sum_{v\sim u}\mu(G\setminus\{uv\},x) + \mu(G\setminus u,x)$".
   Note $v \sim u$ indicates that we sum across all neighbors of u; $G\setminus\{uv\}$ means we delete both vertices u and v from the graph G, and $G\setminus u$ means that we delete vertex u from G.
   Explain how this given theorem equation is an example of the Use/Don't Use Principle (i.e. indicate what it is that we use or don't use).

<u>Assessment 3</u> – <u>Here's Rookin' at You, Kid (Teacher's Version)</u>

1. Given a 10×49 chessboard with no restricted positions, what is the maximum number of rooks you could place on the board? Explain your answer.

<u>Answer</u>: There can be at most 10 rooks on such a board, because there are only 10 rows (or columns). Any more than 10 rooks would require an 11th row in order to avoid the other rooks, but there is no such row. If we tried to put an 11th rook on the board, even if it was on one of the 49 columns, it would hit one of the other rooks already in the 10 rows. Thought of another way, we must always choose a number of rows and columns on which to place our non-attacking rooks. There are at most 10 rows to choose from, so we can't place any more than that.

2. In class we found that the number of configurations of *r* non-attacking rooks on an $m \times n$ board is given by the formula $\binom{n}{r}\binom{n}{r}r!$. Considering the above question (1), use the language of rooks to relate this formula to the convention of letting $\binom{n}{r} = 0$ if $n < r$.

<u>Answer</u>: WLOG say n is the number of rows. Then having $n < r$ it would be like placing more rooks than we have columns. Since there are no ways of doing this (no such configurations), it makes sense that this value should be zero.

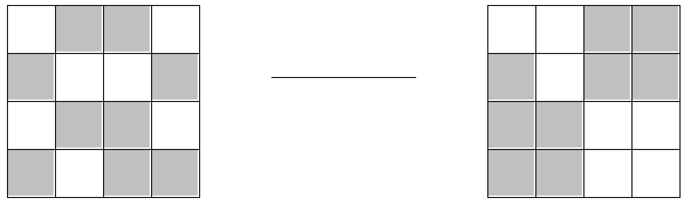
3. Consider the following scenario. Four college students want to go on an exchange program, and their school can send one student each to Spain, New Zealand, and China, and Honduras. There are, of course, some restrictions:

   Ander and Becky don't want to go to a Spanish-speaking country, but John only wants to go to one, while Nick will only be happy if he gets to go to Europe.

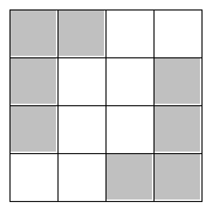
   Use rook rule #1 and rook rule #3 to count the number of ways that these students could be sent to the countries of their choice.

<u>Answer</u>: The board should be able to be drawn, ultimately, like the one on the right below, using rook rule #3.

After this we use rook rule #1 and see that we have disjoint boards. Therefore, we can take the product of the rook polynomials of the disjoint boards, which gives us
$\mu(G, x) = (1 + 3x + x^2)(1 + 4x + 2x^2) = 1 + 7x + 15x^2 + 10x^3 + 2x^4$

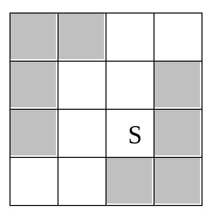


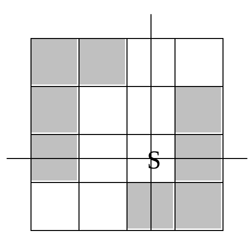
4. For the given board B, apply the **use/don't use** principle (rook rule #2) exactly twice in order to simplify the board. In other words, use this principle to obtain simpler boards (with disjoint sub-boards) that you can easily count using the disjoint board principle (rook rule #1). Then find the board B's rook polynomial. Try to pick a strategic square for S.



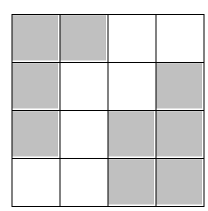
B

Answer:



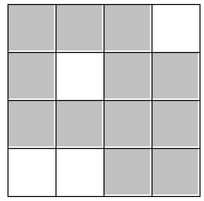


Use S



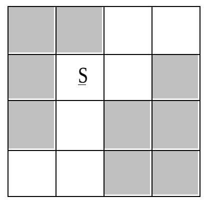
Don't Use S

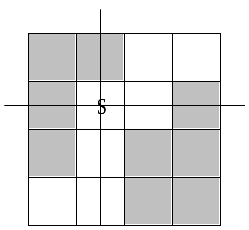
Denote B* as the board B\(rows and columns of S), shown below, that we get if we do use S. Then this graph has the rook polynomial $R(B^*,x) = (1+x)(1+3x+x^2) = 1+4x+4x^2+x^3$. We get this by using rook rule #3 to shift rows and columns, and then applying rook rule #1 since we then have disjoint boards.



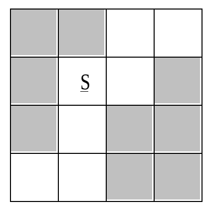
B*

Denote B\S as the board below, which we obtain if we don't use S. It's a good exercise now further to simplify this board. We choose S as the next square to utilize.



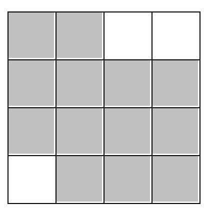


Use S



Don't Use S

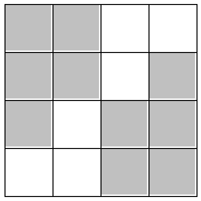
We note that B**, the board that does use S is the board below.



B**

We can easily compute its rook polynomial by using rook rule #1. So
$R(B^{**},x) = (1+x)(1+2x) = 1+3x+2x^2$

And the board B\S is the one below, where we do not use S.



B\S

Again, since there are disjoint sub-boards, this rook polynomial is fairly easy to compute. We get $R(B \setminus \underline{S}, x) = (1+3x+x^2)^2 = 1+6x+11x^2+6x^3+x^4$.

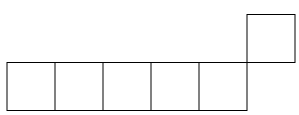
So what's the rook polynomial for the whole board *B*, $R(B,x)$?
We recall, from rook rule #2, that
$R(B,x) = xR(B^*,x) + R(B \setminus S, x)$. But $R(B \setminus S, x) = xR(B^{**},x) + R(B \setminus \underline{S}, x)$.
Substituting in, then we get

$$R(B,x) = xR(B^*,x) + [xR(B^{**},x) + R(B/\underline{S},x)]$$
$$R(B,x) = x(1+4x+4x^2) + x(1+3x+3x^2) + (1+6x+11x^2+6x^3+x^4)$$
$$R(B,x) = x+4x^2+4x^3+x+3x^2+3x^3+1+6x+11x^2+6x^3+x^4$$
$$R(B,x) = 1+8x+18x^2+12x^3+2x^4$$

Thus, since the coefficient of the x^4 term is 2, the answer is 2.

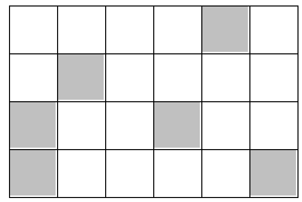
5. Could the following polynomials be rook polynomials for some board? If so, draw a board that represents it. If not, then why not?
   a. $1+2x^2$
   b. $2+4x+4x^2$
   c. $1+6x+5x^2$
   d. $1+3x+3x^2+2x^3$

Answer:
a. Nope – we can't skip a power of x like that. This would imply that we have no one-matchings (so no edges), but we still have two-matchings, which isn't possible.
b. Nope – we must have 1 as our constant term.
c. Sure! – Here's a board that models it



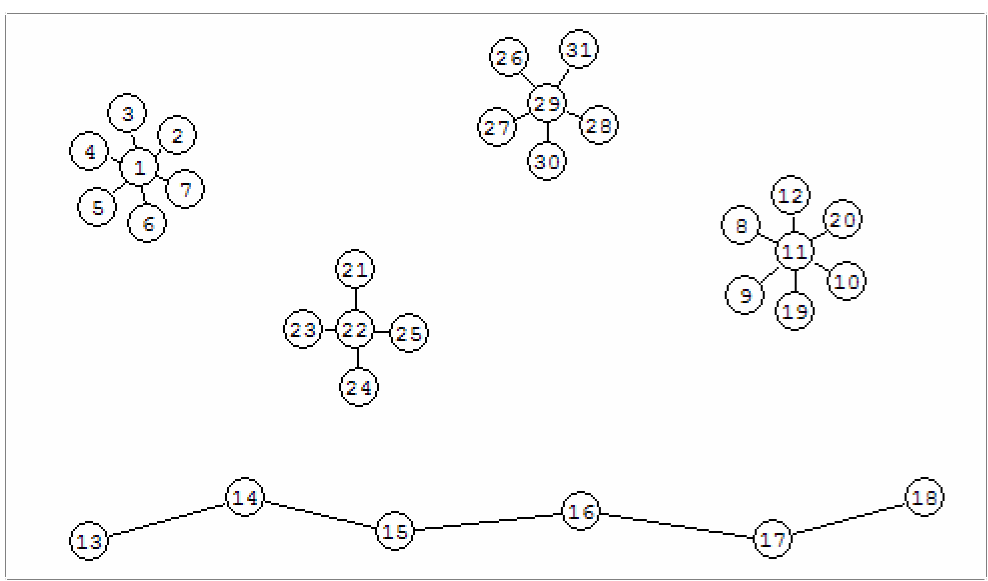
d. Nope – the coefficient of the x term shows that there are 3 allowable squares, and so we couldn't possibly have 2 ways of arranging 3 rooks on these squares.

6. Come up with a story problem that could model the following board.



Answer: Any story problem involving the specified restricted positions would do. We present one such problem. There are six dessert items at a cafeteria (chocolate pudding, cheesecake, carrot cake, apple pie, twinkies, and brownies), and four students (Ander, Brad, Carly, and Derek) must choose exactly one dessert to eat. Some of the students have allergies, however, restricting some of the choices. If we listed them in the order we just described, where the students represent the rows and the desserts represent the columns, then the board models the following restrictions:
Ander refuses to eat twinkies.
Brad is allergic to cream cheese.
Carly doesn't like the consistency of pudding and can't eat apples.
Derek can't have chocolate.

7. Find the matchings polynomial for the following graphs.
   a. A star with n vertices
   b. A path with n vertices
   c. Use these facts to compute the matchings polynomial of the graph given below (yes, those are supposed to be snowflakes).



G

Answer: The matchings polynomial for the components are as follows. Let the stars be denoted S1…S4 from left to right, and let the path be P6. Then

$\mu(S_1, x) = 1 + 6x$

$\mu(S_2, x) = 1 + 4x$

$\mu(S_3, x) = 1 + 5x$

$\mu(S_4, x) = 1 + 6x$

$\mu(P_6, x) = 1 + 5x + 6x^2 + x^3$

So by the disjoint board principle, we can just multiply all of these to get

$$\mu(G, x) = (1+6x)(1+4x)(1+5x)(1+6x)(1+5x+6x^2+x^3)$$
$$= 1 + 26x + 275x^2 + 1511x^3 + 4545x^4 + 7148x^5 + 4884x^6 + 720x^7$$

8. Recall that we had a theorem (Theorem 3 from Activity 7) that states "If u is a vertex of a graph G, then $\mu(G, x) = x\sum_{v\sim u}\mu(G\setminus\{uv\}, x) + \mu(G\setminus u, x)$".
Note $v \sim u$ indicates that we sum across all neighbors of u; $G\setminus\{uv\}$ means we delete both vertices u and v from the graph G, and $G\setminus u$ means that we delete vertex u from G.
Explain how this given theorem equation is an example of the Use/Don't Use Principle (i.e. indicate what it is that we use or don't use).

Answer: This is the Use/Don't Use for a vertex in the graph. If we do use a vertex *u*, then by the definition of a matching we can't also include any of its neighboring vertices. So we must consider all of the sub-graphs that don't include *u* and one of its neighbors, and we look for an $r-1$-matching in that remaining graph. We multiply by *x* since these *r-1*-matchings contribute to the total number of *r*-matchings in *G*. So the reason for multiplying by *x* is to shift the coefficients in the polynomial from the x^{r-1} terms to the x^r term. If we don't use vertex *u*, then we're looking for the number of matchings in the graph *G* minus the vertex *u*. This is how we arrive at the above expression.

Final Reflection

As I reflect back upon this whole process, several salient points come to mind. First, I was utterly amazed at the mathematical connections that came through as I studied this topic. I initially chose to study rook polynomials because I thought they might have an interesting application to counting principles, but I was not prepared for the wide variety of mathematical topics to which these polynomials relate. Indeed, the fact that most textbooks fail to mention such connections would have led me to believe that rooks were a mathematical concept almost entirely independent of the rest of a traditional combinatorics curriculum. It was genuinely exciting, then, to witness all of the inter-connectedness that this particular mathematical topic exhibits.

Enthused by the fascinating mathematics that I investigated, I was eager to see how much of it could be taught to students in a high school class. Although I had little expectation of how much of the mathematics we might actually be able to integrate into curriculum, it has been interesting to see just where points of entry can be made.

Because I initially did not know what to expect of the high school students, I was extremely pleased to have been afforded the luxury of time. In particular, I had an entire week between each meeting with the students. As a result, I could carefully tailor a highly targeted set of activities as we went along; as mentioned above, many of the seven activities (and three assessments) were informed by how the prior activity had gone. Thanks to the flexibility of this time schedule, I was able to be intentional and thoughtful as I planned each exercise, and I believe that this came across to the students. Ultimately I developed a curriculum that spanned all three of the major mathematical topics in the paper: counting principles, generating functions, and matchings.

So how did this grand experiment turn out? I was honestly surprised by how much of the mathematics that I had studied was able to be incorporated into a high school classroom. As I reflect now and evaluate how the entire curriculum process unfolded, I realize that I am very pleased with the overall outcome. Why is this so? Why do I consider it a success?

Superficially, all along the project seemed to be progressing quite smoothly. Many students expressed enthusiasm for our visits; they nodded, smiled, and were attentive in the classroom, and even more explicitly, they took occasions to tell us how much they were learning and enjoying the material. As nice as this was to hear, I was inclined to reserve judgment concerning the success or failure of the curriculum as a whole. In particular, I wanted to wait until I knew more exactly the extent to which these students were able to process the mathematics itself. Specifically, I felt the success of the curriculum should be judged primarily by some measure of how well the students understood the three mathematical aims of counting principles, generating functions, and matchings. Continuing to reflect upon the curriculum, let us focus on each of these mathematical concepts and discuss the didactical successes or drawbacks related to each.

We first consider the question of how well-suited rooks are for teaching counting principles. While, in theory, the answer is unquestionably affirmative, the particulars of our situation somewhat obviated the entire issue. That is to say, the students we taught were simply already quite good at counting before we even walked in the door. Therefore, I do not think that too much can be said regarding the ability of the curriculum to *introduce* counting principles. However, the rooks did provide a helpful context for *discussing* such counting principles. Indeed, as the reflections above indicate, the rook problems gave way naturally to discussions ranging from inclusion/exclusion to the multiplication principle, etc.

Turning now to the question of generating functions and matchings, however, it is clear that students gained a sophisticated understanding of these topics. This fact is evidenced by the speed and skill with which they computed rook and matchings polynomials for small boards. It just recently struck me that this fact, in and of itself, is a great accomplishment – providing evidence that the curriculum was indeed effective in a surprising way. Their ability to handle rook and matchings polynomials is so remarkable because generating functions are notoriously difficult to teach. Indeed, combinatorics professors often struggle with clear and convenient means of explicating this topic. In observing the students and talking with them, it was absolutely apparent that computing and manipulating generating functions was extremely natural for them – so natural, in fact, that I think they took for granted the degree of difficulty of the topic they were studying.

In fact, during a subsequent visit with them, the students presented solutions on the board from the Rook Exam that had been given. This was quite encouraging to see, as nine different students presented correct, clearly-explained solutions to the problems. I was once again struck by the deftness with which they manipulated generating functions. Also in this Rook Exam, they were asked to make connections among concepts they had encountered in several prior activities, and they made these connections effortlessly.

Finally, pedagogically, this experience was beneficial for me personally in another way. I have had little chance in my own teaching experience to experiment with or implement the ideas I have learned in my 'math ed' classes. For instance, I have long wanted to see whether or not students could, in practice, truly come up with sophisticated mathematical ideas. Prior to this project, I had not taken advantage of any occasion to do so. This particular teaching environment, however, provided me with the perfect opportunity to try this out. Indeed, I had the luxury of having almost no time constraints and virtual free rein to see if the students could come up with ideas about rook polynomials and matchings on their own. And in fact, on several notable occasions, they *did*, indeed, come up with some *amazing* results on their own, as the reflections above have indicated. Thus, through the vehicle of this project, I feel that I was able to gain first-hand experience with the potential that students have to develop new ideas on their own.

On the whole, then, this entire project – both the mathematics and the curriculum – have provided me with a wealth of new perspectives and insights. By revealing novel vistas of the higher mathematical terrain, and by offering breathtaking glimpses of an ideal pedagogy, this incredibly formative experience will undoubtedly shape, nourish, and empower the mathematician and educator I am to become.

References

Anderson, I. (1974). *A First Course in Combinatorial Mathematics*. Oxford: Clarendon Press.

Eisen, M. (1969). *Elementary Combinatorial Analysis*. New York: Gordon and Breach.

Godsil, C. D. (1993). *Algebraic Combinatorics*. New York: Chapman and Hall.

Leon, S. (2006). *Linear Algebra with Applications, 7th ed*. New Jersey, Prentice Hall.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, NCTM.

Tucker, A. (2002). *Applied Combinatorics, 4th ed.* New York: Wiley & Sons, Inc.

West, D. (2001). *Introduction to Graph Theory, 2nd ed.* New Jersey: Prentice Hall.

Wilf, H. (1994). *Generatingfunctionology*. Boston: Academic Press.