Eulerian near-triangulations are three-colorable

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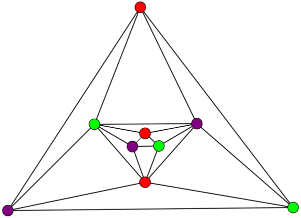
March 1, 2018



A Brief History of Colorability

- The four-color problem
  - Francis Guthrie
  - Arthur Cayley
  - Alfred Kempe
  - Percy Heawood
  - George Birkhoff, Philip Franklin, and Heinrich Heesch
  - Kenneth Apple and Wolfgang Haken

Francis Guthrie-1852 and Arthur Cayley-1878

Francis Guthrie.
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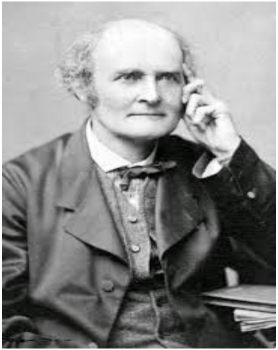


- **Conjecture** - You can color the regions on any planar map, in such a way that if two regions share a non-trivial boarder, then they would be assigned different colors. (using only four colors)

Alfred Kempe-1879 and Percy Heawood-1890

- **Conjecture** - Every planar map can be five-colored.[1]

[1]Source www.people.math.gatech.edu/ thomas/FC/fourcolor.html

George Birkhoff, Philip Franklin, and Heinrich Heesch

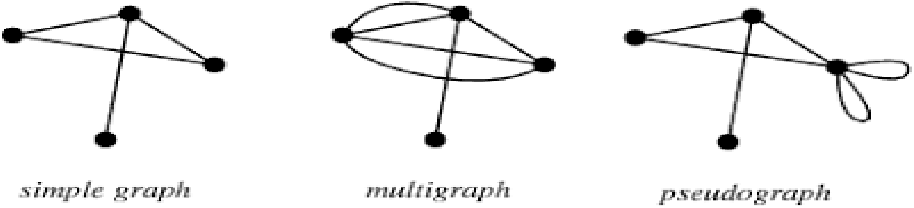
- **Reducibility** - A configuration is *reducible* if it cannot be contained in a triangulation of the smallest graph which cannot be 4-colored.
- 1922 - Proof the conjecture was true for all planar maps with at most 25 regions.
- **Charging (1969)** - A charge is assigned to each face and each vertex of the graph. The charges are assigned so that they sum to a small positive number.

Kenneth Appel and Wolfgang Haken

- In 1976 a computer-aided proof of the four-color theorem was published. [2]
- The computational portion (in theory) is hand-checkable.
- To this day no one has ever verified it in its entirety, as far as we know.
- Several independent algorithms and a number of major reductions in the complexity of the case breakdown.

[2] Source www.people.math.gatech.edu/ thomas/FC/ fourcolor.html

Terminology and Notation

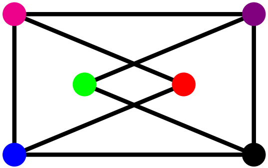


Definition - A simple *graph* G consists of finite set $V(G)$ of *vertices*, together with a collection $E(G)$ of 2-element subsets of $V(G)$, referred to as *edges*.

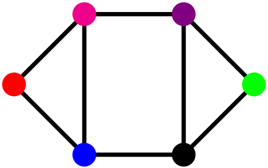
- endpoints
- adjacent
- degree

Definition - A *multigraph* G consists of finite set $V(G)$ of *vertices*, together with a multi-set $E(G)$ of 1-element or 2-element subsets of $V(G)$, referred to as *edges*.

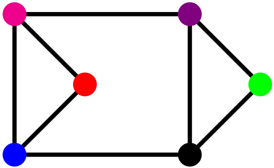
Planar graph (or planar multigraphs)



Planar graph G



Plane graph of G



Different plane graph of G

Definition A *planar* graph is a graph that can be drawn in the plane without any edge crossings. Such a drawing is a *planar embedding*. A *plane* graph is a planar graph together with a fixed embedding. *Planar multigraphs* and *plane mulitgraphs* are defined similarly.

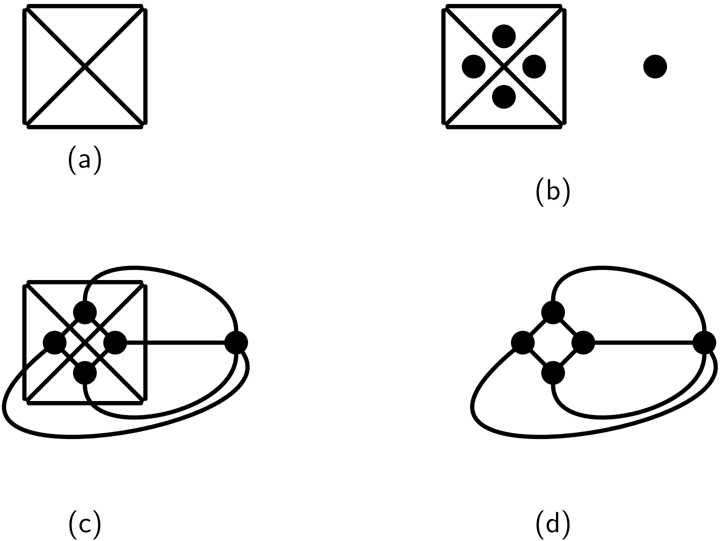
A Walk

Definition A *walk* with endpoints a, b is an alternating sequence of vertices and edges.

$$a = v_0, e_1, v_1, e_2, v_2, ..., e_k, v_k = b$$

Where each edge e_i has endpoints v_{i-1}, v_i. We say G is *connected* if, for every $a, b \in V(G)$, there is a walk with endpoints a, b. A walk is *closed* if its endpoints are identical.

Planar maps into planar graphs



(a) (b) (c) (d)

Vertex Coloring

The way in which we are coloring the vertices in G is as follows:

1) Each vertex receives a color.

2) If $v_i \sim v_j$ then they must receive different colors.

The Big Question

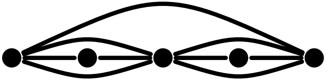
When do we only need three colors to properly color the vertices of a plane graph?

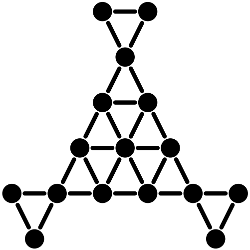
Determining three-colorability of planar graphs

Three Color Problem

- Larry Stockmeyer: NP-complete
- Sufficient conditions for 3-colorability
  - *Grötzsch's* Theorem: Triangle-free planar graphs are 3-colorable.

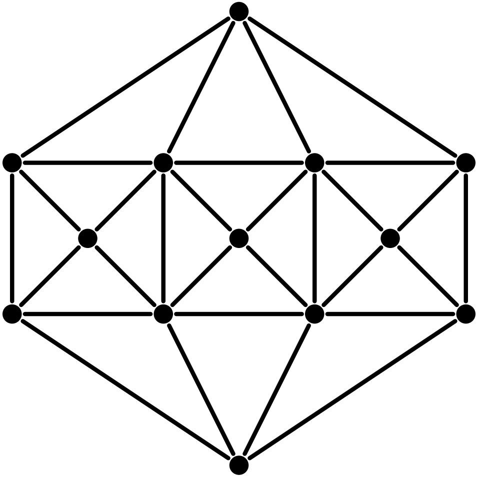
3-colorability of Eulerian triangulations





- **Definition** A *triangulation* is plane graph in which every face is a triangle.
- **Definition** A *near-triangulation* is a planar multigraph whose bounded faces are all 3-cycles.
- **Definition** A connected graph with all vertex degrees even is *Eulerian*, meaning that it has a closed walk traversing all the edges exactly once (an *Eulerian circuit*).

“A new proof of 3-colorability of Eulerian triangulations.” -by Tsai and West



Cycling through the colors 1; 2; 3, consecutively while we walk along the vertices following an Eulerian circuit will produce a proper three-coloring.

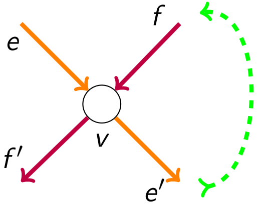
Given an Eulerian near-triangulation graph G...

- First, show that there always exist non-crossing Eulerian circuits in any Eulerian plane multigraph.
- Second, prove that the number of edges in G are divisible by three.
- Third, prove that the length of every sub circuit is divisible by three.

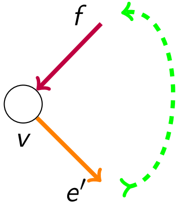
The independent sets in a proper three-coloring of an Eulerian triangulation are unique.

Lemma 1 Every Eulerian plane multigraph has a non-crossing Eulerian circuit.

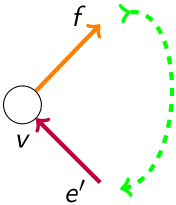
Proof Let G be a Eulerian plane multigraph. Let C be a circuit in G with the least amount of crossings.



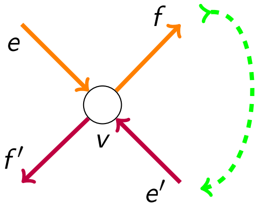
Crossing C_v in C



Subcircuit C_s



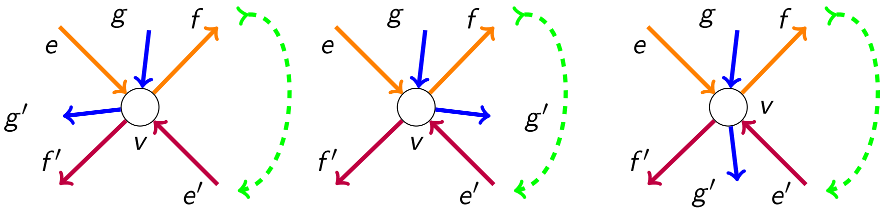
Reversed subcircuit C_s'



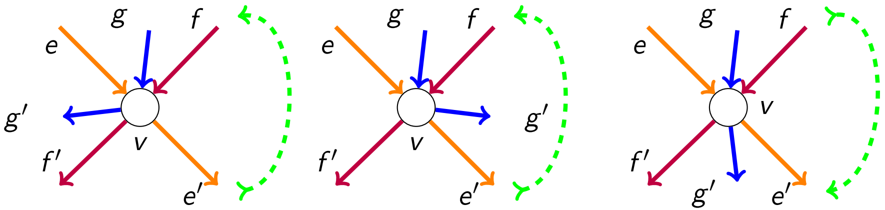
Non-crossing C_v' in C'

Since C_s is a sub-circuit, changing the direction that we walk along its path will not disconnect the graph. Furthermore, it will eliminate this particular crossing at C_v and it will not increase the number of crossings.

For example, if C' has a crossing (e, v, f) with, say, (g, v, g'), there are three cases to consider...

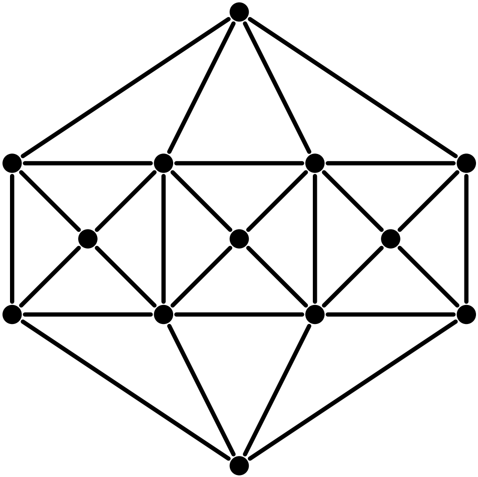


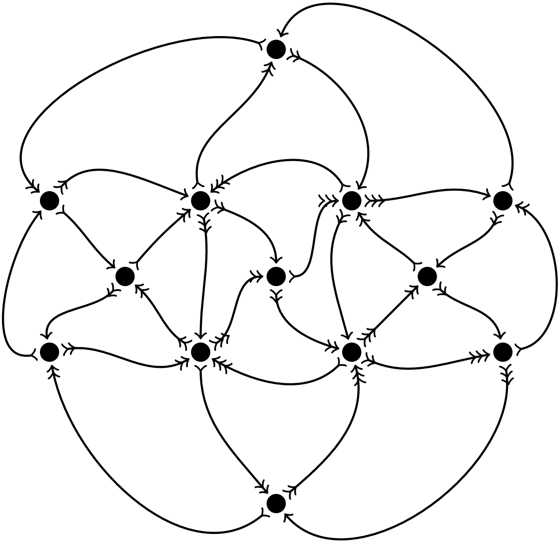
But in each of these cases, either (e, v, e') or (f, v, f') (or both) already formed crossings in C with (g, v, g') as shown:

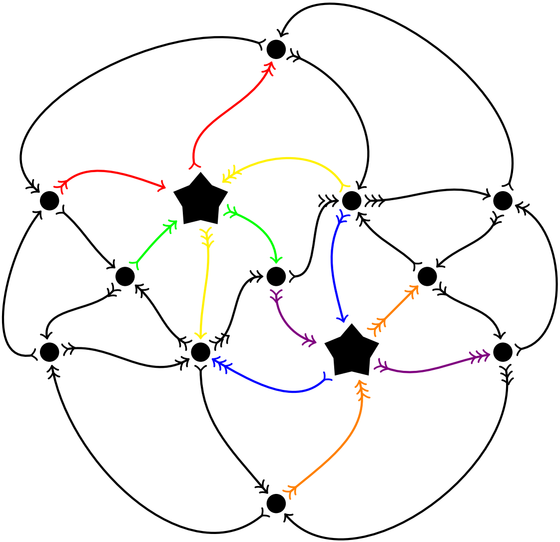


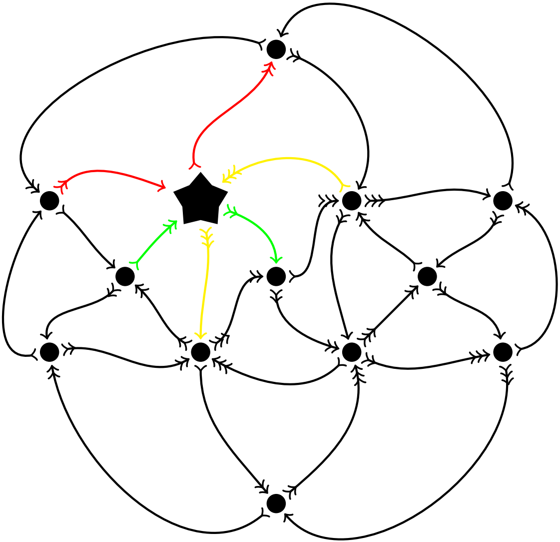
Conclusion...

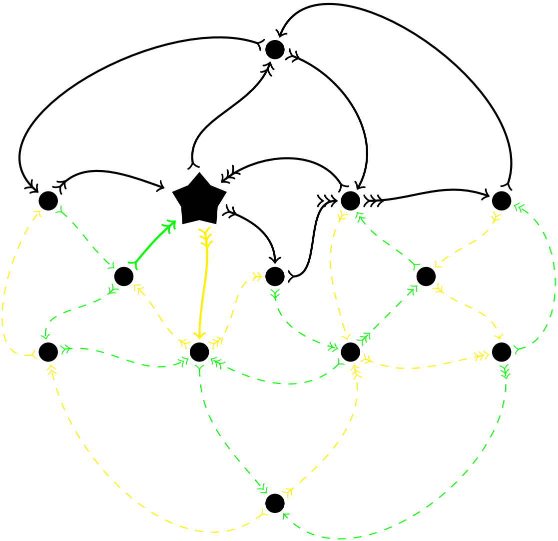
Since C' has fewer crossings than C, and since C was selected to have the fewest crossings possible, we have arrived at the desired contradiction.

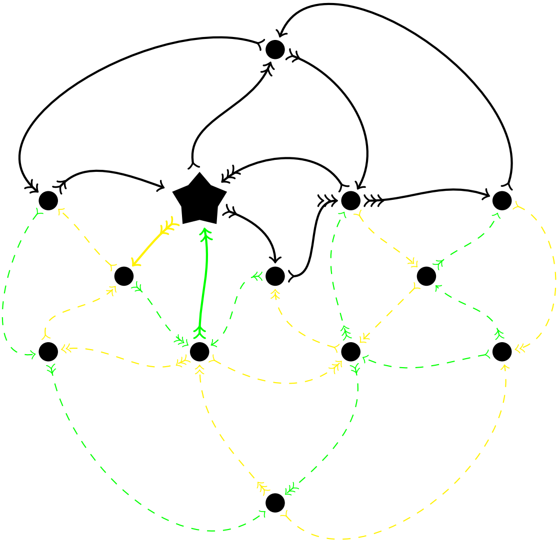


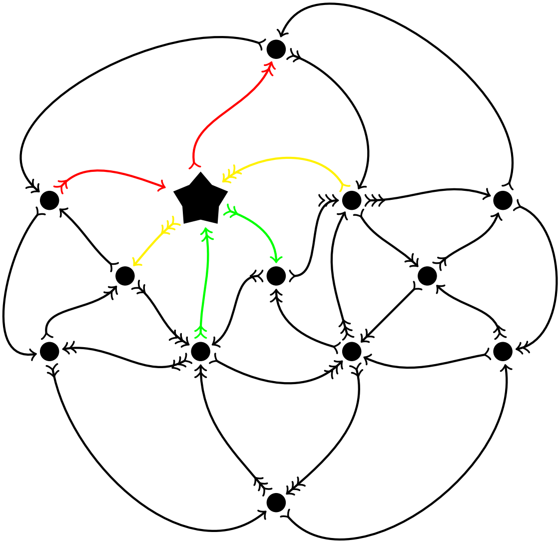


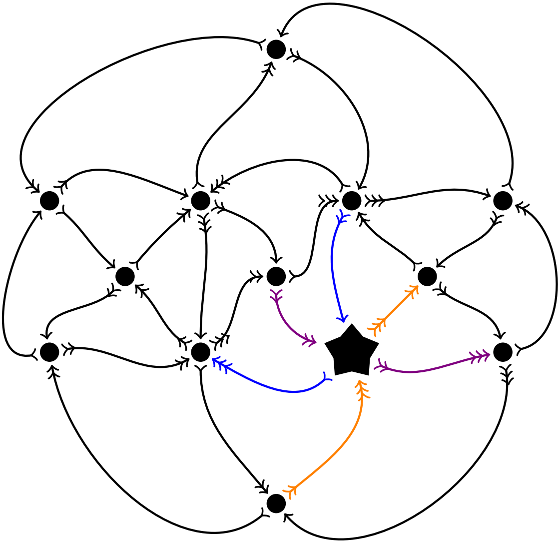


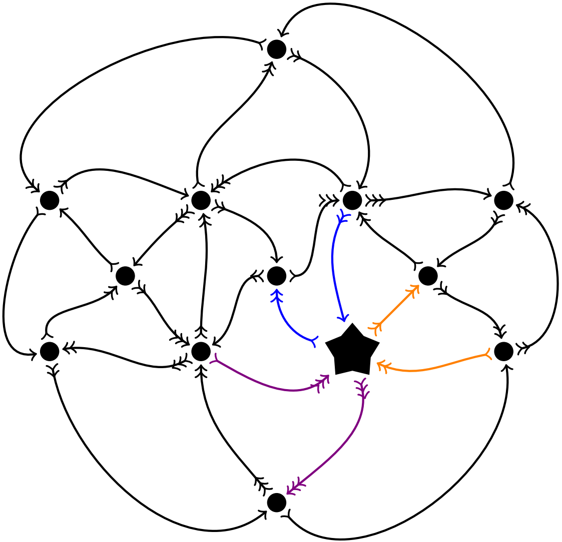


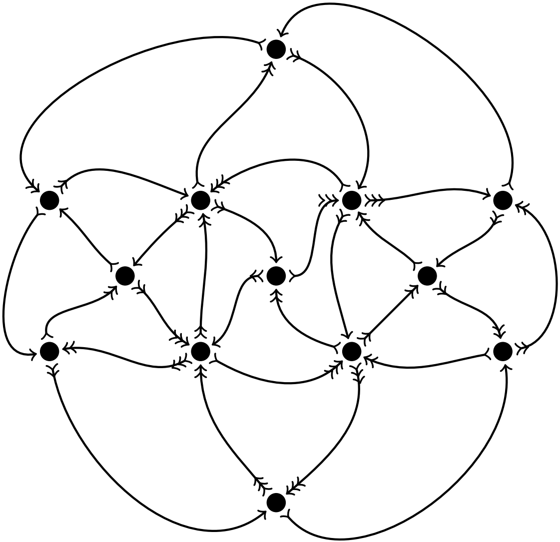








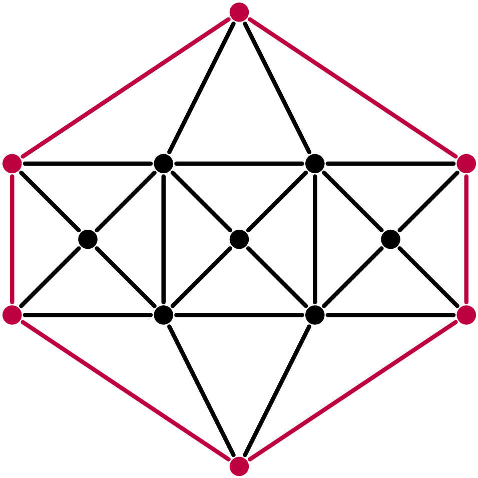




Now that we have a non-crossing Eulerian circuit C, we need to:

- know that the number of edges in G is divisible by three.
- know that the length of every sub-circuit is divisible by three.

The *external edges* and *external vertices* in a plane graph are the edges and vertices incident with the unbounded face. A graph is *trivial* if it has no edges.

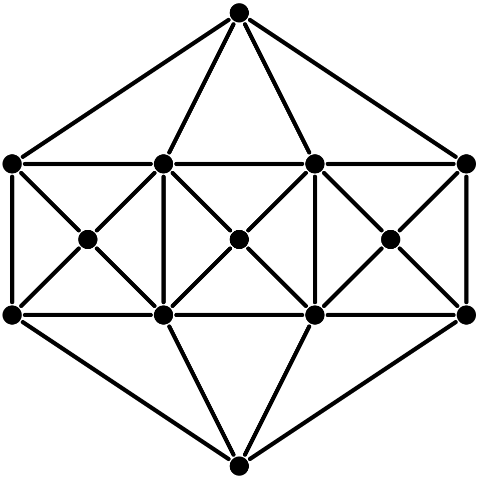


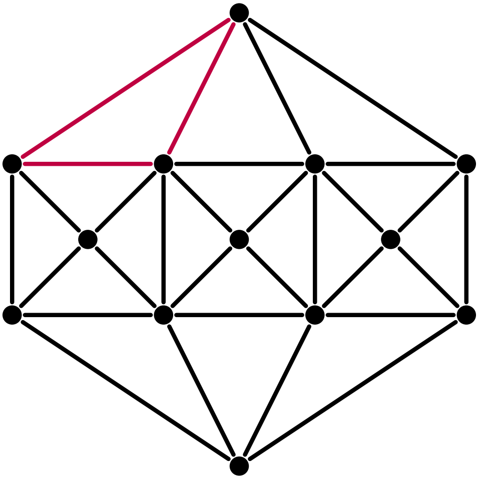
Lemma 2 In every Eulerian near-triangulation, the number of edges is divisible by three.

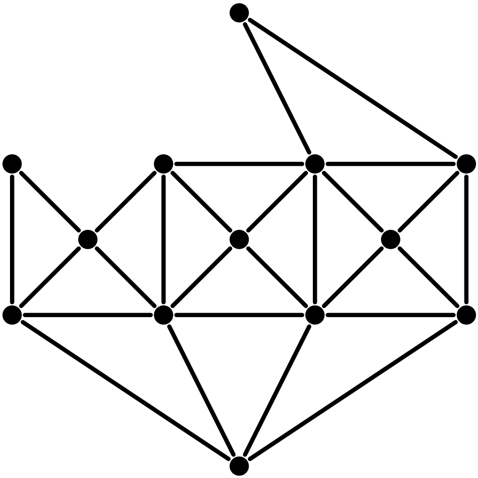
Let n be the number of bounded faces in G, for some $n \in \mathbb{N}$.

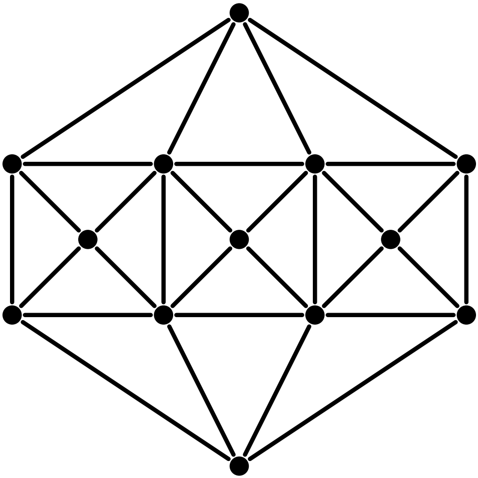
Base Case: When $n = 0$, G is the trivial graph with isolated vertices, which has zero edge count.

Induction hypothesis: In every Eulerian near-triangulation graph H with fewer than n bounded faces, the number of edges in H is divisible by three.



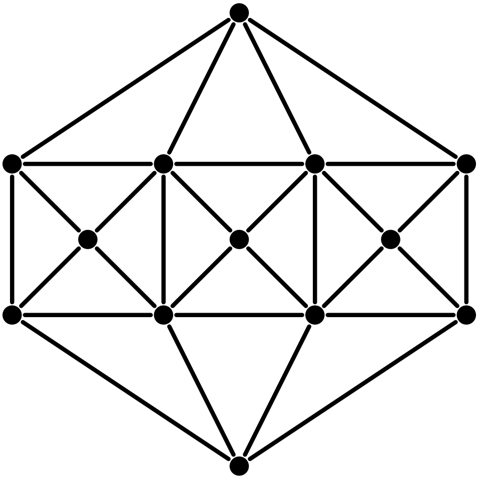


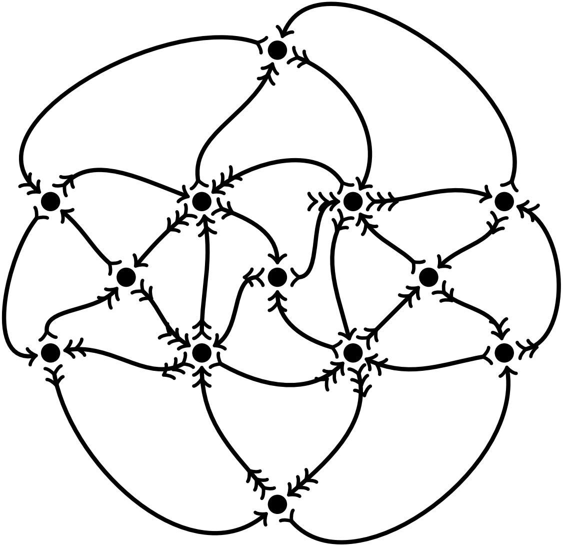


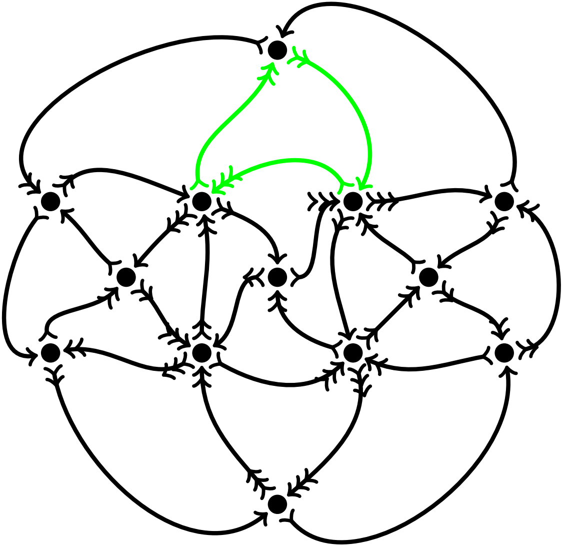


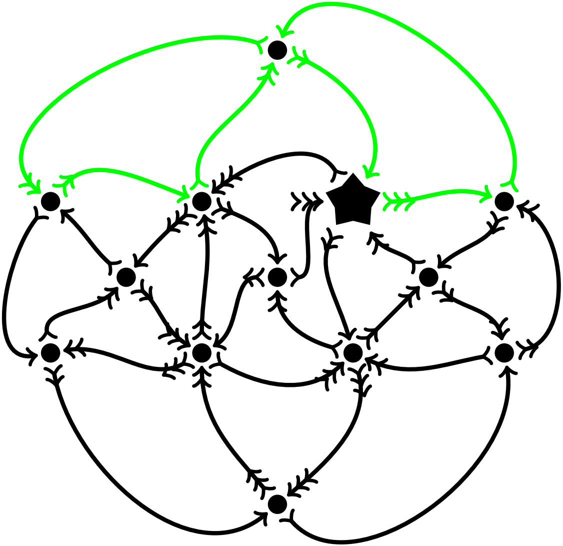
Lemma 3 In a non-crossing Eulerian circuit of an Eulerian near-triangulation, the length of every sub-circuit is divisible by three.

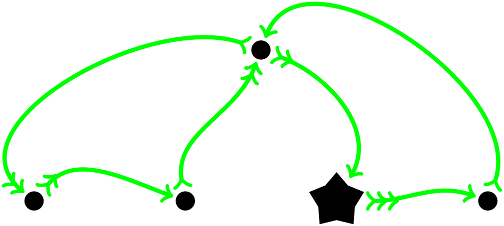
We can use induction on the number of faces that are enclosed by a subcircuit of a non-crossing Eulerian circuit.

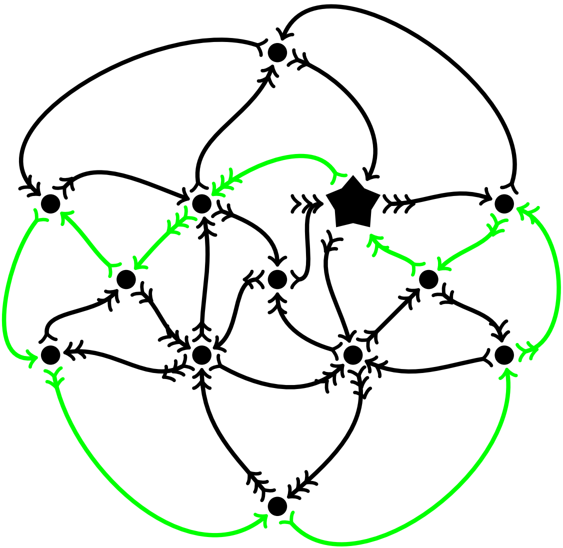


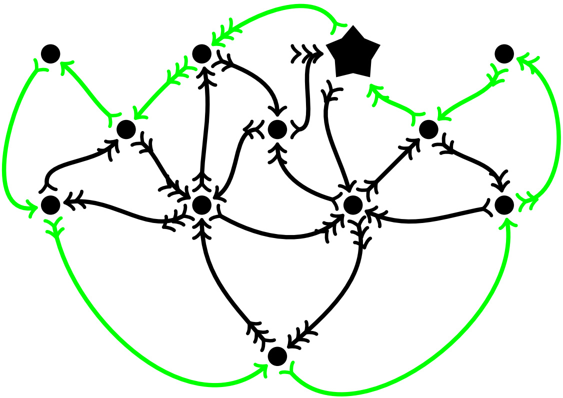


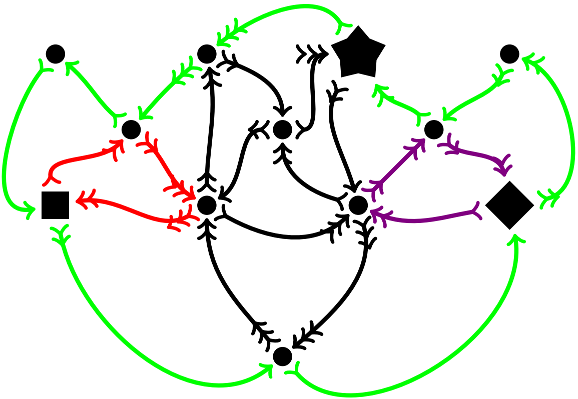


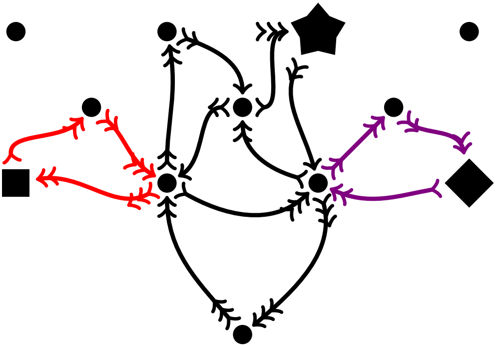


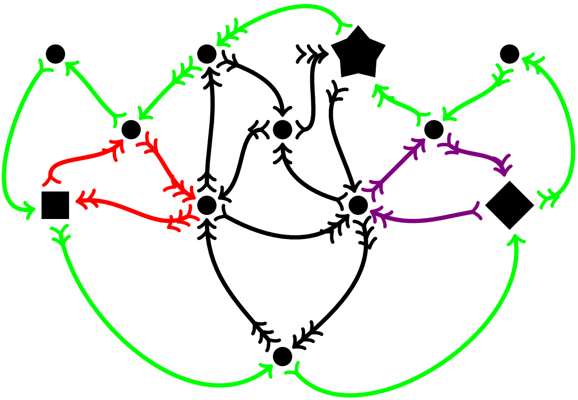


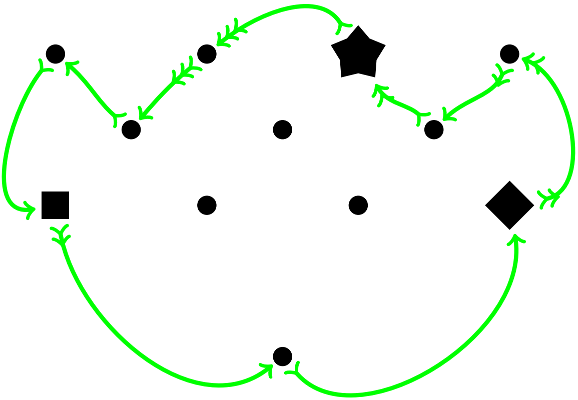












Final Result

Every time you return to a vertex that has already been colored you will have completed a sub-circuit of C. By Lemma 3 the length of the sub-circuit is divisible by three. Therefore, the same color would be assigned to that vertex. In conclusion, the coloring is consistent, and it explicitly assigns distinct colors to the endpoints of every edge. Therefore, every Eulerian near-triangulation is 3-colorable, which completes the proof of our main result.