Proving Eulerian triangulations are 3-colorable

Brittany L. Carter[1]

Portland State University, Portland OR 97207, USA

Under the direction of Dr. John Caughman

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Abstract

In this 501 project we study and present the results of a recent journal article by Tsai and West [14] that uses the existence of non-crossing Eulerian circuits in planar triangulations to establish that such graphs are *3-colorable*.

1. Introduction

We begin by motivating the main result with a short review of some of the history of the problem. We also fix our terminology and we set out the notation to be used in the remainder of the paper.

1.1 The four-color problem

Graph colorability has been a topic of discussion since the 1800's. The original problem was to color the regions on a planar map in such a way that if two regions share a non-trivial (i.e. positive length) border, then they would be assigned different colors. In 1852, Francis Guthrie made a conjecture that you can color any planar map in this fashion using only four colors. The first printed reference of this claim is due to Arthur Cayley (in [5]) in 1878, where he explains the difficulties that lie in attempting to prove the conjecture.

A year later the first 'proof' (in [13]) of this four-color theorem, by Alfred Kempe, had appeared. Kempe received a great amount of acknowledgement from the public for his proof. However, it was pointed out to be incorrect (in [11]) by Percy Heawood eleven years later. Although Heawood showed that Kempe's proof of the four-color problem was wrong, he managed to salvage a proof that every planar map can be 5-colored [17].

The next major contributions came from George Birkhoff, who introduced (in [4]) the concept of *reducibility* (A configuration is *reducible* if it cannot be contained in a triangulation of the smallest graph which cannot be 4-colored.). His work allowed Philip Franklin, in 1922, to prove the conjecture was true for all planar maps with at most 25 regions.

The final idea necessary for the solution of the four-color conjecture was a method called *charging* and was introduced in 1969 by Heinrich Heesch (in [12]). Using this technique, a computer search was developed by Kenneth Appel and Wolfgang Haken, and the result was later confirmed in 1976 when they published (in [3]) their famous computer-aided proof of the

[1]Email:bdody@pdx.edu

four-color theorem. Unfortunately, however, their work was not well received by the math community. This was the first major theorem to be proved in a way that relied critically upon the use of a computer, and although the computational portion (in theory) is hand-checkable, such an undertaking would be extraordinarily complicated and tedious. No one to this day has ever verified it in its entirety, as far as we know.

At present, however, after several independent algorithms have verified the result, and with a number of major reductions in the complexity of the case breakdown, there is no longer any serious doubt concerning the result. Overwhelmingly, the mathematical community has come to accept the four-color theorem as true (see [17]).

We turn now to a brief review of some basic graph theory terminology before we continue to a discussion of our primary focus, the result of the paper "A new proof of 3-colorability of Eulerian triangulation" by Tsai and West [14].

1.2 Terminology and Notation

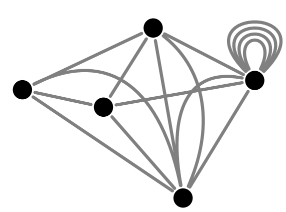
In this section, we collect some of the basic definitions concerning graphs. Our terminology in this paper will remain consistent with the textbooks by Godsil [6] and West [18]. We begin with the formal definition of a graph.

1.2.1 Definition A simple *graph* G consists of finite set $V(G)$ of *vertices*, together with a collection $E(G)$ of 2-element subsets of $V(G)$, referred to as *edges*.

An edge is usually referenced by the two vertices comprising it, and these are called its *endpoints*. For example, we will denote by uv the edge with endpoints u and v. If uv is an edge, then we say the two vertices u and v are *adjacent* and we write $u \sim v$. The *degree* of v is the number of edges with v as an endpoint. For this paper, we will also need to introduce the concept of a multigraph.

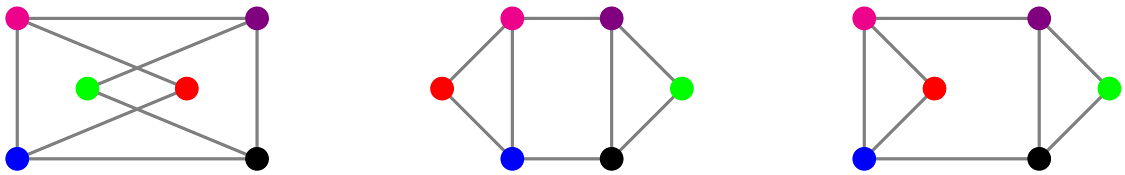
1.2.2 Definition A *multigraph* G consists of finite set $V(G)$ of *vertices*, together with a multi-set $E(G)$ of 1-element or 2-element subsets of $V(G)$, referred to as *edges*.

We may think of a multigraph as a graph G that may have *parallel edges*, which are edges that share the same endpoints, and *loops*, where a loop is an edge that shares an endpoint with itself. When determining the degree of a vertex in a multigraph, we follow the convention that loops count twice.



1.2.3 Figure

1.2.4 Definition A *planar* graph is a graph that can be drawn in the plane without any edge crossings. Such a drawing is a *planar embedding*. A *plane* graph is a planar graph together with a fixed embedding. *Planar multigraphs* and *plane multigraphs* are defined similarly.



Planar graph G Plane graph of G Different plane graph of G

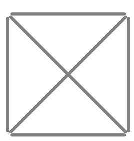
1.2.5 Figure

1.2.6 Definition A *walk* with endpoints a, b is an alternating sequence of vertices and edges.

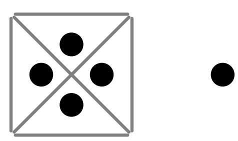
$$a = v_0, e_1, v_1, e_2, v_2, ..., e_k, v_k = b$$

Where each edge e_i has endpoints v_{i-1}, v_i. We say G is *connected* if, for every $a, b \in V(G)$, there is a walk with endpoints a, b. A walk is *closed* if its endpoints are identical.

Returning to the four-color problem, we can see that one can redraw planar maps into planar graphs in the following manner. First, identify each region of the planar map with a vertex (Figure 1.2.7 b). Then let $v_i \sim v_j$ if and only if their corresponding regions share a border of positive length (Figure 1.2.7 c).



(a)



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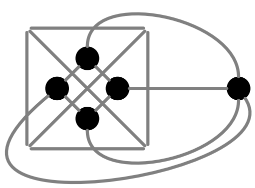

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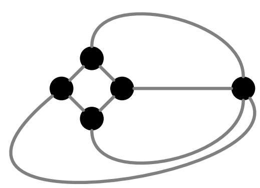


(d)

1.2.7 Figure

In this way, all of the edges in the graph represent borders of adjacent regions in the map. The connected regions in the complement of any plane graph are called *faces*. They are all *bounded* except for the *outer face* which is *unbounded*. The *length* of a face is the number of edge steps taken in a walk around its boundary. In the case of the unbounded face, the length is called the *perimeter* of the graph.

Given any plane graph G, we wish to consider the different ways of coloring the vertices of G. In the context of a given planar map, this corresponds to assigning colors to each region. The way in which we are coloring the vertices in G is as follows:

1) Each vertex receives a color.
2) If $v_i \sim v_j$ then they must receive different colors.

According to the four-color theorem, we can do this by using at most four colors if our graph is given by a planar map. An interesting question that is left unanswered by the four-color theorem is the question: When do we only need three colors to properly color the vertices of a plane graph?

2. Determining three-colorability of planar graphs

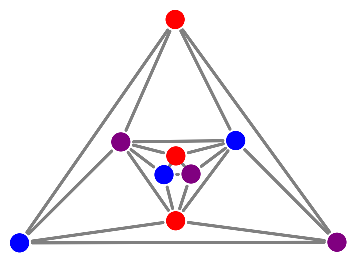
The *Three Color Problem* is the question of determining which planar graphs can be vertex-labeled from a set of three colors so that adjacent vertices have distinct colors, making them three-colorable. Larry Stockmeyer [1] proved that it is NP-complete to determine whether an arbitrary given planar graph is three-colorable, so it is likely that no nice characterization exists. Attention has therefore focused on finding sufficient conditions for three-colorability. For example, *Grötzsch's* Theorem [10] states that triangle-free planar graphs are three-colorable [14].

It was claimed in the 19th century that *triangulations* (plane graphs in which every face is a triangle) are three-colorable if (and only if) every vertex has even degree, but no complete proof was published until much later. The claim holds more generally for near triangulations in which every vertex has even degree, where a *near-triangulation* is a planar multigraph whose bounded faces are all three-cycles. Over the years, this statement has been proved and reproved in a variety of ways. In this project, we follow the exposition of "A new proof of 3-colorability of Eulerian triangulations" by Tsai and West [14], which uses a novel approach that is quite different from the previous proofs.

Recall that a connected graph with all vertex degrees even is *Eulerian*, meaning that it has a closed walk traversing all the edges exactly once (an *Eulerian circuit*). We will need to prove a lemma saying that Eulerian plane graphs have non-crossing Eulerian circuits. In particular, we will see that in a near-triangulation, by cycling through the colors 1; 2; 3, consecutively along the vertices, such a circuit produces a proper three-coloring. The proof will also use a lemma saying that the number of edges of any Eulerian near-triangulation is divisible by three. The method of proof given here can be viewed as providing a fast algorithm to produce the coloring. In fact, once one proves three-colorability, the independent sets in a proper three-coloring of an Eulerian triangulation are immediately unique. Indeed, as one travels from face to face, the vertices that complete two bounded faces sharing an edge must be in the same independent set.

In the next chapter, we begin the proof of the main result of this project, stated below.

2.0.1 Theorem *Every Eulerian near-triangulation is 3-colorable.*



2.0.2 Figure Eulerian near-triangulation with a three-coloring

3. Proof of the main result

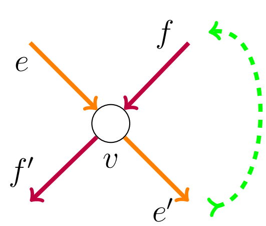
The first task is to show that there always exist non-crossing Eulerian circuits in any Eulerian plane multigraph.

3.0.1 Lemma *Every Eulerian plane multigraph has a non-crossing Eulerian circuit.*

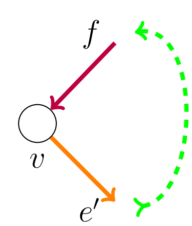
Proof. Let G be a Eulerian plane multigraph. By definition, G has at least one Eulerian circuit. Among all Eulerian circuits, pick the circuit that has the least amount of crossings and call that circuit C. If C has no crossings then we would be done. Then, by way of contradiction, suppose now that C has at least one crossing. Pick a crossing C_v in C on a vertex v with edge sets $\{e, e'\}$ and $\{f, f'\}$. In other words, C contains the sequence

$$..., e, v, e', ..., f, v, f', ...$$

So, C is a Eulerian circuit and we have picked a crossing at the vertex v. We know that there must exist a sub-circuit C_s that starts with v, e' and ends with f, v.



3.0.2 Figure Crossing C_v in C

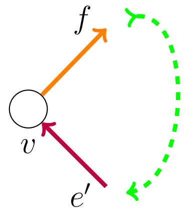


3.0.3 Figure Subcircuit C_s

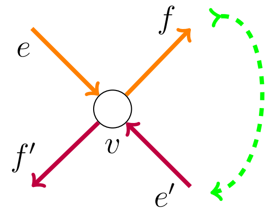
Reverse the direction of the sub-circuit C_s (Figure 3.0.4), and call the resulting Eulerian circuit C'. Now C' follows the path

$$..., e, v, f, ..., e', v, f', ...$$

as shown in Figure 3.0.5.



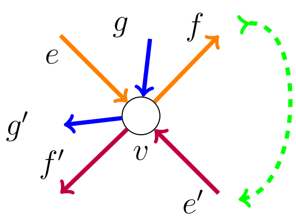
3.0.4 Figure Reversed subcircuit C'_s



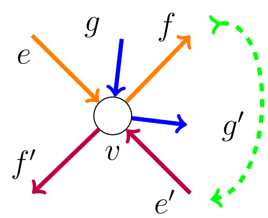
3.0.5 Figure Non-crossing C'_v in C'

Since C_s is a sub-circuit, changing the direction that we walk along its path will not disconnect the graph. Furthermore, it will eliminate this particular crossing at C_v and it will not increase the number of crossings.

To see this, we only need to consider any new crossings formed at v in C' that involve the newly formed sequences (e, v, f) or (e', v, f'). I claim any crossing formed in this way must correspond with a crossing that occurred already in C. For example, if C' has a crossing (e, v, f) with, say, (g, v, g'), there are three cases to consider, depicted below:



3.0.6 Figure



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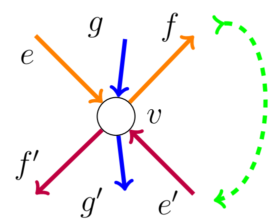
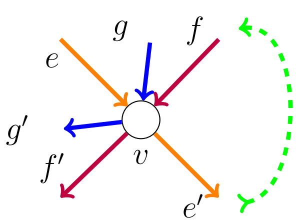
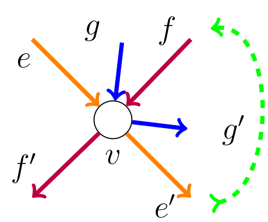


3.0.8 Figure

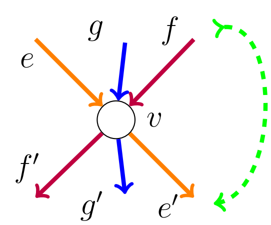
But in each of these cases, either (e, v, e') or (f, v, f') (or both) already formed crossings in C with (g, v, g') as shown:



3.0.9 Figure



3.0.10 Figure



3.0.11 Figure

In Figure 3.0.9 (g, v, g') crossed (e, v, e') in C. In Figure 3.0.10 (g, v, g') crossed (f, v, f') in C. In Figure 3.0.11 (g, v, g') crossed both (e, v, e') and (f, v, f') in C.

There are three more cases that are similar if (e', v, f') forms a new crossing in C' which we omit for brevity. It follows that, since C' has fewer crossings than C, and since C was selected to have the fewest crossings possible, we have arrived at the desired contradiction.

Therefore, every Eulerian plane graph has a non-crossing Eulerian circuit.

■

Now we have a non-crossing Eulerian circuit C. To obtain our coloring, we will assign colors to vertices while traversing the circuit C just described, cycling through colors 1, 2, and 3. In order to do this, it would be helpful to know if the number of edges in G is divisible by three. Also, we would need to know that the length of every sub-circuit is divisible by three. That way, when we are traversing along C, we will not assign more than one color to a vertex. Let us introduce some useful definitions and lemmas.

The *external edges* and *external vertices* in a plane graph are the edges and vertices incident with the unbounded face. A graph is *trivial* if it has no edges.

3.0.12 Lemma *In every Eulerian near-triangulation, the number of edges is divisible by three.*

Proof. Let G be an Eulerian near-triangulation. We will form a smaller near-triangulation G' from G by deleting the three edges of a bounded face that contains at least one external edge. We will then apply induction to this new graph G', which will always just be a disjoint union of smaller Eulerian near-triangulation graphs.

Base Case: Let n be the number of bounded faces in G, for some $n \in \mathbb{N}$.
When $n = 0$, G is the trivial graph with isolated vertices, which has zero edge count.

Induction hypothesis: In every Eulerian near-triangulation graph H with fewer than n bounded faces, the number of edges in H is divisible by three.

Let G be a Eulerian near-triangulation with $n > 0$ bounded faces. Let F be a bounded face containing at least one external edge. I claim when F is removed from G, you will be left with at most $n - 3$ bounded faces. This is because each edge will unbound at most one face. So by removing F you will have unbounded at most three bounded faces, one for each edge.

This leaves us with a graph G' that has at most $n - 1$ and at least $n - 3$ bounded faces. This graph G' is a Eulerian near-triangulations, each with fewer than n bounded faces. Therefore, by our induction hypothesis, the number of edges in G' is divisible by three. When F was removed we removed three edges along with it. Therefore, when F is added back into G' to recreate G, the number of edges in G will also be divisible by three.

Thus, in every Eulerian near-triangulation graph, the number of edges will be divisible by three.

■

3.0.13 Lemma *In a non-crossing Eulerian circuit of an Eulerian near-triangulation, the length of every sub-circuit is divisible by three.*

Proof. Let G be an Eulerian near-triangulation graph. By Lemma 3.0.1 there exists a non-crossing Eulerian circuit C in G. Let C' be a sub-circuit of C. We will use induction on the number of faces that are enclosed by C' to deduce the length of C' is divisible by three.

Base Case: If there is only one face bounded by C', then C' must be a triangle by definition. Therefore, the length of C' is three.

Induction hypothesis: In a non-crossing Eulerian circuit C of an Eulerian near-triangulation G, a sub-circuit C' enclosing fewer than n bounded faces has a length divisible by three, where $n \in \mathbb{N}$.

Let v be the first (and last) vertex of C'. Let H be a subgraph of G such that $E(H)$ and $V(H)$ consist of all the edges and vertices in G that are contained on C' and enclosed by C'.

Case 1: Suppose C' traverses all of H. Since C' is a circuit, H is Eulerian. Since H is a subgraph of G, H is also a near triangulation. So by Lemma 3.0.12 the number of edges in H is divisible by three. Since C' traverses all of H, the number of edges in C' is divisible by three.

Case 2: Suppose C' does not traverse all of H. This means there are portions of C that enter the interior of C' to traverse the parts of H that are contained in C', but that are not included on C'. This can happen at v or another external vertex of H that is visited more than once by C'.

Note that C' is a non-crossing circuit, and each external vertex of H has even degree. Also, since C' is non-crossing, each portion of C that enters the interior of C' must leave the interior at the same external vertex of H where it entered. This contributes even degree on all internal vertices of H and the vertices of H where C enters and exits C'. This tells us that H is Eulerian (and a near-triangulation). So using Lemma 3.0.12 the number of edges is divisible by three.

To restrict now to the portion of H not covered by C', note that each entry by the rest of C into the interior of H enters and departs from the same vertex. So these portions of C create sub-circuits of C that enclose fewer faces than C'. Thus by our induction hypothesis their lengths are divisible by three. So given the number of edges in H, and subtracting the number of edges in these sub-circuits, we are left with the number of edges in C'. The number of edges in H is divisible by three and the number of edges in the subtracted sub-circuits are also divisible by three. This leaves us with C' whose length must be divisible by three.

Therefore, in a non-crossing Eulerian circuit of an Eulerian near-triangulation, the length of every sub-circuit is divisible by three.

■

Every time you return to a vertex that has already been colored you will have completed a sub-circuit of C. By Lemma 3.0.13 the length of the sub-circuit is divisible by three. Therefore, the same color would be assigned to that vertex. In conclusion, the coloring is consistent, and it explicitly assigns distinct colors to the endpoints of every edge. Therefore, every Eulerian near-triangulation is three-colorable, which completes the proof of our main result.

4. Applications of study and new questions to ponder

The problem of coloring a graph arises in many practical areas such as pattern matching, sports scheduling, designing seating plans, exam timetabling, the scheduling of taxis, and solving Sudoku puzzles. The specific question of vertex coloring is useful in many scheduling problems. Given a set of jobs needed to be assigned to time slots, each job requires one such slot. Suppose jobs can be scheduled in any order, but pairs of jobs may be in conflict – in the sense that they may not be assigned to the same time slot. The corresponding graph contains a vertex for every job and an edge for every conflicting pair of jobs. The minimum number of colors assigned leads to a schedule with the optimal time to finish all jobs without any conflicts.

In light of the results of this paper, there are a number of natural questions that could be considered. For example, what other sufficient conditions gives a three-colorable graph? Are there other sufficient conditions that do not rely on triangulation that give a three-colorable graph? Can you use this type of walking around a Eulerian circuit algorithm when you are given a four-colorable graph? What about a five-colorable graph?

Perhaps such further questions could lead to an extension of these results.

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