Reidemeister-Schreier Rewriting Process for Group Presentations

A 501 paper presented for the degree of Master of Mathematics

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The People

Kurt Reidemeister
(13 October 1893 - 8 July 1971)

Loved cats.

Otto Scheier
(3 March 1901 - 2 June 1929)

Hated cats.

Interesting Tidbits

Kurt Reidemeister

- Geometer
- Contributed significantly to Knot Theory
- Leader of the original Vienna Circle of Logical Positivists
- Forced out of Germany in the 1930*s* due to his vocal opposition to the Nazi party.

Otto Schreier

- Algebraist
- Said of Reidemeister in a letter:
  "By his humorous remarks he caused such roaring laughter as has never been heard, so it seems, in the Mathematics Society."
- Musician

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  Ex: the product of cbc and $b^{-1}ba$ is $cbcb^{-1}ba$
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  So multiplication of words in a free group is not commutative.
- We call the word in no (generating) symbols the empty word, and denote it with the symbol e. This is the identity of a free group, F.

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  (Note that the last example shows that
  $(abc)^{-1} = c^{-1}b^{-1}a^{-1}$, so the "socks and shoes" method is valid here.)

Group Presentations

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or as a presentation,

$$D_4 = \langle a, b \mid a^4 = e, b^2 = e, ab = ba^3 \rangle$$

We call the symbols to the left of "|" the generating symbols, and to the right of "|" the defining relations.

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Well, the theory, which we will state without proof, is that if N is the normal subgroup of the free group F generated by the relators, then

$$F/N \cong G.$$

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- Define the homomorphism $\varphi : F \to G$ by

$$\varphi(a^{n_1} b^{n_2} \cdots a^{n_{r-1}} b^{n_r}) = r^{n_1} f^{n_2} \cdots r^{n_{r-1}} f^{n_r},$$

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- Recalling that a requirement for normalcy is that $wNw^{-1} = N$, for all words $w \in F$, N must be the subgroup generated by the words $\{wa^4w^{-1}, wb^2w^{-1}, wababw^{-1}\}$ for all $w \in F$.

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- Note that $ker(\varphi) = N$, and $im(\varphi) = D_4$, since clearly $D_4 \subseteq im(\varphi)$ and $im(\varphi) \subseteq D_4$. Thus by the first isomorphism theorem,

$$F/N = F/ker(\varphi) \cong im(\varphi) = D_4$$

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- But what if we wanted to derive a presentation for V from the presentation for G? Is there a way to do this? The answer may surprise you....
- YES! In fact, the Reidemeister-Schreier rewriting process is a process that will input a group, G, the presentation of G, a subgroup, H, of G, and output a presentation for H.

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- Example: For D_4 with subgroup V, we have

$$D_4/V = \{\{e, a^2, a^2b, b\}, \{a, a^3, ab, a^3b\}\}$$

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- We denote our transversal with $\mathcal{K}$.
- A small restriction for our purposes is that e must be an element of our transversal.

Right Coset Representative Function

- Given a group G, a subgroup H, and a transversal, $\mathcal{K}$, for G/H, a right coset representative function is a function that maps $g \in G$ to $k \in \mathcal{K}$ such that $g \in Hk$.

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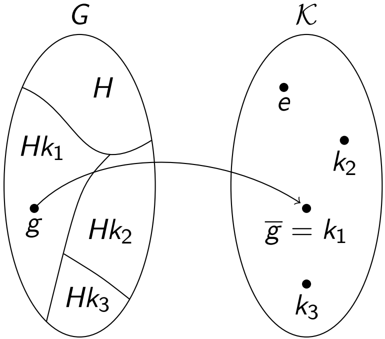

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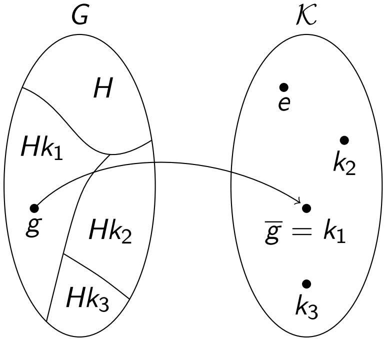

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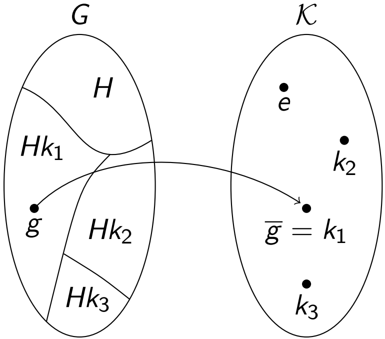

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- Example: If $D_4/V = \{\{e, a^2, a^2b, b\}, \{a, a^3, ab, a^3b\}\}$ and $\mathcal{K} = \{e, ab\}$, then $\overline{b} = e$ and $\overline{a^3b} = ab$.

A Carefully Chosen Set of Generators for a Subgroup

- Suppose $G = \langle a_1, \dots a_r \mid P, Q, R, \dots \rangle$, H is a subgroup of G, and $\mathcal{K}$ is a transversal for G/H. Then H is generated by the set of words

$$S = \left\{ k a_i \overline{k a_i}^{-1} \mid k \in \mathcal{K} \text{ and } a_i \text{ is a generator for } G \right\}.$$

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- Rather than prove this, let's just see how a particular $h \in V$ can be factored into elements of S.

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$$\tau(h) = s_{k_1,a_1}^{\epsilon_1} s_{k_2,a_2}^{\epsilon_2} \cdots s_{k_r,a_r}^{\epsilon_r}$$

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- If $h = aba^{-1}$ (same group element as a^2b) and $\mathcal{K} = \{e, ab\}$, then $k_1 = e$, $k_2 = \overline{a} = ab$, and $k_3 = \overline{aba^{-1}} = e$.

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The Reidemeister-Schreier Rewriting Process

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Theorem

Suppose G has the following presentation:

$$G = \langle a_1, \ldots, a_n \mid P, Q, R, \ldots \rangle, \tag{1}$$

and let H be a subgroup of G. If τ is a Reidemeister-Schreier rewriting process, then H can be presented as

$$\langle s_{k,a_i}, \cdots \mid s_{m,a_\lambda}, \ldots, \tau(kRk^{-1}), \ldots \rangle, \tag{2}$$

where k is an element of a Schreier transversal for G/H, a_i is any generator of G and R is any relator in (1), and m is a Schreier representative and a_λ a generator such that

$$ma_\lambda \text{ is freely equal to } \overline{ma_\lambda}. \tag{3}$$

Note: a Reidemeister-*Schreier* rewriting process, is just a Reidemeister rewriting process in which we've chosen a Schreier transversal for $\mathcal{K}$.

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- We see from the following table that only one pair (m, a_λ) satisfies the conditions of (3):

ma_λ		$\overline{ma_\lambda}$	freely equal?	ma_λ		$\overline{ma_\lambda}$	freely equal?
ea	$\rightarrow$	a	Y	eb	$\rightarrow$	e	N
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- So (2) becomes $\langle s_{e,a}, s_{e,b}, s_{a,a}, s_{a,b} \mid s_{e,a}, \tau(ea^2e^{-1}), \tau(eb^2e^{-1}), \tau(eababe^{-1}),$ $\tau(aa^2a^{-1}), \tau(ab^2a^{-1}), \tau(aababa^{-1}) \rangle$

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- Now might be a good time to note that presentations are not unique, but I hope mine was.