Cycloids and Paths

Why does a cycloid-constrained pendulum
follow a cycloid path?

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Abstract

My MST curriculum project aims to explore the history of the cycloid curve and some of its many interesting properties. Specifically, the mathematical portion of my paper will trace the origins of the curve and the many famous (and not-so-famous) mathematicians who have studied it. The centerpiece of the mathematical portion is an exploration of Roberval's derivation of the area under the curve. This argument makes clever use of Cavalieri's Principle and some basic geometry. Finally, for closure, the paper examines in detail the original motivation for this topic -- namely, the properties of a pendulum constricted by inverted cycloids. Many textbooks assert that a pendulum constrained by inverted cycloids will follow a path that is also a cycloid, but most do not justify this claim. I was able to derive the result using analytic geometry and a bit of knowledge about parametric curves.

My curriculum side of the project seeks to use these topics to motivate some teachable moments. In particular, the activities that are developed here are mainly intended to help teach students at the pre-calculus level (in HS or beginning college) three main topics: (1) how to find the parametric equation of a cycloid, (2) how to understand (and work through) Roberval's area derivation, and, (3) for more advanced students, how to find the area under the curve using integration. Many of these materials have already been tested with students, and so some reflections are included on how to best implement them.

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Part One:

The History and Mathematics of the Cycloid Curve

Chapter 1 – **Introduction and History of the Cycloid**

Section 1.1 – **Introduction**

A cycloid is the elongated arch that traces the path of a fixed point on a circle as the circle rolls along a straight line in two-dimensions. The study of this curve is a subject rich in mathematical meaning, scientific application, and important connections within the history of mathematics. While a child can understand the basic concept of a cycloid, a variety of more advanced mathematical topics -- such as unit circle trigonometry, parametric equations, and integral calculus -- are needed for any real mathematical understanding of the topic. While almost any calculus textbook one might find would include at least a mention of a cycloid, the topic is rarely covered in an introductory calculus course, and most students I have encountered are unaware of what a cycloid is. For the rare teacher with extra time (or for students who are looking for an extra challenge), an exploration of the cycloid is not only meaningful, but also offers an opportunity to use several different mathematical skills together in a problem-solving situation. Indeed, the entire exploration can still remain (primarily) within the realm of "pure math," rather than having to rely heavily on any one specific science or engineering application.

For the mathematician, two main questions arise immediately from the definition of the cycloid. First, what is the length of one cycloidal arch in relation to the size of the circle that generated it? Or, in other words, how far has the point traveled? And second, what is the area between a cycloidal arc and the straight line its "generating circle" rolls along? The answer to both these questions puzzled mathematicians for centuries until the development of calculus brought with it ways to answer them both. And indeed, calculus offered the world some surprisingly tidy answers: the arc length of a single cycloidal arch is precisely 8 times the radius of the generating circle, and the area under such an arch is exactly 3 times the area of the generating circle. We will consider such mathematical properties later in this paper, but we begin with a brief history of the curve. The cycloid has been called the "Helen of Geometers," due to its pull on some of the greatest mathematicians of the time. Indeed the history of the cycloid reads like a "Who's Who" of renaissance and enlightenment era mathematicians.

Section 1.2 – **Early History of the Cycloid**

D.E. Smith, in his *History of Mathematics*, gives the following brief history of the cycloid:

> "This curve, sometimes incorrectly attributed to Nicholas Cusa (c. 1450), was first studied by Charles de Bouelles (1501). It then attracted the attention of Galileo (1599), Mersenne (1628), and Roberval (1634). Pascal (1659) completely solved the problem of its quadrature, and found the center of gravity of a segment cut off by a line parallel to the base. Jean and Jacques Bernoulli showed that it is the brachistochrone curve, and Huygens (1673) showed how its properties of tautochronism might be applied to the pendulum." (327)

The above paragraph, published in 1925, still forms a reasonably accurate outline of the history of the curve's study, but a few more famous names can be added to the list, and we can seek to fill in some of the interesting details. Although the ancient Greeks were aware of a similar phenomenon they called "double motion," there is no evidence that they knew of, or studied the cycloid. Cusa, elsewhere called Cusanus, was said to have discovered the curve in a letter by John Wallis in 1679. However, many scholars, including Cantor (Whitman, 310) have found no evidence of such a discovery, leaving the true discoverer of the curve lost to history. The French mathematician Charles de Bouelles studied the curve, but erroneously thought it was just part of a larger circle, with a radius equal to one-and-a-fourth times that of the generating circle (Whitman, 310).

This brings us to Galileo, who, according to Cantor, both popularized the curve and gave it its name. One of his pupils wrote that he first attempted the quadrature of the cycloid in 1599. (The term quadrature refers to “squaring” a shape by constructing a square with equal area. The quadrature of the circle is one of the great mathematical problems of antiquity, and has long been proven impossible.) As a method of finding the area under the arch, Galileo cut the shape out of a material (some say sheet metal) and compared its weight with that of a generating circle cut from the same material. Several experiments resulted in approximately the same ratio, 3 to 1, before Galileo gave up the study thinking (mistakenly) that the ratio was “incommensurable,” what we now call irrational. The clever approach of using weight to determine area empirically was a hallmark of Galileo's approach to science.

Mersenne, who is also sometimes called the discoverer of the cycloid, can only truly be credited with being the first to give a precise mathematical definition of the curve. However, it was Mersenne who proposed the problem of the quadrature of the cycloid (and the construction of a tangent to a point on the curve) to at least three other very significant mathematicians: Roberval, Descartes, and Fermat. While all three responded with unique constructions, only Roberval was able to conquer the area problem. His ability to do so was based on a new way of finding areas under curves discovered by a student of Galileo. This student was named Bonaventura Cavalieri, and he is the namesake of the well-known Cavalieri's principle. In a later section, we will examine this approach in detail.

Section 1.3 – Breakthroughs and Calculus

Moving on through history, the next major advances in the study of the cycloid were made by Blaise Pascal, the famous French philosopher and mathematician. Although Pascal had given up mathematics in favor of theology, it is documented that a combination of insomnia and a toothache caused his mind to settle on the idea of a cycloid. When the toothache disappeared, he took it as a divine sign that he was permitted once again to engage in mathematical pursuits. Once Pascal had completed a fairly up-to-date study of the curve, he held a contest to answer some of the remaining questions:

1. How to find the area and center of gravity of the region formed between one arch of the cycloid and the x axis.
2. How to find the volume and center of gravity of the solid formed by rotating the cycloid about the x axis.
3. How to find the volume and center of gravity of a solid formed by cutting the above solid with a plane parallel to the x axis.

While only two contestants attempted to answer these questions (and neither did so successfully), by the time the contest was over, three men -- Pascal, Roberval, and Sir Christopher Wren -- had all found satisfactory answers to all the questions. Indeed, Pascal had published *L'Histoire de la Roulett,* (Roulette is the French word for cycloid) bringing this chapter of the story to a close. While Newton was sixteen years old at the time of Pascal's contest, and no calculus (as we know it) was in existence at the time, the solving of these problems was done using new notions of "infinites," or "indivisibles" -- infinitely small slices of shapes that would become the "fluxions" of Newton's calculus.

Some fifteen years later, Christiaan Huygens, a Dutch mathematician, physicist and astronomer, found that constraining a pendulum with two inverted cycloids caused the pendulum to swing in the shape of the same cycloid. With such a pendulum, a wonderful curiosity occurs: the length of the arc each swing follows has no effect on the constant length of time each swing takes. This is due to what is called the "tautochrone" property of inverted cycloids. A ball placed at any point along an inverted cycloid will take the exact same time to roll to its lowest point as a ball placed anywhere else along the cycloid. At the time, this was hailed as a breakthrough that could provide for much more accurate timekeeping. Somewhat disappointingly, however, the limitations of physically implementing this curve did not allow a significant improvement based on this technique alone.

In 1686, Leibniz was able to write the first explicit equation for the curve:

$$y = \sqrt{2x - xx} + \int dx / \sqrt{2x - xx}. \quad \text{(Whitman, 315)}$$

In 1696, the Bernoulli brothers, Jacques and Jean, who had already written some papers on the cycloid, proposed a related mathematical problem known as the brachistochrone problem. The main idea was this: what is the fastest path for a particle pulled by its own weight to travel from one point on a vertical plane to a lower point on the same plane, not directly below it? Famously, the two inventors of calculus, Newton and Leibniz, were both able to answer the question, as were the two brothers who posed it (Dunham, 201-202). The answer, which will come as no surprise at this point, is that the bachristachrone curve is precisely given by the inverted cycloid.

While there is still interesting research being done on cycloids today (see Apostol), most of the important questions concerning the curve were answered before the beginning of the eighteenth century. Therefore, this is where our short survey ends. The rest of this paper, however, will explore Roberval’s quadrature of the cycloid, and will culminate with an examination of Huygen’s cycloid-constrained pendulum.

Chapter 2 – Roberval's Derivation of Area Under a Cycloid

This section is adapted from an exercise in *Mathematics for High School Teachers* by Usiskin, Peressini, Marchisottro and Stanley.

Section 2.1 – Parameters for the Cycloid

A cycloid is the curve produced by tracing the path of a point on a circle as that circle rolls along a straight path. Although the cycloid curve can be given an explicit equation, there also exists a well-known parametric equation that is much simpler to state. In this formulation, we can produce equations for the x- and y-coordinates of the curve in terms of a single parameter t, which denotes the amount of revolutions the circle has turned (in radians). Alternatively, if we assume that the circle is turning at a constant rate, the parameter t could also be regarded as measuring the elapsed time since the circle began rolling.

We will call the radius of our circle a. A graph of the cycloid curve and its generating circle, at t =0, is shown in Figure 2.1. (The x and y coordinates of the grid are scaled as multiples of a.)

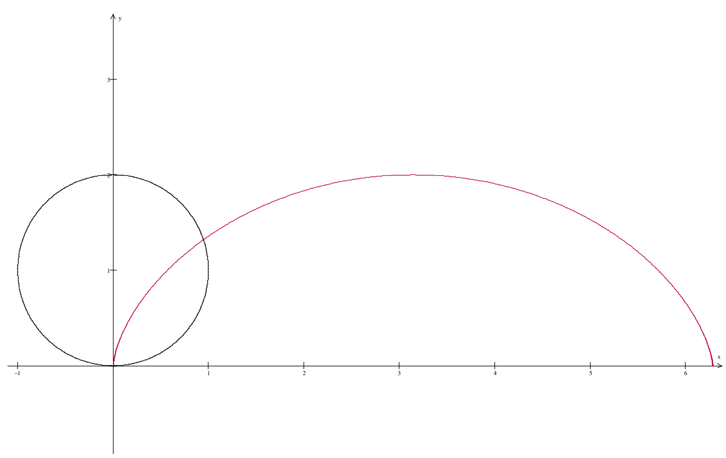


Fig. 2.1: Cycloid curve shown with its generating circle

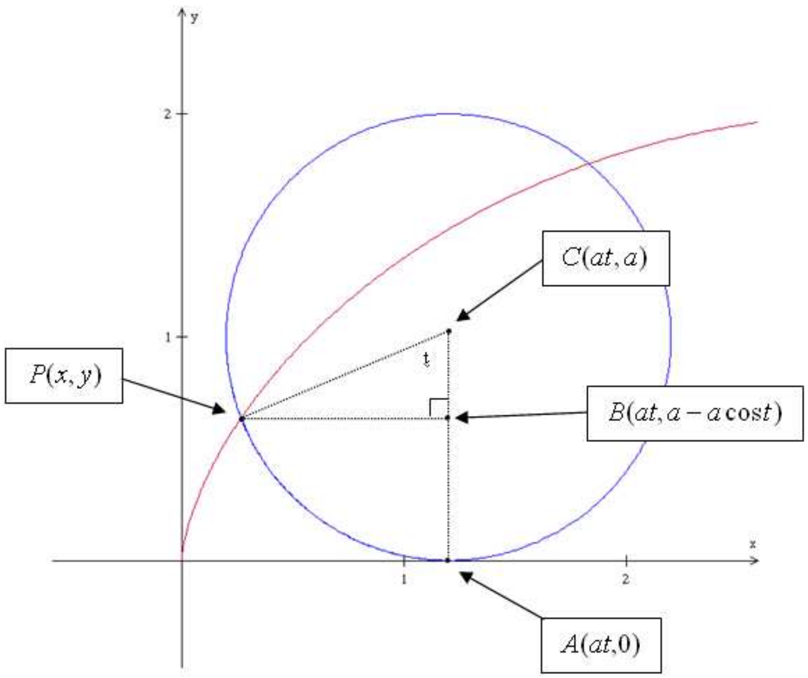


Fig 2.2: Detail of cycloid curve with generating circle

Figure 2.2 shows the left portion of the cycloid curve in red, with its generating circle shown after it has rotated t radians, in blue. The center of the circle has been labeled C, and two radii have been constructed, one to the point (x, y) on the cycloid, and another, vertical radius to a point, A , on the x axis.

Notice the x coordinate of both A and C, is given by the arc length of the circle between A and P, $a \cdot t$. Next, a perpendicular to $\overline{AC}$ through P is constructed and the intersection of it with $\overline{AC}$ is called B. Using right triangle trigonometry on triangle PCB gives us the following:

$$\sin t = \frac{\textit{length of } \overline{PB}}{a} \text{ or } \textit{length of } \overline{PB} = a \cdot \sin t$$

and

$$\cos t = \frac{\textit{length of } \overline{BC}}{a} \text{ or } \textit{length of } \overline{BC} = a \cdot \cos t$$

We can use these lengths to find the coordinates of P in terms of t. Since the x-coordinate of P is simply is the arc length, at minus the length of $\overline{PB}$, and the y-coordinate of P is the radius r minus the length of $\overline{BC}$, some common-term factoring of a gives us the following parametric equation for P:

The x-coordinate of point P is given as

$$x = a(t - \sin t).$$

The y-coordinate is given as

$$y = a(1 - \cos t).$$

<u>Section 2.2</u> – <u>A Companion Curve for the Cycloid</u>

To this parametric graph we now add what Roberval called the "companion curve" of the cycloid. This is a second curve, which has the parametric equation   x= at and   y= a(1-cos(t)). The generating circle, at its starting point, whose parametric equation is given by x = a(t-sin(t)) and y=a(1-cos(t)) also is seen below See Figure 2.3.

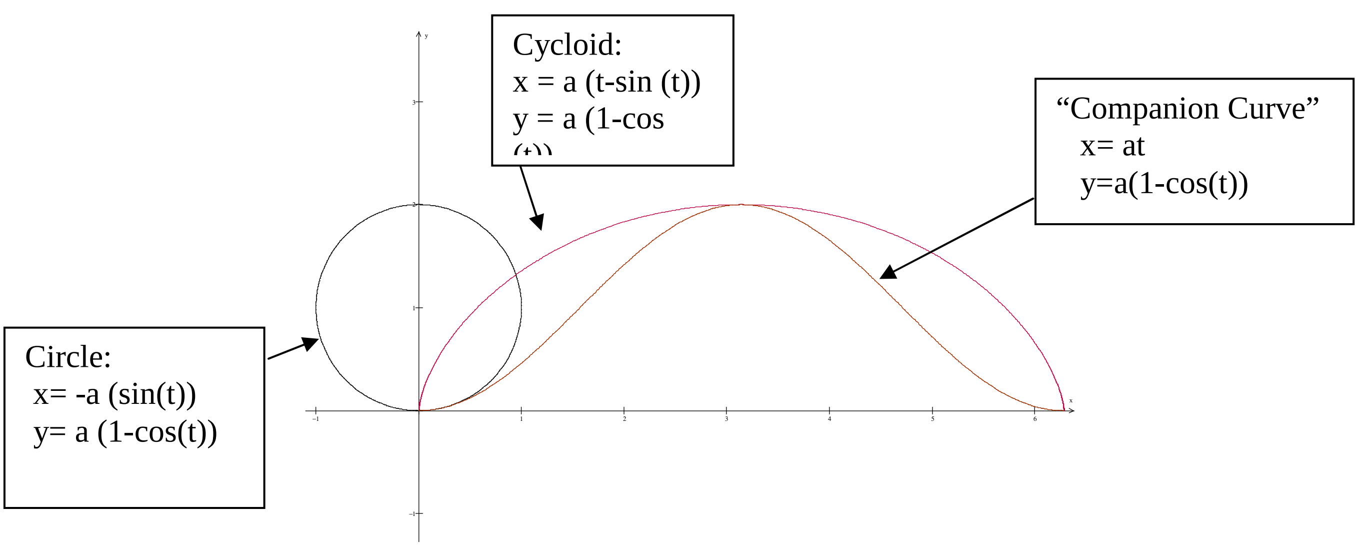


Fig. 2.3: Cycloid curve shown with its generating circle and companion curve

Since all three parametric equations have the same parametric expression for y, we see that at any given t value, the corresponding points on the circle, cycloid and companion curve will have the same y value, and hence, be on the same horizontal line y= a(1-cos(t)).

For a given t value (see Fig. 2.4) the distance between a point P_s on the left half of the generating circle and the y axis is the difference between their x coordinates:

$$\text{dist}(P_s\ , \text{y-axis}) \ = \ 0\text{-}(\text{-}a(\sin(t)) \ = \ a \sin t.$$

At the same value of t, the distance between the corresponding point P_c on the companion curve and the point P_t on the cycloid is given by

$$\text{dist}(P_c\ , P_t\ ) \ = \ at - (at - a\sin(t)) \ = \ a\sin(t)$$

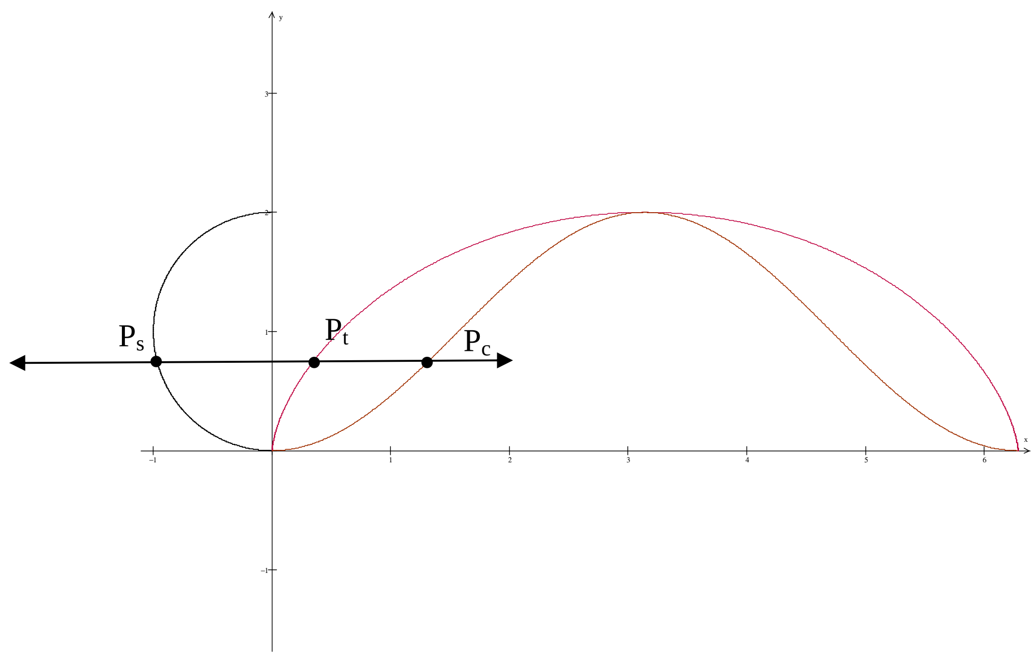


Fig. 2.4: Horizontal line intersecting cycloid curve, generating circle, & companion curve

Cavalieri's principle states that if two regions have the same height for every x in [a, b], then they have the same area in that interval. We are using a rotated version of Cavalieri's principle in which the two regions have the same width for every y in the interval [0, 2a]. The two regions in question are the region between the semicircle and the y axis and the region between the cycloid and the companion curve.

It follows immediately that the region between the cycloid and the companion curve must have the same area as the semicircle, which is, of course, equal to $\frac{\pi a^2}{2}$. In the next section we will use this to deduce the total area.

<u>Section 2.3</u> – <u>Finding the Total Area</u>

Now we consider the rectangle with vertices (0, 0), (πa, 0), (πa, 2a) and (0, 2a). See Figure 2.5 below:

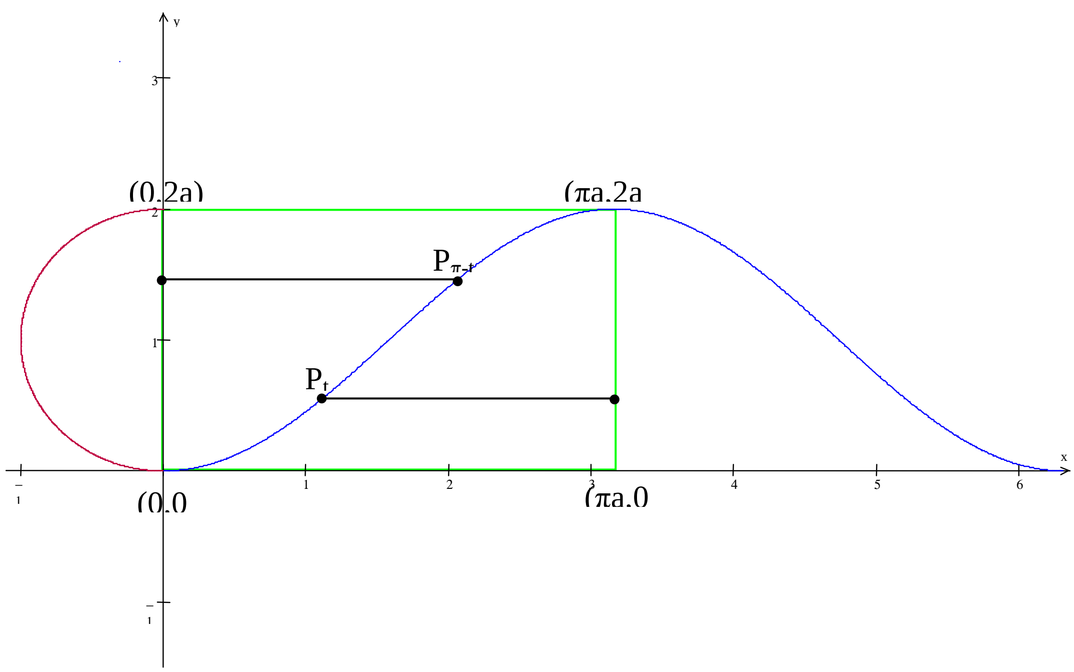


Fig. 2.5: Companion curve and generating semi-circle with dimensions

For any given t, a point on the companion curve, P_t is the same distance from the right side of the rectangle as another point $P_{\pi-t}$ is from the left side of the rectangle.

Distance from P_t to right side of rectangle = $\pi a - at = a(\pi - t)$

Distance from $P_{\pi-t}$ to left side of rectangle = $a(\pi-t) - 0 = a(\pi-t)$

Since the width between P_t and $x=2a$ for any t in the interval $[0, \pi]$ is equal to the width between $P_{\pi-t}$ and $x=0$, we can use Cavalieri's principle again to show that the two regions of the rectangle separated by the companion curve have the same area. Since the regions have the same area, and their union is the rectangle, we conclude that the companion curve divides the rectangle into 2 halves of equal area. See Figure 2.6.

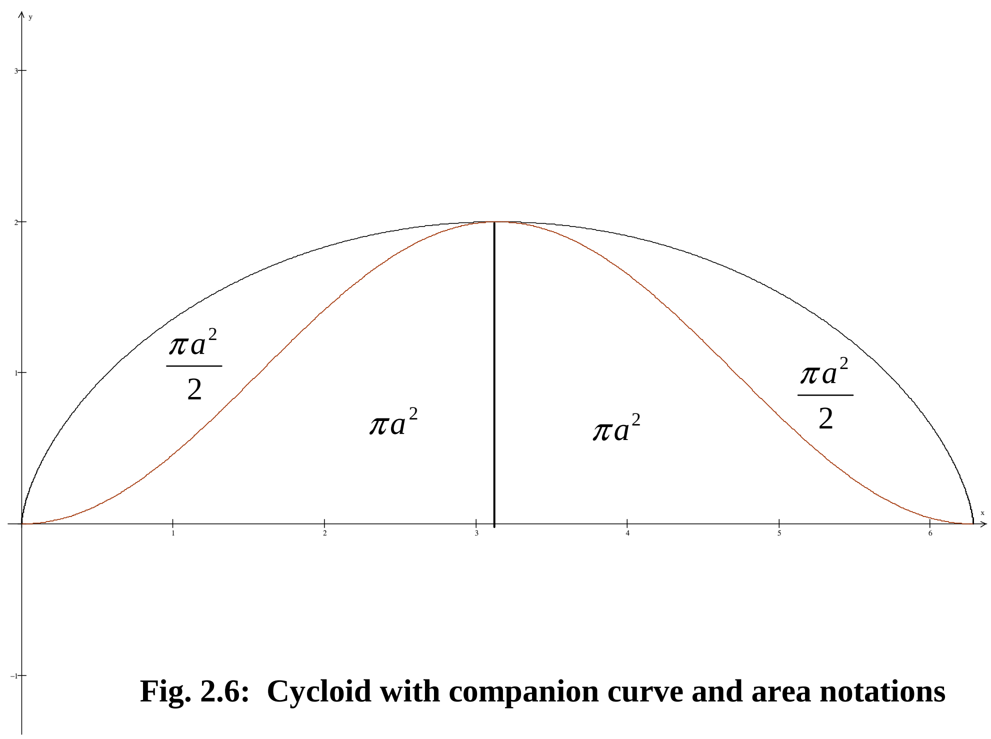


Fig. 2.6: Cycloid with companion curve and area notations

The area of the entire rectangle is πa times $2a$, which equals $2\pi a^2$, half of which is under the companion curve. The area under the companion curve between 0 and πa

equals πa^2 and the area between the companion curve and the cycloid on the same interval is $\frac{\pi a^2}{2}$, making the total area under the cycloid between 0 and πa equal to $\frac{3\pi a^2}{2}$. A similar argument could be made concerning the area under the right side of the cycloid, or we could rely on the symmetry of the cycloid to show the area under the right side of the cycloid is also $\frac{3\pi a^2}{2}$. Therefore the total area under one arch of the cycloid curve is $2\left[\frac{3\pi a^2}{2}\right] = 3\pi a^2$ or 3 times the area of the generating circle.

Chapter 3 – A Pendulum Constrained by Inverted Cycloids

Section 3.1 – Setting up the Pendulum

In this section we consider a pendulum swinging on a cord that is constricted by two inverted cycloids. We will show that such a pendulum swings in the shape of a congruent, inverted cycloid. Consider the Figure 3.1.

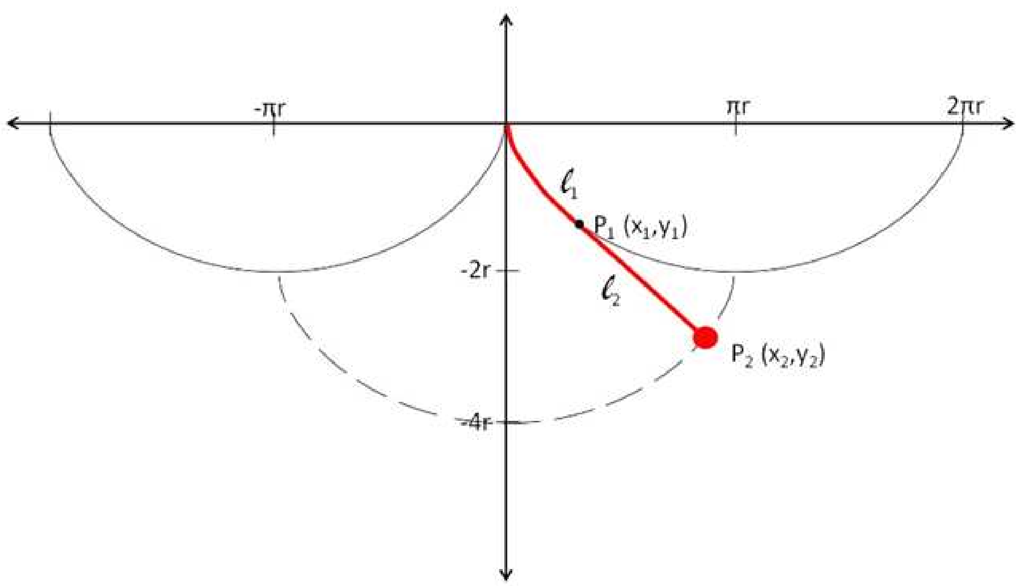


Fig. 3.1: Path of a pendulum constrained by two inverted cycloids

First we need to determine the length of the pendulum's cord. From figure 3.1, we see that it needs to be half the length of the cycloid. This can be shown to be 4r by using the arc length integral formula, as follows.

Begin with the arc length formula, $L = \int_a^b \sqrt{\left(\frac{dx}{dL}\right)^2 + \left(\frac{dy}{dL}\right)^2}\, dt$

Taking the derivatives of our parametric for the cycloid gives us: $\begin{aligned} x' &= a(1-\cos t) \\ y' &= a(\sin t) \end{aligned}$

Substituting into the arc length formula to find the length of the arc between 0 and π

gives us: $L = a\int_0^{\pi} \sqrt{2(1-\cos t)}dt$

$$L = a\int_0^{\pi} \sqrt{4\sin^2\left(\frac{t}{2}\right)}dt$$

$$L = 2a\int_0^{\pi} \sin\left(\frac{t}{2}\right)dt$$

$$L = 2a\left[-2\cos\left(\frac{t}{2}\right)\right]_0^{\pi}$$

$$L = 4a$$

For any point in the path of the pendulum, we can divide the cord into two sections, L_1, the part that is wrapped around the inverted cycloid, and L_2, the part that swings free, and we define $P_1(x_1,y_1)$ as the point that divides the two, where the pendulum leaves the inverted cycloid.

Note that P_1 has the parametric equation of a cycloid except the equation for the y coordinate is negative. That is

$$\begin{aligned} x_1 &= r(\theta - \sin\theta) \\ y_1 &= -r(1-\cos\theta) \end{aligned}$$

Also note that L_2 is tangent to the inverted cycloid at P, so the slope of L_2 is given by the derivative of the inverted cycloid at theta.

$$m_{l2} = -\frac{\sin\theta}{1-\cos\theta}$$

Since the length of L_1 is the arc length of the cycloid at theta, the length of L_2 can be found by subtracting the arc length of the cycloid at theta from 4r, the length of the pendulum.

$$L_2 = 4r - \left(-4r\cos\frac{t}{2} \right)\Bigg|_0^{\theta} = 4r\cos\frac{\theta}{2}$$

We now have a situation where, at any theta, we know the coordinates of P, an endpoint of L_2, we know the length of L_2, and we know the slope of L_2. We should be able to derive a parametric equation for the other endpoint, which happens to be the end, or bob, of the pendulum.

Section 3.2 – Coordinates for the Endpoint

Starting with two well known high school math formulas -- the slope and distance formulas -- we are able to find equations for the coordinates of the missing endpoint of any segment, given a length L, an endpoint (x_1, y_1), and a slope m.

By first solving the slope formula for (y_2-y_1) and substituting into the distance formula, then solving for x_2, we get

$$x_2 = \pm\sqrt{\frac{l^2}{m^2+1}} + x_1$$

Similarly, if we solve the slope formula for (x_2–x_1) and substitute into the distance formula and solve for y_2, we get the slightly more complicated formula below.

$$y_2 = m\left[\pm\sqrt{\frac{l^2}{m^2+1}}\right] + y_1$$

Now it's a matter of substitution and algebra to find a parametric for (x_2, y_2) the moving end of the pendulum. We will begin with x_2.

$$x_2 = \frac{\pm 4r\cos\frac{\theta}{2}}{\sqrt{\left(\frac{\sin\theta}{1-\cos\theta}\right)^2+1}} + r\theta - r\sin\theta$$

$$x_2 = \frac{\pm 4r\cos\frac{\theta}{2}}{\sqrt{\left(\frac{\sin^2\theta + (1-\cos\theta)^2}{(1-\cos\theta)^2}\right)}} + r\theta - r\sin\theta$$

$$= \frac{\pm 4r\cos\frac{\theta}{2}(1-\cos\theta)}{\sqrt{\sin^2\theta + (1-\cos\theta)^2}} + r\theta - r\sin\theta$$

$$= \frac{\pm 4r\cos\frac{\theta}{2}(1-\cos\theta)}{\sqrt{2-2\cos\theta}} + r\theta - r\sin\theta$$

Now we need to use the half angle trig identity, $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$.

$$x_2 = \frac{4r\left(\pm\sqrt{\frac{1+\cos\theta}{2}}\right)(1-\cos\theta)}{\sqrt{2(1-\cos\theta)}} + r\theta - r\sin\theta$$

Now when multiply the numerator and denominator by the denominator, we get some nice cancellation giving us

$$x_2 = \frac{4r\left(\pm\sqrt{1-\cos^2\theta}\right)}{2} + r\theta - r\sin\theta$$

or just

$$x_2 = 2r\left(\pm\sin\theta + r\theta\right) - r\sin\theta$$

which simplifies to

$$x_2 = r(\sin\theta + \theta) \quad or \quad x_2 = r(\theta - 3\sin\theta).$$

Now to y_2:

Our initial substitution utilizing the simplification of the radical expression from the previous work, yields:

$$y_2 = \left(-\frac{\sin\theta}{1-\cos\theta}\right)\left[\pm(2r\sin\theta)\right] - r(1-\cos\theta)$$

which can be simplified to:

$$y_2 = \pm\left(\frac{2r\sin^2\theta}{1-\cos\theta}\right) - r(1-\cos\theta)$$

Substituting using the Pythagorean identity yields:

$$y_2 = \pm\left(\frac{2r(1-\cos^2\theta)}{1-\cos\theta}\right) - r(1-\cos\theta)$$

Now we utilize a "difference of two squares" factoring pattern:

$$y_2 = \pm\left(\frac{2r(1+\cos\theta)(1-\cos\theta)}{1-\cos\theta}\right) - r(1-\cos\theta)$$

$$= \pm 2r(1+\cos\theta) - r(1-\cos\theta)$$

$$= 3r + 3r\cos\theta \quad or \quad y_2 = -3r - r\cos\theta$$

We now have two parametric equations for x_2 and y_2. This situation is due to the fact that our equation to find the other endpoint of a line segment, given its first endpoint and slope, finds two new endpoints by traveling different directions along the line. A quick check with a graphing utility shows that the endpoint we want is the following:

$$x_2 = r(\sin\theta + \theta), \text{ and } \quad y_2 = -3r - r\cos\theta$$

Section 3.3 – **Recognizing the Shifted Cycloid**

Now we want to show that our new parametric equations for (x_2, y_2) show a congruent, but shifted inverted cycloid. It would be a mistake to think that, because we need our new cycloid to be shifted down 2r and left (or right) πr that our new parametric must be shown equal to the parametric of the inverted cycloid with πr subtracted from (or added to) the x formula and 2r subtracted from the y formula. Although those equations would give us the graph we want, they cannot be shown to be equal to our new parametric. This is a curious trait of parametric equations: *two unequal parametric equations can give the exact same graph*.

In order to find an equivalent shift, we notice that at $\theta = 0$, the pendulum is exactly at the halfway point in its cycloid, or at the same position the original, inverted cycloid is at when $\theta = \pi$. So, in order to have this happen, we substitute $\theta + \pi$ for θ in our x_1 equation. To shift the cycloid left πr, we need only to subtract πr.

So now we need to show that our two formulas for x_2, one that we derived earlier, and the other that we obtained by shifting the input of our inverted cycloid, are equal.

$$r((\theta + \pi) - \sin(\theta + \pi)) - \pi r = r(\sin\theta + \theta)$$

We will work to equate the left side with the right. First, distribute the r.

$$r\theta + r\pi - r\sin(\theta + \pi) - \pi r = r(\sin\theta + \theta)$$

Note that the πr terms cancel, and the trig identity that $\sin(\theta+\pi)=-\sin\theta$ gives us:

$$r\theta + r\sin\theta = r(\sin\theta+\theta),$$

or:

$$r(\sin\theta+\theta) = r(\sin\theta+\theta).$$

Now we can shift the y equation of the inverted cycloid the same way, by replacing θ with $(\theta+\pi)$. This time, however, we need to shift it down 2r, so we subtract 2r. Which gives us:

$$y_2 = -r(1-\cos(\theta+\pi)) - 2r$$

We need to show this shift of y_1 is equal to our derived equation, $y_2 = -3r - r\cos\theta$. Once again, this is done by first distributing r.

$$-r + r\cos(\theta+\pi)) - 2r$$

Then using the trig identity, $\cos(\theta+\pi) = -\cos\theta$,

$$-r - r\cos(\theta) - 2r$$

Combining like terms gives us our derived formula for y_2:

$$-3r - r\cos(\theta).$$

So, we have shown that a pendulum constrained by two inverted cycloids will indeed swing in a path of a congruent, but shifted, cycloid.

Part Two:

Teaching Cycloids

Overview of the Curriculum Project

While cycloids are covered in most Calculus and Pre-Calculus texts, they are rarely taught, and it's not difficult to understand why not. In both high school and college level classrooms, time is precious, and essential topics and standards are many. While cycloids may present an excellent example of not only the necessity of parametric equations, but how to integrate and differentiate them, they also require many prerequisite skills and the ability to use those skills to problem solve. Even in a Calculus class, much scaffolding is required so that all the class can keep up.

Sometimes, however, there is extra time, even in an advanced math class. For example, at my high school, there are a handful of non-seniors in the calculus classrooms. Since the seniors graduate one or two weeks early, they are left with several class periods after the final with little material left to cover. These lessons would fit perfectly in this situation. Alternately, a calculus teacher could continue to return to the topic of cycloids throughout the year (or even throughout two years).

While it's often difficult to hold students accountable for complex exploration type activities, a teacher could grade the worksheets provided for completion and, if students were given time to work in groups, he or she could grade each group for effort and participation.

Since the first lesson requires some knowledge of parametric equations, along with right-triangle trigonometry, it would probably be best suited to a pre-calc or calc class, although with some explanation of parametric equations, it could be taught to advanced Algebra 2 students. The remaining three lessons require calculus and could either be taught throughout the year (for example, the second lesson could be taught after covering Cavalieri's Principle, the third after covering integration of parametric equations) or as a culminating activity at the end of the year, as discussed above.

I've taught the first and second lesson twice, once over two class periods to a small group of high school calculus students at Rex Putnam H.S. and once to a community college pre-calc class over one long class period at Longview Community College.

Introduction to Activity 1

Instructional Goal: Students will learn what a cycloid is, label the coordinates of a diagram of a cycloid and its generating circle, and finally use those coordinates to derive the parametric equation of a cycloid.

Time needed: Roughly 45 minutes to one hour.

Prerequisite knowledge: Students should have some knowledge of parametric equations along with good right-triangle trigonometry skills including the use of radian measure as well as some basic algebra and geometry knowledge.

Supplies: Students need only the worksheet provided. The teacher, if possible, should have a graphing utility with the ability to graph parametric equations (Winplot, a free download, and Grapher, which comes with most Apple computers, both work well). A projector to project this graph on a large screen is also highly recommended.

Classroom Organization: While introduction to cycloids at the beginning of the lesson needs to be done in lecture/notes format, the remaining exercises could, and probably should, be done as small group or pair activities.

Activity 1 – Introduction to Cycloids/Deriving a Parametric Equation for a Cycloid

Part 1: Introduction

Worksheet 1.1 is passed out and students are asked to read the definitions of a cycloid. Students are asked to sketch what they think a cycloid will look like. After everyone has a sketch, a cycloid applet is shown on the projector (lots of good applets are available online, ex: http://www.ies.co.jp/math/java/calc/cycloid/cycloid.html) so that the class has a clear understanding of what a cycloid is and what it looks like. Answer the questions about the dimensions of the cycloid as a class.

Part 2: Finding the parametric:

Students are assigned to pairs or groups of 3-4, and are given Worksheet 1.2 to work on. Students should be reminded that the coordinates of the points will be in terms of r, the radius and theta, the amount of rotation of the circle, in radians. Students should also be reminded that when they find the coordinates of P, a point on the cycloid, in terms of r and theta, they will have found a parametric equation for the cycloid. As students work to find the coordinates of the points, the teacher floats about the class, answering questions and giving hints and advice. If the same questions keep coming up, or if the class seems stuck at one place, the teacher may want to address the class as a whole, or allow a group to present their partial finding.

Part 3: Graphing the Parametric:

When the class has all found parametric equations for P, or different groups have different solutions, the teacher can begin graphing them on the projected graphing utility to check if the equations really make cycloid curves. When the class is satisfied that the equations are correct, all students should copy down the correct equations and the exercise is over.

Extensions/Homework:

If an extension or homework assignment is desired, the teacher may want to introduce the concept of a ***hypocycloid***, the path of a point on a circle as it rolls *inside a bigger circle*. Again, applets are available online, and many students will recognize this as being similar to Spirograph, a children's toy. The parametric for a hypocycloid will have 2 constants, R, the radius of the larger circle, and r, the radius of the smaller circle, as well as theta, the amount of rotation in radians. To make finding the parametric easier, situate the origin at the center of the larger circle.

Name__________________ Date_______________

Cycloid Exploration Worksheet 1.1

<u>Part I: What is a cycloid?</u>

Wikipedia:

> the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line.

Webster:

> a curve that is generated by a point on the circumference of a circle as it rolls along a straight line.

Dictionary.com:

> a curve generated by a point on the circumference of a circle that rolls, without slipping, on a straight line.

Exercise 1: Based on the definitions above, sketch a cycloid.

Question: What is the distance along the x axis between the beginning and end of the cycloid?

Question: What is the height of the cycloid?

Part II: The Parametric Equation of a Cycloid

Exercise 2: Find the coordinates of the following points in terms of r, the radius of the circle and θ, the amount of rotation of the circle.

Point A:

(hint: what does the x coordinate of point A represent in terms of the circle?)

Point C:

Now use right triangle trigonometry to find the length of $\overline{PB}$ and $\overline{BC}$ in terms of θ:

Length of $\overline{PB}$:

Length of $\overline{BC}$:

Use the length of the above segments to find the coordinates of B and P.

Point B:

Point P:

The coordinates of point P, a point on the cycloid, represent a parametric equation for the cycloid. After checking them with a graphing utility, write the parametric equation of the cycloid below.

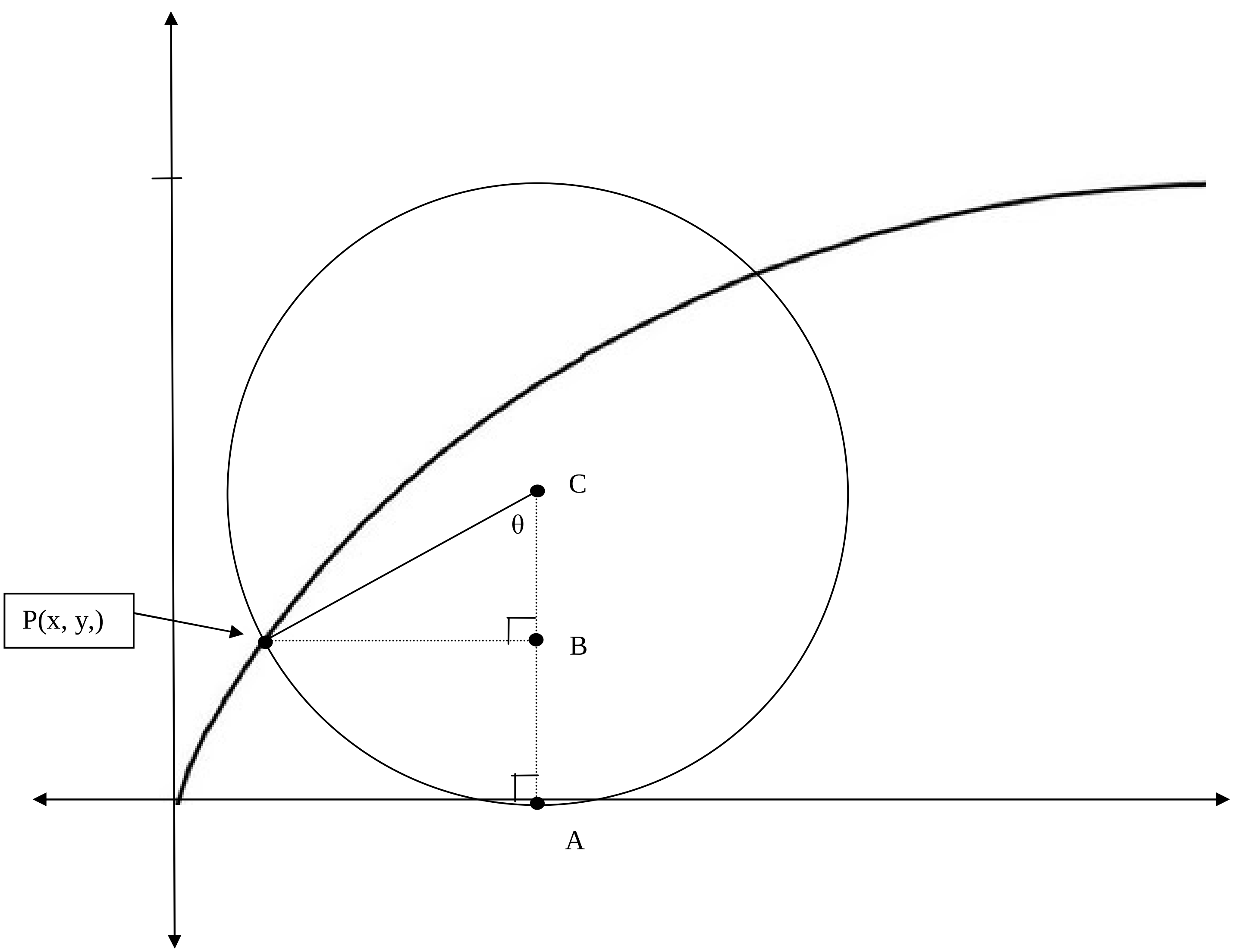
Parametric Equation of a Cycloid of radius r:

x=

y=

Worksheet 1.2

Figure 1: Cycloid with Generating Circle



Activity 1 – Introduction to Cycloids/Deriving a Parametric Equation for a Cycloid (Teacher's Key)

Part 1

Question 1: The length of the cycloid along the x axis is $2\pi r$, the circumference of the generating circle.

Question 2: The height of the cycloid is 2r, the diameter of the generating circle.

Part 2:

Point A: $(r\theta, 0)$ The x coordinate of point A is $r\theta$, the intercepted arc length of θ.

Point C: $(r\theta, r)$

Length of $\overline{PB}$: $r \sin\theta$

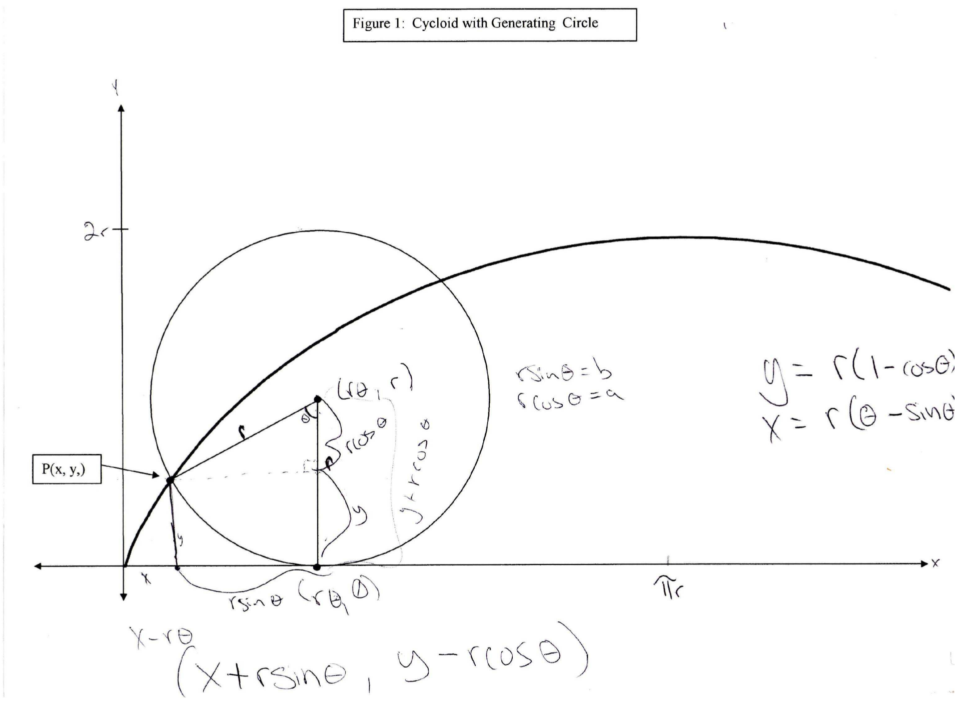
Length of $\overline{BC}$: $r \cos\theta$

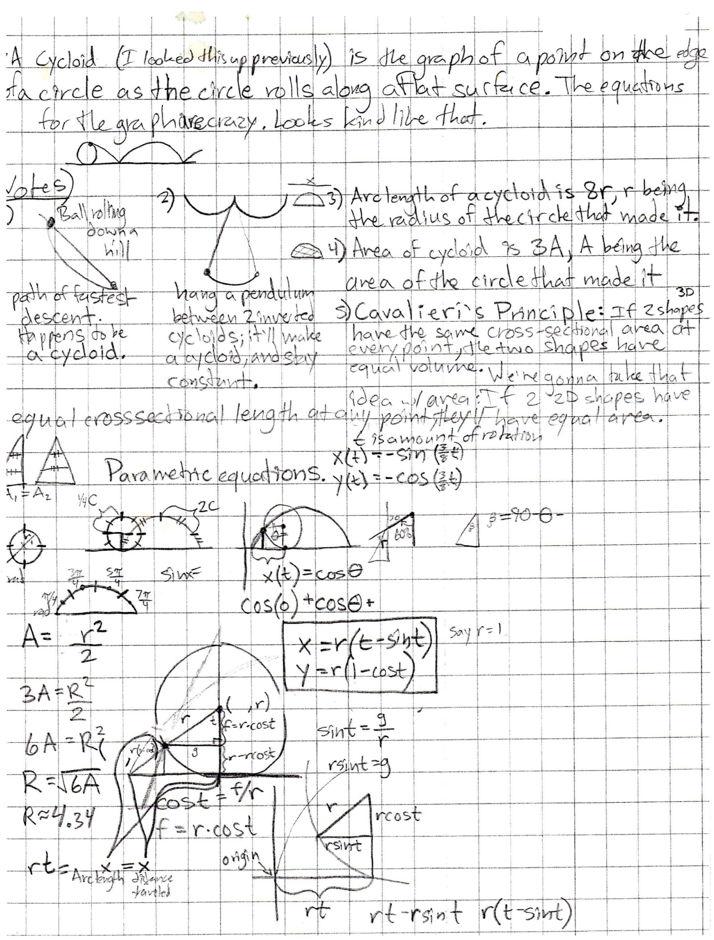
Point B: $(r\theta, r - r\cos\theta)$

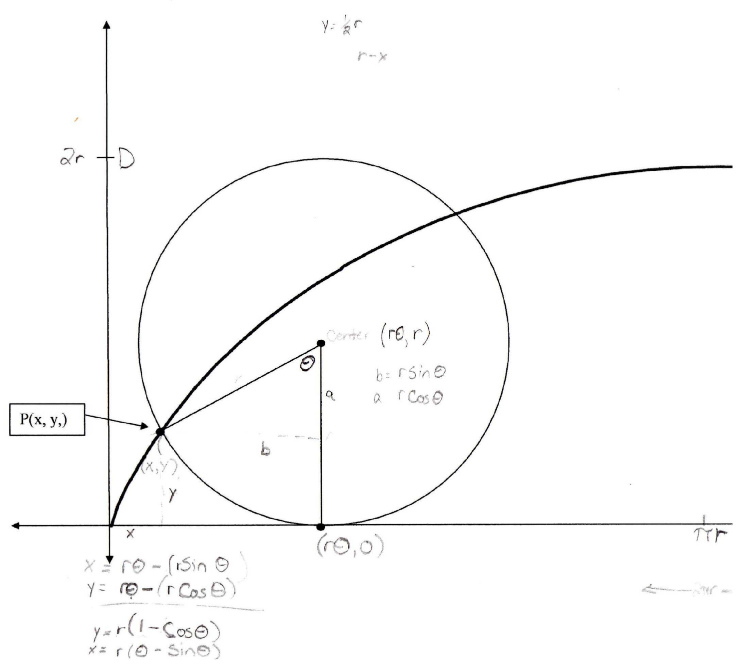
Point P (parametric for the cycloid): $(r\theta - r\sin\theta, r - r\cos\theta)$

or, by factoring out an r from both: $(r(\theta - \sin\theta), r(1 - \cos\theta))$

Activity 1 – Introduction to Cycloids/Deriving a Parametric Equation for a Cycloid (Selected student work)







Reflection on Activity 1

After teaching this lesson once, I developed a worksheet to scaffold the lesson. After teaching it a second time, I modified the worksheet to further scaffold the lesson. So obviously, this is a more difficult lesson than I originally thought. Here are some of the things I think that make this so difficult for students:

First, students are not used to problem solving at this level, and they certainly aren't used to using trigonometry to problem solve, unless it' specifically a trigonometry problem. Students weren't comfortable using trig to find an expression for a side length. But what was really surprising was the number of students who struggled to find the coordinates of P, even *after* we had found the lengths of $\overline{PB}$ and $\overline{BC}$. The idea that we could subtract the lengths of parallel segments to find the coordinates of x and y seemed especially difficult. Many students could not show me where segments of length x and y were on their picture.

On the positive side, students were very engaged by the activity, even though in both classes, it seemed clear that their performance on the exercise would not affect their grade in the class. Also, the concept of a parametric equation was not difficult for them to understand, nor was the difference between a parametric and an explicit equation.

In one class, a student found the y value of the parametric by translating a unit circle up r units. He had difficulty adjusting for the fact that rotation on a unit circle happens in a counter-clockwise direction and starts at "three o'clock" where rotation on a cycloid happens clockwise and begins at "six o'clock." With a new emphasis on transformations in the math curriculum, teachers should be ready for these types of answers.

All in all, I thought the exercise went well and students in both classes told me they enjoyed it, and found it interesting and thought provoking.

Introduction to Activity 2

Instructional Goal: Students will work through Roberval's derivation of the area under a cycloid, furthering their understanding of parametric equations, Cavalieri's Principle, and their ability to problem solve and follow complex mathematical problem solving.

Time needed: Roughly 45 minutes to one hour.

Prerequisite knowledge: Students should have some knowledge of parametric equations. Some familiarity with Cavalieri's Principle may also be helpful. Knowing the area formula for a circle is necessary to success in this project.

Supplies: Students need only the worksheet provided. The teacher, if possible, should have a graphing utility with the ability to graph parametric equations (Winplot, a free download, and Grapher, which comes with most Apple computers, both work well). A projector to project this graph on a large screen is also highly recommended.

Classroom Organization: Depending on the level and motivation of the class, this could be done in a lecture format, or as group work. Probably a mix of the two would be best. The main idea, including the idea of the 'companion curve' could be explained as a lecture. After that, students could work on the worksheet, which leads them through the derivation, in pairs or small groups. It would probably be a good idea to bring the class back once or twice, having students present their answers to various questions on the worksheet to the class, to make sure everyone is up to speed.

<u>Activity 2: Roberval's Derivation of the Area Under a Cycloid</u>

Part 1: Intro and Review

Here the teacher reviews the definition of a cycloid and, if time permits, summarizes some of the history of the study of the curve. A graphing utility is projected so students can see the graph of a cycloid and its generating circle. To this graph a third equation is added, that of the companion curve discussed in Section 2. Students are told that we are going to use the companion curve, along with Cavalieri's Principle to find the area under the cycloid. A brief review of Cavalieri's Principle would be useful, especially if students haven't seen it in awhile.

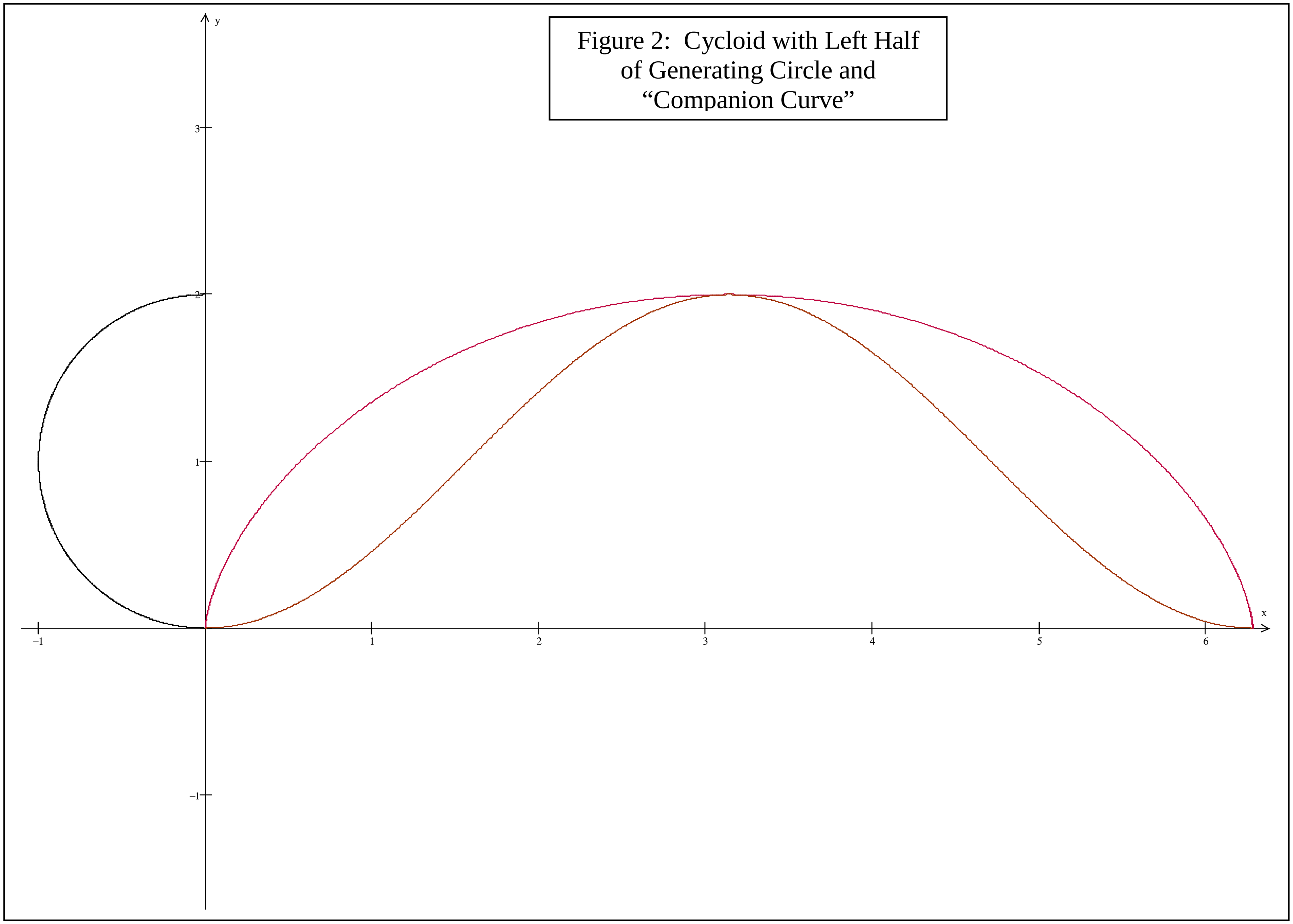
Part 2: Working through the Derivation

The worksheet packet 2.2 is distributed, and students either work in pairs or small groups to answer the questions on the worksheet. The teacher floats about; answering questions and giving hints. Once again, if the same question keeps coming up, or if everyone seems stuck at the same place, the class may need to be called back together.

Part 3: Bringing it all together

When time is winding down, and some, or most, students are finished, the class is called back together. Students can present their answers, or, if time is short, the teacher can go over the questions with the class. Questions are answered, and the teacher makes sure that everyone has access to the correct information.

Worksheet 2.1



<u>Part I: Using Cavalieri's Principle to find the Area under a Cycloid (Roberval's Method).</u>

Label the curves in Figure 2 with their parametric equations:

1. Cycloid, C: See exercise 3 above. $x_C = a(\theta - \sin\theta)$

   $y_C = a(1 - \cos\theta)$

2. Generating Circle (or semicircle), G: $x_G = -a(\sin(\theta))$
   $y_G = a(1 - \cos(\theta))$

3. "Companion Curve," H: $x_H = at$
   $y_H = a(1 - \cos(\theta))$

Label π on the x axis.

Exercise 1: Explain why at any given θ, the points on each of the above curves will lie on the same horizontal line.

Exercise 2: Show that at any θ, the distance between x_G and the y axis is equal to the distance between x_C and x_G.

A two-dimensional version of **Cavalieri's Principle** states:

Suppose two regions in a plane are included between two parallel lines in that plane. If every line parallel to these two lines intersects both regions in line segments of equal length, then the two regions have equal areas.

-From Wikipedia.

Looking back at exercise 2, what two regions must have the same area. What is that area?

Exercise 3. Consider the rectangle with vertices (0, 0), (0, 2r),(πr, 0), (πr, 2r).

a) Determine the area of the rectangle.

b) Show, using Cavalieri's Principle, that the "Companion Curve" divides the rectangle into two equal halves.

Do this by showing that for any θ, the distance between $x_H(\theta)$ and the y axis is equal to the distance between $x_H(\pi-\theta)$ and the line $x = \pi$.

Worksheet 2.2 (p.1 of 3)

Use your results to show that the ***area under any cycloid is equal to 3 times the area of the generating circle***.

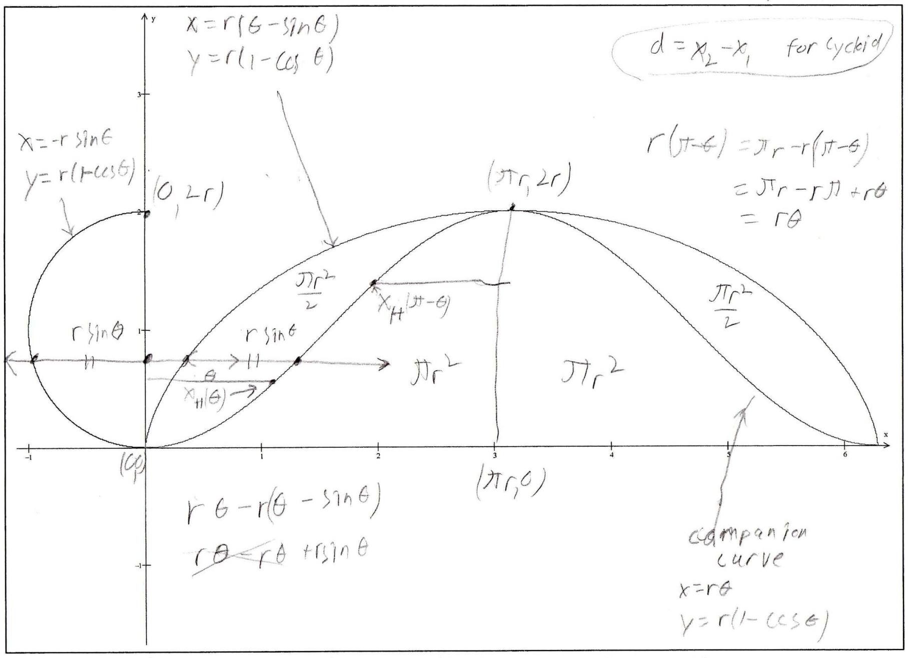
Write any questions or comments you have about today's exploration below, and remember to ask your calculus teacher next year to show you how to find the area under a cycloid using **integration**!

Activity 2: Roberval's Derivation of the Area Under a Cycloid

(Selected Student Work)

Figure 2: Cycloid with Left Half of Generating Circle and "Companion Curve"

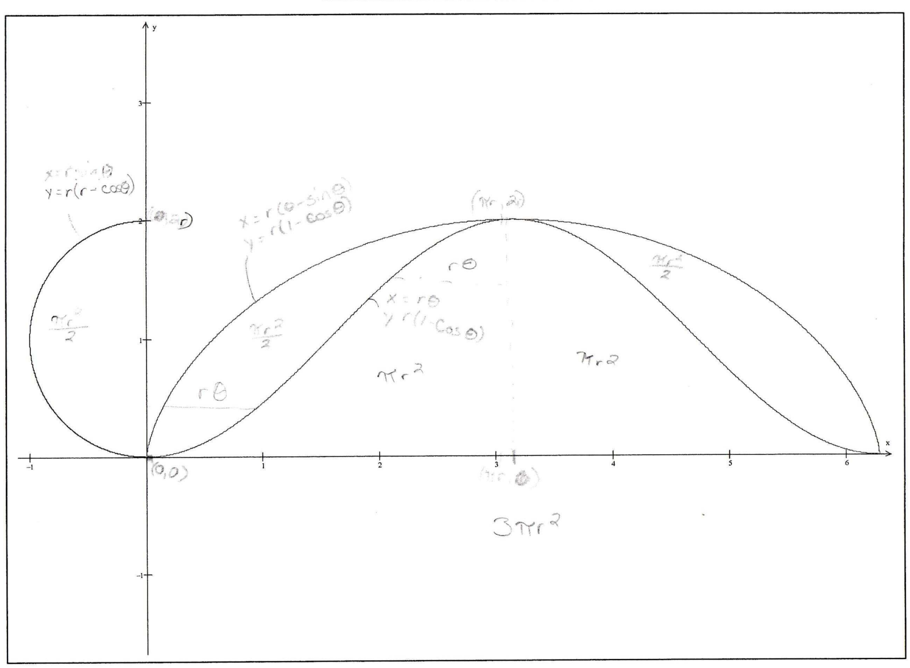
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



$x_C = r(\theta - \sin\theta$ $\quad x_G = -r\sin\theta$ $\quad x_H = r\theta$

$y_C = r(1 - \cos\theta)$ $\quad y_G = r(1 - \cos\theta)$ $\quad y_H = r(1 - \cos\theta)$

Figure 2: Cycloid with Left Half of Generating Circle and "Companion Curve"



Reflection on Activity 2

This activity was very engaging to students. Some students had heard that the area under a cycloid was three times as big as the circle that created it, and students were interested to see the proof. Although most students correctly answered Exercise 1, they were timid about Exercise 2. When I helped them, they had no problem with it, and the idea that the two regions had the same area seemed to make sense to most. Finding the area of the rectangle in Exercise 3a was also no problem either, but, as I suspected, the upside-down use of Cavalieri's Principle in 3b was too difficult for anyone in the class, and I had to walk them through that part.

Afterward, students seemed excited and satisfied with the proof and the activity. Many gave me positive comments on their papers and asked interesting questions about calculus (their next math class). Overall, I was very happy with the activity, and wonder why it's not more widely taught.

Introduction to Activity 3

Instructional Goal: Students will use integration techniques and formulas to find the area under a cycloid and the arc length of a cycloid.

Time needed: Roughly 45 minutes to one hour.

Prerequisite knowledge: Students should have seen integration using substitution and have some knowledge of integration of parametric equations.

Supplies: Students need only the worksheet provided.

Classroom Organization: Depending on the level and motivation of the class, this could be done in a lecture format, or as group work. These are the types of problems generally done on the board, by the teacher in a traditional math classroom. Working in pairs or groups on a scaffolded worksheet gives students more ownership of the problems and their solutions.

Activity 3: Finding the Area Under a Cycloid and the Arc Length of a Cycloid by Integration

This activity is a simple one. Two problems using the same parametric equations are given. In both cases, algebra and trig identities are used, along with integration by substitution to solve the problems. The worksheet provided walks them through the problems step by step.

Depending on the size and motivation of the class, there may be no need for any lecture material at all. This could be a “project” or an “in-class assignment” that is due at the end of the period. Students could work in pairs, small groups or solo, asking questions when appropriate, or checking in at the end of each step.

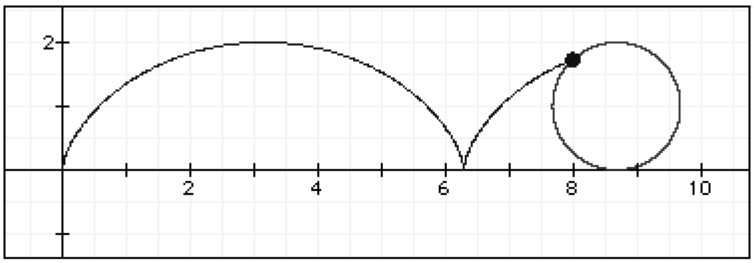
This activity could be done as a review activity after a unit on parametric equations, or as part of a cycloid exploration toward the end of a semester of calculus. I didn’t feel it was in the scope of this curriculum project to spend time on why the formula for the area under a parametric curve works, or deriving the arc length formula. My assumption is that at this point, those lessons have already been taught.

Name______________________ Date__________________

In-Class Assignment: Area Under and Arc Length of a Cycloid

Part 1: Review of Cycloids

Recall that a cycloid is the curve made by a point on a circle as the circle rolls along a flat surface.



A cycloid has the parametric equation
x= r(θ-sinθ) and y= r(1-cosθ) where r is the radius of the generating circle and θ is the amount of rotation of the circle in radians.

Part 1: Area Under the Cycloid

Recall the formula for the area under a parametric curve:

If x = f(t) and y= g(t)

then

$$A = \int_{t1}^{t2} y dx = \int_{t1}^{t2} g(t) \cdot f'(t)$$

Step 1: Find f'(t)

Step 2: Substitute g(t) and f'(t) into the formula above.

Step 3: Determine the limits of integration.

Step 4: Expand $g(t) \cdot f'(t)$

Step 5: Factor any constants out from the integral. How does this change the limits of integration?

Step 6: Use the trigonometric identity $\cos^2 x = \frac{1+\cos(2x)}{2} = \frac{1}{2} + \frac{1}{2}\cos 2x$.

Step 7: Integrate each term. Use a u substitution to integrate the last term.

Step 8: Evaluate for the given limits.

Part 2: Arc Length of a Cycloid

Recall the arc length formula:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Step 1: find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and set the limits of integration.

Step 2: Substitute and expand, factoring out the r^2.

Step 3: Substitute, using the trig identity $1 - \cos t = 2\sin^2\left(\frac{t}{2}\right)$

Step 4: Eliminate the radical, and integrate.

Step 5: Evaluate over the given limits.

Introduction to Activity 4

Instructional Goal: Students will understand and work through some of the proof that a pendulum constrained by two inverted cycloids will swing in the shape of a congruent, inverted cycloid.

Time needed: Roughly 45 minutes to one hour.

Prerequisite knowledge: Students will have worked through the previous 3 activities and have a good understanding of integration, parametric equations, algebra and trig.

Supplies: Students need only the worksheet provided.

Classroom Organization: Depending on the level and motivation of the class, this could be done in a lecture format, or as group work. These are the types of problems generally done on the board, by the teacher in a traditional math classroom. Working in pairs or groups on a scaffolded worksheet gives students more ownership of the problems and their solutions.

Activity 4: Showing that a Pendulum Constrained by Two Inverted Cycloids Swings in the Shape of a Congruent, Inverted Cycloid

Because of the complexity of the algebra involved in the actual derivation (see section 3 of Part 1), I've eliminated some of the more difficult steps. This activity should probably be done as follows:

Intro: Teacher presents Diagram 3.1 and goes over the basic problem.

Students work in pairs or small groups or solo to solve the following problem.

Activity 1: Given the coordinates of one endpoint of a line segment, the length of that segment and the slope of that segment, find a formula to find the other endpoint.

Let (x_1, y_1) be the known endpoint. Let L be the length of the segment, and m be its slope.

(Hint: combine the slope formula and the distance formula. Your "formula" will have two parts, one to find the x coordinate of the missing endpoint, and one to find the y coordinate).

Once everyone has the formulas, we begin activity two.

Activity 2: Looking at the diagram, what expressions can we plug into our formulas for the following variables. Be ready to give a short explanation for your answer.

x_1=

y_1=

m=

L=

Substitute the expressions into the equations:

Teacher explanation: After some involved algebra, we can simplify the above equations into the following:

$$x_2 = r(\sin\theta + \theta) \quad or \quad x_2 = r(\theta - 3\sin\theta)$$

And

$$y_2 = 3r + 3r\cos\theta \;\; or \;\; y_2 = -3r - r\cos\theta$$

Teacher asks, "How do we determine which equations to use?"
Combinations of the two parametrics are graphed on a graphing utility so that students can see the correct equations are:

$$x_2 = r(\sin\theta + \theta), \text{ and } \; y_2 = -3r - r\cos\theta$$

Now a class discussion centers on how to show that these are the graphs of congruent shifted cycloids. If no one comes up with it, the teacher might have to point out that the shifted cycloid is in the middle of its path when the original cycloid has just begun. Students should see that we must add π to the input of both expressions of the shifted cycloid to achieve this effect. Both expressions have been shifted down 2r and students will hopefully know that that is simply a matter of subtracting 2r from both expressions.

Activity 3: The final activity is to show that this works, that is, show that

$$r((\theta+\pi)-\sin(\theta+\pi))-\pi r = r(\sin\theta+\theta)$$

and that
$$-r(1-\cos(\theta+\pi))-2r = -3r - r\cos\theta$$

Final Reflection

Over the several years that I worked on this project off and on, I've never really gotten bored of cycloids. They are an incredibly rich topic mathematically and historically. I've been able to use the math I teach, algebra, geometry, trigonometry, while brushing up on parametric equations and integral calculus. The process has made me a better, more thoughtful and better-rounded mathematician.

I've also found that many students have an intrinsic curiosity about cycloids. I think it's because the idea of a cycloid is so simple, yet the math needed to work with them is very complex. When approached with a cycloid-related problem, it seems that "we should be able to figure this out." Cycloids therefore show the need for complex mathematical ideas such as parametric equations and integrals, and can be used as a motivational tool to students who feel "bogged down" with complex algorithms such as integration by substitution and such.

As we move further into the age of technology, the procedures will lose their importance as computers and calculators can do more and more of them. What will gain in importance is the conceptual understanding of what those procedures are actually finding and the ability to use that conceptual knowledge to solve new problems. I've spoken with many adults who took calculus classes in college, but claim to have not understood what they were doing, even though they earned an A or B in the class. How could they possibly apply calculus to a new problem if that is the case?

As a teacher, I know that teaching problem solving in the mathematics classroom is incredibly difficult. Students are often too afraid to be incorrect to take the risk of attempting to solve a problem. School has become, for many students, a matter of memorizing steps and algorithms and applying them to test questions. When students are given problem solving work samples for state assessments, teachers often show solutions to similar problems before handing them out. Students are able to pass, and even excel at many high school math classes without ever solving a problem in a meaningful way. Many students, and their parents as well, will protest if a student is expected to solve a problem that they haven't seen before, and been shown a correct solution. The emphasis on standardized multiple-choice tests as the main accountability measure for schools doesn't help this phenomenon. Nor does the near constant "raising of the bar" as far as the number and level of mathematical topics students are supposed to master in their high school math curriculum. Now recession era budget shortages add much larger class sizes to the mix, as well as decreased planning and grading time for teachers.

According to many teachers, professors and employers, America having a crisis in problem solving. An important discussion needs to happen in the math education community, and the general education community as well, as to how to teach, and assess this crucial skill. Part of the answer is surely to spend more time on projects and investigations such as the cycloid activities above, which require combining several previously learned skills to solve a complex problem. Unfortunately, this cannot happen without buy-in from parents, politicians, and the public at large, and it will not happen on the cheap.

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