A Finite History of Infinity

An Exploration and Curriculum of the
Paradoxes and Puzzles of Infinity

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Abstract

Acknowledgements

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Part One:

An Exploration of the Concept of Infinity

The infinite! No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet no other concept stands in greater need of clarification than that of the infinite.

- David Hilbert (Maor, p. 7)

Infinity is where things happen that don't.

- anonymous schoolboy (Maor, p. 67)

Chapter 1: Introduction – Why Infinity?

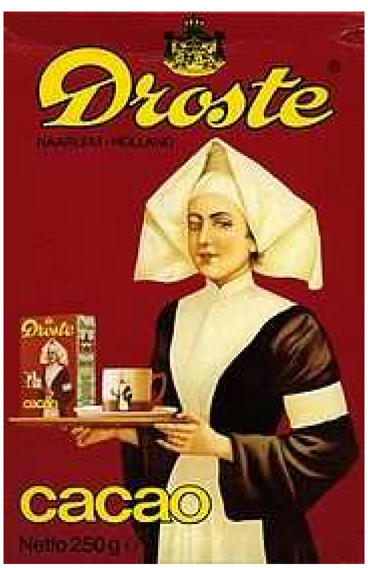
The idea of infinity has been a source of interest, fascination, and occasionally frustration for me as long as I can remember. Infinity seemed to pop up everywhere; it came up not just in mathematics, but also in science, art, and religion.

When I was about six, I wanted to know what the highest number was. My mother explained to me that there was no highest number, that whatever number someone came up with, you could always add one. Although I understood that she was right, it still bothered me. Then a relative gave me a book called something like "The Big Book of Answers" which had questions that children ask and their answers. One of the questions was "What is the largest number?" The book said it was a googol, written as a 1 with 100 zeros after it. Triumphantly, I showed this to my mother who explained that a googol was the largest number with a new name, but that a-googol-and-one was also a number. I knew, even before I showed her the book, that that was what she was going to say. I knew she was right, but it still annoyed me.

The fitting room in the clothing store where my mother would take me had mirrors on all four sides of the cubicles. You could look in the mirror in front of you and see your back. If I looked at the right angle I could see my back, the reflection of my front reflected in the mirror in back of me, and so on. I would wiggle my arms to see all the reflected arms do a perfectly synchronized dance. The idea of my reflection bouncing back and forth forever, giving me infinitely many limbs, intrigued me. A few times I tried counting to see how many arms or shoes I could identify, but my mother got impatient waiting for me.

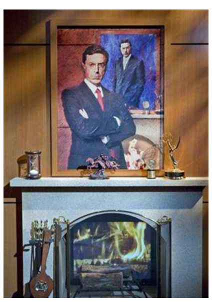
My grandparents had a set of babushkas, or Russian nesting dolls. I'd open the first doll and take out the second doll, inside the second doll was the third doll, and so on. There were mostly about seven dolls, and the last one did not open. Although I realized that these could not go on forever, I was always disappointed when I came to the last doll.

I don't remember what brand of cookies we bought years ago, but the box had a picture of itself on the package. The picture of itself included the picture of itself, which included the picture of itself. At some point in the center it just became some gray spots, but I liked to imagine the pictures going on forever, becoming infinitely small. This is called the "Droste Effect", named for a brand of Dutch cocoa whose box had a picture of itself on it. Other food packages did this as well. Pink Floyd's record "Ummagumma" had an album cover like this.

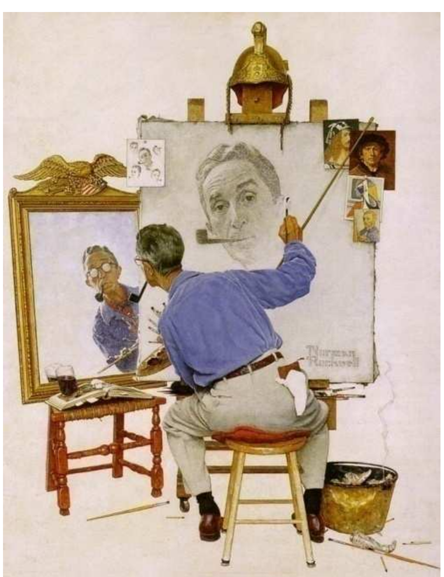


(Droste photograph from Wikipedia, Droste Effect, http://en.wikipedia.org/wiki/Droste_effect. Land O' Lakes photograph from Wikipedia, Land O' Lakes, http://en.wikipedia.org/wiki/Land_O%27Lakes. Pink Floyd Album photograph from Wikipedia, Ummagumma, http://en.wikipedia.org/wiki/Ummagumma)

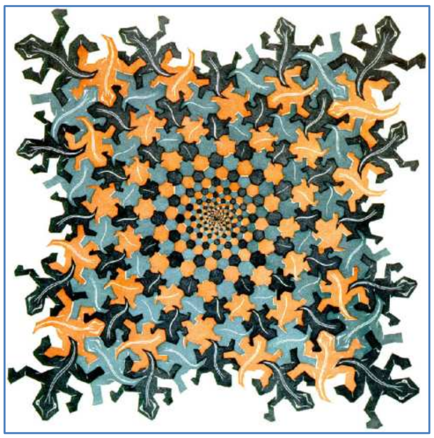
I read about a woman who liked to have her picture taken with celebrities. The first time she had her picture taken with one, she saved the picture. The second time, she held up the first picture while the second picture was being taken. She held up the second picture while the third one was taken. I imagined what the pictures would look like, as the image from the first picture got smaller exponentially.



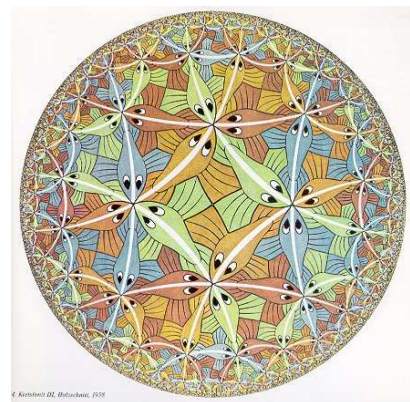
Stephen Colbert, host of The Daily Show, has a portrait of himself standing by his fireplace drawn every year. He then puts the painting on the fireplace mantle, and it gets drawn into the next year's portrait. ("Second Year Portrait" from MSNBC website, http://www.msnbc.msn.com/id/15976666/. Third Year Picture from Smithsonian National Picture Gallery, http://www.npg.si.edu/exhibit/colbert.htm.)



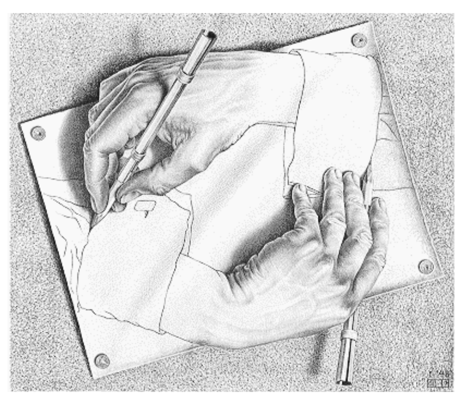
Norman Rockwell painted a picture he called "Triple Self-Portrait". (Photo from The Artchives, http://www.artchive.com/artchive/R/rockwell/rockwell_self.jpg.html) It is a picture of himself looking in the mirror as he draws himself. The picture shows his image three times: once is himself, one is his reflection in the mirror, and one is the picture he is drawing. I think that it would have been nice if someone had taken a photograph of Rockwell painting this picture, giving us four images in the picture. Years ago, I thought this could be continued for many iterations, but then I realized that someone else would have had to hold the camera to take the picture, breaking the chain of Rockwell images. (Even if he used a camera with a timer, he could not be at the camera and at the easel at the same time.)



I have always loved the artwork by M.C. Escher, an artist who used the concept of infinity in many of his drawings. His picture "Development" shows a mass of lizards, getting larger as they come out from the center. (Picture from website M.C. Escher, http://www.cord.edu/faculty/andersod/escher.html) I can imagine the lizards getting infinitely small as get into the center of the picture. Other pictures, like "Fish" have the largest animals in the middle, and it looks like they get infinitely small as we go towards the outside. (picture from website MC Escher: Infinity, http://pirate.shu.edu/~wachsmut/Workshops/Escher/infinity/infinity.html)



Escher's picture "Drawing Hands" gave me a different type of feeling of infinity. (drawing from Drawing Hands on Wikipedia, http://en.wikipedia.org/wiki/Drawing_Hands .) The left hand is drawing the right hand, as the right hand is drawing



the left hand. The left hand can not draw the right hand until the right hand finishes drawing the drawing the left hand, but the right hand cannot draw the left hand until it is drawn. And so on.

Is space infinite? My brother and I wanted to know. I had trouble with the idea of an infinite universe. We learned that there are stars everywhere in the universe. So if the universe is infinite, there must be infinitely many stars. But I felt like the number of stars had to be finite, no matter how large it is. Moreover, if the universe is finite, what is the edge? Could I travel to the edge of the universe and not be able to go any farther? My mother told us that Einstein said that space curves around on itself, but she didn't really understand how. This also frustrated me, much more so than the fact that there is no largest number.

Years later, I learned about the Big Bang theory, that the universe is expanding and everything is getting farther away from each other like raisins in raisin bread as the bread rises. However, parts of the universe could only move at a finite speed. If the Big Bang occurred fourteen billion years ago, objects could only be fourteen billion light-years from the center of the universe. That still left me with the question of what I'd find if I went to the edge of the universe. (This ignores the fact that I could not go faster than the speed of light so I would never catch up with the edge.)

I have felt somewhat less frustrated ever since I read "The Boy Who Reversed Himself" by William Sleator. Omar, a boy in the book, guards the two dimensional universe that exists on the surface of a three dimensional sphere. The two dimensional universe is finite, limited to the surface area of the sphere, but their space curves around on itself, and if someone began travelling, they could go in a straight line around the sphere and end up where they began. Although I cannot picture it, I think we live on the surface of a four dimensional hypersphere. If we travelled long enough and fast enough in a straight line, we would circle around in the fourth physical dimension. The expansion of the universe means the hypersphere is getting larger.

Is time infinite? If the universe began with the Big Bang, did time exist before that? Will time exist after the Big Crunch, assuming there will be one? And if there is no Big Crunch, will time continue forever? I don't know the answers to these questions. I'm also not sure whether the idea of time passing involves science or philosophy. Many scientists say that someday all the energy in the universe will have broken down as far as it can, and the universe will no longer be able to sustain life. If there is no life, does time pass? If all the energy has broken down, will electrons continue to orbit the atom's nucleus, the only way to really keep time?

As I child, I did not encounter much work with infinity in math classes. The closest I came was when my teacher told us we could not divide by zero. This did not bother me. I found that division and fractions made sense if I looked at them as splitting a number of cakes among a number of people. 8÷4=2 means that if you split eight cakes evenly among four people, each person would get two cakes. 8÷0 is meaningless; it asks how many cakes each person would get if you split eight cakes among zero people. Since there are no people to get any cake, there is no answer. I did not feel that the answer should be zero or infinity, as there wasn't anyone getting zero cakes or infinite cakes.

We did look at repeating infinite decimals. I think I was in fourth or fifth grade when we learned how to do long division for decimals that would repeat and never end. I was in eighth grade when I learned how to take a repeating decimal and turn it into a fraction. I remember my teacher showing us that $0.\overline{9}$ is equal to 1.

We covered compound interest briefly in high school. We found that the more often interest is compounded, the more interest you get. In class we looked at interest compounded yearly, quarterly, monthly, and daily. That evening I jokingly said to my mother that on my bank account, I wanted my interest compounded infinitely many times. I was surprised when she told me that this was possible to calculate, and that I would learn how when I learned calculus.

When we studied the circumference of a circle, we were just told the value of pi. I think it would have been interesting to try to estimate it first by inscribing and circumscribing polygons with a circle, and increasing the number of sides of the polygons to get closer to the circle.

The first class I took that really used infinity at all was high school calculus. We learned limits in the first semester. Our teacher showed us that, if we divide a constant by a variable, as the variable approaches zero, the division approaches infinity (or negative infinity). If the variable approaches infinity, the fraction approaches zero. We also learned to look at what other functions would do as the variables approaches infinity.

In later calculus classes, we learned how to calculate the area under a curve using infinitely many inscribed or circumscribed rectangles. We saw shapes that were infinitely long, yet had finite area. We added up infinitely many positive numbers and got a finite sum. What at first seems like a contradiction is possible. I was sucked into a lifetime of loving math.

It was not until after I received my BS in mathematics and was taking some math education classes at a different university that I learned that there are different sizes of infinity. This is very counter-intuitive at first, but the professor made it make sense. At least I left the class understanding the difference between the infinite which is countable and that which is not.

Years later I took a writing class called “writing about ideas”. I thought it would be a class about persuasive writing. The class involved writing about scientific ideas and making them understandable to non-science people. For my term paper, I decided I should write about different sizes of infinity. I did some reading on my own and listened to some explanations from some of the math tutors on the third floor of Neuberger Hall (at a third university) to understand why we need different levels of uncountable infinity.

For two years, while I taught math at Benson High School in Portland, I was the faculty advisor for the math club. I then spent a year teaching middle school and high school math at Deering School, in Deering, Alaska, and was in charge of a once-a-day math elective. For both the math club and the math elective, I was free to work on whatever I wanted with the students. I included some work that involved infinity, including looking at Zeno’s paradoxes, the Hilbert Hotel, and different sizes of infinity.

This gave me a chance to work with students at different levels with varying degrees of interest in math. The math club at Benson included a wide variety of students:

some students just showed up because their teacher gave them extra credit points for it, some were mildly interested, and one boy loved math and wanted to learn all I could teach him at as high a level as I could. At Deering School, the math elective was set up so students could get another credit in math if they needed it, was a combination of high school and middle school students, and was not based on ability or interest. In both groups, I found that topics involving infinity were among those that interested the students the most. I'm sure it is because infinity is somewhat incomprehensible that it is so interesting.

At Deering School and at Bartlett High School in Anchorage, AK, the classes I taught included geometry. I included some topics involving infinity in the curriculum, including finding the shaded area in a shape with an infinite shading pattern, finding the area of a shape with a given circumference as the number of sides increases, and the angle measure of a regular polygon as the number of sides increases.

Currently, I teach college algebra and statistics at the University of Alaska, Anchorage, as well as college algebra and trigonometry at the Elmendorf branch of Embry Riddle Aeronautical University. The curricula of these classes all involve infinity: what happens to rational functions as x approaches infinity, calculating the circumference of a circle using inscribed and circumscribed polygons with increasing number of sides, interest compounded continuously, probability distributions with infinitely many different outcomes, etc.

When I was in school, we were taught that all matter was made up of molecules, molecules were made up of atoms, and atoms were made up of protons, neutrons, and electrons. The end. Protons, neutrons, and electrons could not be divided further. Now scientists are splitting atoms and we have quarks, leptons, and gluons. They remind me of the Russian nesting dolls when I think the smallest one cannot open, and suddenly it does. Can matter be split up infinitely? Or is there some point where it really cannot be subdivided anymore? I'm not sure which seems more incomprehensible to me.

The most incomprehensible subject involving infinity is G-d. But I will just have to accept that G-d is not understandable to the human mind. I'm sure I'm not the only one.

I am doing my thesis on infinity because I find it the most interesting area of mathematics. It is involved in almost every area of math, as well as other subjects. Infinity is somewhat, well, infinite.

Chapter 2: Exploration – A History of Infinity

REVISIONS PENDING…

People have wondered about the infinite ever since they were able to think about the world they lived in. Did the world come into being suddenly or had it always existed? Would it continue to exist infinitely forever? Was space infinite? If one traveled through space in a straight line, could they continue forever? Suppose one cut a piece of wood in half and then cut one of the halves in half and continued doing this. Could they continue forever?

The ancient Greeks were among the first people to consider the infinite. They considered the idea of dividing an object in half forever. The Atomists believed that matter was made up of tiny indivisibles. Others disagreed, and felt that things could be broken down smaller continuously. The Eleatic School of Philosophers discussed many mathematical issues, including the infinite.

Pythagoras felt that there was a finite amount of natural numbers. Aristotle argued against anything actually being infinite, but believed in a potential infinity. While he did not believe in the infinite, he believed that for any finite group, there is a larger finite group. Only a finite number of natural numbers has ever been written down or conceived. If L is the largest number conceived, we can move on to L+1 or L^2, but there are still only a finite amount have been used. Aristotle wrote

> Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the

> untransversable. In point of fact they do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish.[1]

Aristotle believed in a finite universe, consisting of nine celestial spheres centered around the earth. For centuries, many accepted his model.

The most well known of the ancient Greek writings on infinity are by the philosopher Zeno of Elea (495-435 B.C.E.) Zeno was the leading spokesman for the Eleatic School of Philosophers. He felt that science could not grapple with reality unless it took into account the ways infinity seems to appear everywhere in nature. He was known for paradoxes like how can something move through infinitely many points in a finite period of time.

One of his paradoxes discusses a race between Achilles, the mythical runner, and a tortoise, where the tortoise is given a head start. Zeno said that it would be impossible for Achilles to ever catch up to the tortoise. He reasoned that by the time Achilles began running, the tortoise was already a distance ahead. By the time Achilles reached the point where the tortoise had been when Achilles began running, the tortoise had already moved further on. When Achilles gets to this new point, the tortoise has again moved on. This will continue forever, preventing Achilles from ever catching up. This paradox, and Zeno's others like it, are among the first mention of the idea of something continuing forever.

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1 O'Connor, J.J and Robertson, E.F.; "Infinity"; http://www-groups.dcs.stand.ac.uk/~history/HistTopics/Infinity.html#s75

The Eleatic School of Philosophers was one of the first known to estimate the area of a circle by cutting it into triangles and measuring the area of each triangle. As the number of triangles increased and the size of each deceased, the estimation became closer to the actual area. To get the actual area, a person would have to take infinitely many infinitely small triangles. They asked how an infinite number of nothings add up to something like a circle?

Euclid, like Aristotle, also did not consider actual infinity. He is credited for proving, around 300 B.C.E. that there are infinitely many prime numbers. His actual statement, however, was, "Prime numbers are more than any assigned magnitude of prime numbers."[2] This is in line with Aristotle's belief in potential infinity.

In the first century B.C.E., Lucretius considered the idea of an infinite universe. In his poem *De Rerum Natura* he argues for in favor of an infinite universe. If the universe was finite, he argued, there would have to be a boundary. If someone approached the boundary and threw something at it there could be nothing to stop the object. If there was anything to stop the object it would have to lie outside the universe and nothing can be outside the universe. For many centuries this argument was accepted as proof that the universe had to be infinite. Today, many scientists believe in an infinite universe without a boundary.

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[2] O'Connor, J.J and Robertson, E.F.; "Infinity"; http://www-groups.dcs.stand.ac.uk/~history/HistTopics/Infinity.html#s75

The Babylonians were the first to introduce a number system with place value. This enabled people to write larger and larger numbers without being limited by the methods of writing numbers.

Judaism and, later, Christianity believed in an infinite G-d. G-d is omnipotent, omniscient, and omnipresent. G-d always was and always will be.

Saint Augustine, in the fifth century B.C.E., wrote City of G-d where he said

> Such as say that things infinite are past G-d's knowledge may just as well leap headlong into this pit of impiety, and say that G-d knows not all numbers. ... What madman would say so? ... What are we mean wretches that dare presume to limit his knowledge.[3]

Maimonides compiled thirteen principles of Jewish faith. The fourth principle was

> I Believe With Perfect Faith that the Creator is without beginning and without end; He precedes all existence.[4]

The Kabbalah, writings of Jewish mysticism written about 1280, was the first source to suggest different types of infinity, both as an endless collection of discrete

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[3] O'Connor, J.J and Robertson, E.F.; "Infinity"; http://www-groups.dcs.stand.ac.uk/~history/HistTopics/Infinity.html#s75

[4] Jewish America; "The Thirteen Principles of the Jewish Faith, Compiled By Maimonides"; http://www.jewishamerica.com/ja/timeline/thirteen.cfm

items as well as a continuum. This idea was not considered again until Georg Cantor, approximately 700 years later.

In the Kabbalah, G-d is considered infinite. The term "*Ain-Sof*" meaning "Without End" refers to G-d. This phrase is also used to refer to the "Infinite Light" and for the infinite in general.

Explaining that G-d it infinite in volume, the Kabbalah says

> G-d is boundless. This means that there is nothing physical that can hinder His presence. He fills every element of space in all universes that He created, on all levels and there is no place devoid of him.[5]

In a passage referring to the names of G-d, it says

> It is from the name Ehyeh [Will Be] that all kinds of sustenance emanate coming from the Source, which is the Infinite.[6]

Elsewhere it explains that because G-d is infinite, He can not be understood.

> It is necessary to realize that the Infinite Being, the Blessed King of Kings cannot be encompassed by any name or word. It is not correct to speak of any attributes in this Essence since it does not change, and cannot be described.[7]

While this phrase means that G-d has no end, it does not refer to any beginning or lack of one. The Kabbalah explains,

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5 Rabbi Aryeh Kaplan; Meditation and Kabbalah; Jason Aronson Inc.; Northvale, NJ; 1982; p.302
6 Rabbi Aryeh Kaplan; Meditation and Kabbalah; Jason Aronson Inc.; Northvale, NJ; 1982; p.131
7 Rabbi Aryeh Kaplan; Meditation and Kabbalah; Jason Aronson Inc.; Northvale, NJ; 1982; p.184

> The Endless One and not the Beginningless One. If He was called the Beginningless One, it would be impossible to even begin to speak about Him. But to some extent, it is possible to comprehend Him through His creation. This is a beginning but it has no end.[8]

Thomas Aquinas, a Christian theologian and philosopher felt that that actual infinity was not possible. He wrote, in the 13th century,

> The existence of an actual infinite multitude is impossible. For any set of things one considers must be a specific set. And sets of things are specified by the number of things in them. Now no number is infinite, for number results from counting through a set of units. So no set of things can actually be inherently unlimited, nor can it happen to be unlimited.[9]

Beginning with Brahmagupta in the 7th century, Indian mathematicians added zero to their number system over a period of 500 years. They had trouble working with zero as they worked to make zero follow their rules of arithmetic. Their biggest problem was dividing by zero. Bhaskara II wrote

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[8] Rabbi Aryeh Kaplan; Meditation and Kabbalah; Jason Aronson Inc.; Northvale, NJ; 1982; p. 303

[9] O'Connor, J.J and Robertson, E.F.; "Infinity"; http://www-groups.dcs.stand.ac.uk/~history/HistTopics/Infinity.html#s75

> A quantity divided by zero becomes a fraction the denominator of which is zero. This fraction is termed an infinite quantity. In this quantity consisting of that which has zero for its divisor, there is no alteration, though many may be inserted or extracted; as no change takes place in the infinite and immutable God when worlds are created or destroyed, though numerous orders of beings are absorbed or put forth.[10]

This was the first attempt to bring infinity as well as zero into the number system.

Bhaskara II also said that zero times infinity must be equal to every number so all numbers were equal.

Nicholas of Cusa, in the middle of the 15th century, believed that the universe was infinite. He also said that the stars were distant suns. At the time, the Catholic Church was trying to eliminate all heresy that did not view the earth as the center of the universe. Nicholas was brought before the Inquisition. He was tortured for nine years in an attempt to make him say that the universe was finite. He refused, and was burned at the stake in 1600.

Human understanding of infinity was advanced greatly by the writings of Galileo Galilei (1564-1642). He might have had a lot more to say, but, aware of Nicholas's fate, was careful about what he said.

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[10] O'Connor, J.J and Robertson, E.F. "Infinity", http://www-groups.dcs.stand.ac.uk/~history/HistTopics/Infinity.html#s75

In his essay "On Two New Sciences" (1638) he discussed mathematical ideas as a dialogue between Salviati, the intelligent person, and Simplicius, the simpleton. Salviati explains many aspects of infinity to Simplicius.

> Infinity and indivisibles transcend our finite understanding, the former on account of their magnitude, the latter because of their smallness. Imagine what they are when combined.[11]

Galileo differentiated between "potential infinity" and "actual infinity".

He set up a one-to-one correspondence between whole numbers and their squares. Since he could, he reasoned, there must be as many perfect squares as there are whole numbers. In the essay he tried to reason the paradox of a set being equal to a proper subset of itself. To make it more confusing, he wrote, if you take the set of counting numbers, subtract the equally sized set of perfect squares, you are still left with an infinite set of numbers that are not perfect squares.

> ... the totality of all numbers is infinite, and that the number of squares is infinite.; neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and, finally, the attributes "equal", "greater", and "less" are not applicable to the infinite, but only to finite quantities.[12]

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[11] Eli Maor; To Infinity and Beyond: A Cultural History of the Infinite, Boston, MA; Birkhauser Boston Inc.; 1987; p. 179

[12] O'Connor, J.J and Robertson, E.F.; "Infinity"; http://www-groups.dcs.stand.ac.uk/~history/HistTopics/Infinity.html#s75

John Wallis[13] (1616-1703) introduced the symbol ∞ for infinity in 1655. He chose this symbol because it is a curve that can be traced out infinitely many times. It first appeared in his paper *Tract on Conic Sections*. He used it again several months later in his more influential work *Arithmetica infinitorum*.

Calculus was developed independently by two different men. In 1665, Isaac Newton[14] (1643-1727) began to develop calculus. He never had any of his developments published at that time. Gottfried Wilhelm von Leibniz[15] (1646-1716) published a very similar version of calculus in 1684, prompting Newton to publish his in 1704.

Calculus is divided into two parts, differentiation and integration, and they are the reverses of each other. Both involve dividing a finite amount into infinitely many infinitely small parts.

Differentiation is calculating the rate at which one variable in a situation is changing in relation to another variable, either at a given point in space or at a given point in time. In differentiation, we divide a small change in one variable by a small change in another, then let both of these changes shrink until they approach zero, then find the value of the ratio between them approaches as they become infinitely small. This value is

13 picture from http://www-groups.dcs.st-and.ac.uk/history/Mathematicians/Wallis.html
14 picture from http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Newton.html
15 picture from http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Leibniz.html

called a “limit”, and is the answer sought by differentiation- the rate of change at a given point in space or in time.

To work out the rules, Newton imagined two points on a graph sliding towards each other. The average slope between them is the number of vertical units separating the two points divided by the number of horizontal units. As the points approach each other, both of these smaller and become infinitely small as the points merge. The fraction itself, however, does not suddenly become meaningless as the numerator and denominator both approach zero. As the points approach each other, the fraction may be 1/3,000 divided by 1/2,000 and then 1/3,000,000,000 divided by 1/2,000,000,000 but the ratio between them is still 1.5. This limit is the instantaneous slope at the point where the points merge – the rate of change of the vertical, (y), with respect to the horizontal, (x).

Integration works in the other direction. It takes an equation dealing with rate of change and converts it back into an equation in terms of the variables that change.

Galileo’s paradox, where infinity is equal to a subset of itself, is the basis for the story of Hotel Infinity, told by David Hilbert (1862-1943) There is a hotel with infinitely many rooms. On one particular night all the rooms are occupied. A traveler comes looking for a room. The clerk would like to help him, but explains that all the rooms are filled. After thinking for a moment, the clerk has an idea. He tells the person in room 1 to move to room 2, the person in room 2 to move to room 3, and so on. Every person moves to the next room, leaving room 1 empty for this new guest. When a group needing five rooms comes, he has everyone move to the room five higher. He has a new problem when infinitely many people come. After thinking, he has everyone move to the room whose number is twice that of the room they are in. This frees up infinitely many rooms for new guests. (Had he wanted to free up more rooms faster, he could have had everyone move to the room whose number is the square of the one he is in. Infinitely many people would still have kept coming however, so it doesn't really matter.) Hotel Infinity is one example of an infinite set being equal to a subset of itself.

Georg Ferdinand Ludwig Philipp Cantor (1945–1918) was the first to do ant major work with different sizes of infinity. To date, most of our ideas in this area are from him.

The biggest question about infinity still remains. Is the universe infinite, either in time or space?

Scientists have been trying to find out the size of the universe. Einstein said that space curves around on itself. Some scientists view the universe as the three dimensional surface of a four dimensional hypersphere. To help people understand, we can take it down a dimension. Imagine a two dimensional universe on the outside surface of a three dimensional sphere. The space of this universe is finite, yet a two dimensional inhabitant of this universe could travel forever in a straight line. Because their space curves around on itself, if someone traveled long enough, they would end up back where they began. If we move it a dimension (not counting time as a dimension), we are three dimensional in a universe that curves around in a fourth dimension. If space did curve around on itself, the light from other galaxies would have two different routes to earth and the galaxy would appear in two different parts of the sky.

NASA launched a Wilkinson Microwave Anisotropy Probe (WMAP) in 2001 and is measuring temperature ripples in the "cosmic microwave background", the afterglow radiation from the big bang. They feel that if the universe was infinite, the ripples should come in all sizes. The small ripples follow the prediction, but the large scale ripples that scientists expect can not be found. This could just be chance; when scientists run computer models of what ripples they would expect in an infinite universe, once every few hundred models they have no large scale ripples. However, it could be because the universe if finite and not large enough to accommodate the larger ripples. Mathematician Jeffrey Weeks explains, “Just as the vibrations of a bell cannot be larger than the bell itself, any fluctuations in space cannot be larger than space itself.”[16]

Weeks and his colleagues are working together to try to figure out the shape of the universe. Weeks explains

> Whether space is finite is something people have been asking since ancient times, and probably before that. If we resolved this and confirmed that space is finite, this would be an enormous step forward in our understanding of nature.[17]

David Spergel of Princeton University agrees.

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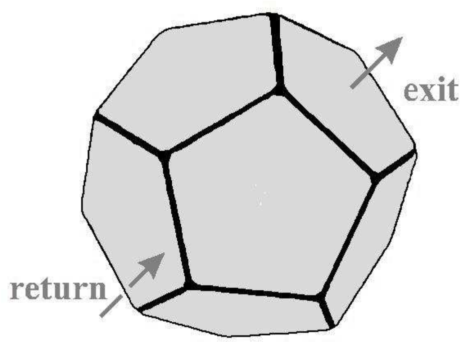
16 Hazel Miur; “Tantalizing Evidence Hints Universe is Finite”; NewScientist.com; October 8, 2003; http://www.newscientist.com/article.ns?id=dn4250

17 Hazel Miur; “Tantalizing Evidence Hints Universe is Finite”; NewScientist.com; October 8, 2003; http://www.newscientist.com/article.ns?id=dn4250

> If we could prove that the Universe was finite and small, that would be earth-shattering. It would really change our view of the Universe.[18]

This team of astrophysicists believe that the WMAP results indicate that the universe is small, about 70 billion light years across. There is no edge to space; the results show that space curves back on itself in a bizarre way. They suggest that the universe might be a dodecahedron, a round shape with a surface of 12 identical pentagons. If someone left the dodecahedron through one pentagon, they would reenter through the opposite side and go through the same galaxies over again.



Weeks feels that the data strongly matches the results he would expect from a universe with this shape. "I was just blown away, the results are far better than I could have imagined."[19]

Other groups of scientists disagree. They argue that if space was shaped like this, they would expect to see certain repeating patterns in the temperature ripples. Spergel

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[18] Hazel Miur; "Tantalizing Evidence Hints Universe is Finite"; NewScientist.com; October 8, 2003; http://www.newscientist.com/article.ns?id=dn4250

[19] Hazel Miur; "Tantalizing Evidence Hints Universe is Finite"; NewScientist.com; October 8, 2003; http://www.newscientist.com/article.ns?id=dn4250

and his colleagues argue that we do not see the patterns we would expect to see. "Weeks's team has a very powerful model that's nice because it makes a very specific prediction about the pattern we should see on the sky," says Spergel. "However, we've looked for it, and we don't see it."[20] Spergel's team and Week's team are working together looking through the WMAP data to look for any possible patterns that might exist. "We're burning up a lot of supercomputer cycles on this,"[21] says Spergel's colleague Neil Cornish of Montana State University.

They are also testing other possible shapes for the universe. Cormish says that his team believes it has already rules out almost half of the possible shapes for the universe, including football and doughnut shapes. He suspects that this work will probably yield nothing, meaning that the universe is either infinite or very large. "We're disappointed because we favored the small-Universe idea, but I guess you've just got to take the Universe you're given."[22]

Is the universe infinite in time? If not, when did it begin? How did it begin? When will it end?

Religious Jews, Christians, and Moslems feel that they can answer the first part. Genesis 1:1 states, "In the beginning G-d created heaven and earth."[23]

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[20] Hazel Miur; "Tantalizing Evidence Hints Universe is Finite"; NewScientist.com; October 8, 2003; http://www.newscientist.com/article.ns?id=dn4250

[21] Hazel Miur; "Tantalizing Evidence Hints Universe is Finite"; NewScientist.com; October 8, 2003; http://www.newscientist.com/article.ns?id=dn4250

[22] Hazel Miur; "Tantalizing Evidence Hints Universe is Finite"; NewScientist.com; October 8, 2003; http://www.newscientist.com/article.ns?id=dn4250

[23] Rabbi Aryeh Kaplan; The Living Torah: The Five Books of Moses and Haftarot; Maznaim Publishing Corporation; New York, NY; 1981; p. 3

Not all scientists agree with the Big Bang theory. Hannes Alfven, winner of the 1970 Nobel Prize, is a proponent of a universe that if infinite in both time and space. It is not a steady-state model like some other infinite models. The universe used to be a uniform hydrogen plasma that had always existed. At some point it began to develop vortices and gravitational instabilities, a process that took trillions of years. Over more trillions of years the universe formed galaxies, stars and planets. The radiation that most scientists attribute to the Big Bang, Alfven attributes to distant galaxies. While most feel that the universe is running down, he feels that the universe is constantly being wound up. He ends his book

> Do we live in an finite universe doomed to decay, where humans are insignificant transitory specks on a tiny planet? Or are we instead the furthest advance of an infinite progress in a universe that has neither beginning nor end? Will our actions today have no meaning in the end of all things, and are we now being swept into that inevitable decay? Or does what we do here and now permanently change the cosmos, a change that will echo through a limitless future?[24]

Infinity has caught many imaginations. People have imagined infinite time, infinite space, infinite life, infinite series.

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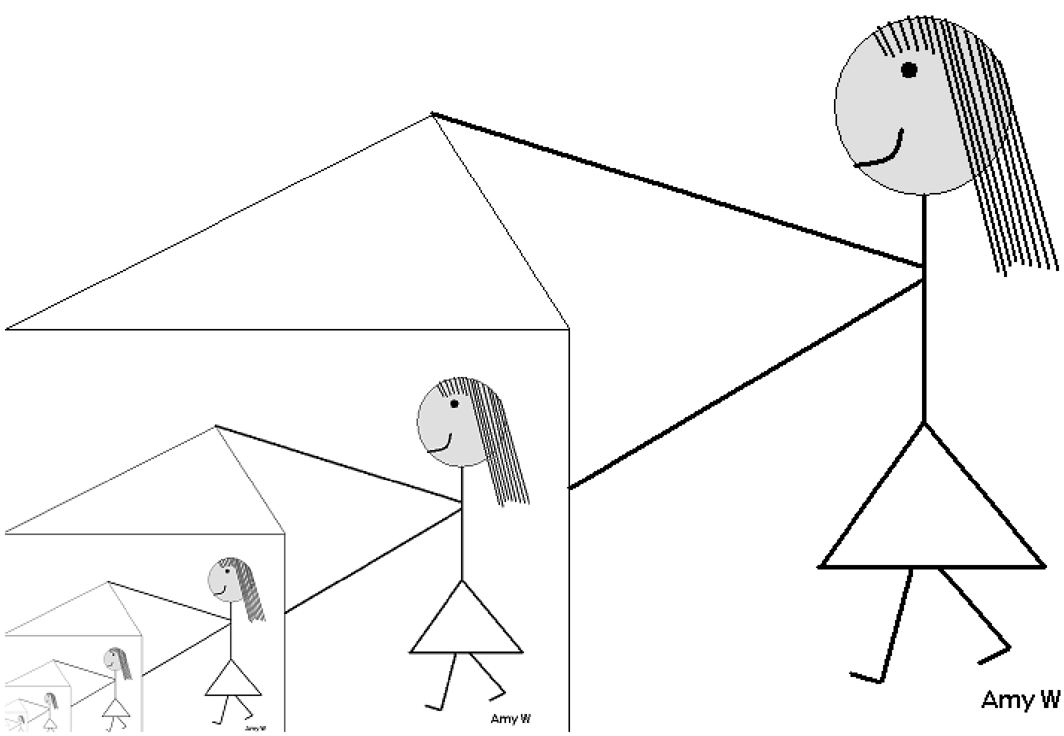
[24] Martin Gardner; Weird Water and Fuzzy Logic: More Notes of a Fringe Watcher; Prometheus Books; Amhurst, NY; 1996; p. 27

If the universe is infinite, and filled with stars throughout, there must be infinitely many livable planets. Even if only a small percentage of them do have intelligent life, that would still make infinitely many planets with life. If the planets with life are infinite, must every possibility exist somewhere? Does everyone have alter egos out there somewhere? Is there a world just like ours except that they have found a cure for cancer?

A somewhat common drawing is a picture that includes a picture of itself, including a picture of itself, including a picture of itself,… an example is this dubious work of art of a picture of a woman holding a picture of herself holding a picture of herself, etc.



Artist Maurits Cornelis Escher (1898-1972) wrote, “The [artist] may want to penetrate all the way into the deepest infinity right on the plane of a simple piece of drawing paper by means of immovable and visually observable images.”[25]

[25] M. C. Escher; Escher on Escher: Exploring the Infinite; Harry N. Abrams, Inc.; 1986; translated from the Dutch by Karin Ford; p. 123

Chapter 3: Exploration – Zeno and Infinity

Under Revision

Chapter 4:   Exploration – Cantor and the Continuum

If I wanted to list all the whole numbers, assuming I will live forever, could I? Yes: 1, 2, 3, 4, 5, 6, and so on. Because they are infinite, I would never finish. But I would, at some point, reach any whole number that you would name, I would get to at some point. 10 would be the tenth number, 1,000,000 would be the one millionth number, etc. We say that these numbers are "countable". Georg Cantor named this size of infinity $\aleph_0$, pronounced aleph-null.

One question I have had students ask me: Are there the same number of even integers as there are integers? Before I answer, I have another question – If you could not count, would you be able to tell whether or not you have the same number of fingers on each hand? How? You can match up your fingers and see that each finger of the left hand can be matched with a finger on the right hand and vice-versa. This works for number too.

We can list all the whole numbers and all the even whole numbers, and match them up in order.

Whole numbers:	1	2	3	4	5	6	7	8	9	…
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
Even whole numbers:	2	4	6	8	10	12	14	16	18	…

When we pair up the numbers between the two sets, each number in the set of whole numbers gets mapped to the number twice as large. Each number in the set of even whole numbers gets mapped to the number half its size. Every number in the top group gets mapped to exactly one number in the bottom group. Every number in the bottom group gets mapped to exactly one number in the top group. So there must be the same amount of numbers in each group.

Any infinite set of numbers is "countable" if we can list it so every number in the set is on the list. The numbers do not have to be listed in size order. An example is the set of integers. We cannot list them in order because there is no lowest integer. We cannot start at 0 and go up, because then the negative integers would never be on the list. But we can list them: 0, 1, -1, 2, -2, 3, -3, 4, -4, 5… We can map them to the whole numbers by mapping the first number in the list of whole numbers to the first number in the list of integers. Map the second numbers on each list together, then the third, and so on.

Whole numbers:	1	2	3	4	5	6	7	8	9	…
	↕	↕	↕	↕	↕	↕	↕	↕	↕	
Integers:	0	1	-1	2	-2	3	-3	4	-4	…

What is $\aleph_0 + 1$? There are $\aleph_0$ whole numbers. If we add 0 to the set, we still have $\aleph_0$, since we can still match them up.

Whole numbers:	1	2	3	4	5	6	7	8	9	…
	↕	↕	↕	↕	↕	↕	↕	↕	↕	
Whole numbers and 0:	0	1	2	3	4	5	6	7	8	…

We could keep adding another element, and the cardinality of the set would still be $\aleph_0$. So $\aleph_0$ plus any finite number is still $\aleph_0$.

What if we double $\aleph_0$? We can take the set of whole numbers and double it by adding in the additive inverse for each whole number. But we saw earlier that it still gives us just $\aleph_0$. $\aleph_0$ times any finite number is still $\aleph_0$.

Are there any sets of numbers that cannot be listed this way, that are not countable? Yes. One example is the real numbers. Another set is the set of all real numbers between 0 and 1.

To show that we cannot list all the numbers between 0 and 1, we can do a proof by contradiction. Let's start by listing all the numbers between zero and 1 in their decimal form, in any order.

$\frac{1}{2}$	=	0	.	**5**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	…
$\frac{1}{3}$	=	0	.	3	**3**	3	3	3	3	3	3	3	3	3	3	3	3	3	…
$\frac{\pi}{4}$	=	0	.	7	8	**5**	3	9	8	1	6	3	3	9	7	4	4	8	…
$\frac{2}{7}$	=	0	.	2	8	5	**7**	1	4	2	8	5	7	1	4	2	8	5	…
$\sqrt{\frac{1}{2}}$	=	0	.	7	0	7	1	**0**	6	7	8	1	1	8	6	5	4	8	…
0.25	=	0	.	2	5	0	0	0	**0**	0	0	0	0	0	0	0	0	0	…
e^{-1}	=	0	.	3	6	7	8	7	9	**4**	4	1	1	7	1	4	4	2	…
log (4)	=	0	.	6	0	2	0	5	9	9	**9**	1	3	2	7	9	6	2	…
$\frac{4}{9}$	=	0	.	4	4	4	4	4	4	4	4	**4**	4	4	4	4	4	4	…
$\frac{1}{100}$	=	0	.	0	1	0	0	0	0	0	0	0	**0**	0	0	0	0	0	…
Listing all natural numbers	=	0	.	1	2	3	4	5	6	7	8	9	1	**0**	1	1	1	2	…
$\sqrt{2}-1$	=	0	.	4	1	4	2	1	3	5	6	2	3	7	**3**	0	9	5	…
i^i	=	0	.	2	0	7	8	7	9	5	7	6	3	5	0	**7**	6	1	…
$\frac{e}{\pi}$	=	0	.	8	6	5	2	5	5	9	7	9	4	3	2	2	**6**	5	…
$\frac{1}{1024}$	=	0	.	0	0	0	9	7	6	5	6	2	5	0	0	0	0	**0**	…

etc.

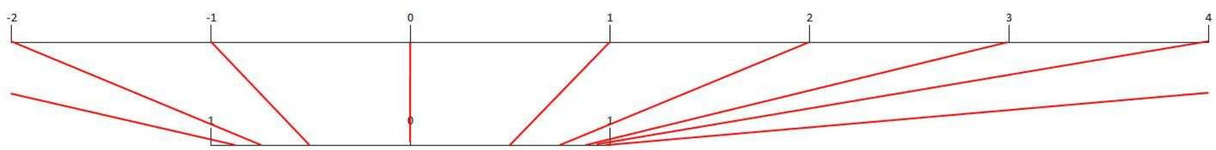
If we kept going, would every number be on the list? No. We can generate a number between 0 and 1 that is not on the list. Take the digits on the diagonal – the first place after the decimal point of the first number, the second place after the decimal point for the second number, the third place after the decimal point for the third number, etc. (They have been bolded in the above table.) To construct our new number, we use the numbers on the diagonal. For the first place after the decimal point, look at the first number on the diagonal. If it is not a 3, the first digit of our new number will be 3. If the number is a 3, we will use a 7 in the new number. For the second digit, go to the second number on the diagonal. If we lined up the numbers we have here on the diagonal, we get 5, 3, 5, 7, 0, 0, 4, 9, 4, 0, 0, 3, 7, 6, 0. So our new number will start with 0.373333333337333. Could our new number already be on the list? No. It cannot be the same as the first number, because the tenths digit is different. It cannot be the same as the second number because the hundredths digit is different. It cannot be the same as any of the infinitely many numbers on our list. We could add it to the top of our list, but then we can repeat this process for another number that is not on the list. So there is no way to list all the numbers between 0 and 1.

If we cannot list all the real numbers between 0 and 1, we certainly cannot list all the real numbers. But are there as many real numbers between zero and 1 as there are on the whole number line?

First, let's compare the interval between 0 and 1 to the interval of the same length between 5 and 6. Do they have the same amount of numbers? Yes. We can map any number n in the first interval to n+5 in the second. The location of the interval on the number line does not affect its cardinality.

Compare the interval between 0 and 1 to the interval between 0 and 2. We can map every number n in the first interval to 2n in the second interval. Every number in the first interval has a number in the second interval to be mapped to, and every number in the second interval has a number from the first interval mapped to it. So the two groups have the same cardinality. The finite length of the interval does not affect this.

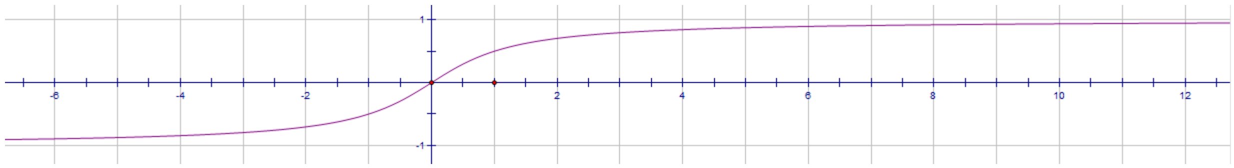
So the points in any finite length can be mapped to the points in any other finite length. But what about the points on a infinite length to the points on an finite length? To show it can be done, we can map the points on the number line to the numbers on the interval from -1 to 1.



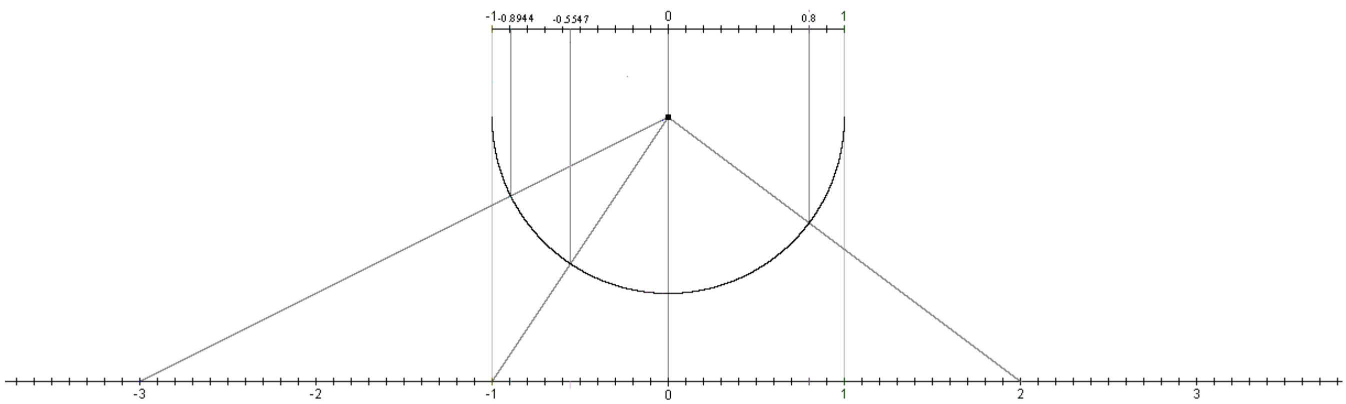
Starting with the number line, map the interval from 0 to 1 onto the interval from 0 to $\frac{1}{2}$.

Map the numbers from the interval 1 to 2 onto the interval from $\frac{1}{2}$ to $\frac{3}{4}$. The numbers from 2 to 3 are mapped to the interval from $\frac{3}{4}$ to $\frac{7}{8}$. Continue, matching each unit length on the number line to half of what is left on the end of the segment from -1 to 1. Reflect the mappings on the negative side.

Another way to map the entire number line onto the interval from -1 to 1 is the equation $y = \frac{2}{\pi} tan^{-1}(x)$, done in radians.



Another imaginative mapping involves a semi-circle that goes from -1 to 1, along with a point in the center.



On this mapping we can map any point on the infinite number line on the bottom to the finite line on the top. Take a point on the infinite line (for example, 2), draw a line from that point on the number line to the point in the center of the semicircle, and where this drawn line crosses the curve draw another line directly upwards to the finite line (here mapping 2 to 0.8)

This maps 0 to 0

1 to 0.55470020

2 to 0.8

3 to 0.89442719 etc.

Since the real numbers cannot be counted, they cannot be matched up with the integers. There must be more real numbers than there are integers. Cantor first labeled the cardinality of the set of real numbers c, for continuum. Then as they followed the cardinality for the set of integers $\aleph_0$, the cardinality of the set of real numbers is $\aleph_1$, aleph-one.

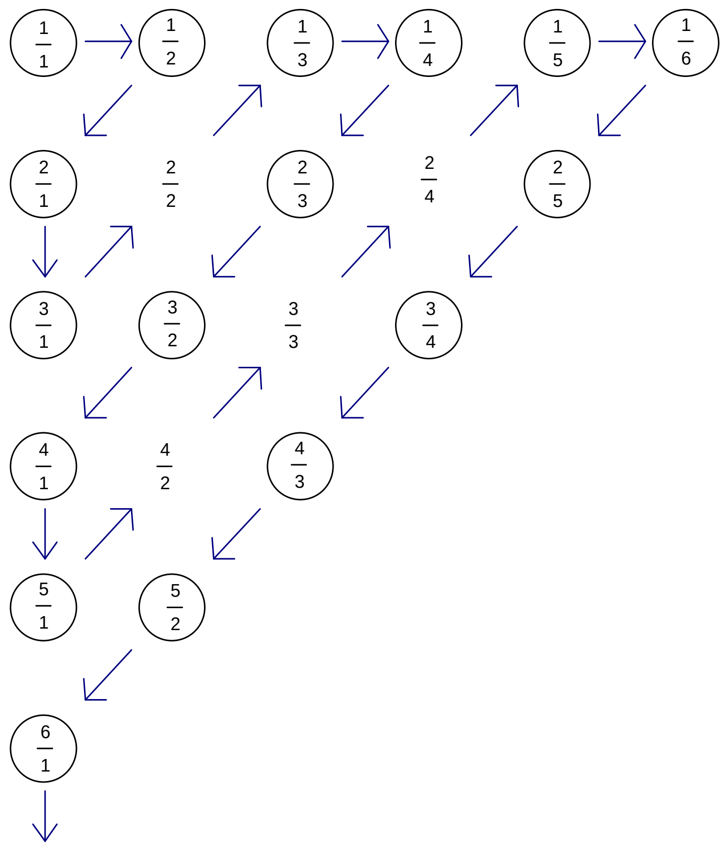
What is the relationship between $\aleph_0$ and $\aleph_1$? First, let's consider how many subsets there are of the integers. If I were buying a pizza from a shop that offered 10 toppings, how many different groups of toppings would I have to choose from? For each topping, I would have two choices – to include it or to not include it – so I would have a choice of 2^{10} or 1024 different groups, representing all the subsets of the toppings. The same goes for the number of subsets of the set of integers. For each of the $\aleph_0$ integers, we have two choices – to include it in the subset or to not include it. This means there are $2^{\aleph_0}$ subsets of real numbers. The set of subsets of a set is the "power set" of any set.

We already know there are $\aleph_1$ real numbers between 0 and 1. Take any one of them, in binary. There are 2 possibilities for the first digit after the decimal point, 0 or 1. There are 2 possibilities for the second digit after the decimal point, also 0 and 1. This is true for every place after the decimal point. Since there are $\aleph_0$ places after the decimal point, there are $2^{\aleph_0}$ real numbers between 0 and 1. Earlier we saw that this is equal to how many real numbers there are. So $\aleph_1 = 2^{\aleph_0}$.

We know that the set of integers is countable, and the set of real numbers is not countable. What about the rational numbers? It seems like they should not be countable. It is pretty easy to see that the integers should be countable. They come in order, one right after the other, one unit apart. Real numbers, however, are infinitely dense; between every two real numbers there are infinitely more real numbers. The same is true for rational numbers; between any two rational numbers there are infinitely many rational numbers. Cantor felt that rational numbers were not countable until he found a way to list them. We had a table of all the positive rational numbers, with each numerator going across a row and each denominator going down each column.

$$
\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\[1em]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \cdots \\[1em]
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \cdots \\[1em]
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \cdots \\[1em]
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

We cannot go across the rows because if we began with the first row we would never get to the second row. We cannot go down the columns because we would never get to the second column. What we can do is go back and forth along diagonals. Since each fraction is in the table many times, we just skip over any that we already had.



Putting the rational numbers in an order (NOT in order of size, which could not be done) we get 1 $\left(as\frac{1}{1}\right)$, $\frac{1}{2}$, 2, 3, (then skip $\frac{2}{2}$ because it is equal to a number we already had and go to) $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{3}$, $\frac{3}{2}$, 4, 5, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{2}{5}$, $\frac{3}{4}$, $\frac{4}{3}$, $\frac{5}{2}$, 6…

If we had two sets, each containing $\aleph_0$ members, and combined them, we would still have just $\aleph_0$ members. If each group is countable, the combination would still be countable by going back and forth between them. An example is combining the $\aleph_0$ odd whole numbers and the $\aleph_0$ even whole numbers to get the set of $\aleph_0$ whole numbers.

The set of real numbers is a combination of rational numbers and irrational numbers. Since the real numbers are not countable but the rational numbers are that means the set of irrational numbers must not be countable. There are more irrational numbers than there are rational numbers.

How many points are there in the three dimensional universe? Cantor realized that the number of points in the universe is equal to the number of points on the number line, $\aleph_1$. To prove this, we need a way to map all the points in the universe to another set of points that we know has a cardinality of $\aleph_1$.

Here is a way we can map all the points in the universe to a point on the number line between 0 and 1. Pick one point to be the center of the universe. Any other point has to have a certain distance from that point along an infinite north-south line, a distance east-west, and a distance up-down. Using any of the methods, map each of these distances to a point between 0 and 1. List each in decimal form. Make a new number between 0 and 1 by getting the places after the decimal point: take, in order, the first place after the decimal point from the north-south number, the first place after the decimal from the east-west number, the first place after the decimal from the up-down number, the second number after the decimal point from the north-south number, the second number after the decimal point from the east-west number, the second place after the decimal point from the up-down number, the third place after the decimal point from the north-south number, etc.

For example, I can consider the end of my left index finger to be the center of the universe. I can use this to map a point several light years away to a point on a one inch number line. I extend an imaginary set of x, y, and z axes from my fingertip, using light years as the unit. The point I'll work with is 2,715.52 on the x axis, -666,777,888 on the y axis, and $0.\overline{3}$ on the z axis. I have to map each of these to a number between 0 and 1. The mapping I'll use is

$$f(a)= \begin{cases} y = \frac{1}{\pi} tan^{-1}(x) & \text{If a is} \geq 0 \\ y = \frac{1}{\pi} tan^{-1}(-x) + 0.5 & \text{If a is} <0 \end{cases}$$

(If a is positive, this maps it to a number between 0 and ½, while if a is negative, this maps it to a number between ½ and 1.) The numbers 2715.52, -666777888, $0.\overline{3}$ get mapped to 0.49988278124…, 0.99999999952…, 0.10241638235… respectively. To find the single point, I take the tenths digit from the first number, the tenths digit from the second, the tenths digit from the third, the hundredths digit from the first, the hundredths digit from the second, etc. The number between 0 and 1 is 0.491990992894891296793898192253425…

Cantor was very surprised when he realized that the number of dimensions did not make any difference on the size of infinity. His comment was "I see it, but I don't believe it." (Aczel, p. 119)

Are there any higher sizes of infinity? Let's look at the subsets of the real numbers. There are $2^{\aleph_1}$ subsets. For any group there must be more subsets of the group than there are elements in the group. A group of 1 item has 2 subsets, one with the item and one empty subset. A group of 2 items has 4 subsets. A group of 5 items has 32 subsets. And so on. But must this still be true with infinite groups?

Suppose it is not. Let's say that there are e elements in a group, whether e is finite or infinite. There are 2^e subsets of the group. We know that there must be at least as many subsets as there are elements in the group, since each element is a subset of 1. (There has to be at least 1 more subset, since the empty group is also a subset. So there must be at least e+1 subsets. But if you add 1 to any size of infinity, it stays the same.) Suppose $2^e = e$. That means that each of the e elements can be matched up with a subset of the e elements. Element e_1 will be matched with $subset_1$, element e_2 will be matched with $subset_2$, and so on. Take any match, let's say element e_n and its match with $subset_n$. Is element e_n in $subset_n$? If yes, fine. If not, include element e_n in $subset_x$. $Subset_x$ cannot be empty since some element must have been mapped to the empty set, and that element would have to be $subset_x$. Is e_x in $subset_x$? It could not be, since $subset_x$ is only for elements not in the subset they are mapped to, so if e_x is in $subset_x$, it should not be in $subset_x$. But if e_x is not in $subset_x$, it should be in $subset_x$. This leads to an impossible contradiction. The only way out of the contradiction is that our original assumption is wrong. So there must be more subsets of a set than there are elements in the set, even if there are infinitely many elements in the set. So $2^n > n$, even if n is infinite.

How many different subsets are there of the set of real numbers? If there are $\aleph_1$ real numbers, there must be $2^{\aleph_1}$ subsets. So there must be a cardinality greater than $\aleph_1$, which is designated $\aleph_2$. How many subsets of subsets of the real numbers are there? $\aleph_3$. And so on.

Cantor believed that, once you have any size of infinity, $\aleph_n$, the next size is $2^{\aleph_n}$ which would be $\aleph_{n+1}$. There is nothing between the cardinality of an infinite set and the cardinality of its power set. He could not prove it, but was certain of it. He named it the "Continuum Hypothesis".

Cantor chose ת, the Hebrew letter taf, to refer to the set of all the alephs. He chose the last letter of the aleph-bet to indicate finality, to say there is nothing above the alephs.

Part Two:

Teaching Aspects of Infinity to Students

Overview of the Curriculum

Under Revision

Activity 1 – Making infinity understandable to young students
Activity 2 – Dividing by zero, part I
Activity 3 – Dividing by zero, part II
Activity 4 – Repeating decimals
Activity 5 – Making infinity with 4 fours
Activity 6 – Infinity and squares
Activity 7 – Angles of shapes with different numbers of sides
Activity 8 – Rational functions
Activity 9 – Compound interest
Activity 10 – Half Life
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Activity 17 – Discussions about infinity
Activity 18 – Infinity in art
Activity 19 – Writing about infinity

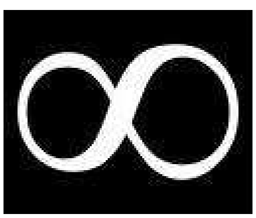
Activity 1:   Making infinity understandable to young students

Making infinity more understandable for younger students

Lesson Plan:

I think I would start with something they can see rather than start with concepts. First I'd have them go outside during the day and look up into space. I would like to then take them outside that night. First just have them lie down and look up. Then have them look through a telescope. Talk to them about distance. How far out does space go? If we kept going forever, would we ever reach the end?

Next I would try talking about numbers. What is the highest number? One billion? How about one billion and one? One billion and two? A googol? How about a googol and one? A googol and two? Do the numbers ever end?



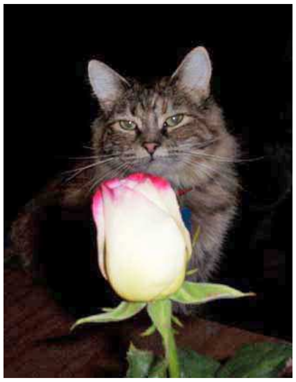
Show them the symbol for infinity. Can they guess why it was chosen? Can we just follow the curve around for ever? (Contrary to what some people suggest, it was not intended to look like a Moebius Strip.)

At a science museum: Show them the "infinite lights" setup, where two mirrors face each other with lights between them that are reflected back and forth.

(photo by Amy Whinston, taken in the Hall of Science, in Queens, NY.) Ask the students

how many times the light goes back and forth if the exhibit is kept up. Have them move the front panel so the angle changes and they can see the path of the lights. Something similar can be done many other places there are mirrors facing each other. (photo "Elevator" at Deviant Art, http://www.deviantart.com)

One method could be using infinite cat pictures. (photos from "Infinite Cat" http://www.infinitecat.com/) It starts with a picture of a cat. The next picture is another cat watching the first cat on a monitor. The third picture is of a third cat watching the second cat watch the first cat. And so on.



The first cat is Frankie B.

Frankie watches Frankie B on a monitor.

Poozy watches Frankie watch Frankie B.

Another Frankie watches Poozy watch the other Frankie who is watching Frankie B.

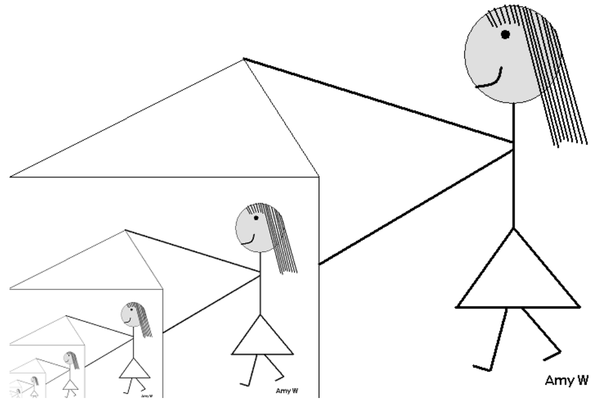
Abby watches Frankie watch Poozy…

Zoot watches Abby watch Frankie who is watching Poozy …

Fritz is watching Zoot watch Abby watch Frankie …

Copper is watching Fritz watch Zoot watch Abby…

Talk to the students - How many cats could there be? Could we keep going? If we do, how big would the original cat be in the picture?



A somewhat common drawing is a picture that includes a picture of itself, including a picture of itself, including a picture of itself, … an example is this dubious work of art of a picture of a woman holding a picture of herself holding a picture of herself, etc. (drawing "Framed Drawing" by Amy Whinston, 2008) Here is another picture, by a somewhat more professional artist, where the room includes a picture of the room. ("Infinity" by Imaginary Creature, http://www.deviantart.com)

<u>Experience with this lesson plan:</u>

I have not had a chance to teach this.

Activity 2: Dividing by zero, part I

Lesson:

Several of the students were having trouble understanding why you cannot divide by zero. To help, I gave a demonstration.

I took 12 disks of colored translucent plastic and put them on the screen of the overhead projector. “Okay, let’s start with 12 divided by 3.” On the board I wrote “12÷3=”. “How many times can I take away three disks before I run out? Count with me.” I took away three disks and counted “one” which the class echoed. I took away another three disks and counted “two”. I took away another group and counted “three”. I took away the last three and counted “four”. “So, what is twelve divided by three?” I asked. “Four,” they answered and I completed the equation on the board. “12÷3=4”.

I put the disks back on the projector and used the same procedure to show 12÷2=6.

I put the disks back again. “Let’s try 12 divided by 0.” On the board I wrote “12÷0”. “How many times can I take away zero disks before I run out? Count with me.” I reached down to the overheard made a large gesture of picking up zero disks. “One.” I repeated the gesture. “Two.” And again. “Three.” A few of the students giggled. “Four.” I paused. “How many times can I do this before running out of disks?” “Forever,” one student answered. “You can keep on going,” said another. “So can I divide by zero?” “No,” the class answered.

My experience with this lesson:

I used this at Mt. Tabor Middle School, Portland, Oregon, 6th grade class

Reflection:

I did not plan to do this, so I don't have any lesson plan for this. I was student teaching, and the teacher was having a difficult time helping the students to understand why you can not divide by zero. This seemed to work really well.

Activity 3: Dividing by zero, part II

Worksheet:

You want a $300 go-cart, but you are not sure you can afford to pay $300. You will decide how much you can afford to spend, and then you will share the purchase with other people who are each willing to pay that same amount.

Cost of go-cart	Amount you are willing to pay	Total number of people who will share the purchase
300	300	
300	150	
300	100	
300	1	
300	0.5	
300	0.01	
300	0.005	
300	0	

Lesson plan:

"You want to buy a go-cart for $300, but can't afford to pay the full price. You decide that you would be OK sharing the go-cart if the others will share the cost equally. Once you figure out how much you can afford to pay, how do you figure out how many people have to share the cost?" (Divide 300 by the amount you can pay.)

total cost		amount you can pay		number of people going in on it
300	÷	300	=	1
300	÷	150	=	2
300	÷	100	=	3
300	÷	1	=	300
300	÷	0.5	=	600
300	÷	0.01	=	30,000
300	÷	0.005	=	60,000
300	÷	0	=	

"If each person can pay $300, how many people need to go in together?" (Just the one buying it.) "And if you can just pay $150?" (2)

Continue this way for a few more dollar amounts, having the amounts get smaller. The students should be able to see that as the amount each person could pay gets smaller, the more people are needed to go in on it.

Ask: "And if I can afford zero?" The students should be able to see that there is no way to have any number of people pay $0 and have it total $300.

"As the amount you can pay goes down, what happens to the number of people who must share the purchase?" The students should realize that as the amount each person pays goes down, the number of people must increase. "What happens as the amount you can pay goes to zero?" There will probably be some students who realize that it goes towards infinity. "But is it infinity? If you had infinitely many people each paying zero, would you get to $300?" (No). So you can't say it is infinity, just that it goes towards infinity.

Experience with this lesson:

Embry Riddle Aeronautical University, class in Beginning Algebra. I teach this class about once a year.

Reflection:

This seemed to help them more than the method I described in the previous lesson.

Activity 4: Repeating decimals

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Activity 19 – Writing about infinity

Activity 5: Making infinity with 4 fours

Used at: The original problem came up in a math elective class at Deering School, Deering, Alaska. Most of the discussion was on the web.

The class was a math elective where I could do pretty much whatever I chose; there was no set curriculum. I had told the students that I would give one point extra credit to the first student who found a way to express each integer 0 through 256 using 4 4's and no other numbers. (For example, 0 was 44-44, 1 was $\frac{44}{44}$, 2 was $\frac{4}{4}+\frac{4}{4}$, 13 was $4!-\frac{44}{4}$, 16 was 4+4+4+4, etc.)

One student asked if she could try to get infinity. I said yes. We had already looked at dividing by 0 and how the quotient approached infinity as the denominator approached 0. she later came to me with $\infty = \frac{44}{4-4}$, saying that she knew it was not really infinity, but this was the best she could do. I said it was fine and gave her credit.

I posted an entry on my blog (I number each entry, and this was number 100.)

44/.44 . I DID IT !!!!!!!!!!!!!!!!!!!!!!!!!!!!

$$\pi = -\sqrt{4} * \sin^{\frac{-4}{4}} \|-.4\|$$

I have finally managed to express all the integers 0 – 100 using only 4 4s (and no other numbers). I've sort of been playing with this off and on since 1976, when Mr. C., my math teacher, showed it to the class. I also managed to get ∞ and π

Can anyone get e?

The following are comments this entry received. I copied them exactly. (I am "the northernmost jew", since Deering is 66◦4' north so that is the name of the blog; "PlanoTX" is someone I never met who lives in Plano, Texas; since "אח שלך, pronounced "ach shelach" is Hebrew for your brother, I assume it was my brother; and "ariel haschnauzer" is, I think, my ex-husband.)

Before I type the comments, I will mention that I know this is not really infinity, and that what I said in the comments was sort of silly. However, my brother gets very upset when people do not follow the rules, and it can be fun to annoy him.

PlanoTX **said...**

Infinity is 44/(4-4), but what is pi?

אח שלך said...

44/(4-4) is not infinity but undefined. It may seem logical to call it infinity, but that would lead to contradictions. I would think that to get infinity you'd need to use limits.

PlanoTX **said...**

That's true. The division of any number by zero is "undefined". However, I don't visualize a calculus expression using only four digits of '4' that would be evaluated as infinity.

Ariel HaSchnauzer said...

Wow I didn't realize the cabin fever was THIS bad

The Northernmost Jew **said...**

.

Yes, for infinity I used 44/(4-4) which I say is undefined because it is INFINITE! Using limits is not allowed because that would involve using other numbers or i or x or something.

.

For pi, I used -sqrt(4)*(sin^(-4/4))(int(-.4)) which is a lot easier to follow if you look at it in the original posting.

אח שלך said...

It's not entirely accurate to say that 44/(4-4) is undefined because it is infinite. Afterall, the limit as x approaches 4 of (44/|x-4|)) is infinity. No mathematician will say it is "undefined" simply because it is infinite. So simply being infinite (in the sense that 44/(4-4) is infinite) isn't sufficient grounds for calling an expression undefined.

If you take infinity as the value of a positive real divided by zero, then you end up with contradictions. So while infinity works as the limit of the value of a fraction (with positive constant numerator) as the denominator approaches zero from the right, it does not work as the value of a fraction with the same positive real numerator and a zero denominator.

ךלש חא said...

Oh, and by the way...

When I had Mr. C, and he gave my class the same kind of puzzle (it was to create the numbers using the digits in whatever year it was -- though I forget what year it was), he allowed me to use summations.

I bring that up because summations use a letter (typically i, though I suppose you can use any symbol as long as you don't create ambiguities) as the index of summation. So your reason for rejecting limits would apply to summations as well.

Now I acknowledge the other reason for rejecting limits -- that it makes it too easy to get to any integer (e.g., 5 would be the limit as x goes to 44/44 of x+x+x+x+x). And if you're asking the question, you can set whatever restrictions you want on what can be included in the solutions.

But I still object to the use of 44/(4-4) as infinity, because it is not mathematically correct.

PlanoTX **said...**

Ahhhh....the inverse sine function for pi! Very clever.

ךלש חא said...

One more thing about infinity...

The Northernmost Jew **said...**

44/(4-4) is infinity. If I wanted negative infinity, I would switch the two 4s in the denominator.

חא שלך said...

I realize you are joking about just switching the 4s in the denominator. But the humor doesn't get us by the fundamental fact that division by 0 doesn't give you infinity. One demonstration of this is that you don't even get infinity in the limit, because the limit of a ratio (as the denominator approaches 0) doesn't even exist, much less equal infinity. The right limit and the left limit are not equal.

The Northernmost Jew **said...**

Fine, put the 44/(4-4) in absolute value bars!

Followup: I asked in the post if anyone can get e. I posted a followup a few months later, with the solution someone sent me.

$$e = \sqrt[4*\ln 4]{4^4}$$

Activity 6: Infinity and squares

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Activity 19 – Writing about infinity

Activity 7: Angles of shapes with different numbers of sides

Note:

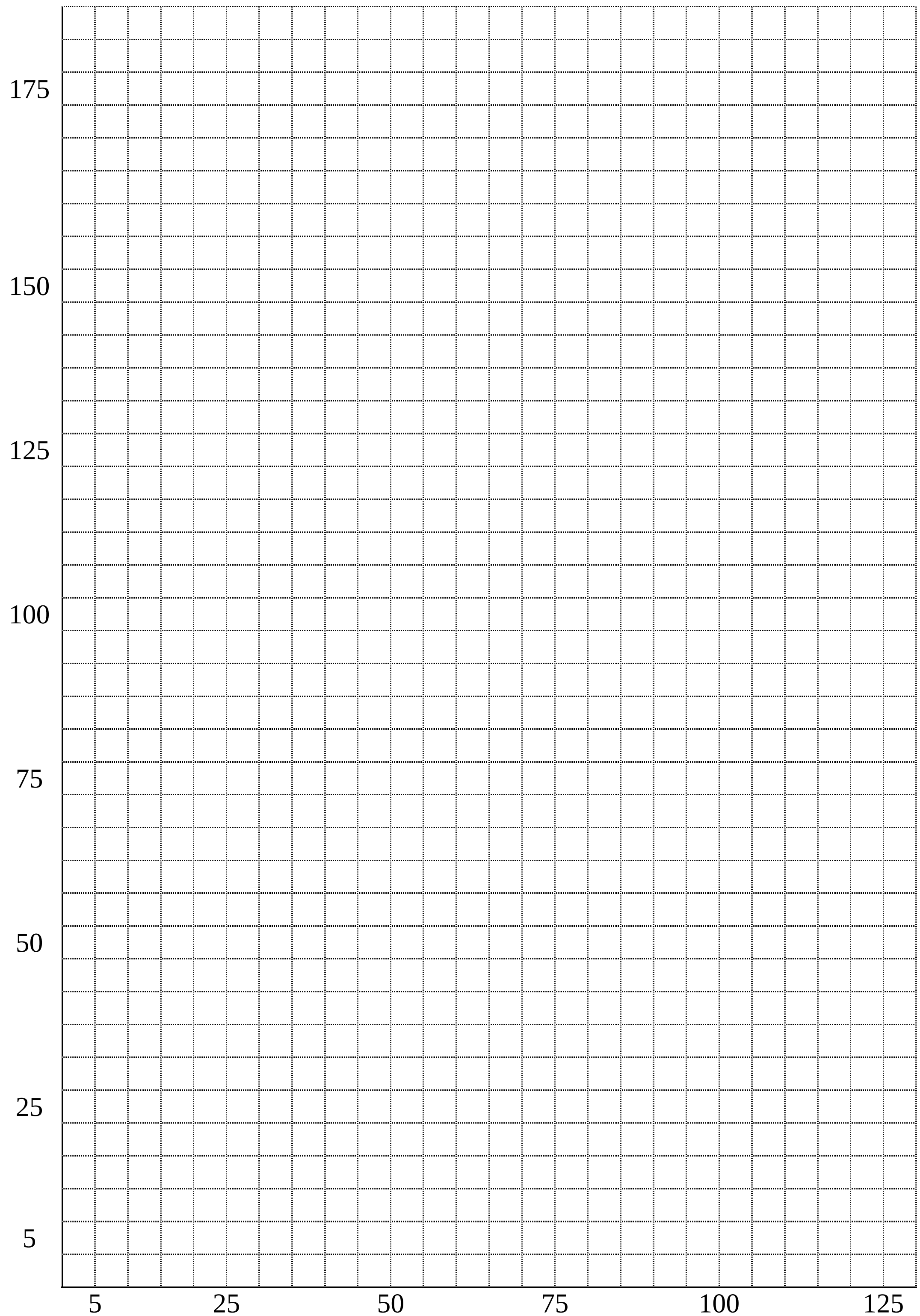
The students already knew that the sum of the angles of a triangle total 180˚.

Worksheet:

number of sides	number of triangles	formula to find total degrees	total degrees in polygon	degrees in each angle of a regular polygon
3				
4				
5				
6				
9				
12				
20				
30				
60				
90				
120				
300				

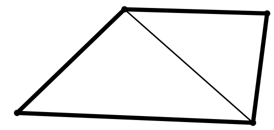
600				
1,200				
1,800				
3,000				
5,000				
8,000				
12,000				
20,000				
30,000				
60,000				
120,000				
600,000				
5,000,000				

Graph them below

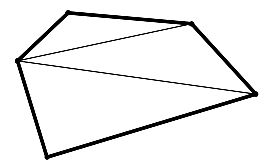


<u>Lesson Plan:</u>

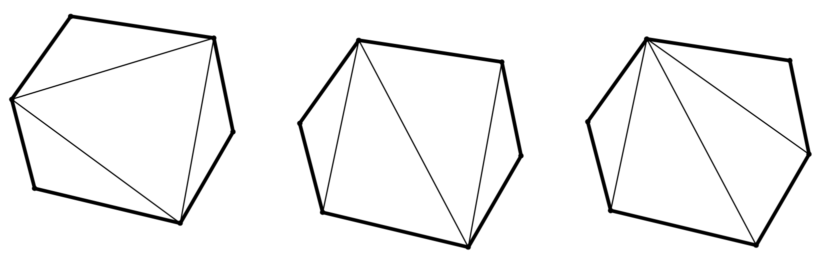
Ask the class "What is the total of the angles of a quadrilateral?" I will probably have few of the students say 360, because a rectangle has 4 right angles which totals 360˚, so if all quadrilateral must total the same amount, it must be 360.



"When we draw a polygon with more than 3 sides we can divide it into triangles, with every vertex fitting into a corner of the polygon. Here is a quadrilateral. How many triangles is it divided into?" (2) "How many degrees in each of the triangles?" (180.) "So the total of the degrees in a quadrilateral is 2 times 180 which is 360˚."



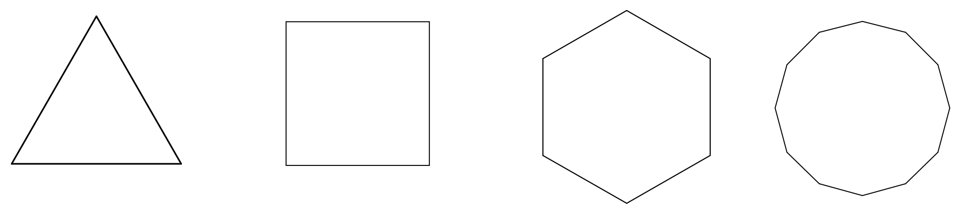
"Let's try a pentagon. If we divide the pentagon into triangles, how many do we get?" Show the students that you can't have the diagonals cross, because then not all the vertices of the triangle are in the angles of the polygon, so we can't add them. (3) "So how many degrees total in the pentagon?" (540)



"Let's divide up the hexagon into triangles." Draw a few hexagons and have students divide them into triangles different ways. "Whichever way you divide them up, you get three triangles. So how many degrees in the hexagon?" (720)

"We can make a table with the number of sides, the number of triangles we can make, and the number of degrees total." Make the table. "What do we notice about the number of sides and the number of triangles?" The students should notice that the number of triangles is 2 fewer than the number of sides. From there, we can come up with the formula for the total degrees in an n-gon as 180(n-2). Fill in the triangle for some polygons with more sides. Together, come up with the number of degrees in each angle of a regular n-gon as 180(n-2)/n. To help with the lesson, write it as $d = \frac{180n - 360}{n}$.

"What happens to the individual angles as we add more sides?" (They become larger.) Have a transparency with different regular polygons to put up to show them the angles. They can see that the shapes with more sides have larger angles.



If the number of sides doubles, does the measure of the angles double? They can see from the table that the answer is no. Also, the angles in a regular hexagon are each $120°$, but the angles of a regular dodecagon are certainly not $240°$.

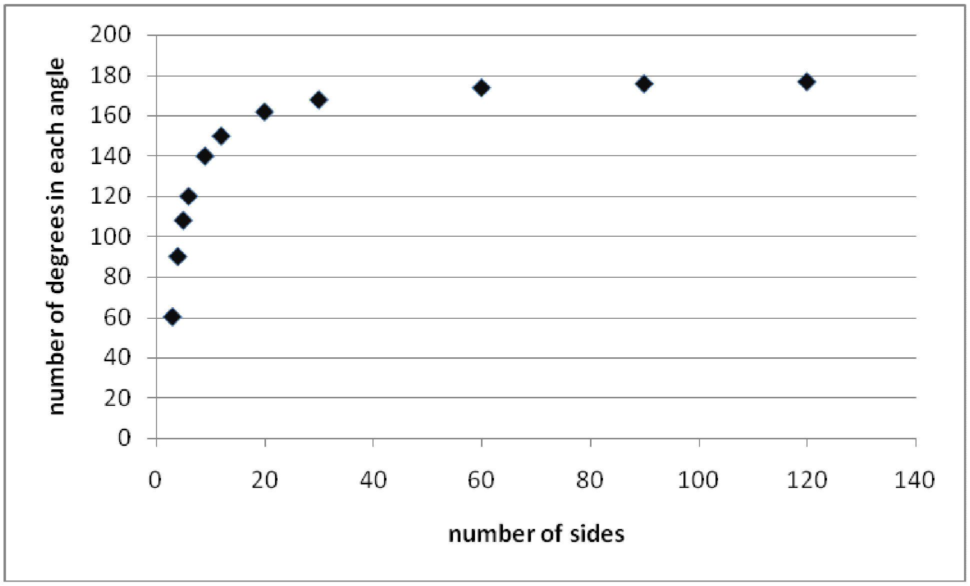
Calculate the angles of polygons with a lot of sides, going into the ten-thousands. "If you look at the measure of an angle, does it seem to be getting closer to something?" They should be able to see that it is getting closer and closer to 180.

number of sides	number of triangles	formula to find total degrees	total degrees in polygon	degrees in each angle of a regular polygon
3	1		180	60
4	2	180*2	360	90
5	3	180*3	540	108
6	4	180*4	720	120
9	7	180*7	1,260	140
12	10	180*10	1,800	150
20	18	180*18	3,240	162
30	28	180*28	5,040	168
60	58	180*58	10,440	174
90	88	180*88	15,840	176
120	118	180*118	21,240	177
300	298	180*298	53,640	178.8
600	598	180*598	107,640	179.4
1,200	1,198	180*1,198	215,640	179.7
1,800	1,798	180*1,798	323,640	179.8
3,000	2,998	180*2,998	539,640	179.88
5,000	4,998	180*4,998	899,640	179.928
8,000	7,998	180*7,998	1,439,640	179.955

12,000	11,998	180*11,998	2,159,640	179.970
20,000	19,998	180*19,998	3,599,640	179.982
30,000	29,998	180*29,998	5,399,640	179.988
60,000	59,998	180*59,998	10,799,640	179.994
120,000	119,998	180*119,998	21,599,640	179.997
600,000	599,998	180*599,998	107,999,640	179.9994
5,000,000	4,999,998	180*4,999,998	899,999,640	179.9999

"Can it ever reach 180?" Many students should realize that it can not, from a geometric view. If the angles are 180 degrees, the whole shape would be a straight line, and could not be a polygon.

"Let's look at it another way. The formula for the measure of each angle is $d = \frac{180n - 360}{n}$. As n gets larger and larger, the 360 becomes less significant. So as n gets larger, this gets closer to $\frac{180n}{n}$, which is 180. But it never actually reaches 180 because, while the 360 becomes less significant in comparison to the n, it is still there."



Experience with this lesson:

Deering School, Deering, Alaska, high school Geometry class; Bartlett high school, Anchorage, Alaska, high school Geometry class

Reflection:

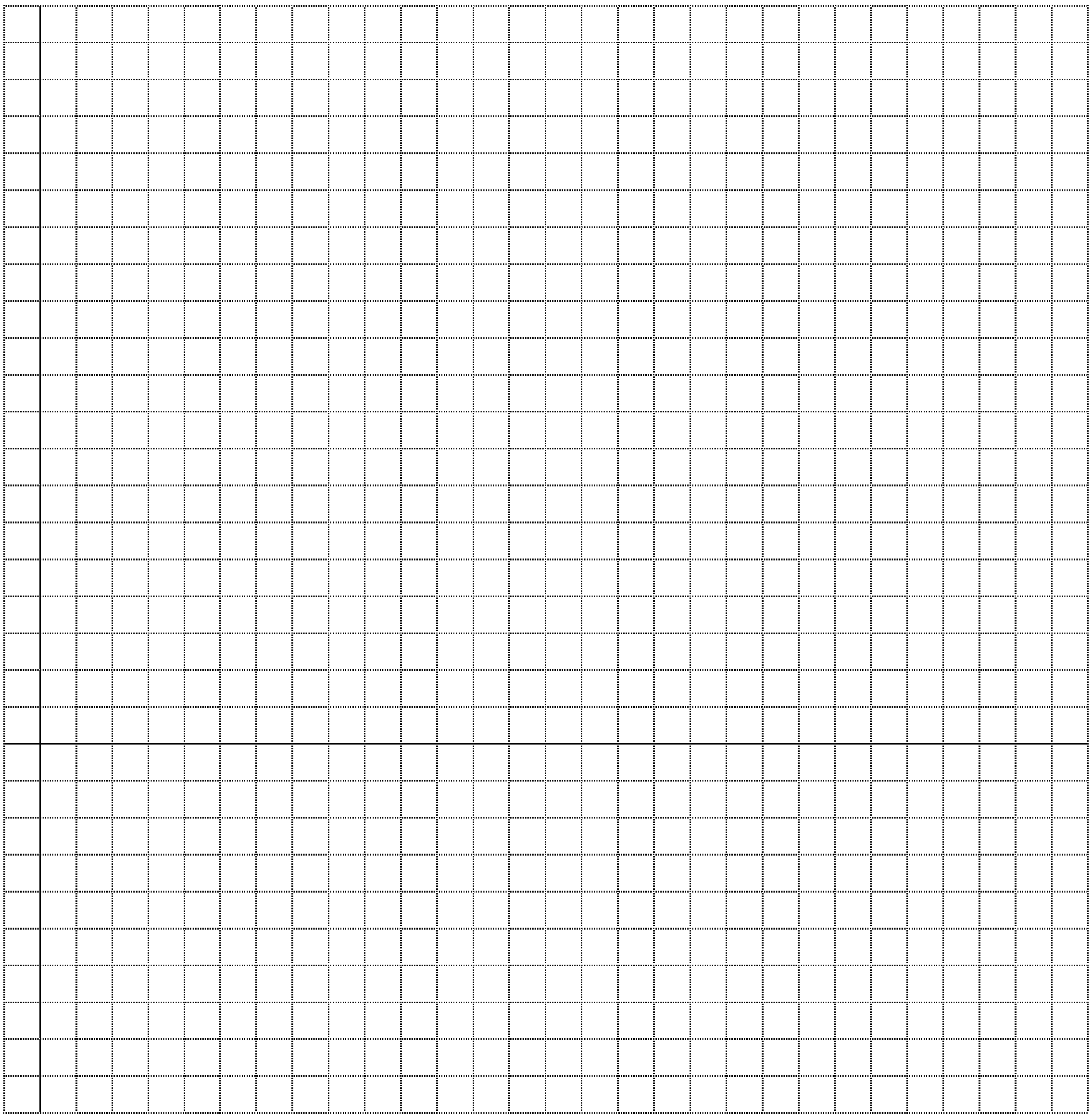
This lesson has always gone smoothly. Then students understand logically that the angles of a polygon can not be greater than 180 degrees. Most could see from the equation that the measure of each angle will approach 180 degrees as the number of sides increases.

Activity 8:   Rational functions

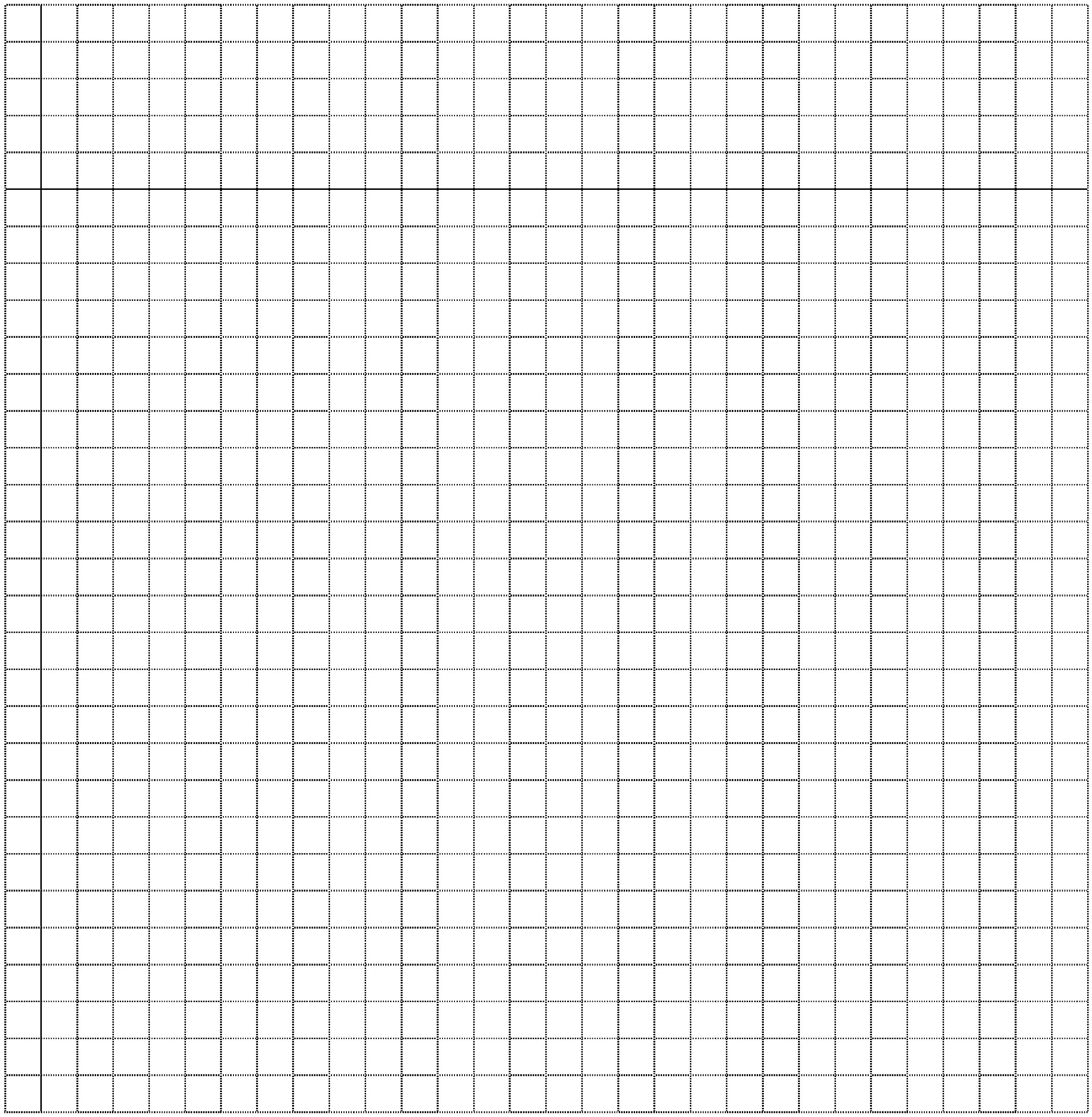
Rational Functions

Worksheet:

1. Joe spent $9 for a lawn mower, and mows lawns for $3 per hour. What is his profit? Graph the profit depending on the hours he works.

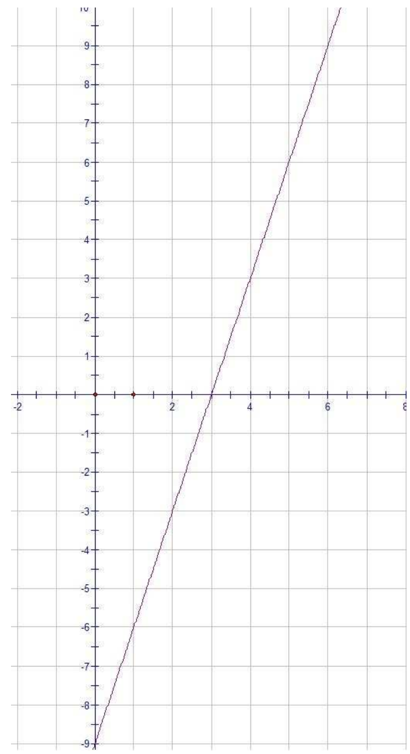


2. Joe spent $9 for a lawn mower, and mows lawns for $3 per hour. What is his profit PER HOUR WORKED? Graph the profit depending on the hours he works.



Lesson plan

Put the first problem on the board:
Joe spent \$9 for a lawn mower, and mows lawns for \$3 per hour. What is his profit?
The students should realize that it depends on the numbers of hours he worked. Have them give me some points. Ask them if they can give me the equation and then graph it.



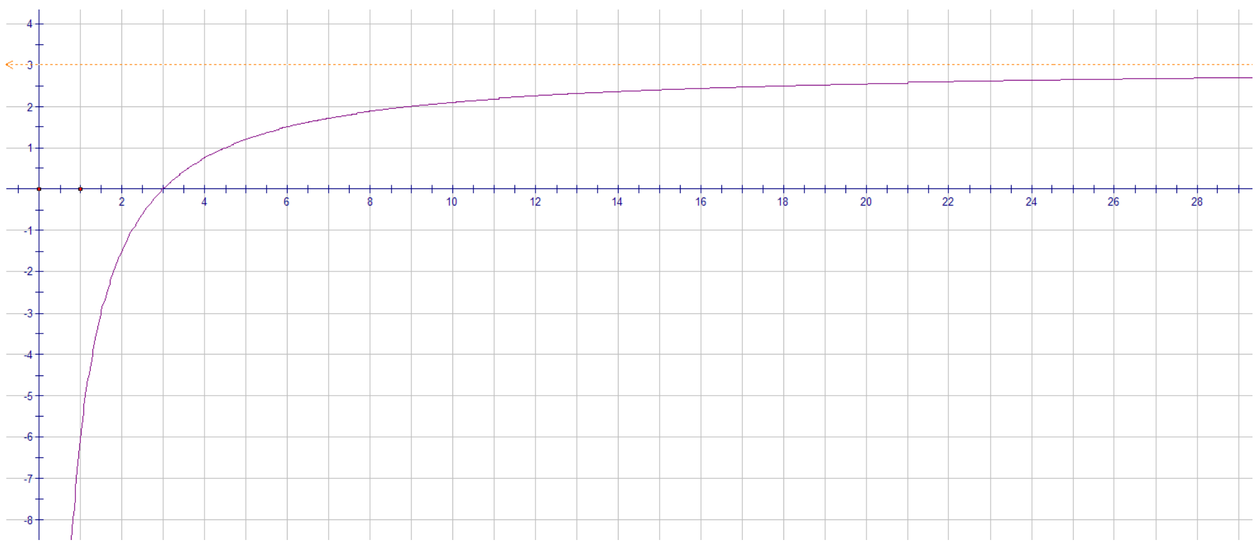
x	y
0	-9
1	-6
3	0
4	3
10	21

y=3x-9

Then give the class a new problem:
Joe spent \$9 for a lawn mower, and mows lawns for \$3 per hour. What is his average profit per hour?
Find the y value for several x values before coming up with an equation. Then give them some larger values of x.
Ask the class what happens to his average wage per hour as the number of hours increases. They should realize that as the number of hours he worked went up, the average profit per hour approached \$3. Show them that it would never reach \$3, since he was changing \$3 per hour, but we still had to take out the \$9 he paid for the lawn mower. However, as he worked more hours, the \$9 was divided up among more hours, so it got closer to \$3. Use the points make the graph, and have them come up with the function.

x	y
0	
0.1	-87
0.5	-15
1	-6
3	0
4	0.75
10	2.1
20	2.55
50	2.82
100	2.91
200	2.955
500	2.982
1000	2.991

$$y = \frac{3x-9}{x}$$



After finding the value for some very large numbers, show them that as x gets larger, the 9 becomes less significant, and this gets closer to $\frac{3x}{x}$ which is 3. So the graph will get closer and closer to 3. Will it ever equal 3? No, because he still did spend the $9 for the lawnmower. The line y=3 is an asymptote, a line that our graph will get closer and closer to, but will never touch.

<u>Experience with this:</u>

I use this College algebra classes at University of Alaska, Anchorage, Alaska and Embry Riddle Aeronautical University, Elmendorf, Alaska.

Comments:

I used this as an introduction to graphing rational functions. The students were already accustomed to graphing linear functions. This lesson also introduced horizontal asymptotes. It tends to go smoothly. The students seem to have an easier time understanding this word problem than they do with a generic rational function. There has always been at least one student who can explain to the rest of the class why this function approaches 3 asymptotically.

Notes:

I took this lesson from the textbook College Algebra by Gustafson and Frisk.

Activity 9: Compound interest

Worksheet:

If we invest $1,000,000 in the bank at 6% annual interest, how much will we have in our account years later?

	Interest is compounded					
	annually	2x a year	monthly	daily	every second	continuously
0						
1/365	XXXXX	XXXXX	XXXXX			
1/12	XXXXX	XXXXX				
1/2	XXXXX					
1						
2						
5						
10						
100						

Lesson plan:

"If we put one million dollars in the bank at 6% interest, how much interest would we get in the first year?" ($60,000) Most students should be able to answer this, even though the class had never dealt with interest before. "If we leave it in another year, will we get another 60,000?" (No, more than that.) "Why?" Most students will probably know that we should now be getting interest on the interest we received the first year. "So how much interest will we get the second year?" ($63,600)

Draw a table and calculate how much would be in the bank at the end of different years. Help they students see where the formula $A = P(1+r)^n$ comes from.

"Now suppose I decide that the second half of the year I want to earn interest on the interest I received the first half of the year. How much interest do we get the first half of the year?" ($30,000) "So we begin the second half of the year with $1,030,000 and get

6% interest on that for half the year. How much do we now end the year with?” ($1,060,900) “So do we get more money this way?” (Yes, $900 more.) Calculate how much money we would have at the end of different lengths of time.

“What about getting interest the second month on the interest we earned the first month?” Go through the same process.

Explained the phrase ’compound interest’ and we will generate the formula

$A = P\left(1+\frac{r}{n}\right)^{rt}$.

“What happens if we compound the interest more often?” (We make more money.) “So we would like to compound the interest as often as possible, right? How often can we compound it?” Students will probably suggested daily, hourly, or every second. Calculate the amounts we’d have if interest was compounded daily and every second, and add the amounts to the table.

“As we increase the frequency of the compounding, what happens to the amount of money we have?” The students should realize that it increases.   “But does it increase very fast? Look at the amount of money we have if it is compounded every month or every second. The amount of times it is compounded goes up a lot. Does the amount of money we have go up as fast?” The students will probably see that it is leveling off.

“How much more often could we compound the interest?” There will probably be a student who suggests continuously. “When we compound continuously, we would need to make n infinity. Will that work in the formula?” Students will probably see that the formula will not work. Explain that we will have to use a different formula.

“Start with $\left(1+\frac{1}{n}\right)^n$. If n is 1, what do we get?” (2) “If n is 2, what do we get?” “2.25.” “If n is 3?” (About 2.37) Start a table. “Now we don’t want to do this for every integer, so let’s jump ahead. If n is 10?” (2.59) “If n is 100?” (2.70) “If n is fifty thousand?” (About 2.7183) “What happens as n gets larger? “ The students should be able to see that it gets larger. “But our n is getting larger very fast. What about the value for the function?” Most students should see that it gets larger, but very slowly and is leveling off. “As n goes to infinity, this gets closer and closer to…” Write the number on the board 2.7182818285… and add it to the table. “Just like pi goes on forever after the decimal point without ever forming a repeating pattern, so does this number. Just like we represent pi with the Greek letter pi, we represent this with a lower case e.”

n	$\left(1+\frac{1}{n}\right)^n$
1	2.0000000000
2	2.2500000000
3	2.3703703704
4	2.4414062500
5	2.4883200000
10	2.5937424601
20	2.6532977051
100	2.7048138294
1,000	2.7169239322
50,000	2.7182546461
1,000,000	2.7182804692
1,000,000,000,000	2.7185234960
∞	2.7182818285

“The formula for the amount of money we would have if we compounded the interest continuously is $A = Pe^{rt}$. How much money do we have in the bank after one month if the interest is compounded continuously?” Calculate it together. ($1,000,164.40) “So after one month, we don’t see any difference between compounding every second and compounding continuously if it is rounded to the nearest penny.” We calculated it for the other amounts of time. “After a year, we have one penny more. After 100 years, we have about $144 more.”

	Interest is compounded					
	annually	2x a year	monthly	daily	every second	continuously
0	1,000,000.00	1,000,000.00	1,000,000.00	1,000,000.00	1,000,000.00	1,000,000.00
1/365				1,000,164.38	1,000,164.40	1,000,164.40
1/12			1,005,000.00	1,005,012.11	1,005,012.52	1,005,012.52
1/2		1,030,000.00	1,030,377.51	1,030,451.99	1,030,454.53	1,030,454.53
1	1,060,000.00	1,060,900.00	1,061,677.81	1,061,831.31	1,061,836.54	1,061,836.55
2	1,123,600.00	1,125,508.81	1,127,159.78	1,127,485.73	1,127,496.84	1,127,496.85
5	1,338,225.58	1,343,916.38	1,348,850.15	1,349,825.53	1,349,858.78	1,349,858.81
10	1,790,847.70	1,806,111.23	1,819,396.73	1,822,028.95	1,822,118.74	1,822,118.80
100	339,302,083.51	369,355,815.22	397,442,318.65	403,229,913.15	403,428,649.58	403,428,793.49

Experience with this lesson:

I teach it at the University of Alaska, Anchorage, college algebra class. I teach this class about three times a year.

Reflection:

The students seemed fine with this formula. I was surprised that no one asked where it came from. I later mentioned that formulas often give us trouble when things become zero or infinity. In this case, the length of time between compounding becomes zero, and the number of times it is compounded becomes infinite. I told them that anyone who takes calculus will learn how to come up with formulas like this one.

I would like to find a way to transition to compounding continuously so that the students see where the formula comes from. Since the students have not yet had calculus, I cannot have them find the limit as the number of times it is compounded approaches infinity. I have not yet found a way to transition. I have spoken to a few other professors, but they also did not know how to do this.

How I would do this differently for a calculus class:

My first change is that I would assume the students know what e is. After calculating the formula for compound interest, $A = P\left(1+\frac{r}{n}\right)^{rt}$ we would find what happens when n approaches infinity. (We don't need to include the P. That does not get affected, so we can multiply it later.) r is just treated like a constant.

$\lim\limits_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}$ gives us $(1+0)^{\infty} = 1^{\infty}$ which is indeterminate. We can solve it using L'Hospital's Rule, if we can get it in the form of 0/0 or ∞/∞.

We have $A = \left(1+\frac{r}{n}\right)^{nt}$. We can take the natural log of each side.

$$\ln(A) = \left(1+\frac{r}{n}\right)^{nt} = nt\cdot\ln\left(1+\frac{r}{n}\right) = \frac{t\cdot\ln\left(1+\frac{r}{n}\right)}{\frac{1}{n}}$$

$$\lim_{n\to\infty}\frac{t\cdot\ln\left(1+\frac{r}{n}\right)}{\frac{1}{n}} = \frac{t\cdot\ln\left(1+\frac{r}{\infty}\right)}{\frac{1}{\infty}} \to \frac{t\cdot\ln(1+0)}{0} = \frac{t\cdot\ln(1)}{0} = \frac{0}{0}.$$

Let's find the derivative of numerator and the denominator.

$$der\left[t\cdot\ln\left(1+\frac{r}{n}\right)\right] = der\left[t\cdot\ln\left(1+rn^{-1}\right)\right] = t\cdot\frac{1}{1+\frac{r}{n}}\cdot(-1)\cdot n^{-2}\cdot r$$

$$der\left(\frac{1}{n}\right) = der\left(n^{-1}\right) = (-1)\cdot n^{-2}$$

Divide the derivative of the numerator by the derivative of the denominator

$$\frac{t \cdot \frac{1}{1+\frac{r}{n}} \cdot (-1) \cdot n^{-2} \cdot r}{(-1) \cdot n^{-2}}$$

The denominator cancels out. We cancel what we can and simplify.

$$\frac{t \cdot \frac{1}{1+\frac{r}{n}} \cdot (-1) \cdot n^{-2} \cdot r}{(-1) \cdot n^{-2}} = \frac{rt}{1+\frac{r}{n}}$$

Find the limit as n approaches infinity.

$$\lim_{n\to\infty}\left(\frac{rt}{1+\frac{r}{n}}\right) = \frac{rt}{1+\frac{r}{\infty}} = \frac{rt}{1+0} = rt$$

Now we can go back. $A = \left(1+\frac{r}{n}\right)^{nt}$ so $\ln(A) = \ln\left[\left(1+\frac{r}{n}\right)^{nt}\right]$ so

$\lim_{n\to\infty}\{\ln(A)\} = \lim_{n\to\infty}\{\ln\left[\left(1+\frac{r}{n}\right)^{nt}\right]\}$ so $\lim_{n\to\infty}\{\ln(A)\} = rt$

$\ln(A) = rt$ so $A = e^{rt}$. Since we divided out the principal and did this work for a principal of $1, the amount you would have in the bank if you deposited P at rate r compounded continuously for time t

$A = Pe^{rt}$.

Activity 10: Half Life

worksheet:

Carbon-14 has a half-life of 5,700 years. A bone is found and the amount of carbon is compared to the amount it would have had when the animal died.

1. It has 90% of the carbon that it would have had when the animal died. How old is the bone?

2. It has 50% of the original carbon. How old is it?

3. It has 25% of the original carbon. How old is it?

4. It has 10% of the original carbon. How old is it?

5. The bone is 20 years old. What proportion of its original carbon does it have?

6. The bone is 7,500 years old. What proportion of its original carbon does it have?

7. Write an equation to find t, the age of the object, given the proportion of the original carbon, P.

8. When will all the carbon be gone?

9. Graph the fraction of the original carbon still in the bone, by year. Show the points we already found.

Lesson plan:

Tell the class:

All radioactive elements have a half-life. The half-life is the amount of time it takes for half the amount of the element to decay. Medications also have a half life; when you take medication, the half life is the amount of time it takes for the medication in your body to decrease by half.

$$A = A_0 2^{-\frac{t}{h}}$$

A = amount of the element or medication

A_0 = the original amount

t = time

h = half life

Carbon-14 is an element in all living things. When a living organism dies, the carbon-14 starts to decay. It has a half-life of 5,700 years. When old bones are found, archeologists can tell how old they are by measuring the carbon in the bone and comparing it to the amount the bone would have had when the animal died.

A bone is found, and the amount of Carbon-14 is compared to the amount it had originally. (Let the class work on each question for a bit before running through it with them.)

1. It has 90% of the carbon that it would have had when the animal died. How old is the bone?

$$.9 = 2^{-\frac{t}{5700}}$$

$$\log(.9) = \log\left(2^{-\frac{t}{5700}}\right)$$

$$\log(.9) = \left(\frac{-t}{5700}\right)\log(2)$$

$$\frac{\log(.9)}{\log(2)} = \frac{-t}{5700}$$

$$\frac{-5700 \cdot \log(.9)}{\log(2)} = t$$

$$t = 866\ years$$

2. It has 50% of the original carbon. How old is it?

   For 50% , we just take the half life, 5700 years.

3. It has 25% of the original carbon. How old is it?

   25% is half of half. This would be two half lives, 11,400 years.

4. It has 10% of the original carbon. How old is it?

   Same process as the first question. $t = 18{,}935$

5. The bone is 20 years old. What proportion of its original carbon does it have?

$$2^{-\frac{20}{5700}} = 99.76\%$$

6. The bone is 7,500 years old. What proportion of its original carbon does it have?

    40.17%

7. Write an equation for t.

    $$P = 2^{-\frac{t}{h}}$$

    $$P = 2^{-\frac{t}{5700}}$$

    $$\log(P) = \log\left(2^{-\frac{t}{5700}}\right)$$

    $$\log(P) = -\frac{t}{5700}\log(2)$$

    $$t = \frac{-5700 \cdot log(P)}{\log(2)}$$

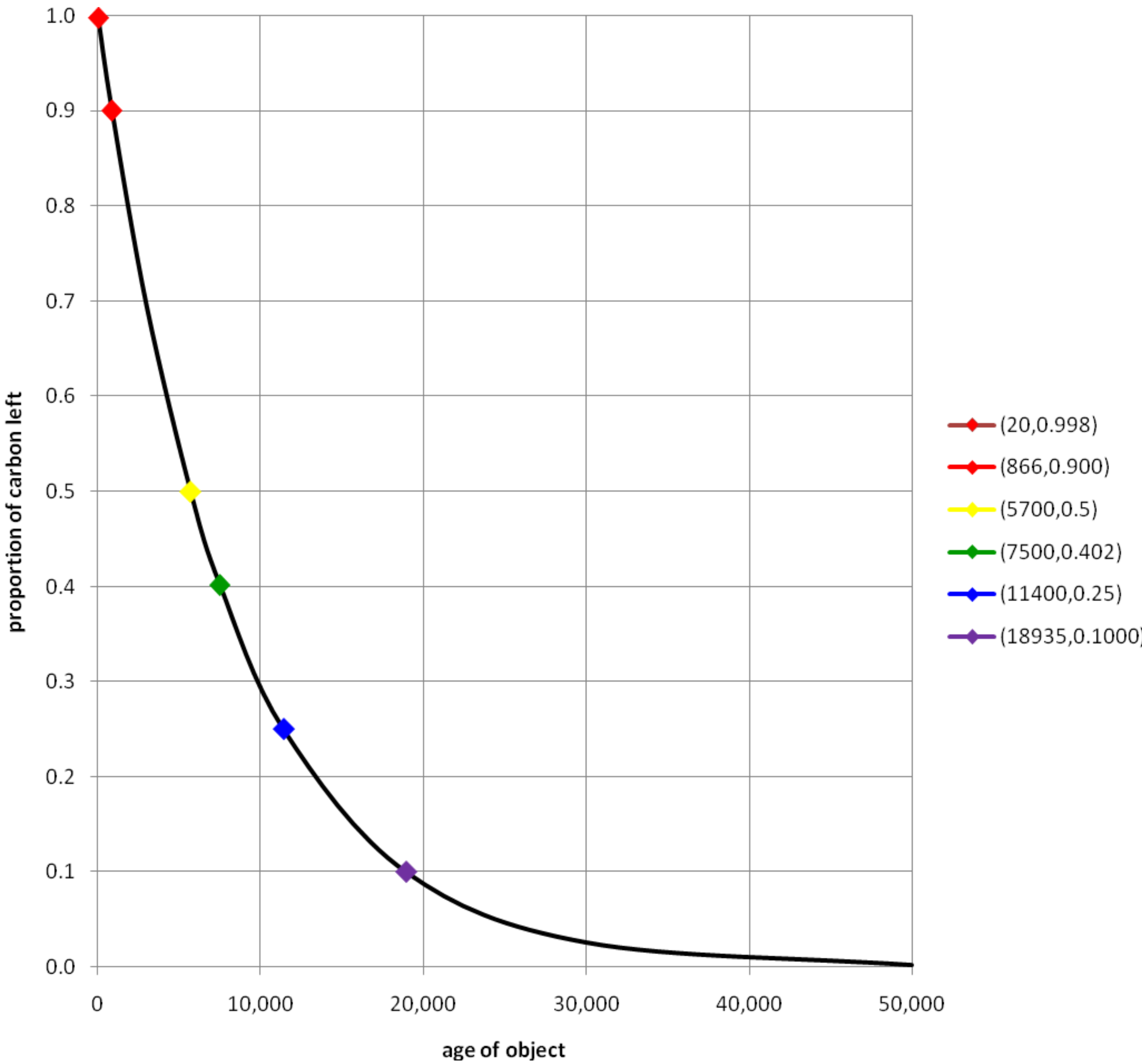
8. When will all the carbon be gone?

    $$t = \frac{-5700 \cdot log(P)}{\log(2)}$$

    $$t = \frac{-5700 \cdot log(0)}{\log(2)}$$

    What does this give us? There is no log of 0. As x gets closer and closer to 0 from the positive side, the log of x goes lower and lower towards negative infinity. Since we have a negative in this, we can say the value of the time goes towards positive infinity. So it will take infinitely long for all the carbon to disappear. It will never be all gone.

9.



1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
proportion of carbon left
0
10,000
20,000
30,000
40,000
50,000
age of object
(20,0.998)
(866,0.900)
(5700,0.5)
(7500,0.402)
(11400,0.25)
(18935,0.1000)

Show the students that the proportion of carbon approaches zero, but never reaches it. The x axis is a horizontal asymptote.

Experience with this lesson:

I use this lesson when I teach college algebra at University of Alaska and at Embry Riddle Aeronautical University.

Reflection:

Some of the students realize right away that if the proportion of carbon keeps declining by half every half-life, then it will never completely go away. Some others realize it once we work with the equation. Everyone has always understood it once we do the graph. A few times I have had students argue that we cannot have a fraction of an atom. I have responded that the number of atoms is extremely high, so that idea is pretty much just theoretical, since the universe is not expected to last indefinitely.

Comment:

I took most of this lesson from the textbook I use for the UAA class. The textbook does not ask about when there will be no carbon left; I added that in myself.

How I would change this for a calculus class:

10. What happens to the amount of carbon as the time approaches infinity?

$$\lim_{t \to \infty} \left(2^{-\frac{t}{5700}}\right)$$

$$2^{-\frac{t}{5700}} = \frac{1}{2^{\infty}} = \frac{1}{\infty} = 0$$

So as the age of the item gets older, the amount of carbon approaches 0.

Activity 11:   Estimating pi

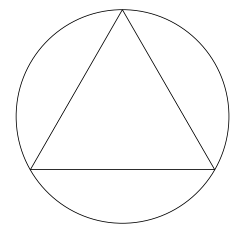
Class this is intended for:

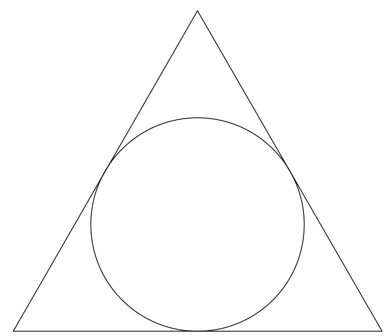
Trigonometry

Worksheet:

1. A circle has a diameter of 1.

    a. Draw an equilateral triangle inscribed in the circle. What is the perimeter of the triangle?

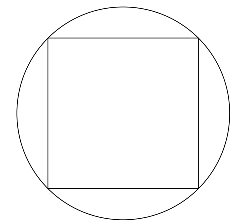




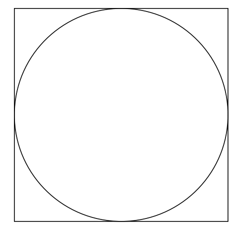
    b. Draw an equilateral triangle circumscribed around the circle. What is the perimeter of the triangle?

2. A circle has a diameter of 1.

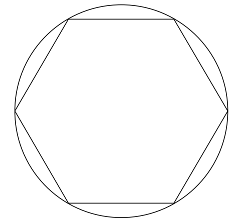
    a. Draw a square inscribed in the circle. What is the perimeter of the square?



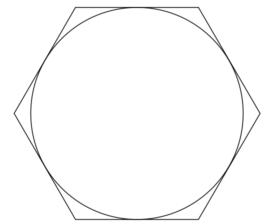
    b. Draw a square circumscribed around the circle. What is the perimeter of the square?



3. A circle has a diameter of 1.

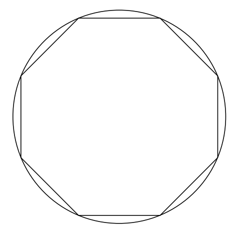


   a. Draw a regular hexagon inscribed in the circle. What is the perimeter of the hexagon?

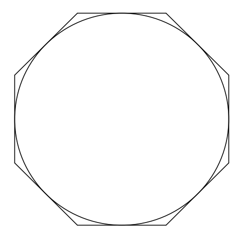


   b. Draw a regular hexagon circumscribed around the circle. What is the perimeter of the hexagon?

4. A circle has a diameter of 1.

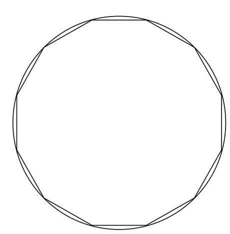


   a. Draw a regular octagon inscribed in the circle. What is the perimeter of the octagon?



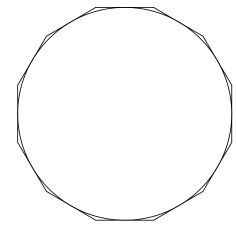
   b. Draw a regular octagon circumscribed around the circle. What is the perimeter of the octagon?

5. A circle has a diameter of 1.



   a. Draw a regular dodecagon inscribed in the circle. What is the perimeter of the dodecagon?

b. Draw a regular dodecagon circumscribed around the circle. What is the perimeter of the dodecagon?



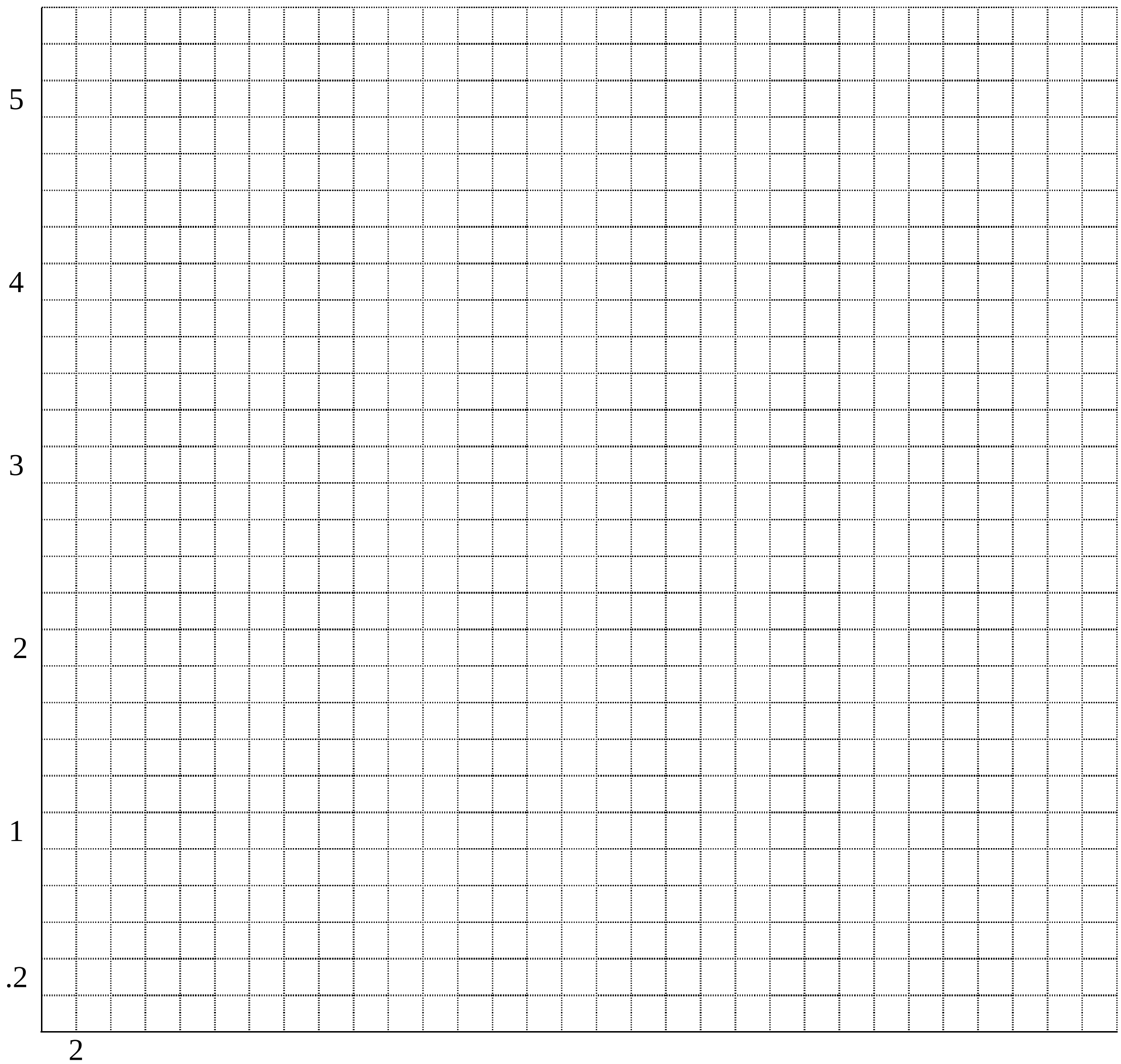
6. Write an equation that gives the perimeter of a shape with n sides inscribed in a circle with a diameter of 1.

7. Write an equation that gives the perimeter of a shape with n sides circumscribed around a circle with a diameter of 1.

8. Fill in the table:

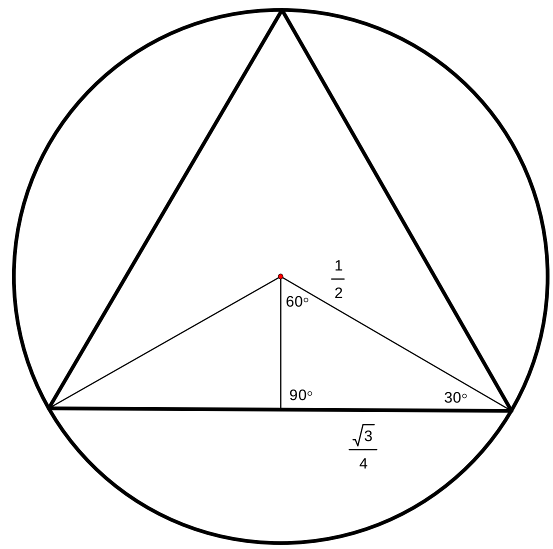
Number of sides	Perimeter of inscribed polygon	Perimeter of circumscribed polygon
3		
4		
6		
8		
12		
20		
30		
60		

9. Fill in the points and graph two curves, one for inscribed polygons and one for circumscribed polygons,



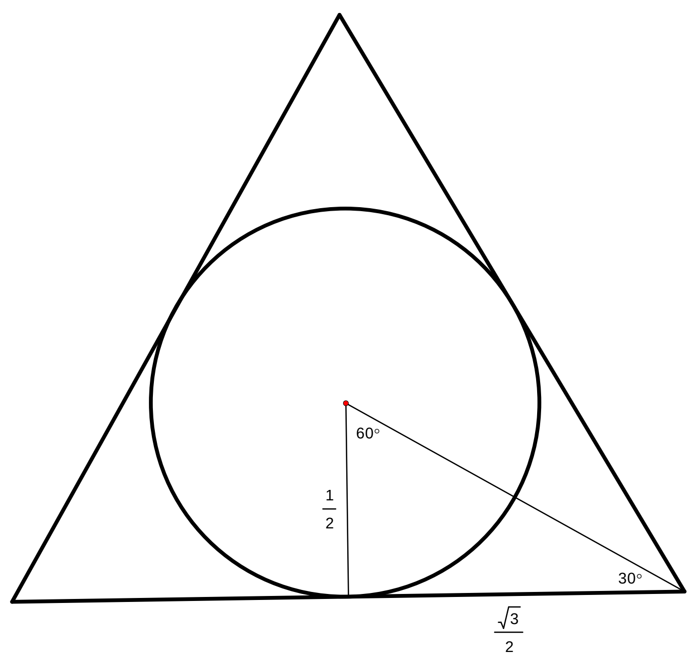
Lesson plan:

Draw a circle on the board and tell the students it has a diameter of 1. Draw an equilateral triangle inscribed in the circle and have the students find the perimeter of the triangle. Give them some time to work on their own, and then do it together with the class.

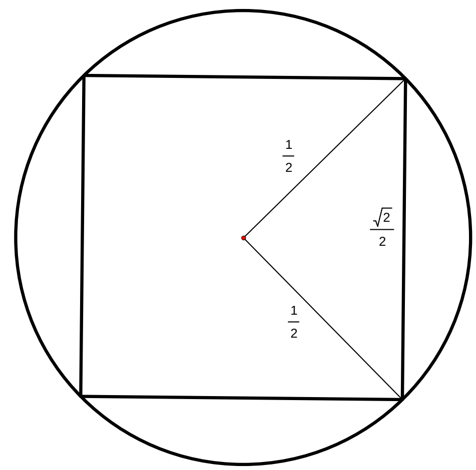


We can draw lines from the center of the circle to two angles of the triangle. Each is a radius of the circle, so their lengths are ½. The angle between them is 60°. Draw an angle bisector between them to the side of the triangle. We now have a 30-60-90 triangle, with a hypotenuse of ½. The side opposite the 60° angle is $\frac{\sqrt{3}}{4}$. It is half a side of a triangle, so the perimeter of the triangle is 6 times that, $\frac{3\sqrt{3}}{2}$, approximately 2.5981.

Repeat the process (let the students work on it themselves, then do it together with the class) for a triangle circumscribed around the circle.

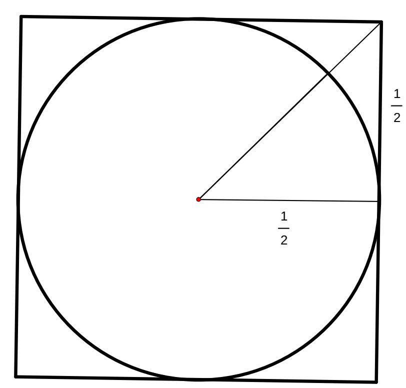


Draw a line from the center of the circle to the midpoint of one of the triangle sides. Since this is also the radius of the circle, its length is $\frac{1}{2}$. Add a line from the center of the circle to an angle of the triangle next to the side we used. We have a 30-60-90 triangle, where the length of the side opposite the 30° angle is ½. This makes the side opposite the 60° angle $\frac{\sqrt{3}}{2}$. Since it is also the length of half the side of the triangle, the perimeter of the triangle is $3\sqrt{3}$, approximately 5.1962.



Repeat with the inscribed square.

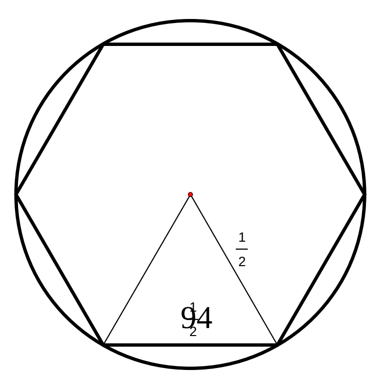
Make a triangle by drawing lines from the center of the circle to adjacent corners of the square. Those lines are also radii of the circle. Since the diameter of the circle is 1, each radius is ½. This is a right triangle and the hypotenuse is a side of the square with length $\frac{\sqrt{2}}{2}$. Since there are 4 of them, the perimeter of the square is $2\sqrt{2}$, approximately 2.8284.



And the circumscribed square.

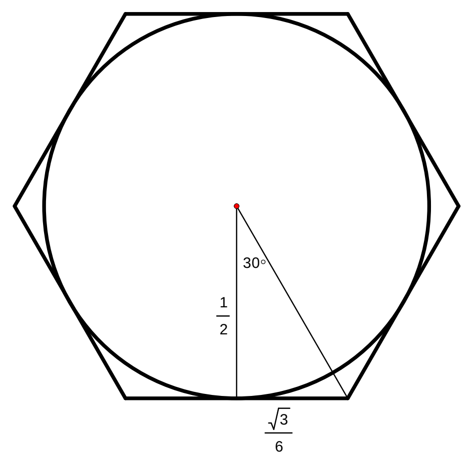
Draw a line from the center of the circle an angle of the square. Draw another line from the center of the circle to the center of an adjacent side. We have a 45-45-90 triangle where a leg is ½. The other leg is also ½ so the perimeter of the square is 4.

Work with the inscribed hexagon.



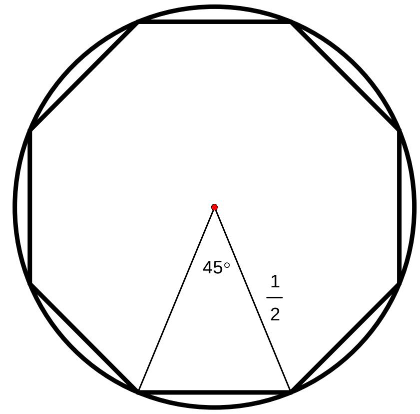
Draw lines from the center of the circle to two adjacent angles on the hexagon. Each of these are radii of the circle, so they are ½ units long. Since the angle between them is 60°, we have an equilateral triangle. Each side of the hexagon is ½ so the perimeter is 3.

Work with the circumscribed hexagon.



Draw a line from the center of the circle to the center of side of the hexagon where the circle meets the hexagon. This is a radius of the circle, so the length is ½. Draw a line from the center of the circle to one end of this side of the hexagon. This is a 30-60-90 triangle, where the radius of the circle is the side opposite the 60° angle. This makes the part of the triangle opposite the 30° angle equal to $\frac{\sqrt{3}}{6}$. Since this segment is half of a side of the hexagon, the perimeter of the hexagon is 12 times this, $2\sqrt{3}$, approximately 3.4641.

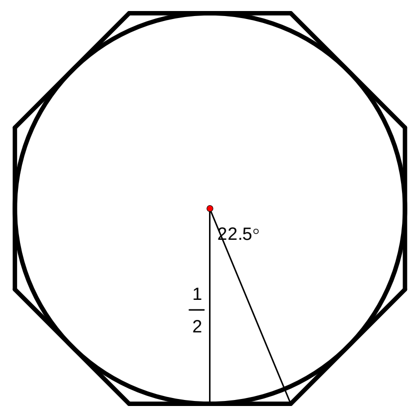
Work with the inscribed octagon.



Draw lines from the center of the circle to two adjacent angles of the octagon. With the side of the octagon, we have an isosceles triangle where two sides have length of ½, and the angle between them is 45°. The other angles are each 67.5°. We can use the law of sines to find the length of the side of the octagon.

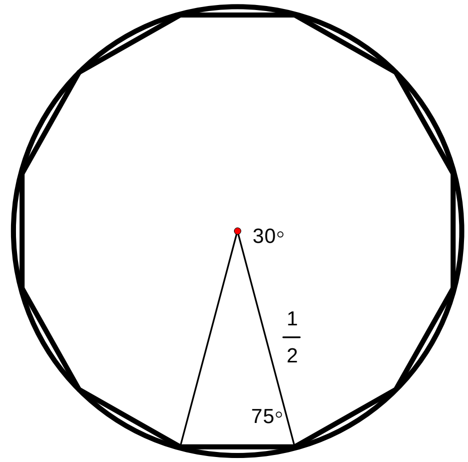
$\frac{x}{\sin 45°} = \frac{\frac{1}{2}}{\sin 67.5°}$. We know the sin of 45° is $\frac{\sqrt{2}}{2}$ so the length of one side of the hexagon is $\frac{\frac{1}{2}\cdot\frac{\sqrt{2}}{2}}{\sin 67.5°}$. The octagon has eight sides, so the perimeter is $\frac{2\sqrt{2}}{\sin 67.5°}$, approximately 3.0615.

Go to the circumscribed octagon.



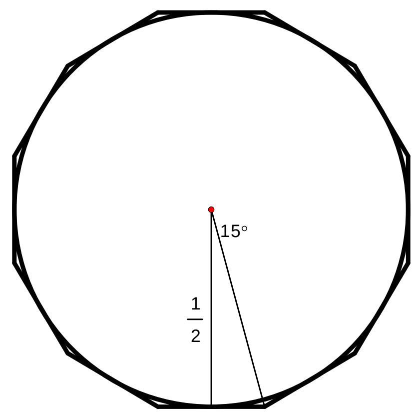
Draw a line from the center of the circle to the midpoint of a side, where the side meets the circle. This is a radius, so the length is ½. Draw a line from the center of the circle to an end of the same side. This is a right triangle and the angle in the center of the circle is 22.5°. If the length of half the side of the octagon is x, the tangent of 22.5° is equal to x over ½. $\tan(22.5°) = \frac{x}{\frac{1}{2}}$ so $x = \frac{1}{2}\tan(22.5°)$. The perimeter is 16x, so it is $8 \cdot \tan(22.5°)$ which is approximately 3.3137.

On to the inscribed dodecagon.



Draw an isosceles triangle by drawing lines from the center of the circle to adjacent corners of the dodecagon. The angle in the center is 30° and each base angle is 75°. The equal sides are also radii of the circle, so their length is $\frac{1}{2}$. Let x be the length of a side of the dodecagon. By the law of sines, $\frac{x}{\sin 30°} = \frac{\frac{1}{2}}{\sin 75°}$. Since the sin of 30° is $\frac{1}{2}$, x is equal to $\frac{1}{4 \sin 75°}$. There are 12 sides, so the perimeter is $\frac{3}{\sin 75°}$, approximately 3.1058

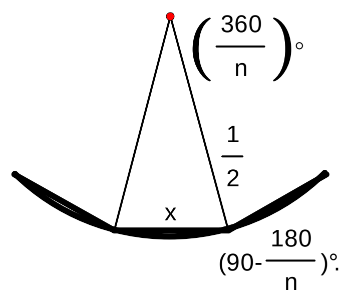
For the circumscribed dodecagon:



Draw a line from the center of the circle to the middle of one of the sides. This is also the radius of the circle, so it has length of $\frac{1}{2}$. Draw another line, this one from the center of the circle to an angle of the dodecagon, The angle between them is 15°. Let x be the length of half a side of the dodecagon. Tan 15° is equal to $\frac{x}{\frac{1}{2}}$, so x is equal to $\frac{1}{2}tan 15°$. Since there are 12 sides, and x is equal to half a side, the perimeter is $12 tan 15°$, approximately 3.2154.

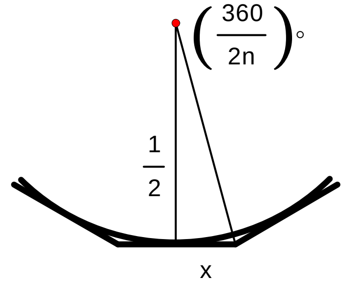
Ask the students: As the polygons get more and more sides, what do they get closer to? The students should notice that the polygons look more and more like circles. If they get closer to circles, what should their perimeters get closer to? Pi.

Can we develop a formula for the perimeter of a polygon with n sides inscribed in a circle with a diameter of 1?



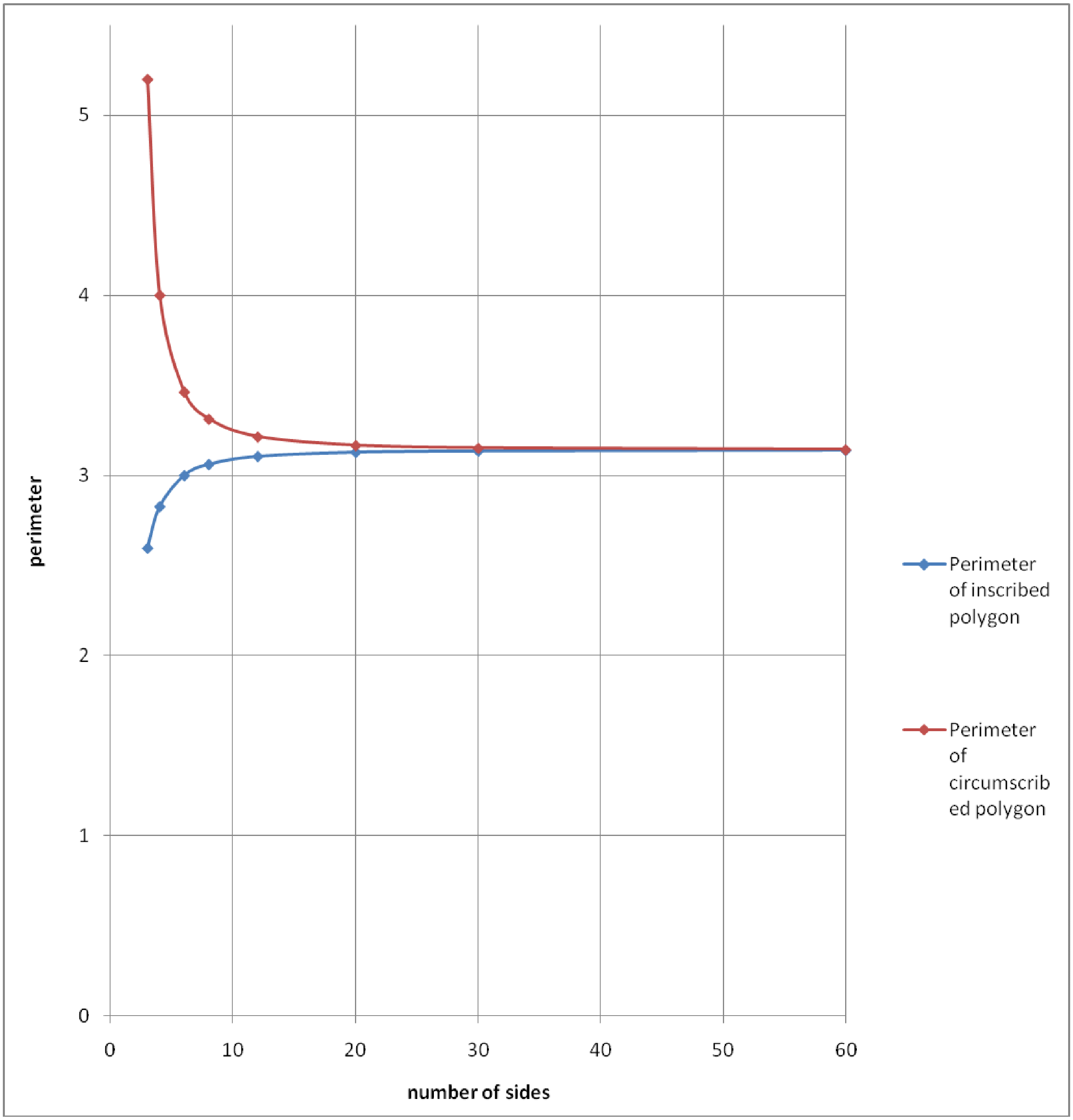
Let x be the length of a side. We have an isosceles triangle where two of the sides are each ½ and the third side is x. The angle opposite the side of x is $\left(\frac{360}{n}\right)^{\circ}$, while the other angles are each $\left(\frac{180-\frac{360}{n}}{2}\right)^{\circ}$, which simplifies to $\left(90-\frac{180}{n}\right)^{\circ}$. We can use the law of sines to find the length of x. $\frac{x}{\sin\left(\frac{360}{n}\right)^{\circ}}=\frac{\frac{1}{2}}{\sin\left(90-\frac{180}{n}\right)^{\circ}}$. So $x=\frac{\sin\left(\frac{360}{n}\right)^{\circ}}{2\cdot\sin\left(90-\frac{180}{n}\right)^{\circ}}$. Since there are n sides of the polygon, the perimeter is $\frac{n\cdot\sin\left(\frac{360}{n}\right)^{\circ}}{2\cdot\sin\left(90-\frac{180}{n}\right)^{\circ}}$.

What would the formula be for the perimeter of a circumscribed polygon with n sides?



Let x= half the length of a side of the polygon. Draw a line from the center of the circle to the midpoint of a side of the polygon, where the polygon touches the circle. This is a radius of the circle, so the length is ½. Draw another line from the center of the circle to an angle of the polygon adjacent to the side we used. We now have a right triangle. The measure of the angle in the center of the circle is $\left(\frac{360}{2n}\right)^{\circ}$. The tangent of the angle is $\frac{x}{\frac{1}{2}}$. So $x = \frac{1}{2}\tan\left(\frac{180}{n}\right)^{\circ}$, and the perimeter of the polygon is $n \cdot \tan\left(\frac{180}{n}\right)^{\circ}$.

Number of sides	Perimeter of inscribed polygon	Perimeter of circumscribed polygon
3	2.5981	5.1962
4	2.8284	4
6	3	3.4641
8	3.0615	3.3137
12	3.1059	3.2154
20	3.1287	3.1677
30	3.1359	3.1531
60	3.1402	3.1445



5
4
3
2
1
0
perimeter
0
10
20
30
40
50
60
number of sides
Perimeter of inscribed polygon
Perimeter of circumscribed polygon

Point out to the class that as we increase the sides of the polygons, the perimeters approach pi. This is how the value of pi was originally estimated.

<u>My experience with this lesson:</u>

I go through this with my trigonometry class at Embry Riddle Aeronautical University. They seem to find it interesting that this is how the value of pi was first calculated.

<u>What I would add to this for a calculus class:</u>

If this were a calculus class, we would find the angles in radians,

10. What do the perimeters of the polygons approach as the number of sides approach infinity?

The perimeter of the inscribed polygon is $\dfrac{n \cdot sin\left(\frac{2\pi}{n}\right)}{2 \cdot sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right)}$

Find $\lim\limits_{n \to \infty} \dfrac{n \cdot sin\left(\frac{2\pi}{n}\right)}{2 \cdot sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right)}$ We get $\dfrac{\infty \cdot \sin\left(\frac{2\pi}{\infty}\right)}{2 \cdot \sin\left(\frac{\pi}{2} - \frac{\pi}{\infty}\right)} = \dfrac{\infty \cdot sin(\mathbf{0})}{2 \cdot \sin\left(\frac{\pi}{2}\right)} = \dfrac{\infty \cdot \mathbf{0}}{\mathbf{2}}$.

$\infty \cdot \mathbf{0}$ is indeterminate, so we use L'Hospital's rule. We have to change it, so that we have $\frac{\mathbf{0}}{\mathbf{0}}$ or $\frac{\infty}{\infty}$. Change to $\dfrac{sin\left(\frac{2\pi}{n}\right)}{\frac{2}{n} \cdot sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right)}$ and take the derivative of the top over the derivative of the bottom. $\frac{dy}{dn} sin\left(\frac{2\pi}{n}\right) = \frac{-2\pi \cdot \cos\left(\frac{2\pi}{n}\right)}{n^2}$ and

$\frac{dy}{dn}\left[\frac{2}{n} \cdot \sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right)\right] = \frac{2\pi \cdot sin\left(\frac{\pi}{n}\right) - 2n \cdot \cos\left(\frac{\pi}{n}\right)}{n^3}$.

$$\frac{-2\pi \cdot \cos\left(\frac{2\pi}{n}\right)}{n^2} \div \frac{2\pi \cdot sin\left(\frac{\pi}{n}\right) - 2n \cdot \cos\left(\frac{\pi}{n}\right)}{n^3} = \frac{\pi n \cdot cos\left(\frac{2\pi}{n}\right)}{n \cdot cos\left(\frac{\pi}{n}\right) - \pi \cdot sin\left(\frac{\pi}{n}\right)}$$

We can also write it as $\frac{\pi \cdot \cos\left(\frac{2\pi}{n}\right)}{cos\left(\frac{\pi}{n}\right) - \frac{\pi}{n} \cdot sin\left(\frac{\pi}{n}\right)}$. $\lim_{n \to \infty} \frac{\pi \cdot \cos\left(\frac{2\pi}{n}\right)}{cos\left(\frac{\pi}{n}\right) - \frac{\pi}{n} \cdot sin\left(\frac{\pi}{n}\right)}$

we get $\frac{\pi \cdot cos\left(\frac{2\pi}{\infty}\right)}{\cos\left(\frac{\pi}{\infty}\right) - \frac{\pi}{\infty} \cdot \sin\left(\frac{\pi}{\infty}\right)} = \frac{\pi \cdot \cos(0)}{\cos(0) - 0 \cdot \sin(0)} = \frac{\pi \cdot 1}{1 - 0} = \pi$

So as the number of sides of the inscribed polygon goes to infinity, the perimeter approaches pi.

What about the perimeter of the circumscribed polygon?

The perimeter is $n \cdot \tan\left(\frac{\pi}{n}\right)$. $\lim_{n \to \infty}\left[n \cdot \tan\left[\left(\frac{\pi}{n}\right)\right]\right] = \infty \cdot \tan(0) = \infty \cdot 0$ so we need to use L'Hospital's rule. Make it $\frac{tan\left(\frac{\pi}{n}\right)}{\frac{1}{n}}$. $\frac{dy}{dn} \tan\left(\frac{\pi}{n}\right) = \frac{-\pi}{cos^2\left(\frac{\pi}{n}\right) \cdot n^2}$. $\frac{dy}{dn}\left(\frac{1}{n}\right) = \frac{-1}{n^2}$.

$$\frac{-\pi}{cos^2\left(\frac{\pi}{n}\right) \cdot n^2} \div \frac{-1}{n^2} = \frac{\pi}{cos^2\left(\frac{\pi}{n}\right)}$$

$$\lim_{n \to \infty} \frac{\pi}{cos^2\left(\frac{\pi}{n}\right)} = \frac{\pi}{cos^2\left(\frac{\pi}{\infty}\right)} = \frac{\pi}{cos^2(0)} = \frac{\pi}{1} = \pi$$.

So as the number of sides of the circumscribed side approaches infinity, the perimeter approaches pi.

Activity 12: Markov chains

Worksheet

Andrea decided to become a foster mother for homeless schnauzers. She decided she has enough space for 6 schnauzers. On the first day of the month she will take in one new dog from the shelter, assuming she has room. Each dog has a 30% chance of being adopted each month. At the end of each month, she will check to see how many dogs she has.

What is the probability distribution for the number of dogs she will have at the end of the first month?

What is the probability distribution for the number of dogs she will have at the end of the first month?

Lesson Plan

Give the class the problem:

Andrea decided to become a foster mother for homeless schnauzers. She decided she has enough space for 6 schnauzers. On the first day of the month she will take in one new dog from the shelter, assuming she has room. Each dog has a 30% chance of being adopted each month. At the end of each month, she will check to see how many dogs she has.

What is the probability distribution for the number of dogs the first month? The second month? The third? What will the probably approach as the number of months approaches infinity?

This can be done using Markov chains. This calculates the probability distribution for the next step, given the value for this step. It assumes that the probability for the next step is based on the results for this step only and is not affected by any previous steps. The probability distribution for the number of dogs she'll have at the end of next month is affected by this month only, not by previous months.

Suppose she ends a month with 0 dogs. What is the probability distribution for the end of the next month? She will get a new dog the next day. There is a 30% chance he will be adopted in the next month leaving her with no dogs at the end of the month, and a 70% chance he won't be adopted and she'll still have him. If she has 0 dogs at the end of a month, the expected number of dogs at the end of the next month is 0.7.

If she ends a month with 1 dog, she will get a new dog the next day. The number of dogs she will have left is a binomial distribution. $P(0)=(.3)^2=.09$, $P(1)={}_2C_1(.3)(.7)=0.42$, $P(2)=(.7)^2=.49$. The expected number is 1.4.

If she ends a month with 2 dogs, she will get a new dog the next day. The number of dogs she will have left is a binomial distribution. $P(0)=(.3)^3=.03$, $P(1)={}_3C_1(.3)^2(.7)=0.19$, $P(2)={}_3C_2(.3)(.7)^2=0.44$, $P(3)=(.7)^3=.34$. The expected number is 2.1.

And so on. If she ends a month with 6 dogs, she does not get any new dogs, but starts the next month with those six.

Here is a table. Each row is the probability distribution for the number of dogs she'll have at the end of the next month, based on the number of dogs she has the end of this month.

		probability distribution - number of dogs at the end of next month							expected number end of next month
		0	1	2	3	4	5	6	
	0	0.3000	0.7000	0.0000	0.0000	0.0000	0.0000	0.0000	0.70
today	1	0.0900	0.4200	0.4900	0.0000	0.0000	0.0000	0.0000	1.40
	2	0.0270	0.1890	0.4410	0.3430	0.0000	0.0000	0.0000	2.10

3	0.0081	0.0756	0.2646	0.4116	0.2401	0.0000	0.0000	2.80
4	0.0024	0.0284	0.1323	0.3087	0.3602	0.1681	0.0000	3.50
5	0.0007	0.0102	0.0595	0.1852	0.3241	0.3025	0.1176	4.20
6	0.0007	0.0102	0.0595	0.1852	0.3241	0.3025	0.1176	4.20

From this we make a transition matrix.

$$\begin{bmatrix} 0.30 & 0.70 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.09 & 0.42 & 0.49 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.03 & 0.19 & 0.44 & 0.34 & 0.00 & 0.00 & 0.00 \\ 0.01 & 0.08 & 0.26 & 0.41 & 0.24 & 0.00 & 0.00 \\ 0.00 & 0.03 & 0.13 & 0.31 & 0.36 & 0.17 & 0.00 \\ 0.00 & 0.01 & 0.06 & 0.19 & 0.32 & 0.30 & 0.12 \\ 0.00 & 0.01 & 0.06 & 0.19 & 0.32 & 0.30 & 0.12 \end{bmatrix}$$

We can start on the first day of month 0 – tomorrow she will get the first dog. Today she has zero dogs. On the last day of month 1 there is a 0.3 chance of having zero dogs and a 0.7 chance of one dog. This has an expected value of 0.7 dogs. What is the probability distribution for the last day of month 2?

We can make a 1 x 6 matrix of the probability distribution for the end of month 1 and multiply it by the transition matrix. The result is the distribution for the end of month 2 (looking forward from the end of month 0).

$$\begin{bmatrix} 0.30 & 0.70 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \end{bmatrix} * \begin{bmatrix} 0.30 & 0.70 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.09 & 0.42 & 0.49 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.03 & 0.19 & 0.44 & 0.34 & 0.00 & 0.00 & 0.00 \\ 0.01 & 0.08 & 0.26 & 0.41 & 0.24 & 0.00 & 0.00 \\ 0.00 & 0.03 & 0.13 & 0.31 & 0.36 & 0.17 & 0.00 \\ 0.00 & 0.01 & 0.06 & 0.19 & 0.32 & 0.30 & 0.12 \\ 0.00 & 0.01 & 0.06 & 0.19 & 0.32 & 0.30 & 0.12 \end{bmatrix}$$

$$= \begin{bmatrix} 0.15 & 0.50 & 0.34 & 0.00 & 0.00 & 0.00 & 0.00 \end{bmatrix}$$

There is a 0.17 chance of 0 dogs, 0.50 chance of 1 dog, and 0.34 chance of 2 dogs. (Probability does not add up to 1 due to rounding.) This has an expected value of 1.19 dogs.

We can multiply again to get the distribution for the end of month 3.

$$\begin{bmatrix} 0.15 & 0.50 & 0.34 & 0.00 & 0.00 & 0.00 & 0.00 \end{bmatrix} * \begin{bmatrix} 0.30 & 0.70 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.09 & 0.42 & 0.49 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.03 & 0.19 & 0.44 & 0.34 & 0.00 & 0.00 & 0.00 \\ 0.01 & 0.08 & 0.26 & 0.41 & 0.24 & 0.00 & 0.00 \\ 0.00 & 0.03 & 0.13 & 0.31 & 0.36 & 0.17 & 0.00 \\ 0.00 & 0.01 & 0.06 & 0.19 & 0.32 & 0.30 & 0.12 \\ 0.00 & 0.01 & 0.06 & 0.19 & 0.32 & 0.30 & 0.12 \end{bmatrix}$$

$$= \begin{bmatrix} 0.10 & 0.38 & 0.40 & 0.12 & 0.00 & 0.00 & 0.00 \end{bmatrix}$$

This has an expected value of 1.533 dogs.

We can keep doing this forever.

month number	number of dogs she might have: 0	1	2	3	4	5	6	expected number
0	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.3000	0.7000	0.0000	0.0000	0.0000	0.0000	0.0000	0.7000
2	0.1530	0.5040	0.3430	0.0000	0.0000	0.0000	0.0000	1.1900
3	0.1005	0.3836	0.3982	0.1176	0.0000	0.0000	0.0000	1.5330
4	0.0764	0.3156	0.3947	0.1850	0.0282	0.0000	0.0000	1.7731
5	0.0635	0.2754	0.3814	0.2203	0.0546	0.0047	0.0000	1.9412
6	0.0561	0.2505	0.3690	0.2392	0.0741	0.0106	0.0006	2.0588

What happens as the number of months approaches infinity? What does this approach?

We want to try to find probabilities that will stay the same from month to month.

$$\begin{bmatrix} a & b & c & d & e & f & g \end{bmatrix} * \begin{bmatrix} 0.30 & 0.70 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.09 & 0.42 & 0.49 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.03 & 0.19 & 0.44 & 0.34 & 0.00 & 0.00 & 0.00 \\ 0.01 & 0.08 & 0.26 & 0.41 & 0.24 & 0.00 & 0.00 \\ 0.00 & 0.03 & 0.13 & 0.31 & 0.36 & 0.17 & 0.00 \\ 0.00 & 0.01 & 0.06 & 0.19 & 0.32 & 0.30 & 0.12 \\ 0.00 & 0.01 & 0.06 & 0.19 & 0.32 & 0.30 & 0.12 \end{bmatrix} = \begin{bmatrix} a & b & c & d & e & f & g \end{bmatrix}$$

This has 7 unknowns. We can make 7 equations.

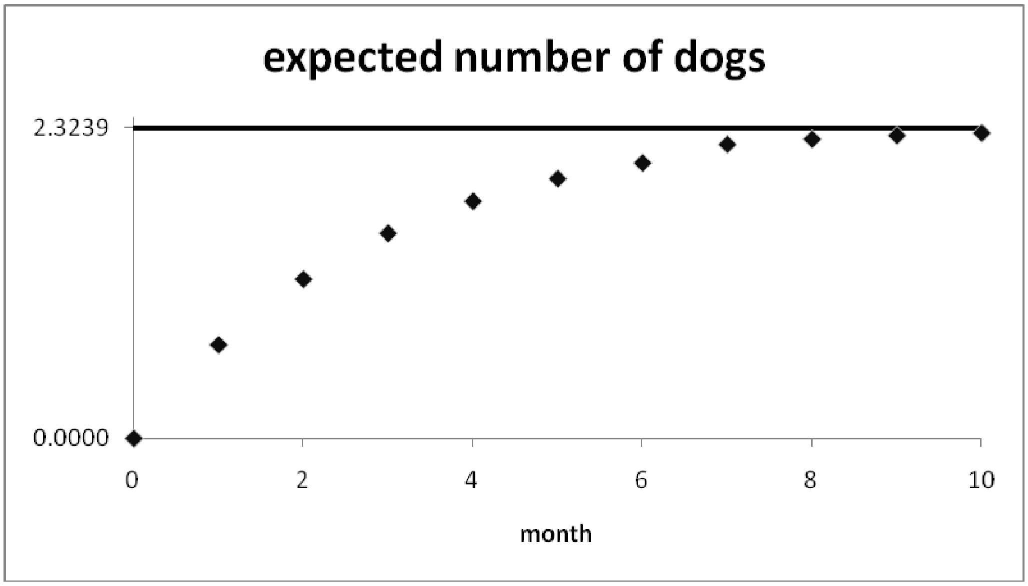
$0.30a + 0.09b + 0.03 + 0.01d + 0.00e + 0.00f + 0.00g = a$ and we subtract a from both sides to get $-0.70a + 0.09b + 0.03c + 0.01d + 0.00e + 0.00f + 0.00g = 0$.

$0.07a + 0.42b + 0.19c + 0.08d + 0.03e + 0.01f + 0.01g = b$ and we subtract b from both sides to get $0.07a - 0.58b + 0.19c + 0.08d + 0.03e + 0.01f + 0.01g = 0$.

We do this for each row. This gives us 7 equations to solve for 7 unknowns. However, each equation is equal to zero, so we need another. Since this is a probability distribution, $a+b+c+d+e+f+g =0$.

Solving them gives us the probability distribution as the month number approaches infinity.

month number	number of dogs she might have: 0	1	2	3	4	5	6	expected number
0	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.3000	0.7000	0.0000	0.0000	0.0000	0.0000	0.0000	0.7000
2	0.1530	0.5040	0.3430	0.0000	0.0000	0.0000	0.0000	1.1900
3	0.1005	0.3836	0.3982	0.1176	0.0000	0.0000	0.0000	1.5330
4	0.0764	0.3156	0.3947	0.1850	0.0282	0.0000	0.0000	1.7731
5	0.0635	0.2754	0.3814	0.2203	0.0546	0.0047	0.0000	1.9412
6	0.0561	0.2505	0.3690	0.2392	0.0741	0.0106	0.0006	2.0588
7	0.0485	0.2240	0.3521	0.2564	0.0972	0.0199	0.0020	2.1976
8	0.0465	0.2169	0.3470	0.2603	0.1037	0.0230	0.0026	2.2369
9	0.0452	0.2121	0.3434	0.2629	0.1081	0.0252	0.0030	2.2640
10	0.0443	0.2089	0.3410	0.2646	0.1112	0.0267	0.0033	2.2827
∞	0.0424	0.2018	0.3354	0.2682	0.1179	0.0302	0.0040	2.3239



Experience with this lesson plan:

I have not had a chance to teach this.

Comments:

This is intended for a college level probability class. The students should have already learned how to work with matrices.

Notes:

I learned Markov chains when I was an undergraduate student. I based this on what I learned when I took a class called “Math for Computers” at Touro College, taught by Professor Meyer Peikes in 1983.

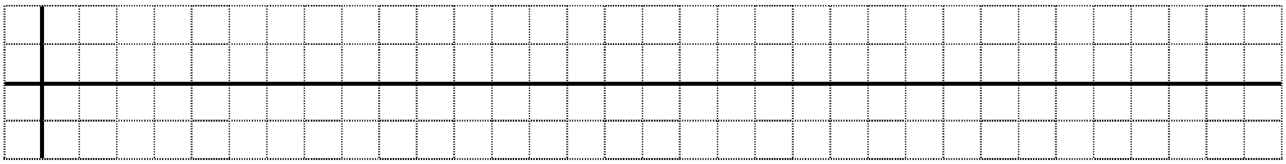
Activity 13: Areas under curves

Activity 1 – Making infinity understandable to young students
Activity 2 – Dividing by zero, part I
Activity 3 – Dividing by zero, part II
Activity 4 – Repeating decimals
Activity 5 – Making infinity with 4 fours
Activity 6 – Infinity and squares
Activity 7 – Angles of shapes with different numbers of sides
Activity 8 – Rational functions
Activity 9 – Compound interest
Activity 10 – Half Life
Activity 11 – Estimating pi
Activity 12 – Markov chains
Activity 13 – Areas under curves
Activity 14 – Infinitely long solids of rotation
Activity 15 – Infinity minus infinity, part I
Activity 16 – Infinity minus infinity, part II
Activity 17 – Discussions about infinity
Activity 18 – Infinity in art
Activity 19 – Writing about infinity

Activity 14:   Infinitely long solids of rotation

Worksheet:

Show part of the graph of the function $f(x) = \frac{1}{x}$ from 1 to infinity.



Rotate this around the x axis to make a solid.

Use integration to find the volume of this solid.

Use integration to find the surface area of this solid.

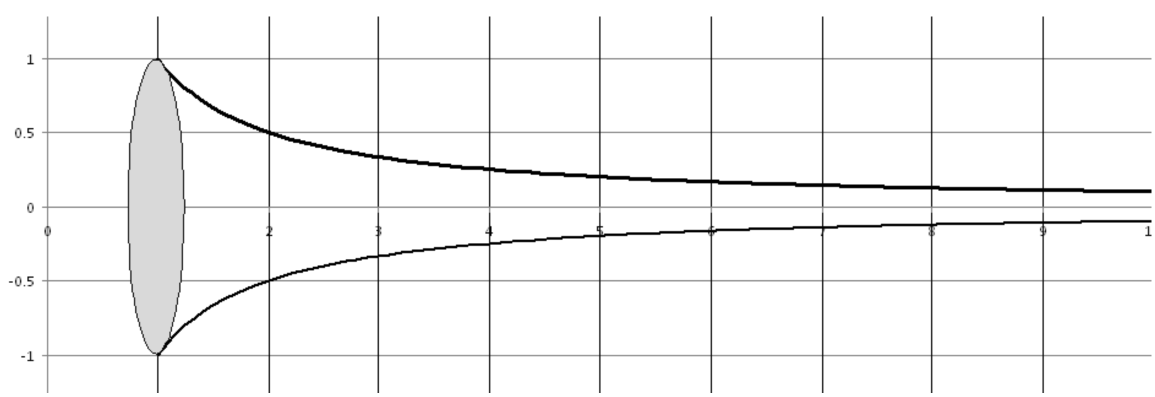
Experience with this:

I used this lesson in a graduate-level math education (MST) class I was taking at Portland State University where each student had to give a math lesson to the class.

Lesson Plan:

Hand out the worksheets. Give the students a few minutes to do the graph themselves. Then graph it on the board to work with, and draw the solid we get if we rotate it around the x axis.

Graph $f(x) = \frac{1}{x}$ from 0 to ∞ and rotate it around the x axis. We get a shape like a trumpet.



Ask: What is the volume of the solid of rotation? Give the students some time to work on it, then go over it together.

The volume of a solid of rotation is $A = \pi\int_a^b f(x)^2\,dx$. If we put this function in, we get

$A = \pi\int_1^\infty \left(\frac{1}{x}\right)^2 dx$. If we integrate, we get $\pi\left[-\frac{1}{x}\right]_1^\infty$, which simplifies to $\pi\left(-\frac{1}{\infty}+\frac{1}{1}\right)$ which becomes π. So even though it is infinitely long, we have a finite volume, since it gets thin very quickly and keeps getting thinner.

Ask: What is the surface area of this solid? This is tricky, have the class do it together.

The surface area of a solid of rotation is $A = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$. If we put the function in, we get $A = 2\pi\int_1^\infty \frac{\sqrt{1+\frac{1}{x^4}}}{x}dx$. This does not fit any of our integration patterns. We can integrate something smaller than this, and realize that our answer must be larger.

Since $1 < \sqrt{1+\frac{1}{x^4}}$ for all values of x above 0, $2\pi\int_1^\infty \frac{\sqrt{1+\frac{1}{x^4}}}{x}dx \ge 2\pi\int_1^\infty \frac{1}{x}dx$. If we integrate the second part, we get $2\pi\int_1^\infty \frac{1}{x}dx = 2\pi[\ln x]_1^\infty = 2\pi[\ln(\infty)-\ln(1)] = 2\pi\ln(\infty) = \infty$. Since the surface area of our shape must be at least this large, the surface area is infinite.

We have a shape with a finite area of pi units, but infinite surface area. If we want to fill this with paint, we need less than 4 cubic units. But if we want to paint the outside, there is not enough paint in the world.

Ask the class if they had any comments on this. The comment I am hoping for is "But if you fill it with paint, doesn't that include painting the surface from the inside?"

Answer: "If we paint the outside, is there a minimum to the thickness of the layer of paint? (Yes) It would have to be at least as thick as a paint molecule. But if we fill the fill the shape with paint, it is not always going to be as thick as a paint molecule. So I guess we are not filling it all the way. We are just filling it to the point where it gets too thin for the paint."

Reflection

This lesson worked well with this group. I realize that it might be a little more difficult for a lower level class, but anyone who is taking a calculus class that would include this should not have much trouble with the ideas.

Yes, a student did ask the question I was hoping for about painting the shape from the inside.

Notes

When I taught at Benson High School, I asked Michael Ball, the teacher who taught AP calculus, about lessons that used infinity, and he suggested this to me.

I got some help with the integration from Wikipedia's page "Gabriel's Horn".

Activity 15: Infinity minus infinity, part I

Used at: An online question and answer website. Someone posted the question and several people posted answers. The asker chose my answer as the best. (I copied her writing exactly.)

Her main question: Why is infinity-infinity infinity?

Her question details:

I suddenly thought of what infinity-(infinity-1) would be.

And I concluded that if you leave the brackets, you get
infinity - infinity = 0
since infinity -1 is still infinity.

But if you open the brackets, you get
infinity - infinity + 1 = 1

But then, I emailed my math teacher (in the middle of the hols!), and he replied, saying that, surprisingly, the answer is infinity.
he said that infinity - infinity= infinity.
How can that be?

I understand infinity is not a number, but isn't anything taking away itself 0?

:/

My answer:

It depends on what you subtract from what.

There are infinitely many integers, infinitely many even integers, and infinitely many odd integers.

If you take all the integers, and subtract all the integers, you have nothing left, or zero. But if you take all the integers, and subtract all the even integers, you are left with all the odd integers, which is infinite.

To make it worse, if you take all the integers and subtract all the integers except 7, 8, 9 you are left with three. You could get any number this way.

Since infinity is not a number, but a concept, you cannot use it like a number.

What math are you taking? When you get to calculus you'll learn how to use limits as numbers approach infinity, which should help you. If you keep going in math, you can learn about different sizes of infinity!

Her response:

great! thanks for all the answers. my math isn't very good, and I didn't read up on this so, yea, I didn't know you can't do that. :D

(What is infinity minus infinity, Yahoo Answers, http://answers.yahoo.com/question/index;_ylt=AghI.dsoa6XE7PO8OE.xdwDty6IX;_ylv=3?qid=20080615103625AADX11v&show=7#profile-info-8I8cJcSLaa)

Activity 16: Infinity minus infinity, part II

Lesson plan:

Hand out a sheet with the following quote:

It is known that there is an infinite number of worlds, but that not every one is inhabited. Therefore, there must be a finite number of inhabited worlds. Any finite number divided by infinity is as near to nothing as makes no odds, so if every planet in the Universe has a population of zero then the entire population of the Universe must also be zero, and any people you may actually meet from time to time are merely the products of a deranged imagination.

- -Douglas Adams, The Hitchhiker's Guide to the Galaxy radio show (Quoteland, http://www.quoteland.com/search.asp)

Talk to the students. (First tell them that this is certainly not coming from an astronomy lesson, that the author does not mean this to be taken seriously, and that we don't know that there are infinitely many worlds to begin with.)

First just ask the students for any comments on the quote.

Ask questions:

- If there are infinitely many worlds, and some are not inhabited, does that mean that only a finite number are inhabited? Suppose there are infinitely many worlds and five are not inhabited. That leaves infinitely many that are inhabited.

- Suppose there really are infinitely many worlds.   If there are infinitely many that are uninhabited, would that mean that the number of inhabited worlds had to be finite? Or could they both be infinite? There are infinitely many counting numbers, with infinitely many even numbers and infinitely many odd numbers.
- If the universe was infinite, and the amount of people finite, would the population density of the universe be zero? No, but it would approach zero.
- If the universe was infinite, and the amount of people finite, could you have places in the universe where the population density does not approach zero? Yes, since the people would not be evenly spread out throughout the universe, but would be concentrated on planets that support life.

Experience with this:

I used this twice for a session of the after school math club at Benson High School in Portland, OR, once each year. I also used it at Deering School, during the free math elective.

Reflection:

The students in the math club had good ideas on this, and we had interesting discussions. In Deering School, the students seemed more confused. I think the difference is that the math club had a self-selected group of students who were more interested in the subject. In Deering School, the elective consisted mostly of students who did not want to take gym.

Activity 17:   Discussions about infinity

Activity 1 – Making infinity understandable to young students
Activity 2 – Dividing by zero, part I
Activity 3 – Dividing by zero, part II
Activity 4 – Repeating decimals
Activity 5 – Making infinity with 4 fours
Activity 6 – Infinity and squares
Activity 7 – Angles of shapes with different numbers of sides
Activity 8 – Rational functions
Activity 9 – Compound interest
Activity 10 – Half Life
Activity 11 – Estimating pi
Activity 12 – Markov chains
Activity 13 – Areas under curves
Activity 14 – Infinitely long solids of rotation
Activity 15 – Infinity minus infinity, part I
Activity 16 – Infinity minus infinity, part II
Activity 17 – Discussions about infinity
Activity 18 – Infinity in art
Activity 19 – Writing about infinity

Activity 18: Infinity in art

Infinity in art

Lesson plan:

I would like to ask students in an art class to bring in examples of pictures that give them a feeling an infinity. These are some very different ones that do for me.

Starry Night by Vincent Van Gogh

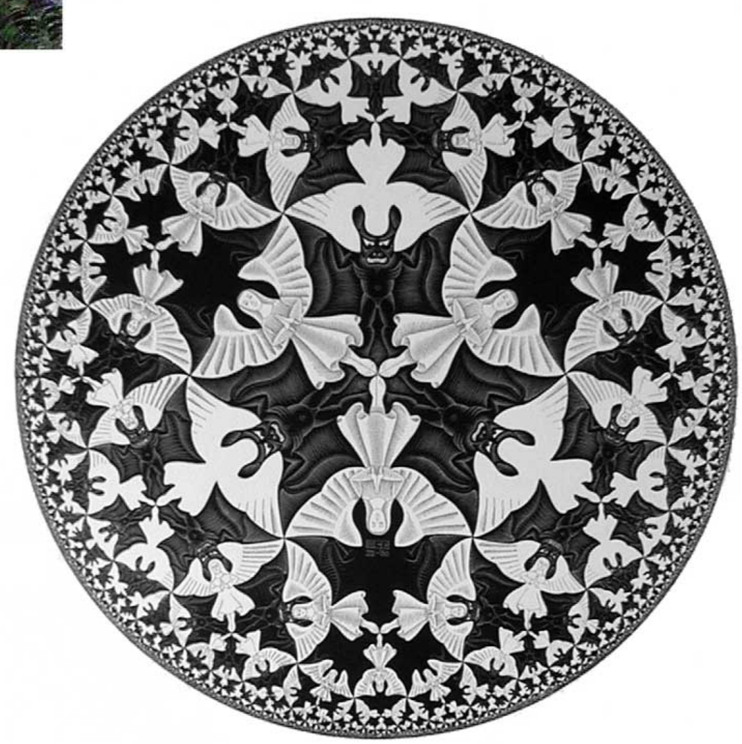
(photo from

http://www.vangoghgallery.com/painting/starryindex.html)

Circle Limit by M. C. Escher

(photo from

http://www.worldofescher.com/gallery/A8.html)



I asked some art students outside of their class. A few said Starry Night, one I had already picked myself. Most could not give me an answer. One said "The Creuse, Sunset" by Claude Oscar Monet (picture from Classic Reproductions,

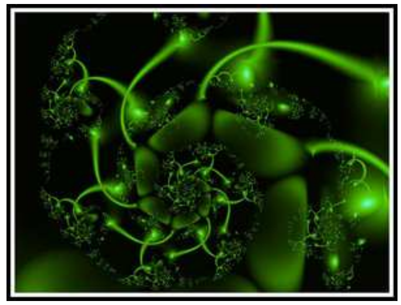
http://www.classicartrepro.com/ready_to_ship.iml?painting=123)

She explained "I just imagine turning that corner, and finding more and more forever, and I'm just riding down the never ending river, admiring the sunset."

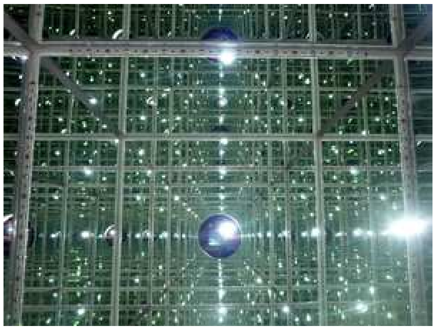
A project for an art class would be to make a picture expressing infinity. The students would not have to state what aspects of infinity it makes them think of or justify their art.

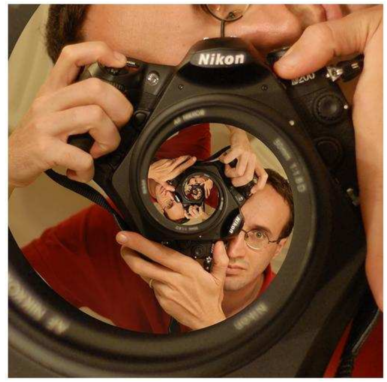
Here is my art: ("Paint" by Amy Whinston)

Here are some examples I've found on the internet, whose artists categorized them by tagging them "infinity".

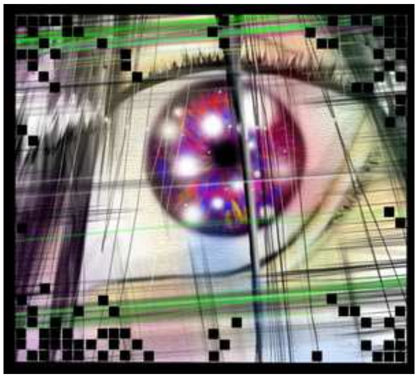
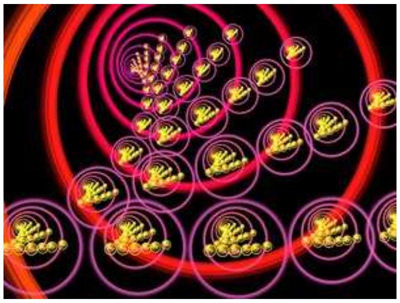


- "Infinity" by Imaginary Creature.   (http://www.deviantart.com/#)
- "Infinity" by Illusion Island. (http://www.deviantart.com/#)
- "102 - Infinity" by Dragonfly113. (http://dragonfly113.deviantart.com/art/102-Infinity-89394312com/)





- (http://www.deviantart.com/#)
- “Nikon Spiral” by Seb Przd. (http://www.flickr.com/photos/Sbprzd)
- “Today We Escape” by Manu. The artist said he was inspired by “Nikon spiral” to the right. (http://www.flickr.com/photos/Manuperez)

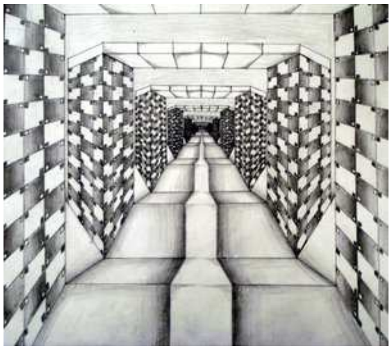
- (http://www.deviantart.com/#)
- (http://www.deviantart.com/#)
- “Birth of Infinity” by Agemosu. Artist’s comment: “It eventually turned into this, light competing with dark over dominance as everything swirls in and expands outward. I think it might be what the big bang could've looked like in one of those parallel dimensions they advertise on QVC.” (http://agemosu.deviantart.com/art/Birth-of-Infinity-12140580)



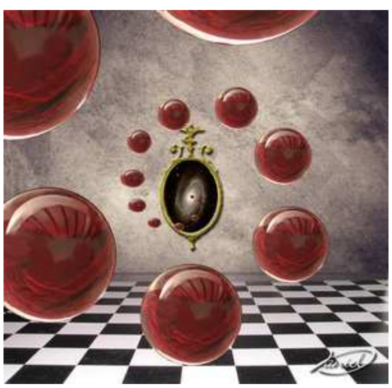
- “COLOUR INFINITY” by Geaannunziata (http://geaannunziata.deviantart.com/art/COLOUR-INFINITY-72197275)
- “Sea and Storm” by Patrick Smith. Carmel by the Sea, CA. (http://www.flickr.com/photos/patrick-smith-photography/3182878595/)
- “End of Infinity” by Corvidae 65 Author’s comment: “Did you know at the end of infinity there was a bunny?” (http://corvidae65.deviantart.com/art/End-Of-Infinity-90704824)



- “Infinity” by Azarius. **(**www.flickr.com/photos/azariusrex/2264127612/)
- “Reflected Infinity” by DeDaniel. (www.flickr.com/photos/26432908@N00/2172443230/)
- “Infinity Docs Mac” by Js1stuff. This is the same idea as mine. He used a Mac, while I used a PC (www.flickr.com/photos/js1stuff/2734548861/)

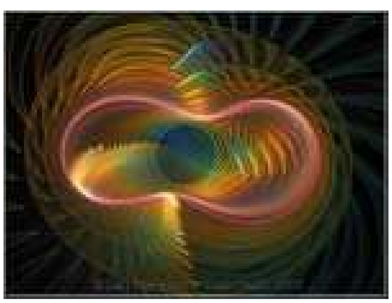
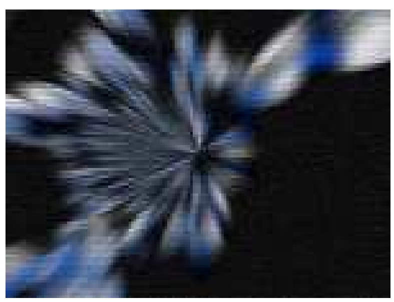
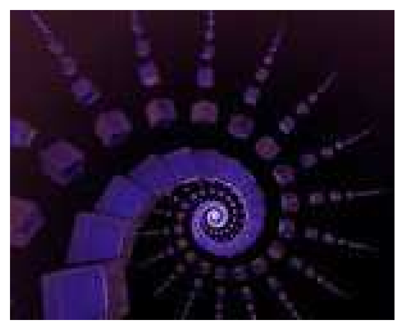


- “Infinity” by Keylley. Artist’s comment: “This took me aaaaggggggggeeeeeesssss to complete it. was literally ‘infinity’.” (http://keylly.deviantart.com/art/infinity-65162207)
- “Infinity” by Gabriele Danes (http://gabrieledanes. deviantart. com/art/Infinity-104469686)
- “Crossing Infinity” by dicalva (http://dicalva.deviantart.com/art/crossing-infinity-74537674)

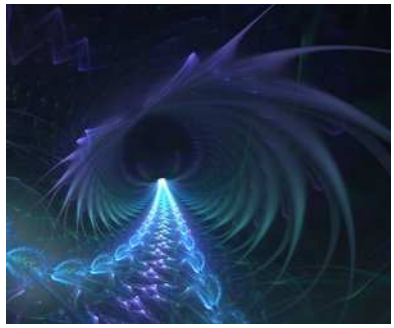


- “Infinity” by dehouse42 (http://dehouse42.deviantart.com/art/Infinity-56239676)
- “Infinity” by Sanjab (http://sanjab.deviantart.com/art/Infinity-59561009)
- “Infinity” by tombstone (http://tombst0ne.deviantart.com/art/infinity-22924901)

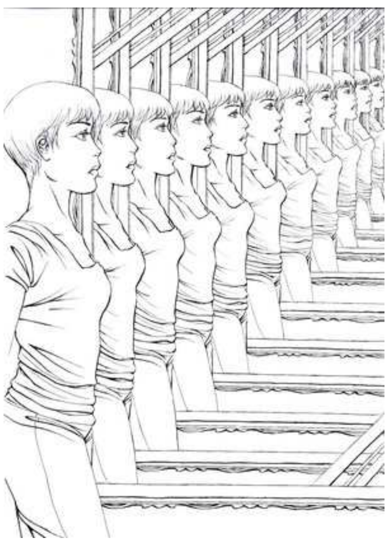

- "Infinity" by Tru Z. (http://truzt.deviantart.com/art/Infinity-18029303)
- "To Infinity …and beyond" by Si Du Bexter. (http://sidiusbexter.deviantart.com/art/To-Infinity-and-Beyond-64584789)
- "Infinity atmosphere" by Ichaea. (http://ichnaea.deviantart.com/art/Infinity-atmosphere-100146986)

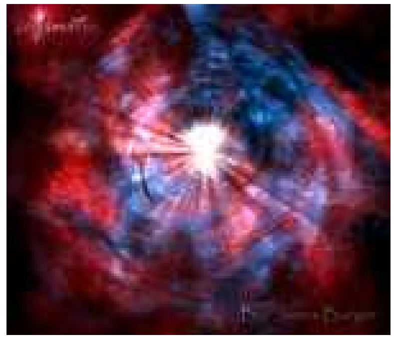
- Infinity by Booleann Angel. (http://www.deviantart.com/#)
- "Infinity" by Bush 0. (http://bush0.deviantart.com/art/infinity-75955693)
- "For Infinity" by Inter Light. (http://interlight.deviantart.com/)

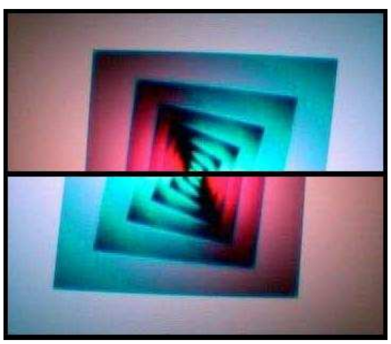
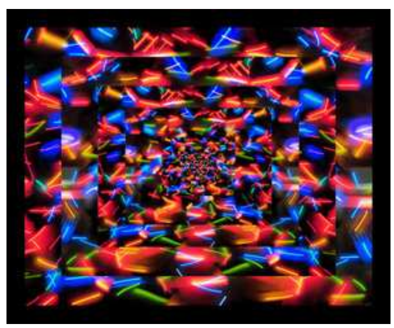
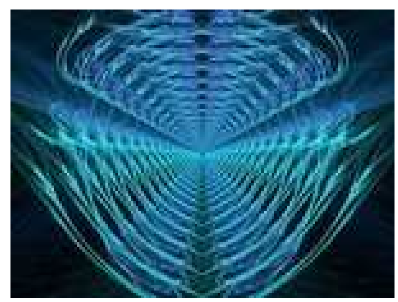
- "Infinity" by nullmorphem. (http://nullmorphem.deviantart.com/gallery/)
- "The Road to Infinity" by Nightmares 06. (http://nightmares06.deviantart.com/art/The-Road-to-Infinity-79093971)
- "Infinity" by Without Remorse. Artist's comment: "To infinity and beyond!" (http://withoutremorse.deviantart.com/art/Infinity-16262258)

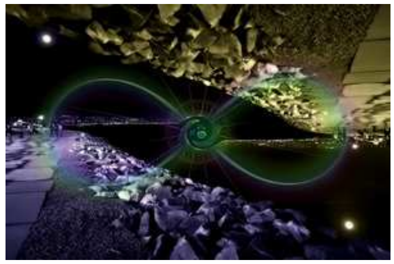
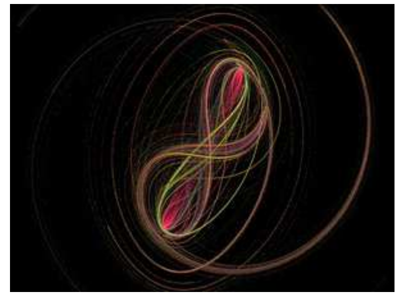
- Curlique. (http://mushroomcurlique.deviantart.com/art/Infinity-112336740)
- "Infinity" by J Brown 500. (http://jbrown500.deviantart.com/art/Infinity-27600915)
- "Infinity" by Widerkizer. (http://wizerkizer.deviantart.com/art/Infinity-68478194)


- “Infinity” by fanatic-alien. It's a lightning conductor in the tourist centre in Russia (http://fanatic-alien.deviantart.com/art/infinity-32398476)
- “Infinity” by J P Burger. (http://jpburger.deviantart.com/art/Infinity-16402252)
- “Infinity” by Lord Eth. Artist’s comment: “Does this pathway ever end? Or am I just traveling through infinity?” (http://lordieth.deviantart.com/art/Infinity-22291269)

- “Infinity” by Lord Penguin. (http://lordpenguin.deviantart.com/art/Infinity-23805355)
- “Infinity” by Mr. Parts. Artist’s comment: “Just keeps on going …” (http://mrparts.deviantart.com/art/Infinity-48526725)
- “Infinity” by Pobeli. Artist’s comment: It looks kind of like some ladders to me. They go forever. Maybe it's a ribcage though. (http://pobeli.deviantart.com/art/Infinity-24103329)

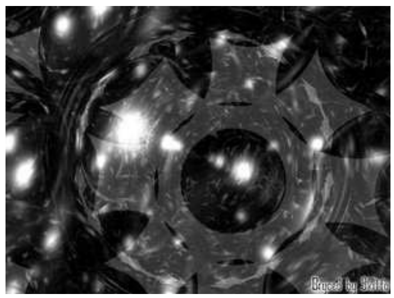
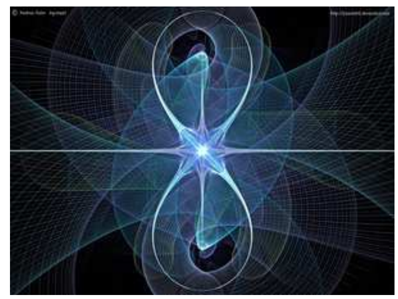
- “Infinity” by M-016. Artist’s comment: “This is the first work of mine to use a technique to create a Jupiter-like gas giant. I have also stepped back from a lot of bright stars, which seems to create more depth. This spawned the title ‘Infinity’.” (http://m-016.deviantart.com/art/Infinity-33393512)
- “Infinity” by Pamukkale. (http://pamukkale.deviantart.com/art/Infinity-72250514)
- “Infinity” by psychotri33. Artist’s comment: “Through all the twists and turns...time goes on...never failing....never dying...never forming.....just warping and twisting.... “ (http://psychotri33.deviantart.com/art/Infinity-54767210)



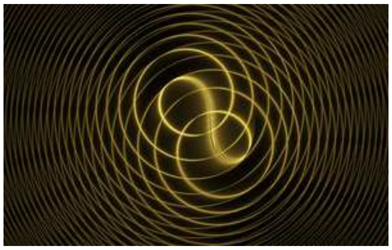
- “Trinity Infinity” by Sequential. (http://sequential.deviantart.com/art/Trinity-Infinity-38848792)
- “Touch of Infinity” by Gucken. Artist’s comment: “Oh, and, those spaceships.. they are made out of an highly reflective material, and they also have their own light somewhere, so there’s no false lightning.” (http://gucken.deviantart.com/art/Touch-of-Infinity-70320801)
- “To Infinity” by Tom Bailey 16. (http://tombailey16.deviantart.com/art/To-Infinity-81897291)

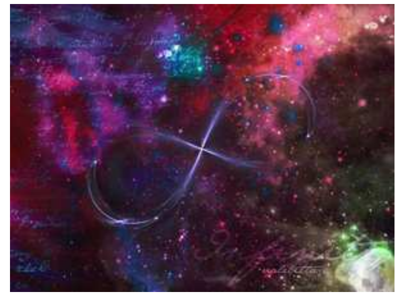


- “Map to Infinity” by Lizard Jedi. (http://lizardjedi.deviantart.com/art/Map-to-Infinity-10752063)
- “Into Infinity” by Ratafluke. (http://ratafluke.deviantart.com/art/Into-infinity-24043831)
- “Into Infinity” by MiseryIndex. (http://miseryindex.deviantart.com/art/Into-Infinity-71417851)

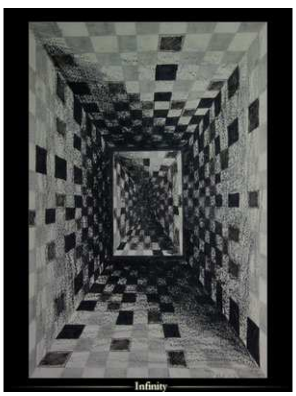




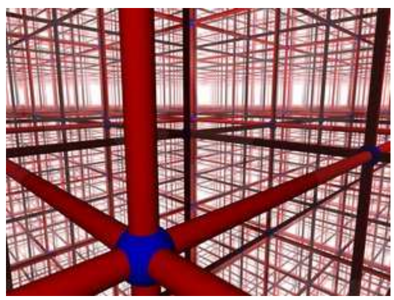
- “Infinity” by dilekt. Artist’s comment: “The setting takes place on an alien world somewhere out there in space, you imagine.” (http://dilekt.deviantart.com/art/Infinity-10031080)
- “Infinity Spheres” by Skitto. Artist’s comment: “Gaze into infinity with 1000x zoom” (http://skitto.deviantart.com/art/Infinity-Spheres-10721672)
- “Infinity Inc.” by Psion 005. (http://psion005.deviantart.com/art/Infinity-Inc-54384713)



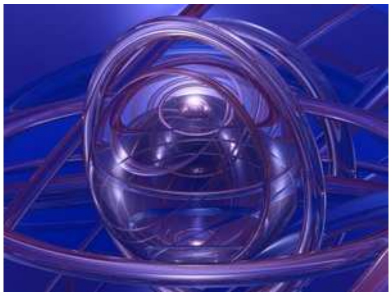


- “Infinity” by Thadius 33. Artist’s comment: “Bubble and swirl…” (http://thadius33.deviantart.com/art/Infinity-68047740)
- “Infinity” by Svarci. Stairs in a hotel, St. Paul's Bay, Malta. (http://svarci.deviantart.com/art/Infinity-42316030)
- “Infinity” by Stormy Night 83. (http://stormynight83.deviantart.com/art/Infinity-83256576)






- “Infinity” by Sphereuk. (http://sphereuk.deviantart.com/art/Infinity-20900611)
- “Infinity” by Roxor 128. (http://roxor128.deviantart.com/art/Infinity-46220373)
- “Infinity” by RoadCaesar. (http://roadcaesar.deviantart.com/art/infinity-49759591)

- “Infinity” by Road Buster. (http://roadbuster.deviantart.com/gallery/#)
- “Column of Infinity” by Kazumi Alex. Targu-Jiu, Romania. (http://kazumi-alex.deviantart.com/art/infinity-84231806)
- “Infinity” by Gaelic. Artist’s comment: “This is perhaps one of the most amazing images in the night sky. This image of the deer lick group contains no fewer than ten different galaxies that I could count, all of varying sizes and distances, representing an unbelievably vast area of space. Truly incredible.” (http://gaelic.deviantart.com/art/Infinity-40892849)

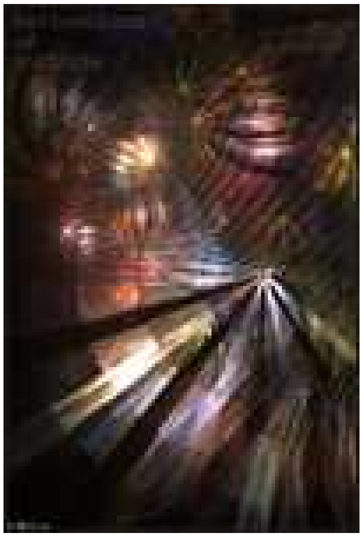
- “Infinity” by VL Web 3D. (http://vlweb3d.deviantart.com/art/Infinity-91395924)
- “Roars of Infinity” by Einhjar. (http://www.deviantart.com/#)
- “Lost in Infinity” by Cyber Angel 8. (http://cyberangel8.deviantart.com/art/Lost-In-Infinity-119016206)

- “Infinity” by Vlad I Mirovic. (http://vladimirovic.deviantart.com/art/Infinity-41604369)
- (http://www.deviantart.com/#)
- “Infinity” by rocamiadesign. (http://rocamiadesign.deviantart.com/art/Infinity-85769839)

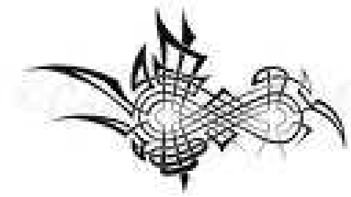
- “Infinity” by thebullfrog Endless sea of colors and faces, during a ceremony held in a small village next to Jaipur, India (http://thebullfrog.deviantart.com/art/Infinity-50273045)
- “Belgium – Infinity II” by lux69aeterna Artist’s comment: The feeling of infinity along the coast, by a pier which is lost in time... Belgium. (http://lux69aeterna.deviantart.com/art/Belgium-Infinity-II-77085316)
- “Stairway to infinity” by foureyes.   Downtown Houston (http://foureyes.deviantart.com/art/Stairway-to-infinity-2042693)

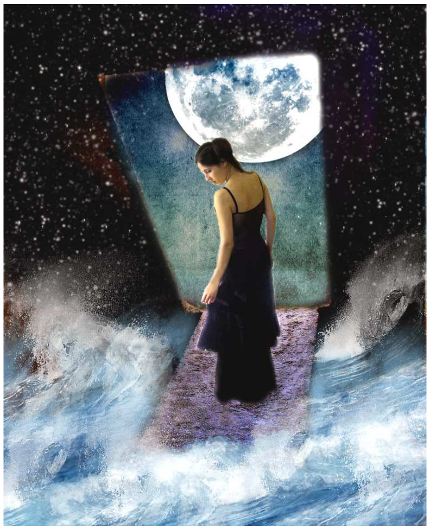
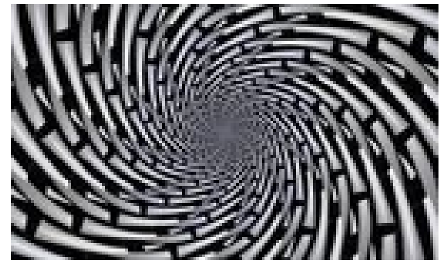
- “Infinity” by FrozenStarRo. Artist’s comment: “This was originally created to look like a dream, I mean who hasn't dreamt of flying like at least once in ther entire life? And that's kinda what I tried to portray with this.” (http://frozenstarro.deviantart.com/art/Infinity-88632210)
- “Infinity” by fgoellner (http://fgoellner.deviantart.com/art/Infinity-84003660)
- “Infinity” by Blassed (http://blassed.deviantart.com/art/Infinity-5267422)

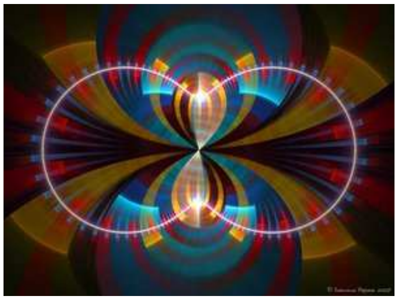
- “Reflections of Infinity” by IDeviant (http://ideviant.deviantart.com/art/Reflections-of-infinity-33595973)
- “INFINITY” by gilad. Arlozorov train station, Israel. (http://gilad.deviantart.com/art/INFINITY-6430290)
- “Into Infinity” by gilad Running through the inside court of the Jerusalem high supreme court, Israel. (http://gilad.deviantart.com/art/Into-Infinity-83877319)



- “Infinity” by GrayAliEN. My comment: If you see it on the computer screen it grows and seems to be coming at you, so more and more mice come at you. (http://grayalien.deviantart.com/art/Infinity-28296717)
- “Infinity” by GravityGlitch. Artist’s comment: This is a tattoo design for a friend of a friend, from what I know she has a tattoo that she would like to be more complex. (http://gravityglitch.deviantart.com/art/Infinity-98542566)
- “To Infinity” by herbstkind. düsseldorf, germany 2006 (http://herbstkind.deviantart.com/art/to-infinity-39266278)

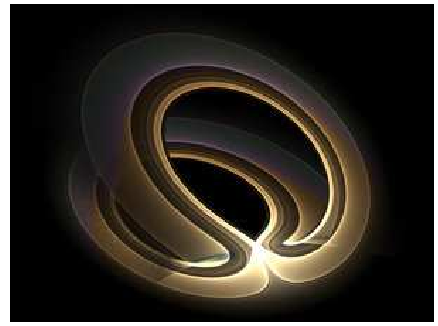
- “Inner Infinity” by HiddenCaitastrophe. Artist’s comment: I learned to love the infinite stars and the crashing of the waves. / But now, it seems I crave a different sky and firm earth beneath my feet. / Now, it seems I will open that door and find a world unknown to me. (http://hiddencaitastrophe.deviantart.com/art/Inner-Infinity-58407688)
- “Infinity” by bloodied-hollow-soul. (http://bloodied-hollow-soul.deviantart.com/art/Infinity-96489549)
- Apo-Infinity by R C Page (http://rcpage.deviantart.com/art/Apo-Infinity-115503828)



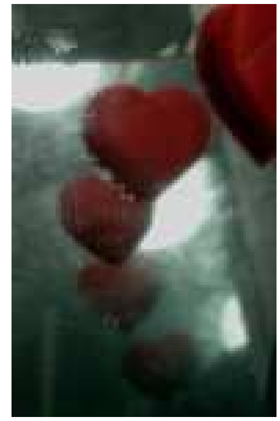
- "Infinity" by aerphis. (http://aerphis.deviantart.com/art/Infinity-59654431)
- "Infinity pt III" by Photogramme. (http://photogramme.deviantart.com/art/Infinity-pt-III-68019425)
- "Infinity" by ogg hunter. Artist's comment: "Interesting. I have discovered infinity!" (http://ogghunter.deviantart.com/art/Infinity-83762796)

- “Infinity and Beyond” by ayamteksa. (http://ayamteksa.deviantart.com/art/Infinity-and-beyond-111110981)
- “Infinity” by j14v6. Artist’s comment: “*‘To see a World in a Grain of Sand / And a Heaven in a Wild Flower, / Hold Infinity in the palm of your hand / And Eternity in an hour.’*/ - William Blake” (http://j14v6.deviantart.com/art/Infinity-68605874)
- “Infinity” by Lonely Enigma. (http://lonely-enigma.deviantart.com/art/Infinity-121779744)



- “Infinity” by Ian Weller. Artist’s comment: “Infinity is a really fun thing to attempt to grasp, especially in the wonderful world of fractals. Fractals are, in a very basic sense, taken out to infinity. But how much is infinity? To the renderer, infinity is finite, so it is able to grasp it.” (http://ianweller.deviantart.com/art/Infinity-72974140)
- “Looking into Infinity” by Gabriella Xavier. (http://gabriella-xavier.deviantart.com/art/look-at-infinity-110113615)
- “Bent Infinity” by Jaedle. Artist’s comment: “I had completely forgotten that I had folded up infinity to get it out of my way for a bit, and left it on the windowsill. By the time I found it, it had begun to corrode. It still has a bit of shine, though.” (http://jaedle.deviantart.com/art/Bent-Infinity-86035012)

- “Infinity” by lennart. Artist’s comments: I wanted to make a work you'd feel unhappy holding and watching it. That's why I came up with this concept of the "droste effect" which could go on outside the picture. But if it would, it would mean the spectator should mutitlate itself. ^_^ I know, a little farfetched, but it was fun to do. (http://lennart.deviantart.com/art/Infinity-32719102)
- “Infinity” by On The Wall. (http://onthewall.deviantart.com/art/Infinity-28803570)
- “Love Infinity II” by Rock Gem. Artist’s comment: “A random heart shape and a pair of mirrors decided to provide some artistic entertainment for me to create this” (http://rockgem.deviantart.com/art/Love-Infinity-II-106774064)

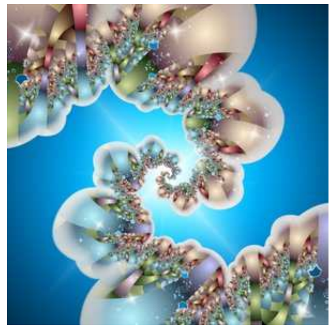
  

- “Infinity” by Lew Rosenberg. Artist’s comment: “Infinity ongoing iteration between masculine and feminine elements in nature.” (http://lew-rosenberg.deviantart.com/art/Infinity-57041997)
- “Rails to Infinity” by Bian Bian. (http://bian-bian.deviantart.com/art/Rails-to-Infinity-84775006)
- “Infinity Cove” by japaslavian. Berkeley, CA. (http://japaslavian.deviantart.com/art/Infinity-Cove-85182763)

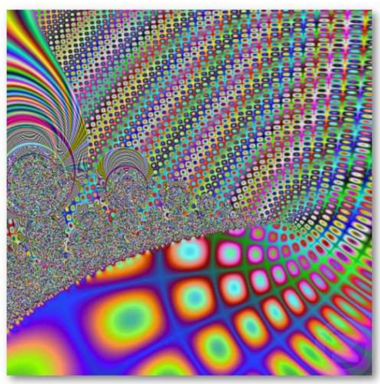
- “Beast of Infinity” by Edalbsanatak.   My comment: If he swallows his tail, and keeps swallowing, then what happens? (http://edalbsanatak.deviantart.com/art/Beast-of-Infinity-94992239)
- “To Infinity and Beyond” by Intricate Illusion. (http://intricate-illusion.deviantart.com/art/to-infinity-and-beyond-98339048)
- “Infinity of the Tao” by Helen Baq (http://helen-baq.deviantart.com/art/Infinity-of-the-Tao-63593845)

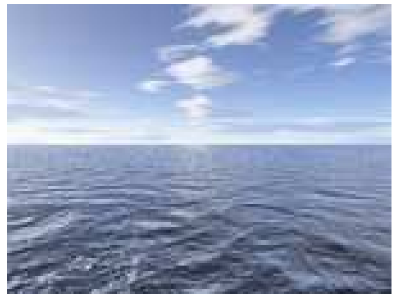
- "A Glimpse Into Infinity" by Blackmage9. Artist's comment: "I did this during an on-the-spot poster making contest..And I landed 1st place with it..^__^" (http://blackmage9.deviantart.com/art/Science-A-Glimpse-to-Infinity-4772092)
- "Infinity is the Word" by tasawa69. Artist's comment: This is one of two towers for the new luxurious condominiums know as The Infinity. The property is located in the Embarcadero area of San Francisco. (http://www.flickr.com/photos/cordell28/3412994491/)
- "Doorway to Infinity" by Mr. Mark. (http://www.flickr.com/photos/mark_boucher/116767393/)


- “Fractal Dreams” by Nullmorphem. (http://nullmorphem.deviantart.com/art/Fractal-Dreams-46151935)
- “Fractal Dreams” by Magnusti78. (http://magnusti78.deviantart.com/art/Fractal-Dreams-76121907)
- “Damn, What Floor Are We On?” by Josh Sommers. (http://www.flickr.com/photos/joshsommers/357909166/)


- “Entrance to Infinity” (zazzle.com)
- “Infinity” by Squidoo. (http://www.squidoo.com/SuperFantastic)
- “Plane of the Infinite Infinities” by Michael Bonnell (zazzle.com)



- “Eternity” by V L Web 3D (http://vlweb3d.deviantart.com/art/Eternity-90638474)
- “Infinity” by Geoff 123. Artist’s comment: “One thing that goes on forever: the ocean.” (http://geoff123.deviantart.com/art/Infinity-14309031)
- “Infinity 3” by Geoff 123. Artist’s comment: “Yet ANOTHER thing that goes on.....and on........and on. The mountains.” (http://geoff123.deviantart.com/art/Infinity-3-14340824)

Comment:

I was never in a position to test this.

Note:

I did not plan to include so many. I found it fascinating how many different pictures there are; not the number of pictures, but how different they are from each other. This is one reason I find infinity so interesting; it can be taken so many different ways.

If the number of pictures I have included seems excessive, consider that I left out the other 5,000 plus pictures I looked at.

Activity 19: Writing about infinity

This could be a lesson for a writing class.

Lesson plan:

1. Ask the students if they can mention a piece of writing that makes them think of infinity.
2. Have them write something that would make someone think of infinity.

Here is a poem I would mention for a piece of writing I read.

Great fleas have little fleas upon their backs to bite ‘em,

And little fleas have lesser fleas, and so ad infinitum.

And great fleas, themselves, in turn, have greater fleas to go on,

While these again have greater still, and greater still, and so on.

- Augustus de Morgan (Wells, Mathematics, p. 179)

Another well known poem by William Blake:

To see a World in a Grain of Sand,
And a Heaven in a Wild Flower,
Hold Infinity in the palm of your hand
And Eternity in an hour.
(Barrow, p. 13)

Another, less well known piece of writing was the winning entry in a British writing contest. The contest was to write on the topic “What would you most like to read on opening the morning newspaper?”

Our Second Competition

The First Prize in the second of this year’s competitions goes to Mr. Arthur Robinson, whose witty entry was easily the best of those we received. His choice of what he would like to read on opening his paper was headed “Our Second Competition” and was as follows: “The First Prize in the second of this year’s competition goes to Mr. Arthur Robinson, whose witty entry was easily the best of those we received. His choice of what he would like to read on opening his paper was headed “Our Second Competition,” but owing to paper restrictions we cannot print all of it.

(Gardner, p. 318)

And this is from the book Lila by Robert Pirsig:

He watched her for a long time and she knew that he was watching her and he knew that she knew he was watching her, and he knew that she knew that he knew; in a kind of regression of image that you get when two mirrors face each other and the images go on and on and on in some kind of infinity.

(Barrow, p. 12)

This is what I wrote:

"Escher" by Amy Whinston

Mary scowled. She had been trying to write her story for a while and was having trouble. She had decided that the main character in her story, Charles, was a writer and was writing a book. But that was all she had decided so far. With a sigh, Mary picked up her pen and began a rough draft:

He opened his laptop and faced a blank Word page. Charles decided to type and see what came to him. He had always been advised to "write what you know", so he would write about someone writing:

Eric was working on his story, Charles wrote, and was coming along slowly. Eric's main character, Mary, was a writer working on her story but having a difficult time. Eric had decided to make Mary rather old-fashioned. In keeping with this, he had Mary writing with a pen and then using a typewriter:

She was doing better. Mary had finally managed to get something down on paper. She decided that Charles would be writing a book about a writer named Eric.

Mary had not written much yet, but things were going smoothly. Mary decided to give Charles a case of writer's block:

Charles took the pencil out of his mouth. His mother had told him that the yellow paint had lead, that he should not chew on pencils. He did it anyway when he could not think of what to write. He just could not decide what Eric should do.

Eric felt like he was lacking any direction in life. He was trying to write about Mary, but just was getting nowhere. He decided to sit down at the computer and just write anything at all. It would be better than nothing, no matter what he wrote. He would just write whatever came to him.

Things seemed to be going really strangely lately, Mary thought. She just felt like nothing made sense in her life anymore….

Here are some other writings I found where the writers had given them the tag of infinity.

"Infinity" by Fallen Midnight Stars

My love is infinity,

Even all the eternity,

Wouldn't be able to keep up

With this overflowing feelings.

I wonder if your love to me,

Is everlasting,

Neverchanging,

Infinity too?

Is that why you kept saying those 3 words,

I love you,

Everytime you could,

Just to make me blush,

And realize?

(http://fallenmidnightstars.deviantart.com/art/INFINITY-102683076)

“Infinity” by Drool in Terror

Infinity's a lonely place

Where shapeless clouds stitch countless silver seams,

Like many needles of eclipse

And turn to night all hopes and dreams.

Infinity can be a state of mind where one,

With swiftly unclosed eyes

Can witness shocking truths; and chance upon

Those many, painful, unchanged lies.

Perhaps infinity has voice

That knocks and creaks and howls,

A voice so fierce and fear-provoking that

It's heard from sky to earth's deep bowels.

Or maybe it is endless colour

A sea of ever-changing faces,

Like a circus disappearing act

They appear in different places.

But I believe infinity is more

Than a place where spirits go,

Perhaps we'll wander there some day.

Perhaps, we'll never know...

writer’s comment: I joined the writer's club at school, and our first assignment was to choose two words: one that represented ourselves and one that we just liked. (I chose vampire and mastication**, but that's besides the point ^^) Anyway, after we wrote those words, we had to exchange them with someone else. So I got "infinity" and "uncomfortable". After this, we were told to write a poetic piece, either including or describing these two words. It didn't have to rhyme, but it couldn't be a prose. So this is the result. I decided not to use the word "uncomfortable", but I made the atmosphere "uncomfortable" instead. (At least I tried to.) Also, this is one of my first true attempts at rhyming...

(http://drool-in-terror.deviantart.com/art/Infinity-37384213)

“Tale of Infinity” by Haro Rioko/Shiana Nunn

Infinity, my love

The future, awaits

The sun is dull

You are my light

Colours are plain

You are vibrant

Art is unattractive

You are a masterpiece

A story.

Sometimes real, other times not

A fantasy.

Something we wish to happen

But what if they were to combine?

Add, two lovers? A dream? A memory? A setting? A plot? Problem? Solution? A thought? A wish? A want? A need? Events? Celebrations? Moments? Emotions? Senses? A breath? A heartbeat? Something lost? Something found? Night? Day? Time? Questions? Answers? Body? Mind? Soul? A name or two?

They would make a beginning, middle, but never an end

Two paths bound to intertwine

Together a path leading to Infinity

A connection meant to last for Eternity

You *are* my love

And what I feel for you,

Is something far greater...

Haro Rioko/Shiana Nunn

Author's comment: "Infinity! Hahh! Wonderful. My life practically revolves around it."

(http://axroh-infinity.deviantart.com/art/Tale-Of-Infinity-79841813)

"The Universe" by whoa

bang,spew,forever expanding

the end

(answers.yahoo.com)

Comment:

I have never had an opportunity to try this lesson.

Overview of Presentation

Is it possible to add together infinitely many positive numbers and get a finite sum?

Yes. An example: Suppose you have a cake. Today you eat half the cake. Tomorrow you eat half of what is left. The day after you eat half of what is left that day. The day after, you eat half of what is left that day. And so on. You are getting closer and closer to having eaten one whole cake. You can add up the fraction of the cake you eat each day. The first day you eat $\frac{1}{2}$ a cake. The next day it is half of the half that is left, so it is $\frac{1}{4}$ of a cake. The next day you eat $\frac{1}{8}$ and the day after that is $\frac{1}{16}$. We can add these up: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}\cdots$ When you add up a series of numbers where there is a common ratio between each term and the next, it is a geometric series. The sum of an infinite geometric series is $\frac{first-term}{1-ratio}$. The common ratio here is $\frac{1}{2}$, so the sum is $\frac{\frac{1}{2}}{1-\frac{1}{2}}$, which equals 1, and we are approaching 1 whole cake.

The probability that my softball team will win any particular game is 0.6. We are playing 5 games this season. Can I find the probability we will win exactly 4 games? Yes. This is the binomial distribution. The probability we win 4 out of 5 games is $\binom{5}{4}0.6^4\,0.4^1$ which equals 0.010. This is an easy situation to work with; there are 6 possible outcomes. We could win 0, 1, 2, 3, 4, or 5 games. We can figure out the probability of each, which should total 1 if we add them together.

# of games	probability
0	0.010
1	0.077
2	0.230
3	0.346
4	0.259
5	0.078

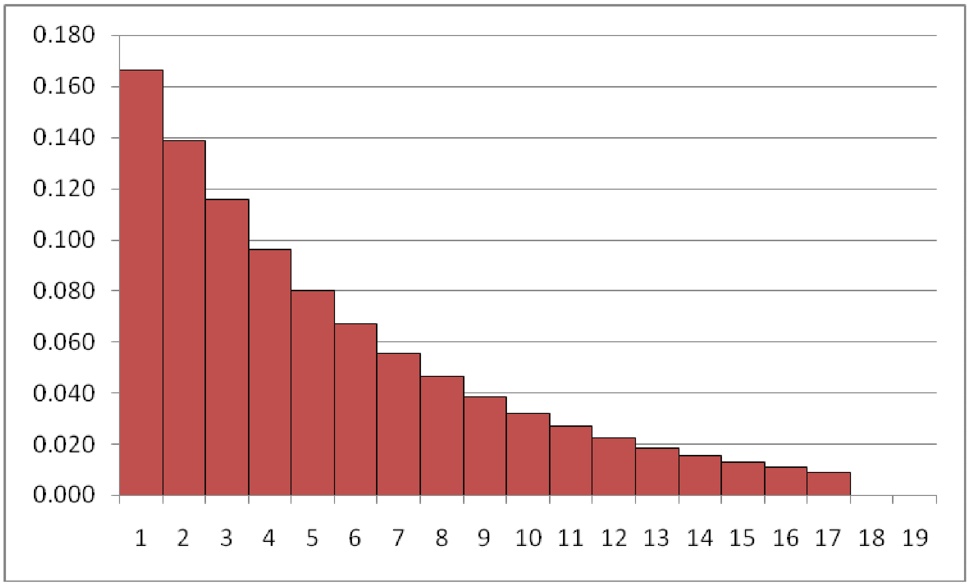
But suppose we have infinite possibilities. Can we do the same?

That cake you are eating – Suppose before I baked it, I put a tiny bit of poison that will kill you as soon as you eat it, but I don't know where in the cake it is. What is the probability you would die on the first day? It is equal to the part of the cake you eat that day. So the probability you would die on the first day is $\frac{1}{2}$, the probability you die the second day is $\frac{1}{4}$, the probability you die the third day is $\frac{1}{8}$, and so on. We saw before that these total 1.

I will roll a die until I roll a 5. This is the Geometric Distribution. What is the probability I roll it only once? That means I roll a 5 on the first try, and the probability is $\frac{1}{6}$. What is the probability I roll the die exactly twice? That means that first I roll a not-five and then I roll a 5, so the probability is $\frac{5}{6} \cdot \frac{1}{6} = \frac{5}{36}$. The probability I'd roll it exactly 3 times is $\left(\frac{5}{6}\right)^2 \cdot \frac{1}{6} = \frac{25}{216}$, the probability I'd roll it 4 times is $\left(\frac{5}{6}\right)^3 \cdot \frac{1}{6} = \frac{125}{1296}$. The probability I'd roll it n times is $\frac{5^{n-1}}{6^n}$.

What does this add up to? The ratio of one term to the previous term is $\frac{5}{6}$, so they add up to $\frac{\frac{1}{6}}{1-\frac{5}{6}}$ which is equal to 1.

Here is a histogram.

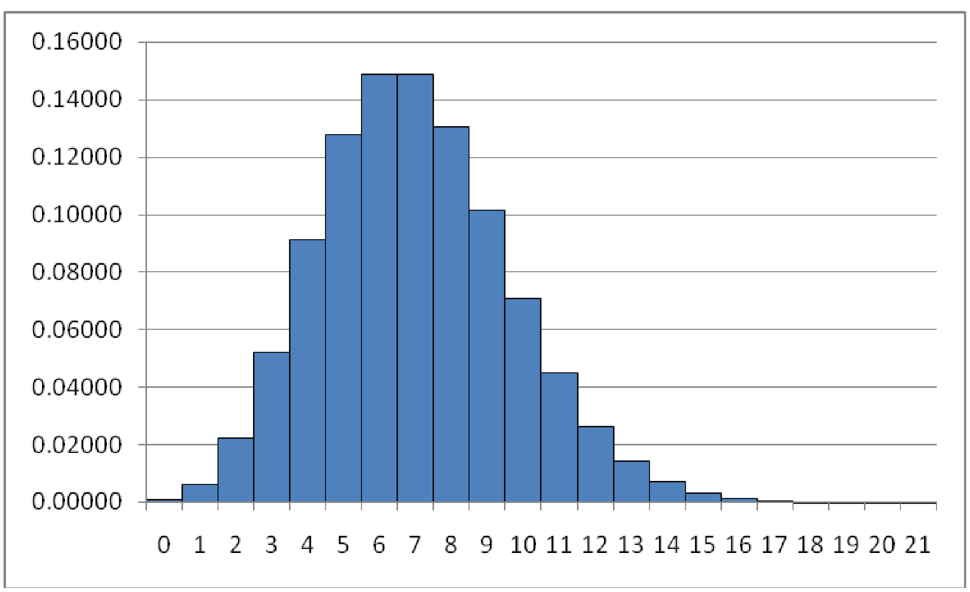


How is this skewed? To the right. Since there are infinitely many possibilities, the probability has to keep getting lower, but it will keep on going to the right forever, making it skewed to the right.

Another distribution with infinitely many possibilities is the Poisson Distribution. Let's assume that on a certain night of a meteor shower, the number of shooting stars you see in one minute is Poisson distributed with a mean of 7. How many could we see in one minute? Possibly zero, possibly one, possibly seven, possibly ten, and so on. The probability of any particular number is $(n) = \frac{e^{-7}7^n}{n!}$. If we plug in different values of n, we get

0	0.00091
1	0.00638
2	0.02234
3	0.05213
4	0.09123
5	0.12772
6	0.14900
7	0.14900
8	0.13038
9	0.10140
10	0.07098
11	0.04517
12	0.02635
13	0.01419
14	0.00709
15	0.00331
16	0.00145
17	0.00060
18	0.00023
19	0.00009
20	0.00003
21	0.00001

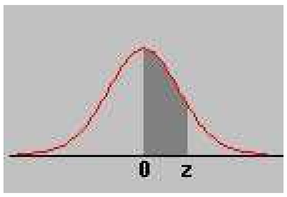
Here is a probability histogram



How is this skewed? The histogram goes up, then goes down, so there is a tail on either side. But the tail on the left is very short, while the tail on the right is infinitely long, so it is skewed to the right.

Let's look at the normal distribution. Here is a table of the distribution.

Area between 0 and z



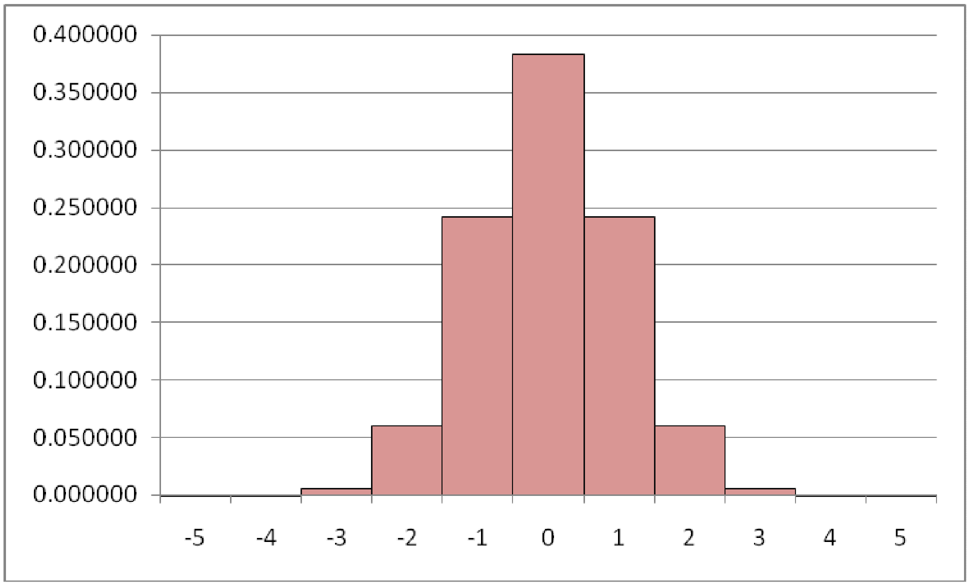
Area between 0 and z										
	0.00	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986

3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

Suppose there is a city, Freezeburg, where the temperature is normally distributed with a mean of zero degrees and a standard deviation of 1. What is the probability the temperature is 2° ?

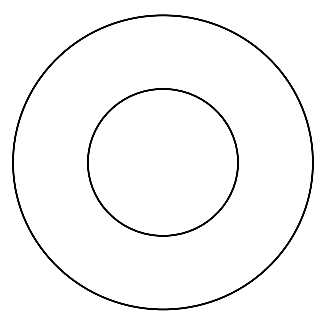
If I say the temperature is 2, I really mean that when I round the temperature to the nearest degree, I get 2. So a temperature of 2 really means from 1.5 to 2.5. These are the z values, so we can look them up in the table. The area between 0 and 1.5 is 0.4332, and the area between 0 and 2.5 is 0.4938. 0.4938 – 0.4332=0.0606 So the probability the temperature is 2° is 0.0606. The table goes on forever in either direction, although below -3 and above 3, the probabilities are very low.

temperature	is really from			probability
-5	-5.5	to	-4.5	0.000003
-4	-4.5	to	-3.5	0.000229
-3	-3.5	to	-2.5	0.005977
-2	-2.5	to	-1.5	0.060598
-1	-1.5	to	-0.5	0.241730
0	-0.5	to	0.5	0.382925
1	0.5	to	1.5	0.241730
2	1.5	to	2.5	0.060598
3	2.5	to	3.5	0.005977
4	3.5	to	4.5	0.000229
5	4.5	to	5.5	0.000003



This works when we round everything to the nearest degree. But what is the probability the temperature is EXACTLY 0°? We would be looking for the area of a sliver with width of zero, so we would get the probability of that is zero. What is the probability the temperature is exactly 1.453908125 degrees? This would also calculate to be zero. Since there are infinitely many possibilities, the probability of each exact temperature is zero. But the temperature must equal something.

The gestation period for humans is normally distributed with a mean of 266 days, and a standard deviation of 16 days.[26] Can we find the probability that the length of a pregnancy is exactly 263.45555566 days? No. We would have to divide the bell curve up into infinitely thin slivers, and each would give us a probability of zero.



Suppose I am playing darts on this dartboard. The radius of the inner circle is half the radius of the outer circle. I am a poor darts player, my dart could go anywhere on the board with equal probability. Since I am a poor player, I will keep on throwing the dart until I hit the dartboard. The

[26] Weiss, Neil A.; Introductory Statistics, seventh edition; Pearson Education, Inc.; Boston; 2005; p. 279.

probability I hit the inner circle is $\frac{1}{4}$ since that is a fourth of the total area. I can find an area of the board and find the probability my dart ends up somewhere in that area. But I cannot pick a point on the board and find the probability that the center of the tip of my dart hits that point. Since a point has zero area, there are infinitely many points on the dartboard, so we can't give each one a probability. But it must hit some point.

We can give a probability to each number of times I might roll a die, or each number of shooting stars we might see, even though there are infinite possible outcomes. But we cannot give a probability to an exact temperature, an exact length of a pregnancy, or an exact point on the dartboard. Why some but not others?

If I wanted to list all the whole numbers, I could start listing them: 1, 2, 3, 4, 5, 6, 7 … I would never finish, but, assuming I live long forever, I would get to any given whole number. 10 would be the tenth number I'd list, 1000 would be the thousandth, 25,000,000 would be the twenty-five-millionth, and so on.

Are there the same amount of even whole numbers as there are whole numbers? We can also list the even whole numbers forever: 2, 4, 6, 8, 10, 12, 14…

Suppose you could not count, and someone asked if you have the same number of fingers on each hand. How could you figure this out without counting? You can just put your hands together so your fingers pair up. If every finger on your left hand matched up with exactly one finger on your right hand, you would know you have the same number of fingers on each hand.

We can do the same thing with the whole numbers and the even whole numbers. Each whole number is matched to the even whole number twice as large, each even whole number is matched to the whole number half its size. This way, each number in either set is matched to exactly one number in the other set.

1	2	3	4	5	6	7	8	9	10	…
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	
2	4	6	8	10	12	14	16	18	20	…

We can list all the numbers in either set. We would never finish, but any number in the set would come up at some point.

We don't have to list the numbers in numerical order. Suppose we wanted to list all the integers, including zero, positive and negative integers. We could not start at zero and keep going up, because then we would never get to the negative integers. We can't go down, because then we would not get to the positives. But we could list them 0, 1, -1, 2, -2, 3, -3, 4, -4… This way we would get to any given integer at some point. We could also use this order to match them up to the whole numbers or the even whole numbers. So there are also the same number of integers as there are whole numbers. That means they are "countable". Mathematician Georg Cantor named the cardinality of the set of whole numbers $\aleph_0$, the Hebrew letter aleph with a subscript of 0, called "aleph-null"

Can we list all the real numbers this way? Is there any way to arrange all the real numbers so that you can list them and get to each one at some point? No.

To prove it, let's try listing just the real numbers between 0 and 1. If we can't list them, we certainly can't list all the real numbers. So let's try listing them. We know we can't list them in numerical order, since between any two real numbers is another real number. I'll start listing them, and writing out the decimal form.

$\frac{1}{2}$ = 0 . 5 0 0 0 0 0 0 0 0 0 0 0 0 0 0 …

$\frac{1}{3}$	=	0	.	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	…
$\frac{\pi}{4}$	=	0	.	7	8	5	3	9	8	1	6	3	3	9	7	4	4	8	…
$\frac{2}{7}$	=	0	.	2	8	5	7	1	4	2	8	5	7	1	4	2	8	5	…
$\sqrt{\frac{1}{2}}$	=	0	.	7	0	7	1	0	6	7	8	1	1	8	6	5	4	8	…
0.25	=	0	.	2	5	0	0	0	0	0	0	0	0	0	0	0	0	0	…
e^{-1}	=	0	.	3	6	7	8	7	9	4	4	1	1	7	1	4	4	2	…

etc.

Can we put every number between zero and 1 on this list? No. We can always come up with another number that is not on the list. Take the list and box the numbers on the diagonal after the decimal point.

0	.	$\boxed{5}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	…
0	.	3	$\boxed{3}$	3	3	3	3	3	3	3	3	3	3	3	3	3	…
0	.	7	8	$\boxed{5}$	3	9	8	1	6	3	3	9	7	4	4	8	…
0	.	2	8	5	$\boxed{7}$	1	4	2	8	5	7	1	4	2	8	5	…
0	.	7	0	7	1	$\boxed{0}$	6	7	8	1	1	8	6	5	4	8	…
0	.	2	5	0	0	0	$\boxed{0}$	0	0	0	0	0	0	0	0	0	…
0	.	3	6	7	8	7	9	$\boxed{4}$	4	1	1	7	1	4	4	2	…

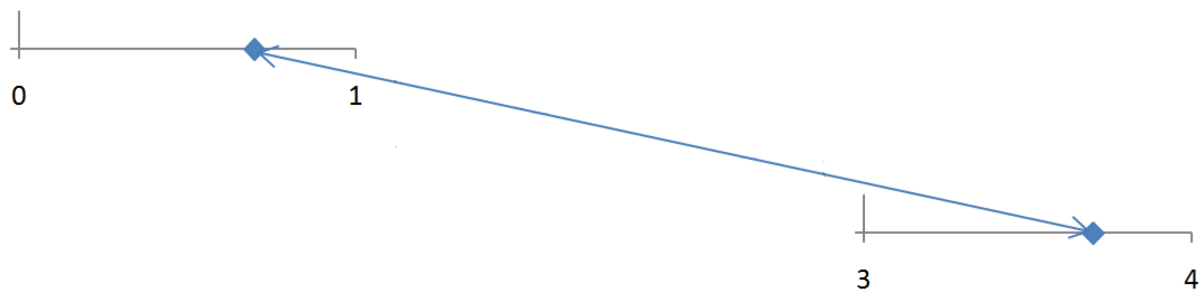
etc.

We can now make a new number, by adding 2 to each of these. The number in the first space in the first row is 5, so we will use 7. The number in the second space of the second row is 3, so we will use 5. The number in the third space of the third row is 5, so we will use 7. And so on. (If the number in the box was 8, we would use 0. If the number in the box was 9, we would use 1.)

Our new number is 0.7579226… This cannot equal the first number on our list because the tenths digit is different. It can't equal the second number on the list, because the one hundredths place is different. And so on. So we have another number between zero and one that is not on our list. We could just add this number to the top of our list, but then we could just repeat the process for another new number. Therefore, it is not possible to list all the real numbers between 0 and 1 in any order. They are not countable.

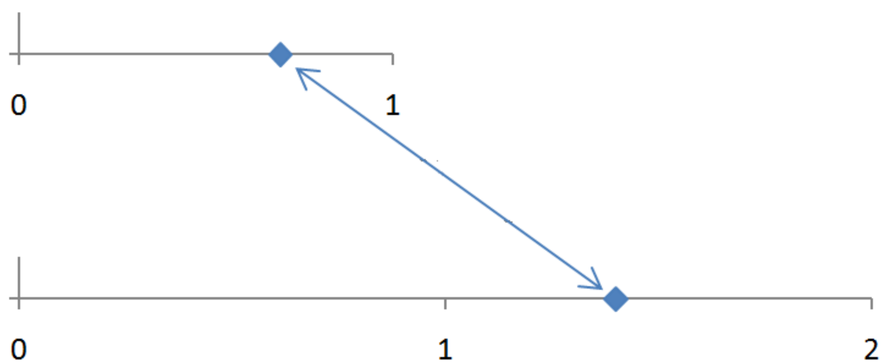
Are there the same amount of real numbers between 0 and 1 as there are on the whole number line? Could we match them up? Let's look at some easier mappings first.

Could we map each point between 0 and 1 to a point between 3 and 4? Yes, just map each n to n+3.



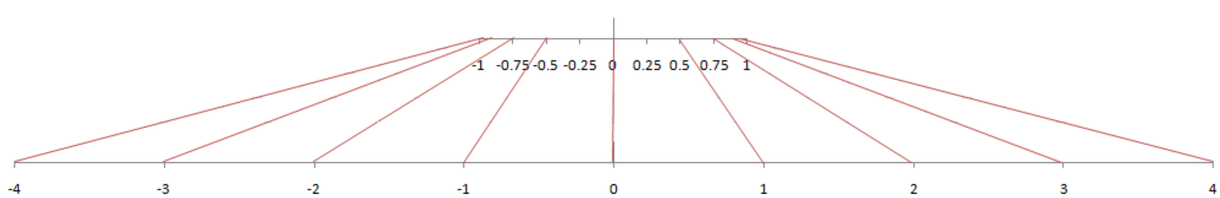
Could we match up the numbers between 0 and 1 with the numbers between 0 and 2?

Yes. We can just map n to 2n.



We can map the points from any finite length to any other finite length. Can we map between a finite length and an infinite length?

Let's map from (-1,1) to (-∞,∞).



On the positive side, we can map everything from $\left[\frac{0,1}{2}\right]$ to [0,1]. Then we can map half of what is left, from $\left[\frac{\frac{1}{2,3}}{4}\right]$ to [1,2], then half of what is now left on that side to [2,3], half of what is still left to [3,4] and so on forever. Reverse it for the negative numbers. This maps every point from -1 to 1 to a point on the real number line.

Since we cannot list all the real numbers, we cannot match them up with the integers. Cantor called the cardinality of this set $\aleph_1$

If I were making a pizza, and had my choice of 10 toppings, how many different groups of toppings could I make? Assume I can use as many or as few of the toppings as I wanted, and the order in which I put them on the pizza does not matter. For garlic, I have two choices; I can either use include it or not include it. For basil, I have the same two choices, as I do for the other toppings. I can make 2^{10} or 1024 subsets of the toppings.

If we were to make a subset of all the whole numbers, how many different subsets are there? $2^{\aleph_0}$. This set of subsets is the Power Set of the set of whole numbers.

Just For Fun

Just for Fun

The longest songs:

I know a song that gets on everybody's nerves,

Everybody's nerves, everybody's nerves.

I know a song that gets on everybody's nerves,

And this is how it goes:

I know a song that gets on everybody's nerves,

Everybody's nerves, everybody's nerves. ….

This is a song that doesn't end.

Yes, it goes on and on my friend.

Some people started singing it, not knowing what it was,

And they'll continue singing it forever just because

This is a song that doesn't end.

Yes, it goes on and on my friend. …

Aleph-null bottles of beer on the wall,

Aleph-null bottles of beer.

Take one down, pass it around,

Aleph-null bottles of beer on the wall.

Aleph-null bottles of beer on the wall,

Aleph-null bottles of beer….

Quotes:

Black holes are where G-d divided by zero.

- bumper sticker

Infinity is a number that is impossible to count to

- Fourth grader Glen Schuster of Altoona, Wisconsin (Maor, p. 232)

Only two things are infinite – the universe and human stupidity. And I'm not sure about the former.

- Albert Einstein (http://quotes.prolix.nu/Authors/?Albert_Einstein)

Interestingly, according to modern astronomers, space is finite. This is a very comforting thought-- particularly for people who can never remember where they have left things.

- Woody Allen (http://www.quotationspage.com/quotes/Woody_Allen)

The range of focus of your telescope is from 15 feet to infinity and beyond.

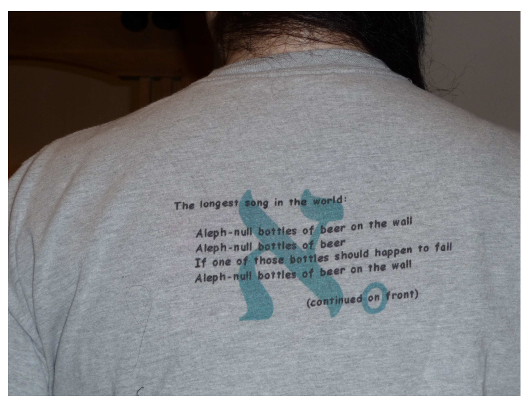
- telescope manual (Maor, p. 68)

<u>T-shirts</u>

(Mental Floss t-shirts ,

http://www.mentalfloss.com/store/home.php?cat

=103)



At the last Mensa annual gathering, someone told me my shirt was “infinitely awesome”.

Poems:

Pi goes on and on and on…

And e is just as cursed.

I wonder: Which is larger

When their digits are reversed?

- Martin Gardner (Darling, p. 101)

Big whorls have little whorls,

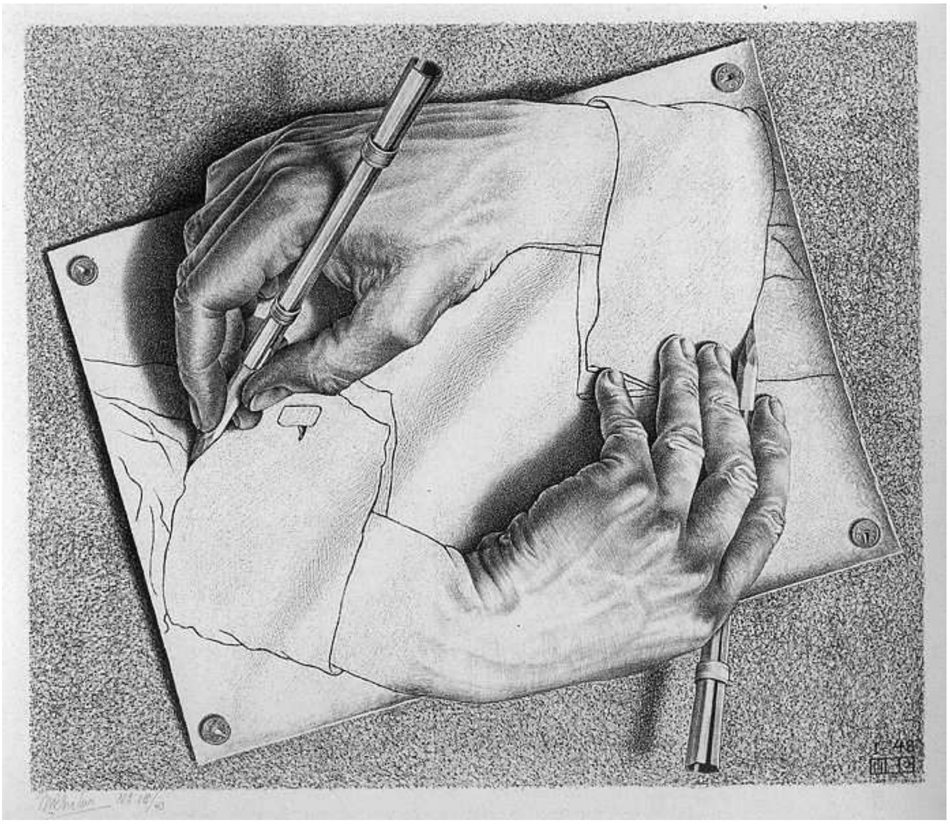
Which feed on their velocity;

And little whorls have lesser whorls,

And so on to viscosity.

- Lewis Richardson (Wells, Mathematics, p. 180)

Pictures that get into paradoxes with infinite loops if you think about them long enough:



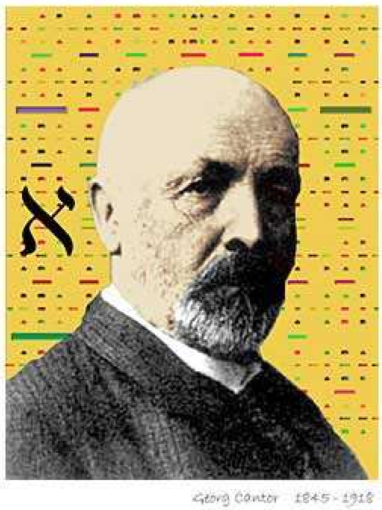
Drawing Hands by M. C. Escher (Drawing Hands, Wikipedia, http://en.wikipedia.org/wiki/File:DrawingHands.jpg)

Headline reads: Woman spotted yesterday reading today's paper.

(The Droste Effect, Arts on Squidoo, http://www.squidoo.com/droste)

<u>My Rosh Hashanah cards:</u>

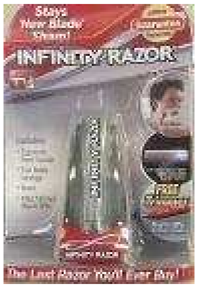
Front:

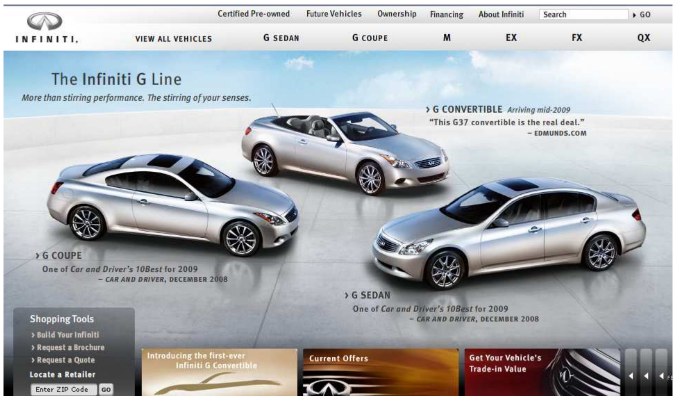


Inside:

A Cantor for Rosh Hashanah	May The Infinite One grant transfinite good wishes to aleph your friends and family. Amy Whinston

Brand names:





Infinity razor Infiniti cars Infinity Sanitary Napkins

math joke:

After explaining to a student through various lessons and examples that:

$$\lim_{x \to 8} \frac{1}{x-8} = \infty$$

I tried to check if she really understood that, so I gave her a different example. This was the result:

$$\lim_{x \to 5} \frac{1}{x-5} = 5$$

(picture from Detronizator.org, http://www.detronizator.org/2006/09/)

Infinity poster:

The Infinity Symbol or Lemniscate looks like the number 8 lying on its side. The lemniscate represents the cycles of infinity and creation, as one universe grows into and becomes another. It is past, present, and future, all in one, and represents being in the eternal Now. In spiritual terms, the lemniscate represents eternity, the numinous and the higher spiritual powers. The Magus, the first card in the Major Arcana of the tarot, is often depicted with the lemniscate above his head or incorporated into a wide-brimmed hat, signifying the divine forces he is attempting to control. The lemniscate often appears in Russian tarot designs dating from the early twentieth century, also in association with the Magician or Strength cards.

(photo from Zazzle, http://www.zazzle.com/infinity_symbol_info_poster-228941271650183568)

Miscellaneous:

A host on a radio station said he saw a car with the license plate " 1 OVER 0 ". What type of car was it? (It was a Ford Infinity)

2 listings in the index of a textbook:

endless loop: see loop, endless

loop, endless: see endless loop

"Infinity Bookcase" by Dutch artist Job Koelewijn

(photo from Neat-O-Rama,http://www.neatorama.com/2008/11/15/the-infinity-bookcase/)

A bicycle's gas mileage

(photo from Zazzle, http://www.zazzle.com/)

A photograph from the Hilton Hotel:

The man in the picture is the president of the Hilton chain. The man in the picture in the picture is the previous president. I asked in the office whether they were planning to continue this when there was a new president, but they did not know. (photo taken at the Anchorage Hilton)

A road sign in Sydey, Australia, by an arts festival

Photo by Triniti 101 (http://www.flickr.com/photos/87322603@N00/55990342/)

Song: Hotel Infinity by Lawrence Mark Lesser (can be sung to the tune of "Hotel California)

On a dark desert highway -- not much scenery

Except this long hot, stretchin' far as I could see.

Neon sign in front read "No Vacancy,"

But it was late and I was tired, so I went inside to plea.

The clerk said, "No problem. Here's what can be done--

We'll move those in a room to the next higher one.

That will free up the first room and that's where you can stay."

I tried understanding this as I heard him say:

CHORUS: "Welcome to the HOTEL INFINITY --
Where every room is full (every room is full)
Yet there's room for more.
Yeah, plenty of room at the HOTEL INFINITY --
Move 'em down the floor (move em' down the floor)
To make room for more."

I'd just gotten settled, I'd finally unpacked
When Isaw 8 more cars pull into the back.
I had to move to room 9; others moved up 8 rooms as well.
Never more will I confuse a Hilton with a Hilbert Hotel!

My mind got more twisted when I saw a bus without end
With an infinite number of riders coming up to check in.
"Relax," said the nightman. "Here's what we'll do:
Move to the double of your room number:
    that frees the odd-numbered rooms."   (Repeat Chorus)

Last thing I remember at the end of my stay--
It was time to pay the bill but I had no means to pay.
The man in 19 smiled, "Your bill is on me.
20 pays mine, and so on, so you get yours for free!"

("Hotel Infinity" http://www.math.utep.edu/Faculty/lesser/hotelinfinity.html)

Ambigram:

(http://www.johnlangdon.net/forsale/turntableth.jpg)

Why I can't lose much weight – by Amy Whinston

Before my weight loss can equal a pound, I have to lose half a pound. And then I have to lose half of what is left of that first pound. And then I have to lose half of what is still left of that first pound. And then half of what is STILL left of that first pound. And so on, and so on…

Bibliography

Under Revision

Tables and Materials

Normal Distribution Table
Penrose Tiles
Magnetic Sheet for Tiles