Distances, Diameter, Girth, and Odd Girth in Generalized Johnson Graphs

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Abstract

Let $v > k > i$ be non-negative integers. The generalized Johnson graph, $J(v, k, i)$, is the graph whose vertices are the k-subsets of a v-set, where vertices A and B are adjacent whenever $|A \cap B| = i$. In this project, we present the results of the paper "On the girth and diameter of generalized Johnson graphs," by Agong, Amarra, Caughman, Herman, and Terada [1], along with a number of related additional results. In particular, we derive general formulas for the girth, diameter, and odd girth of $J(v, k, i)$. Furthermore, we provide a formula for the distance between any two vertices A and B in terms of the cardinality of their intersection. We close with a number of possible future directions.

1. Introduction

In this project, we present the results of the paper "On the girth and diameter of generalized Johnson graphs," by Agong, Amarra, Caughman, Herman, and Terada [1], along with a number of related additional results.

Let $v > k > i$ be non-negative integers. The *generalized Johnson graph*, $X = J(v, k, i)$, is the graph whose vertices are the k-subsets of a v-set, with adjacency defined by

$$A \sim B \Leftrightarrow |A \cap B| = i. \tag{1}$$

Generalized Johnson graphs were introduced by Chen and Lih in [4]. Special cases include the Kneser graphs $J(v, k, 0)$, the odd graphs $J(2k + 1, k, 0)$, and the Johnson graphs $J(v, k, k - 1)$. The Johnson graph $J(v, k, k - 1)$ is well known to have diameter $\min\{k, v - k\}$, and formulas for the distance and diameter of Kneser graphs were proved by Valencia-Pabon and Vera in [8].

Generalized Johnson graphs have also been studied under the name *uniform subset graphs*, and a result in [5] offers a general formula for the diameter of $J(v, k, i)$. However, that formula gives incorrect values when $i > \frac{2}{3}k$, an important case that includes the Johnson graphs. In [1], we corrected and extended these expressions for the diameter of generalized Johnson graphs, and we present those results below.

In addition, we provide a formulas for the girth and odd girth of $J(v, k, i)$. The general formula for odd girth was proved by Denley in [6] for *generalized Kneser graphs*, which are defined similarly to generalized Johnson graphs; with adjacency condition (1) replaced by $A \sim B \Leftrightarrow |A \cap B| \leq i$. However, the proofs in [6] imply that the same expression for odd girth also holds for generalized Johnson graphs (although this is not stated explicitly). We give here a new proof of this result aimed specifically at generalized Johnson graphs.

There are still a large number of open questions regarding generalized Johnson graphs. These include general expressions for independence number and chromatic number. While both of these are known for the special case of Kneser graphs, they are both *unknown* in the special (and well-studied) case of Johnson graphs.

It is possible to extend the definition of $X = J(v, k, i)$ to include $v \geq k \geq i$. However, X is an empty graph when $k = i$ or $v = k$. If $v = 2k$ and $i = 0$, then X is isomorphic to the disjoint union of copies of K_2. Furthermore, by taking complements, the graphs $J(v, k, i)$ and $J(v, v - k, v - 2k + i)$ are easily seen to be isomorphic (see [7, p.11]). For the remainder of this article, we will be concerned with generalized Johnson graphs that are connected, so we make the following global definition.

Definition 1.1. Assume $v > k > i$ are nonnegative integers, and let $X = J(v, k, i)$ denote the corresponding generalized Johnson graph. To avoid trivialities, further assume that $v \geq 2k$, and that $(v, k, i) \neq (2k, k, 0)$.

2. Girth

In this section we derive an expression for the girth $g(X)$ of a generalized Johnson graph, X. We begin with a lemma that characterizes when two vertices have a common neighbor.

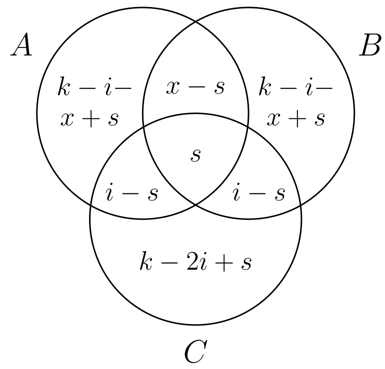

Figure 1: Vertices with a common neighbor

Definition 2.1. The *girth* of a graph, G, is the length of the shortest cycle that is a (not necessarily induced) subgraph of G.

Lemma 2.2. *With reference to Definition 1.1, let A and B be vertices and let $x = |A \cap B|$. Then A and B have a common neighbor if and only if $x \geq \max\{-v + 3k - 2i, 2i - k\}$.*

Proof. Vertices A and B have a common neighbor C if and only if there exists a nonnegative integer s, such that every region in the above diagram (Figure. 1) has nonnegative size.

By simplifying the resulting inequalities, we find that A and B have a common neighbor if and only if there exists $s \in \mathbb{Z}$, such that

$$\max\{0,\ i + x - k,\ 2i - k\} \leq s \leq \min\{x,\ i,\ v - 3k + 2i + x\}.$$

Such an integer s exists if and only if the expression on the left-hand side above does not exceed the expression on the right-hand side. Under our global assumptions, this is equivalent to $x \geq \max\{-v + 3k - 2i, 2i - k\}$. □

Lemma 2.3. *With reference to Definition 1.1, the girth $g(X) = 3$ if and only if $v \geq 3(k - i)$.*

Proof. The graph X contains a 3-cycle if and only if there exist adjacent vertices A and B that have a common neighbor. By Lemma 2.2, this occurs if and only if $i \geq \max\{-v + 3k - 2i, 2i - k\}$. Since $i \geq 2i - k$ holds in all $J(v, k, i)$ graphs, this condition is equivalent to $v \geq 3(k - i)$. □

A sufficient condition for the girth to be at most 4 is the existence of a 4-cycle.

Lemma 2.4. *With reference to Definition 1.1, if $(v, k, i) \neq (2k + 1, k, 0)$ then $g(X) \leq 4$.*

Proof. We proceed in two cases.

Case 1: $i \geq 2$ or $v > 2k + 1$. In this case we get that $v \geq 2k - i + 2$. So we can find disjoint sets, A_1, A_2, A_3, A_4, and B_1, B_2, and C such that $|A_1| = |A_2| = |A_3| = |A_4| = 1$, and $|B_1| = |B_2| = k - i - 1$, and $|C| = i$. Then

$$A_1 \cup B_1 \cup C, \quad A_2 \cup B_2 \cup C, \quad A_3 \cup B_1 \cup C, \quad A_4 \cup B_2 \cup C$$

is a 4-cycle in X.

Case 2: $i = 1$. In this case, since $v \geq 2k$, we can find disjoint sets A_1, A_2, A_3, A_4 and B_1, B_2 such that $|A_1| = |A_2| = |A_3| = |A_4| = 1$ and $|B_1| = |B_2| = k - 2$. Then

$$A_1 \cup A_2 \cup B_1, \quad A_2 \cup A_3 \cup B_2, \quad A_3 \cup A_4 \cup B_1, \quad A_4 \cup A_1 \cup B_2$$

is a 4-cycle in X. □

Combining the above lemmas, we obtain a general expression for the girth.

Theorem 2.5. *With reference to Definition 1.1, the girth of X is given by*

$$g(X) = \begin{cases} 3 & \text{if} \quad v \geq 3(k-i); \\ 4 & \text{if} \quad v < 3(k-i) \text{ and } (v,k,i) \neq (2k+1,k,0); \\ 5 & \text{if} \quad (v,k,i) = (5,2,0); \\ 6 & \text{if} \quad (v,k,i) = (2k+1,k,0) \text{ and } k > 2. \end{cases}$$

Proof. The first two cases follow from Lemmas 2.3 and 2.4. The remaining cases are odd graphs, for which the girth is well-known. (See, for example, [2, p.58].) □

3. Distance

In this section we derive a general expression for the distance between two vertices in terms of the size their intersection, Theorem 3.5.

Lemma 3.1. *With reference to Definition 1.1, let A and B be vertices and let $x = |A \cap B|$. Suppose $x < i$. If $x < -v + 3k - 2i$, then*

$$\text{dist}(A, B) = 3.$$

Proof. Since $x < i$, $\text{dist}(A, B) \geq 2$. By Lemma 2.2, $\text{dist}(A, B) > 2$. Let $A' \subseteq A \setminus B$, such that $|A'| = i - x$. Let $B' \subseteq B \setminus A$, such that $|B'| = k - i$. Let $C = A \cap B$, and let $D = C \cup A' \cup B'$. Then $|D| = x + (i - x) + (k - i) = k$, and $|A \cap D| = x + (i - x) = i$, so D is a vertex adjacent to A. Note that $|D \cap B| = k - i + x \geq -v + 3k - 2i$. Also, since $x < -v + 3k - 2i$, we have $2i - k < -(v - 2k) - x \leq 0$, so $|D \cap B| \geq 2i - k$. Hence by Lemma 2.2, $\text{dist}(D, B) \leq 2$. Hence $\text{dist}(A, B) = 3$. □

Together with the previous lemma, the next result characterizes the distance between vertices whose intersection is less than i.

Lemma 3.2. *With reference to Definition 1.1, let A and B be vertices and let $x = |A \cap B|$. Suppose $x < i$. If $x \geq -v + 3k - 2i$, then*

$$\text{dist}(A, B) = \left\lceil \frac{k - x}{k - i} \right\rceil.$$

Proof. We proceed in two cases.

Case 1: $x \geq 2i - k$. Since $x < i$, we know $\text{dist}(A, B) \geq 2$. Since $x \geq 2i - k$, Lemma 2.2 implies that $\text{dist}(A, B) = 2$. Note that the above inequalities imply $k - i < k - x \leq 2(k - i)$. Hence $\lceil \frac{k-x}{k-i} \rceil = 2$.

Case 2: $x < 2i - k$. In this case, $k - x > 2(k - i)$. Therefore, there exist positive integers q, m such that $k - x = (q + 1)(k - i) + m$ with $0 < m \leq k - i$. Let $C = A \cap B$. Then we can write A and B as disjoint unions

$$A = A_1 \cup \cdots \cup A_{q+2} \cup C \quad \text{and} \quad B = B_1 \cup \cdots \cup B_{q+2} \cup C,$$

where $|A_j| = |B_j| = k - i$ for $j \in \{1, \ldots, q + 1\}$ and $|A_{q+2}| = |B_{q+2}| = m$. Define

$$X_j = (B_1 \cup \cdots \cup B_j) \cup (A_{j+1} \cup \cdots \cup A_{q+2}) \cup C$$

for each $j \in \{1, \ldots, q\}$. Then $A, X_1, \ldots, X_q$ is a path of length q. Note that $|X_q \cap B| = x + q(k - i) = i - m$, so $2i - k \leq |X_q \cap B| < i$ and therefore Case 1 applies. Thus, $\text{dist}(X_q, B) = 2$ and so

$\mathrm{dist}(A,B) \le q+2 = \lceil \frac{k-x}{k-i} \rceil$. On the other hand, since adjacent vertices differ by $k-i$ elements, $\mathrm{dist}(A,B) \ge \lceil \frac{k-x}{k-i} \rceil$. □

We now address the case where the intersection between A and B is greater than i. The following lemma adapts Lemmas 1 and 2 in [9] to generalized Johnson graphs.

Lemma 3.3. *With reference to Definition 1.1, let A and B be vertices and let $x = |A\cap B|$. Suppose $x > i$ and assume there is an AB-path of length d.*

(i) If $d = 2p$, then

$$p \ge \left\lceil \frac{k-x}{v-2k+2i} \right\rceil.$$

(ii) If $d = 2p+1$, then

$$p \ge \left\lceil \frac{x-i}{v-2k+2i} \right\rceil.$$

Proof. For brevity, let $\Delta = v-2k+2i$. If $d=0$, then $A=B$ so, $x=k$ and $p = 0 \ge \lceil \frac{k-x}{\Delta} \rceil$. If $d=1$, then $x=i$, so $p=0 \ge \lceil \frac{x-i}{\Delta} \rceil$. If $d=2$, then by Lemma 2.2, $x \ge -v+3k-2i$, which implies $k-x \le \Delta$. Hence, $p = 1 \ge \lceil \frac{k-x}{\Delta} \rceil$. Assume $d \ge 3$ and that the claim holds for all paths of length less than d. We proceed in two cases.

Case 1: $d = 2p$. We can find a vertex C such that $\mathrm{dist}(A,C) = 2(p-1)$ and $\mathrm{dist}(C,B)=2$. By the inductive hypothesis, $k-|A\cap C| \le (p-1)\Delta$ and $k-|C\cap B| \le \Delta$. Therefore, $k-x = |A\setminus B| \le |A\setminus C| + |C\setminus B| = (k-|A\cap C|)+(k-|C\cap B|) \le p\Delta$. Hence $p \ge \lceil \frac{k-x}{\Delta} \rceil$.

Case 2: $d = 2p+1$. We can find a vertex C adjacent to B and such that $\mathrm{dist}(A,C) = 2p$. By the inductive hypothesis, $|A\setminus C| \le p\Delta$. Therefore, $x-i = |A\cap B| - i \le |A\setminus C| + |B\cap C| - i \le p\Delta$. Hence $p \ge \lceil \frac{x-i}{\Delta} \rceil$. □

The previous lemma implies a lower bound on the distance. The next result will show that this bound is sharp.

Lemma 3.4. *With reference to Definition 1.1, let A and B be vertices and let $x = |A\cap B|$. Suppose $x > i$. Then*

$$\mathrm{dist}(A,B) = \min\left\{ 2\left\lceil \frac{k-x}{v-2k+2i} \right\rceil, 2\left\lceil \frac{x-i}{v-2k+2i} \right\rceil + 1 \right\}.$$

Proof. For brevity, let $\Delta = v-2k+2i$. When $x=k$ the result is trivial, so assume $x<k$. Let $C = A\cap B$ and $D = \overline{A\cup B}$; it follows that $|C| = x$ and $|D| = v-2k+x$. There exist non-negative integers q,m such that $k-x = q\Delta + m$, with $0 < m \le \Delta$. We can write A and B as disjoint unions $A = C\cup\{a_1,\ldots,a_{k-x}\}$ and $B = C\cup\{b_1,\ldots,b_{k-x}\}$. If $q=0$, then $k-x = m \le \Delta$, which implies $x \ge -v+3k-2i$. Since $x>i$, we also have $x > 2i-k$. Hence, by Lemma 2.2, $\mathrm{dist}(A,B)=2$ as needed. Now, assume $q\ge 1$. For $j \in \{1,\ldots,q\}$, let

$$A_j = \{a_1,\ldots,a_{(j-1)\Delta+i}\} \quad \text{and} \quad A'_j = \{a_{j\Delta+1},\ldots,a_{k-x}\},$$

$$B_j = \{b_1,\ldots,b_{j\Delta}\} \quad \text{and} \quad B'_j = \{b_{j\Delta-i+1},\ldots,b_{k-x}\},$$

and define

$$X_{2j-1} = D\cup A_j\cup B'_j \quad \text{and} \quad X_{2j} = C\cup B_j\cup A'_j.$$

Then $A, X_1,\ldots,X_{2q}$ is a path of length $2q$ (see Figure. 2). Note that $|X_{2q}\cap B| = k-m \ge k-\Delta = -v+3k-2i$. Also, since $m \le k-x$, we have $|X_{2q}\cap B| = k-m \ge x > i \ge 2i-k$. Hence

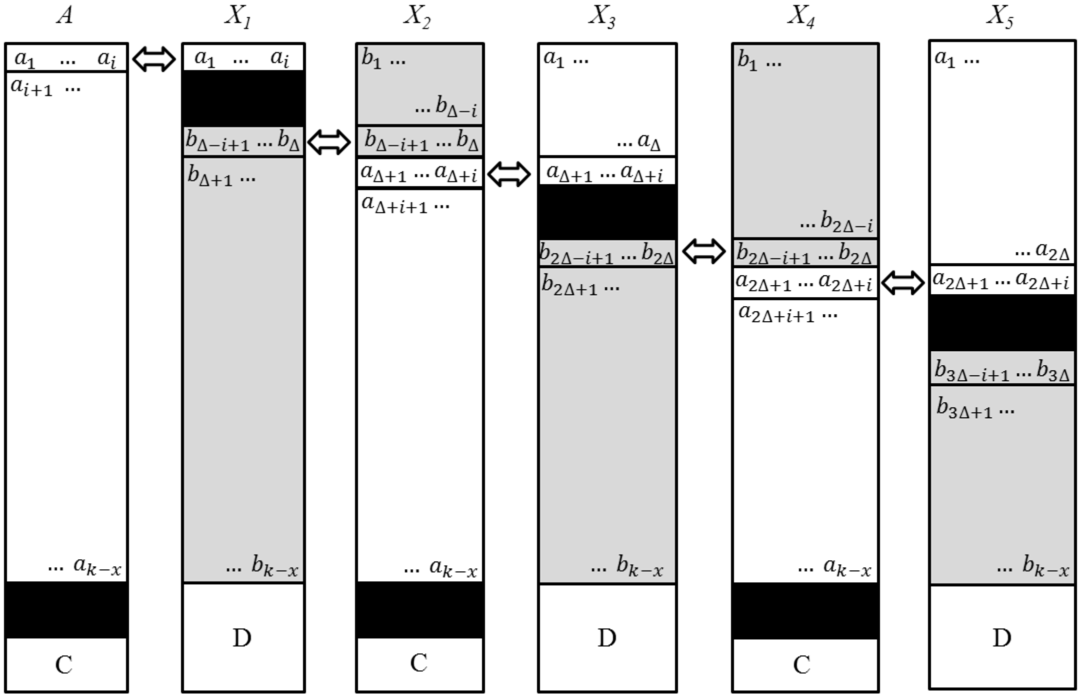

Figure 2: Start of path from A

$\text{dist}(X_{2q}, B) = 2$, by Lemma 2.2. Thus, there is an AB-path of length $2(q+1) = 2\lceil (k-x)/\Delta \rceil$ from A to B.

Now, let $D' \subseteq D$, $C' \subseteq C$ be such that $|D'| = |C'| = x - i$. Let $A' = (B \setminus C') \cup D'$. Then A' is a vertex adjacent to A. Further, $|A' \cap B| = k - x + i > i$. By applying the previous argument to A' and B, there is an $A'B$-path of length $2\lceil \frac{k-(k-x+i)}{\Delta} \rceil = 2\lceil \frac{x-i}{\Delta} \rceil$. By Lemma 3.3, $\text{dist}(A, B) = \min\{2\lceil \frac{k-x}{v-2k+2i} \rceil, 2\lceil \frac{x-i}{v-2k+2i} \rceil + 1\}$. □

From the above results, we obtain a general formula for the distance between two vertices.

Theorem 3.5. *With reference to Definition 1.1, let A and B be vertices and let $x = |A \cap B|$. Then*

$$dist(A, B) = \begin{cases} 3 & \text{if} \;\; x < \min\{i, -v + 3k - 2i\}; \\ \lceil \frac{k-x}{k-i} \rceil & \text{if} \;\; -v + 3k - 2i \le x < i; \\ \min\{2\lceil \frac{k-x}{v-2k+2i} \rceil, 2\lceil \frac{x-i}{v-2k+2i} \rceil + 1\} & \text{if} \;\; x \ge i. \end{cases}$$

Proof. Apply Lemmas 3.1, 3.2, and 3.4. Note that when $x = i$, we have $\text{dist}(A, B) = 1 = \min\{2\lceil \frac{k-x}{v-2k+2i} \rceil, 2\lceil \frac{x-i}{v-2k+2i} \rceil + 1\}$. □

4. Diameter

In this section, we will use Theorem 3.5 to derive a general expression for the diameter of generalized Johnson graphs. The following lemma determines the maximum value of the expression in Lemma 3.4.

Definition 4.1. The *diameter* of a graph is the maximum distance between any pair of vertices.

Lemma 4.2. *Assume $k > i+1$ and let $f(x) = \min\left\{2\left\lceil\frac{k-x}{v-2k+2i}\right\rceil, 2\left\lceil\frac{x-i}{v-2k+2i}\right\rceil + 1\right\}$. Then*

$$\max_{x\in\mathcal{I}} f(x) = \left\lceil\frac{k-i-1}{v-2k+2i}\right\rceil + 1,$$

where $\mathcal{I} = \{i+1, \ldots, k\}$.

Proof. For brevity, let $\Delta = v - 2k + 2i$ and let $x \in \mathcal{I}$. There exist $\epsilon \in \{0,1\}$ and non-negative integers q, m such that $k-i-1 = (2q+\epsilon)\Delta + m$ and $0 < m \leq \Delta$. We prove $\max_{x\in\mathcal{I}} f(x) = 2q+\epsilon+2$.

Let $x_0 = (q+\epsilon)\Delta + i$. If $x > x_0$, then $2\lceil\frac{k-x}{\Delta}\rceil \leq 2\lceil\frac{k-(x_0+1)}{\Delta}\rceil = 2(q+1) \leq 2q+\epsilon+2$. If $x \leq x_0$, then $2\lceil\frac{x-i}{\Delta}\rceil + 1 \leq 2\lceil\frac{x_0-i}{\Delta}\rceil + 1 = 2(q+\epsilon)+1 \leq 2q+\epsilon+2$. Hence, $f(x) \leq 2q+\epsilon+2$.

Let $x_1 = q\Delta + i + 1 + \epsilon(m-1) \in \mathcal{I}$. It follows that $\lceil\frac{k-x_1}{\Delta}\rceil = q+\epsilon+1$ and $\lceil\frac{x_1-i}{\Delta}\rceil = q+1$. Therefore, $f(x_1) = \min\{2(q+\epsilon+1), 2q+3\} = 2q+\epsilon+2$. It follows that $\max_{x\in\mathcal{I}} f(x) = 2q+\epsilon+2$. □

We now present our main result, which extends and corrects the diameter expression in [5].

Theorem 4.3. *With reference to Definition 1.1, we have*

$$diam(X) = \begin{cases} \lceil\frac{k-i-1}{v-2k+2i}\rceil + 1 & \text{if } v < 3(k-i)-1 \text{ or } i = 0; \\ 3 & \text{if } 3(k-i)-1 \leq v < 3k-2i \text{ and } i \neq 0; \\ \lceil\frac{k}{k-i}\rceil & \text{if } v \geq 3k-2i \text{ and } i \neq 0. \end{cases}$$

Proof. We will use the distance expression from Theorem 3.5. We proceed in three cases.

Case 1: $v < 3(k-i)-1$ or $i = 0$. If $i = 0$, the result is proved in [8]. Assume $v < 3(k-i)-1$. In this case $\lceil\frac{k-i-1}{v-2k+2i}\rceil + 1 \geq 3$. Also, $2k \leq v < 3(k-i)$, so $\lceil\frac{k}{k-i}\rceil \leq \lceil\frac{3}{2}\rceil = 2$. Hence, $\lceil\frac{k}{k-i}\rceil \leq 3 \leq \lceil\frac{k-i-1}{v-2k+2i}\rceil + 1$. Since $0 \leq i < k < v < 3(k-i)-1$ by Definition 1.1, it follows that $k > i+1$. By Lemma 4.2, there exist vertices A and B such that $\text{dist}(A,B) = \lceil\frac{k-i-1}{v-2k+2i}\rceil + 1$. From Theorem 3.5, it follows that $\text{diam}(X) = \lceil\frac{k-i-1}{v-2k+2i}\rceil + 1$

Case 2: $3(k-i)-1 \leq v < 3k-2i$ and $i \neq 0$. Since $v \geq 3(k-i)$, we have $\lceil\frac{k-i-1}{v-2k+2i}\rceil + 1 \leq 2$. Since $2k \leq v < 3k-2i$, we have $\lceil\frac{k}{k-i}\rceil \leq 2$. By Theorem 3.5, if A and B are disjoint vertices, $\text{dist}(A,B) = 3$; hence $\text{diam}(X) = 3$.

Case 3: $v \geq 3k-2i$ and $i \neq 0$. In this case $\lceil\frac{k-i-1}{v-2k+2i}\rceil + 1 \leq 2$. Since $v \geq 3k-2i$, we have $-v+3k-2i \leq 0$, so the first case in Theorem 3.5 does not occur. Since $i \neq 0$, we have $\lceil\frac{k}{k-i}\rceil \geq 2$. If A and B are disjoint vertices, $\text{dist}(A,B) = \lceil\frac{k}{k-i}\rceil$, by Theorem 3.5. Hence $\text{diam}(X) = \lceil\frac{k}{k-i}\rceil$. □

5 Odd Girth

Definition 5.1. The *odd girth* of a graph is the length of its shortest odd cycle.

We now derive an expression for the odd girth of $X = J(v,k,i)$, denoted $og(X)$. In the case that $g(X)$ is odd, it is immediate that $g(X) = og(X)$. Therefore, the girth formula for generalized Johnson graphs, given above, also gives the odd girth for the cases $g(X) = 3$ and $g(X) = 5$. Below, we will work out the remaining cases $g(X) = 4$ and $g(X) = 6$. It is quite surprising that, while the girth formula (and distance and diameter formulas) given above all require at least 3 cases, a single formula is sufficient to give the odd girth for all generalized Johnson graphs!

The odd girth is of particular interest when studying graph homomorphisms, since $G \to H$ implies that $og(H) \leq og(G)$. This is because the homomorphic image of an odd cycle must contain an odd cycle of no greater length. There are several well known results regarding the homomorphism structure of Kneser graphs (e.g. [7]), but less is known about the homomorphism structure of generalized Johnson graphs. The results below contribute to the understanding of this structure by precluding the existence of homomorphisms between certain pairs of generalized Johnson graphs.

Theorem 5.2. *With reference to Definition 1.1, the odd girth of X is given by*

$$og(X) = 2\left\lceil \frac{k-i}{\Delta} \right\rceil + 1. \tag{2}$$

Theorem 2.5 tells us that the girth of X satisfies $3 \leq g(X) \leq 6$.

5.3 Girth 3 or 5

When $g(X) = 3$ or 5, the odd girth equals the girth, so $og(X) = 3$ or 5, respectively.

5.4 Girth 6

When $g(X) = 6$, Theorem 2.5 tells us that X is an odd graph, with $(v, k, i) = (2k+1, k, 0)$. In this case, the odd girth is given by the following well-known result.

Lemma 5.5. *Let $X = J(2k+1, k, 0)$ where $k > 2$. Then $og(X) = 2k + 1$.*

Proof. Suppose $\mathcal{O}$ is any odd cycle in X with length $2t + 1$. We aim to show that $t \geq k$. Fix any three consecutive vertices A, B, C along $\mathcal{O}$. By Theorem 2.5, the girth of X is 6, so $A \not\sim C$. Therefore, $dist(A, C) = 2$ and, by Theorem 3.5, we know $|A \cap C| = k - 1$. Along $\mathcal{O}$, there exists an AC-path of length $2t - 1$. By Lemma 3.3, we have $t \geq k$ as desired.

Next we show that a closed walk of length $2k+1$ exists. If k is even, let $k = 2d$ and consider the vertices $A = [2d]$, $B = [3d] \setminus [d]$, and $C = [4d] \setminus B$. If k is odd, let $k = 2d+1$ and consider $A = [2d+1]$, $B = [3d+2] \setminus [d+1]$, and $C = [4d+3] \setminus (B \cup \{1\})$. In either case, $dist(A, B) = dist(A, C) = k$, and $B \sim C$ by Theorem 3.5, as desired. □

5.6 Girth 4

It remains to consider the case when $g(X) = 4$. For brevity, we will again abbreviate $\Delta = v - 2k + 2i$.

Lemma 5.7. *With reference to Definition 1.1, assume $g(X) = 4$. Fix any vertices A,B and let $r = \lceil \frac{k-i}{\Delta} \rceil$. Then the following hold.*

(i) $1 < \Delta < k - i$.

(ii) *If r is odd and $|A \cap B| = \lfloor \frac{k+i-\Delta}{2} \rfloor$ or $\lceil \frac{k+i-\Delta}{2} \rceil$, then $dist(A, B) = r$.*

(iii) *If r is even and $|A \cap B| = \lfloor \frac{k+i}{2} \rfloor$ or $\lceil \frac{k+i}{2} \rceil$, then $dist(A, B) = r$.*

Proof. (i). Note that $\Delta = 0$ iff $(v, k, i) = (2k, k, 0)$ which is excluded by our hypotheses. Also, $\Delta = 1$ iff $(v, k, i) = (2k+1, k, 0)$. Therefore, by Theorem 2.5, and since $g(X) = 4$, we have $\Delta < k - i$ and $\Delta \neq 1$, as desired.
(ii). Let $x = |A \cap B|$ and $\lceil \frac{k-i}{\Delta} \rceil = 2d + 1$. Then $\Delta \geq 2$ implies that $2d - 1 + \frac{2}{\Delta} < \frac{k-i}{\Delta} \leq 2d + 1$. It follows, by our assumptions, that $\Delta(d-1) + 1 + i \leq x \leq \Delta d + i$. This implies both $\lceil \frac{k-i}{\Delta} \rceil = d$ and $x > i$. Note that

$$\left\lceil \frac{k-x}{\Delta} \right\rceil + \left\lceil \frac{x-i}{\Delta} \right\rceil \geq \frac{(k-x) + (x-i)}{\Delta} = \frac{k-i}{\Delta} > 2d,$$

so $\lceil \frac{k-x}{\Delta} \rceil \geq d + 1$. Now, by Theorem 3.5, we have $dist(A, B) = 2d + 1$ as desired.

(iii). Similar to (ii). □

Lemma 5.8. *With reference to Definition 1.1, assume $g(X) = 4$. Fix any $r \geq 1$ and suppose that $og(X) \geq 2r+1$ and $\frac{k-i}{\Delta} > r-1$. Then*

$$og(X) = 2r+1 \qquad \textit{if and only if} \qquad \frac{k-i}{\Delta} \leq r.$$

Proof. Case $r = 2d+1$: (⇐) Assume $\frac{k-i}{\Delta} \leq r$, so that $\lceil \frac{k-i}{\Delta} \rceil = r$. Let $x = \lfloor \frac{k+i-\Delta}{2} \rfloor$, $y = \lceil \frac{k+i-\Delta}{2} \rceil$. It follows from $2k \leq v < 3(k-i)$ that $0 \leq x, y \leq k-i$. Fix any adjacent vertices $A \sim B$. Choose $A_0 \subseteq A \setminus B$ and $B_0 \subseteq B \setminus A$ with $|A_0| = x$ and $|B_0| = y$. Let $C_0 = [v] \setminus (A \cup B)$, and $C = A_0 \cup B_0 \cup C_0$. Then $|C| = x+y+(v-2k+i) = k$, so $C \in V(X)$. Also, $|A \cap C| = x$ and $|B \cap C| = y$. By Lemma 5.7(ii), $dist(A, C) = dist(B, C) = r$ and X has a closed walk of length $2r+1$.

(⇒) Assume $og(X) = 2r+1$ and fix adjacent vertices A, B on a $(2r+1)$-cycle. Let C be the opposite vertex on that cycle, so that $dist(A, C) = dist(B, C) = r$. Now $|A \cap C|, |B \cap C| \leq \Delta d + i$ by Theorem 3.5. Therefore, $v - 2k + i \geq |C \setminus (A \cup B)| \geq k - 2(\Delta d + i)$, which implies $\frac{k-i}{\Delta} \leq r$.

Case $r = 2d$: (⇐) Assume $\frac{k-i}{\Delta} \leq r$, so that $\lceil \frac{k-i}{\Delta} \rceil = r$. Let $x = \lfloor \frac{k+i}{2} \rfloor$, $y = \lceil \frac{k+i}{2} \rceil$ and notice that $0 \leq x-i, y-i \leq k-i$. Fix any adjacent vertices $A \sim B$. Choose $A_0 \subseteq A \setminus B$ and $B_0 \subseteq B \setminus A$ with $|A_0| = x-i$ and $|B_0| = y-i$. Let $C_0 = A \cap B$, and $C = A_0 \cup B_0 \cup C_0$. Then $|C| = k$, so $C \in V(X)$. Also, $|A \cap C| = x$ and $|B \cap C| = y$. By Lemma 5.7(iii), $dist(A, C) = dist(B, C) = r$ and X has a closed walk of length $2r+1$.

(⇒) Assume $og(X) = 2r+1$ and fix adjacent vertices A, B on a $(2r+1)$-cycle. Let C be the opposite vertex on that cycle, so that $dist(A, C) = dist(B, C) = r$. Now $|A \cap C|, |B \cap C| \geq k - \Delta d$ by Theorem 3.5. So $k = |C| \geq |A \cap C| + |B \cap C| - |A \cap B| \geq 2(k - \Delta d) - i$, and thus $\frac{k-i}{\Delta} \leq r$. □

Theorem 5.9. *With reference to Definition 1.1, the odd girth of X is given by*

$$og(X) = 2\left\lceil \frac{k-i}{\Delta} \right\rceil + 1. \tag{3}$$

With all of the possible cases for the girth of X having been considered, we are now ready to put the pieces together.

Proof. By Theorem 2.5 and Lemma 5.5, equation (3) holds whenever $g(X) \neq 4$. Now assume $g(X) = 4$ and let $r = \lceil \frac{k-i}{\Delta} \rceil$. We have $r \geq 1$ and $\lceil \frac{k-i}{\Delta} \rceil \geq r-1$. By way of contradiction, suppose $og(X) < 2r+1$. Then $og(X) = 2\hat{r}+1$ for some $1 \leq \hat{r} < r$, and clearly $\lceil \frac{k-i}{\Delta} \rceil > \hat{r} - 1$, so Lemma 5.8 applies to $\hat{r}$, giving $\frac{k-i}{\Delta} \leq \hat{r} \leq r-1$, which is a contradiction. Hence $og(X) \geq 2r+1$. Now Lemma 5.8 applies to r, yielding $og(X) = 2r+1$, as desired. □

6 Future Directions: Independence Number

Definition 6.1. The independence number, $\alpha(X)$, of a graph, X, is the size of the largest set of vertices that induces an empty graph.

As mentioned in the introduction, no general expression is known for the independence number of the generalized Johnson graphs (or even the Johnson graphs). While the independence number for Kneser graphs is given by the well-known Erdos-Ko-Rado theorem; even the specific case of Johnson graphs is still not known. For specific values of v, k and i, there are interesting techniques known for constructing large independent sets. Below we outline one such case whose method comes from chapter 11, theorem 1 of [3].

Lemma 6.2. *Let F be the field of order q; $S, T \subset F$. Let $f_S(z) = \Pi_{s \in S}(z - s)$ and define $S =_j T$ to mean that the first j power sums of the elements of S and T are equal. Let $A, B, C \subset F$ be pairwise disjoint with $|A| = |B| = k$ and $|A \setminus B| = |B \setminus A| = j$. The following are equivalent.*

(i) $f_A = f_B$

(ii) $f_{A \setminus B} = f_{B \setminus A}$

(iii) $A \setminus B =_j B \setminus A$

(iv) $A =_j B$

Proof. Let $C = A \cap B$. Then $f_A = f_{A \setminus B} f_C$ and $f_B = f_{B \setminus A} f_C$. Therefore, (i) $\Leftrightarrow$ (ii) follows from the fact that $F[z]$ is a domain. Since (iii) $\Leftrightarrow$ (iv) is trivial, it is sufficient to show (ii) $\Leftrightarrow$ (iii).

For $S \subset F$, let $p_m(S)$ and $e_m(S)$ denote the m^{th} power sum of S and the m^{th} symmetric polynomial in $|S|$ variables evaluated at S, respectively. Then (ii) holds if and only if $e_m(A \setminus B) = e_m(B \setminus A)$ for $m = 1, \ldots, j$; and (iii) holds if and only if $p_m(A \setminus B) = p_m(B \setminus A)$ for $m = 1, \ldots, j$. It follows from Newton's identities that $p_m(S)$ is a polynomial in $e_1(S), \ldots, e_m(S)$; and $m! e_m(S)$ is a polynomial in $p_1(S), \ldots, p_m(S)$. Since $j < q$, we get that $m! \neq 0$ for $m = 1, \ldots, j$. It follows immediately that (ii) $\Leftrightarrow$ (iii). □

Corollary 6.3. *Let F be the field of order q. Let $A, B \subset F$ with $A \neq B$ and $|A| = |B| = k$. If $A =_{k-i} B$ then $|A \cap B| < i$.*

Proof. Let $j = |A \cap B|$. Using 6.2 and unique factorization, we get $A \neq B \Rightarrow f_A \neq f_B \Rightarrow A \neq_{k-j} B$. It follows that $k - i < k - j$, or equivalently $j < i$. □

Corollary 6.4. *Let F be the field of order q. Let $p : \mathcal{P}(F) \to F^{k-i}$ be defined by $p(X)_j = \Sigma_{x \in X} x^j$ for $j = 1, \ldots, k - i$. Then, identifying F with $[q]$, the fibres of $p \upharpoonright \binom{F}{k}$ are independent sets in $J(q, k, i)$. Hence, by the PHP, $J(q, k, i)$ has an independent set of size $\binom{q}{k}/q^{k-i}$*

Proof. Let A and B be adjacent vertices in $J(q, k, i)$. Suppose, towards a contradiction, that A and B are in the same fibre of p. Let $C = A \setminus B$, $D = B \setminus A$. Then C and D are disjoint sets of cardinality $k - i$ and $p(C) = p(D)$. Let $f(z) = \Pi_{c \in C}(z - c)$ and $g(z) = \Pi_{d \in D}(z - d)$. Then, for $j = 0, \ldots, k - i - 1$, the coefficient in f (resp. g) of z^j is $e_{k-i-j}(C)$ (resp. $e_{k-i-j}(D)$). Since the first $k - i$ power sums of C and D are equal, Newton's identities (over F) imply that $e_j(C) = e_j(D)$ for $j = 1, \ldots, k - i$. Hence $f = g$, so f and g must have the same roots in F; thus $C = D$, a contradiction. □

Theorem 6.5. *Fix k and i, and let $h(v) = \alpha(J(v, k, i))$ for all $v \geq 2k$. Then $h(v) \in \Omega(v^{\max(k-i-1, i)})$.*

Proof. Let A be an $(i + 1)$-subset of $[v]$. Then, the set of all k-subsets of $[v]$ containing A is an independent set in $J(v, k, i)$ with size $\binom{v-i-1}{k-i-1}$. Hence $h(v) \geq \binom{v-i-1}{k-i-1} \in \Omega(v^{k-i-1})$.

By Bertrand's postulate, there is a prime, q, between $\lfloor v/2 \rfloor$ and v. Let F be the field with q elements. By 6.4, $J(q, k, i)$ has an independent set of size $\binom{q}{k}/q^{k-i} \in \Omega(q^i)$. Since, $q \geq \lfloor v/2 \rfloor$, $\alpha(J(v, k, i)) \geq \alpha(J(q, k, i)) \in \Omega(v^i)$. □

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