ON THE PLANARITY AND HAMILTONICITY OF HANOI GRAPHS

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1 Overview

The Tower of Hanoi puzzle was created by French number theorist Édouard Lucas in 1883, and consists of eight wooden discs of varying size, and three vertical pegs on a wooden base. The legend of the origin of the puzzle may have been created by Lucas, or the legend may have been his inspiration for creating the game.

As the legend goes, in the Kashi Vishvanath Temple in the Indian city of Varanasi, beneath a dome that marks the center of the world, there rests a brass plate with three diamond needles. At the beginning of the world, 64 gold discs were placed on one of the needles, the largest resting on the brass plate and the others placed in order of decreasing size from bottom to top. Brahmin priests have been moving the discs day and night since the beginning of time. As the discs are fragile, the priests may not place any larger disc atop any smaller disc. When they have moved all 64 discs to one of the other needles, the temple will crumble to dust and the world will end in a clap of thunder. Because of this legend, the puzzle is also known as the Tower of Brahma [3].

The optimal solution for the Tower of Hanoi puzzle with n discs on 3 pegs is known to be $2^n - 1$ moves, so if the Brahmin priests moved one disc per second in the optimal way, it would take $2^{64} - 1$ seconds, or about 585 billion years to complete their work.

As mentioned above, the optimal solution for the Tower of Hanoi puzzle with n discs on 3 pegs is known to be $2^n - 1$ moves, and there is a recursive algorithm for solving the puzzle in the minimum number of moves. Though there are many variations of the classic Tower of Hanoi puzzle, a natural extension is to consider the Tower of Hanoi puzzle on more than three pegs. However, increasing the number of pegs even by one drastically increases the complexity of analyzing the optimal solutions of the puzzle. The puzzle on four discs, known as the Reve's puzzle, was first discussed in 1907 by H. E. Dudeney [5], and the Frame-Stewart algorithm for solving the puzzle on four or more pegs was discovered in 1941 [3, p.46]. This is the supposed optimal solution to the puzzle, and the claim that the Frame-Stewart algorithm gives the optimal solution is known as the Frame-Stewart Conjecture. The optimal solution for four pegs was not proved until 2014 [1], and the conjecture for more than four pegs is still an open problem.

A major component in the study of the Tower of Hanoi puzzle and its variations is the study of the associated family of graphs, called Hanoi graphs. In this paper, we will examine the results presented by Hinz and Parisse in their article "On the Planarity of Hanoi Graphs" [2], settling the question of two basic properties of graphs with respect to the Tower of Hanoi puzzle. In particular, we will see that these graphs are hamiltonian, and that, with only a few exceptions, they are generally non-planar.

We begin by giving a precise mathematical description of the Tower of Hanoi puzzles.

1.1 The Tower of Hanoi Puzzles

A Tower of Hanoi puzzle consists of n discs arranged on $3 + m$ vertical pegs, where both n and m are nonnegative integers. While the pegs of the puzzle are all identical, the discs each have a different size. A *regular* state of the puzzle is any state in which the discs are distributed among the pegs such that if multiple discs are on the same peg, they are arranged in decreasing size from bottom to top. A *perfect state* is a regular state in which all of the discs are on the same peg. The object of the puzzle is to move from one perfect state to another by moving one disc at a time from the topmost position on one peg to the topmost position on another peg. The *divine rule* governing movement of the discs is that no larger disc may be placed on top of any smaller disc. It can be shown that regular state of the puzzle is reachable under this restriction. Indeed, we will see this as a consequence of some of our work.

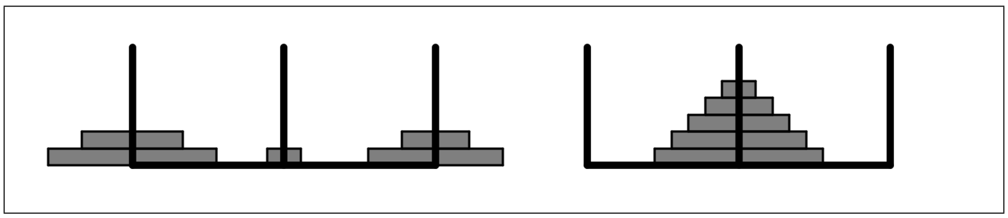

FIGURE 1.1: Regular and perfect states in the Tower of Hanoi puzzle with 5 discs and 3 pegs.

2 Preliminaries

In this section we present some definitions and concepts that will be used throughout the paper. We begin with some basic definitions from graph theory, then discuss the ideas of hamiltonicity and planarity of graphs in general, two concepts which are central to our main results regarding Hanoi graphs. Finally, we will explain the construction of the Hanoi graphs from the Tower of Hanoi puzzles and discuss some properties of these graphs.

2.1 Graphs

A *graph* G consists of a finite *vertex set* $V(G)$ and an *edge set* $E(G)$, consisting of unordered pairs of elements of $V(G)$. In particular, we will be dealing only with finite undirected graphs with no loops or multiple edges. The elements of $V(G)$ and $E(G)$ are called the *vertices* and *edges* of G, respectively. The two vertices of an edge are called *endpoints*. If a vertex x is the endpoint of an edge e, we say that x and e are *incident*. The *degree* of a vertex is the number of edges incident to it. Two vertices x and y are *adjacent* if they are the endpoints of the same edge. In this case, we write $x \sim y$ and call the common edge xy. If x and y are not adjacent, we write $x \nsim y$.

In a graph G, a *walk* is an alternating list of vertices and edges, $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$, such that edge e_i has endpoints v_{i-1} and v_i, for $1 \leq i \leq k$. A *path* is a walk that does not repeat any edges or vertices. A *cycle* is a walk

$$v_0, e_1, v_1, e_2, \dots, e_k, v_0$$

where $v_i \neq v_j$ and $e_i \neq e_j$ for all $i \neq j$. The *length* of a path or cycle is the number of its edges.

A *subgraph* H of a graph G is any graph that has a vertex set $V(H) \subset V(G)$ and an edge set $E(H) \subset E(G)$. A *spanning subgraph* of G is a subgraph of G that has vertex set $V(G)$. A subgraph H is an *induced subgraph* of G if for any $x, y \in V(H)$ we have $x \sim y$ in H if and only if $x \sim y$ in G. If $V(H) = S$, we write $G[S]$ for the subgraph of G induced by the vertex set of H.

Two graphs G and H are *isomorphic* if there exists a bijection $f : V(G) \to V(H)$ such that, for any $x, y \in V(G)$, $x \sim y$ in G if and only if $f(x) \sim f(y)$ in H. In this case, we write $G \cong H$, and we say f is an *isomorphism*. An isomorphism from a graph to itself is an *automorphism*, and the group of all such functions is the *automorphism group* of G, denoted $Aut(G)$.

The *complete graph* K_n is the graph on n vertices with all possible edges. That is, $x \sim y$ for every pair $x, y \in V(K_n)$ such that $x \neq y$. K_5 is shown in Figure 2.1.

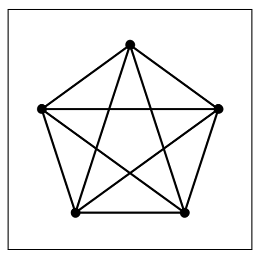

FIGURE 2.1: The complete graph K_5.

2.2 Hamiltonicity of Graphs

A *hamiltonian path* in a graph G is a spanning path, that is, a path in G whose vertex set is $V(G)$. A graph G is called *hamiltonian* if it contains a cycle that is a spanning subgraph of G, that is, a cycle that goes through every vertex in $V(G)$. Such a cycle is called a *hamiltonian cycle* and necessarily has length equal to the size of the vertex set. Although there are necessary and sufficient conditions for determining whether a graph may be hamiltonian, a straightforward way to show that a graph is hamiltonian is simply to identify a hamiltonian cycle contained in the graph. An example of a hamiltonian graph is the dodecahedron; a hamiltonian cycle in this graph is illustrated in Figure 2.2.

Note that the complete graph K_n is hamiltonian for any $n \in \mathbb{N}$. To see this, begin by labeling the vertices of K_n as $x_1, x_2, \dots, x_n$. Since $x_i \sim x_j$ for each $i, j \in \{1, \dots n\}$ with $i \neq j$, the graph K_n contains the spanning cycle $x_1, x_2, \dots, x_n, x_1$. This simple fact will be useful in one of the main results of the paper.

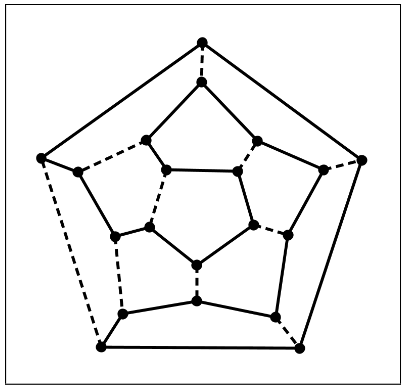

FIGURE 2.2: Hamiltonian cycle (solid edges) in the graph of the dodecahedron.

2.3 Planarity of Graphs

A graph can represented visually by a drawing in the plane, where vertices are dots and edges are line segments. Such a drawing of a graph is by no means unique; the same graph can be drawn in many different ways. A *crossing* in a drawing of a graph is a point in the plane where two edges intersect that is not a common endpoint of the edges. A graph is called *planar* if it can be drawn without crossings. Such a drawing is called a *planar embedding*. The *faces* of a planar embedding are the polygonal regions of the plane bounded by the edges of the graph, along with the one unbounded *outer face*. Note that if a graph contains any subgraph that is non-planar, then the graph itself is non-planar.

Consider the *complete graph* K_4, the complete graph on 4 vertices. This graph can be drawn with or without crossings, as shown in Figure 2.3, where the faces of the planar embedding are labeled 1-4, face 4 being the outer face. Since a planar embedding exists, K_4 is planar. On the other hand, K_5 cannot be drawn without crossings, and so is not planar. This can be verified easily using a result in [6] that follows from Euler's Formula and gives an upper bound for the number of edges in a planar graph. In particular, if G is a planar graph with at least 3 vertices, then $|E(G)| \leq 3|V(G)| - 6$. We see that K_5 violates this bound, since $3|V(K_5)| - 6 = 3(5) - 6 = 9 < |E(K_5)| = 10$, and thus is not planar.

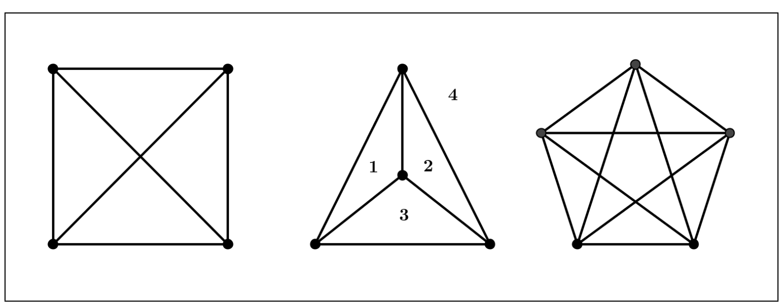

FIGURE 2.3: K_4 drawn with and without crossings, K_5 cannot be drawn without crossings.

2.4 The Hanoi Graphs

A Tower of Hanoi puzzle consists of n discs on $3+m$ pegs, where both n and m are nonnegative integers. The corresponding *Hanoi graph* H_m^n represents all regular states and legal moves of the corresponding Hanoi puzzle by vertices and edges, respectively, and is defined as follows. Label the pegs of the puzzle as $0, 1, \ldots, (2+m)$ from left to right. As each disc is a different size, we define disc i to have radius i, for $i = 1, 2, \ldots n$. Let x_i be the position (peg) of disc i. Then each regular state of the Tower of Hanoi puzzle with n discs on $3+m$ pegs can be uniquely represented as an n-tuple $(x_1, x_2, \ldots, x_n)$, with $x_i \in \{0, 1, \ldots, (2+m)\}$ for each i. These n-tuples make up the vertex set of the Hanoi graph. That is,

$$V\big(H_m^n\big) = \Big\{(x_1, x_2, \ldots, x_n) \mid x_i \in \{0, 1, \ldots, (2+m)\}\Big\}.$$

(Note that we will sometimes represent an n-tuple $(x_1, x_2, \ldots, x_n)$ as $x_1 x_2 \ldots x_n$, in cases where each x_i is a single digit number and no confusion will arise.) It follows that H_m^n has $(3+m)^n$ vertices when $n > 0$. When $n = 0$, we have the Tower of Hanoi puzzle with $3+m$ pegs and no discs. The corresponding Hanoi graph is the *null graph*, the graph with no vertices and no edges. The null graph is trivial with regard to our main results, and for this reason we will consider only cases where n is strictly positive.

An edge in the Hanoi graph represents a legal move of a single disc, so two vertices are adjacent if the corresponding regular states can be achieved from one another through a legal move of exactly one disc. In particular, adjacent vertices differ in exactly one entry of their n-tuples. We summarize formally with the following definition.

Definition 2.1. The Hanoi graph H_m^n is the graph with vertex set $V\big(H_m^n\big)$ given by

$$V\big(H_m^n\big) = \Big\{(x_1, x_2, \ldots, x_n) \mid x_i \in \{0, 1, \ldots, (2+m)\}\Big\}$$

and where

$$(x_1, x_2, \ldots, x_n) \sim (y_1, y_2, \ldots, y_n) \Rightarrow \big|\{i \mid x_i \neq y_i\}\big| = 1.$$

□

To illustrate, in Figure 2.4 we have the Tower of Hanoi puzzle H_m^5 with five discs and $3+m$ pegs. The first image shows a regular state that has all five discs on peg 1, and corresponds to the vertex $(1, 1, 1, 1, 1)$ in the graph H_m^5. If we move the smallest disc to peg 0, we have the state that corresponds to vertex $(0, 1, 1, 1, 1)$, depicted in the second image of Figure 2.4. Since we can achieve the second state from the first by moving exactly one disc, and vice versa, we have $(1, 1, 1, 1, 1) \sim (0, 1, 1, 1, 1)$. If we now move the second smallest disc to peg $2+m$, we have the state that corresponds to vertex $(0, 2+m, 1, 1, 1)$, depicted in the third image of Figure 2.4. This third state can be achieved from the second via a single legal move, but not from the first, so, in the graph H_m^5,

$$(0, 2+m, 1, 1, 1) \sim (0, 1, 1, 1, 1) \text{ and } (0, 2+m, 1, 1, 1) \nsim (1, 1, 1, 1, 1).$$

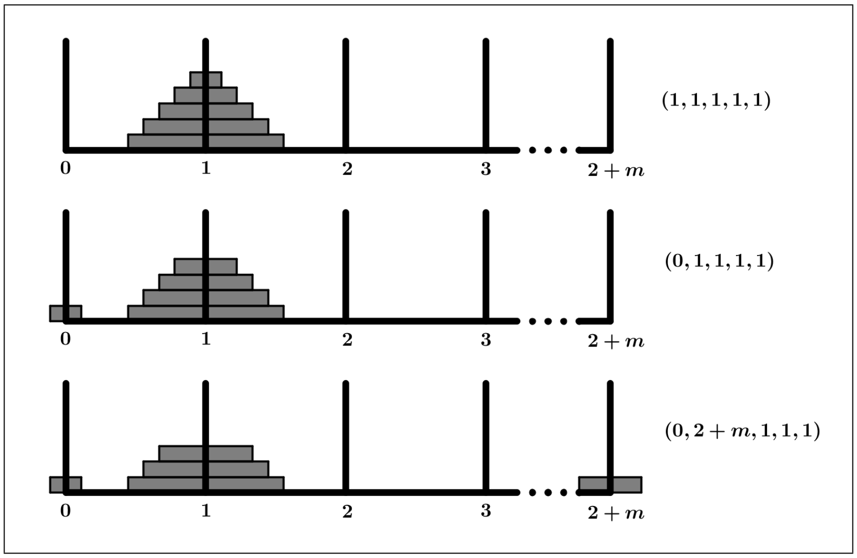

FIGURE 2.4: Three states of the Tower of Hanoi puzzle with 5 discs and $3 + m$ pegs, and their corresponding vertices in H_m^5.

Before moving on to some properties of Hanoi graphs, in Figure 2.5 we provide illustrations of the Hanoi graphs H_0^1, H_0^2, and H_0^3; these are the Hanoi graphs corresponding to 1, 2, and 3 discs, respectively, on 3 pegs. Note that these three graphs are planar, since they can be drawn without crossings, as shown.

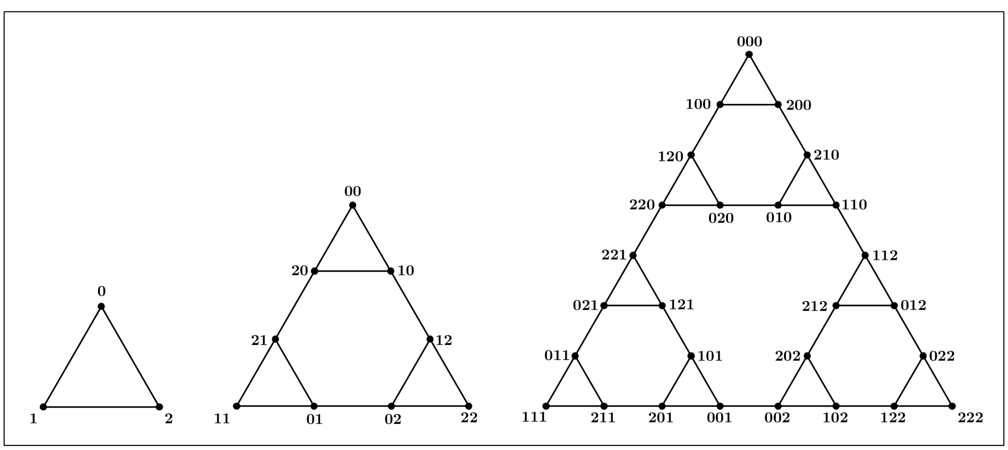

FIGURE 2.5: Hanoi graphs H_0^1, H_0^2, and H_0^3.

2.5 Basic Properties of Hanoi Graphs

To analyze the properties of Hanoi graphs more closely, we give an alternate characterization of adjacency.

Lemma 2.2. Let $x, y \in V(H_m^n)$, with

$$x = (x_1, x_2, \ldots, x_i, \ldots, x_n) \quad \text{and} \quad y = (y_1, y_2, \ldots, y_i, \ldots, y_n).$$

Then $x \sim y$ if and only if there exists i $(1 \leq i \leq n)$ such that

$$\{j | x_j = y_j\} = \{j | j \neq i\} \text{ and } \{x_i, y_i\} \cap \{x_1, \ldots, x_{i-1}\} = \emptyset.$$

Proof. $(\Rightarrow)$ Suppose $x \sim y$. Then only one disc can switch pegs between states x and y, so there exists an i $(1 \leq i \leq n)$ such that $x_j = y_j$ if and only if $j \neq i$ for all j $(1 \leq j \leq n)$. Also, in the regular state represented by vertex x it is possible to move disc i from peg x_i, so disc i is the topmost disc and peg x_i does not have any discs of radius less than i. Then none of discs $1, \ldots, i-1$ are on peg x_i, so $x_i \notin \{x_1, \ldots, x_{i-1}\}$. Moreover, it is possible to move disc i to peg y_i, so peg y_i cannot already have any discs with radius less than i. Then none of discs $1, \ldots, i-1$ are on peg y_i, so $y_i \notin \{x_1, \ldots, x_{i-1}\}$. Thus $\{x_i, y_i\} \cap \{x_1, \ldots, x_{i-1}\} = \emptyset$.

$(\Leftarrow)$ Suppose there exists i $(1 \leq i \leq n)$ such that $\{j | x_j = y_j\} = \{j | j \neq i\}$ and $\{x_i, y_i\} \cap \{x_1, \ldots, x_{i-1}\} = \emptyset$. Then neither peg x_i or y_i contains any discs of radius less than i, so disc i is free to move between pegs x_i and y_i. Thus $x \sim y$. ■

Note that H_0^2 is a subgraph of H_0^3, and that H_0^1 is a subgraph of both H_0^2 and H_0^3. We can see that this should be true by considering the corresponding Tower of Hanoi puzzles. For each of H_0^1, H_0^2, and H_0^3, the puzzle occurs on 3 pegs. In the puzzle for H_0^1, there is one disc and it can move freely between each peg. In the puzzle for H_0^2, there are two discs. By holding the largest disc fixed on any one peg, the smaller disc is free to move among all the pegs, reducing the puzzle to 1 disc on 3 pegs; H_0^1. So we see that there is one copy of H_0^1 in H_0^2 for each peg. That is, for each possible position of the largest disc. Similarly, H_0^3 corresponds to 3 discs on 3 pegs. By holding the largest disc fixed on any one peg, the two smaller discs can move exactly as if the largest disc were not there, reducing the puzzle to 2 discs on 3 pegs. So H_0^3 contains one copy of H_0^2 for each peg.

The same argument can be used to show that H_0^n contains H_0^k as a subgraph for any $k < n$. Indeed, if the discs with radii $k+1, \ldots, n$ are all on the same peg, then the k smaller discs can move exactly as they would if the larger discs were not there. We generalize this for all Hanoi graphs in the following lemma.

Lemma 2.3. Fix any $m, n \in \mathbb{N}$ and any $k \in \mathbb{N}$ such that $k < n$. Fix any $l \in \{0, 1, \ldots, 2+m\}$. Let $S = \{(x_1, x_2, \ldots, x_n) | x_{k+1} = x_{k+2} = \ldots = x_n = l\}$. Then the subgraph of H_m^n induced by S is isomorphic to H_m^k. That is,

$$H_m^n[S] \cong H_m^k.$$

Proof. Define the function $f : S \to V(H_m^k)$ by

$$f\big((x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_n)\big) = (x_1, x_2, \ldots, x_k).$$

Then f is clearly a bijection. Moreover, fix any $x, y \in S$, with $x = (x_1, x_2, \ldots, x_k, l, \ldots l)$ and $y = (y_1, y_2, \ldots, y_k, l, \ldots l)$. Then $f(x) = (x_1, x_2, \ldots, x_k)$ and $f(y) = (y_1, y_2, \ldots, y_k)$.

Suppose $x \sim y$ in $H_m^n[S]$. Then by Lemma 2.2, there is $i \in \{1, 2, \ldots, n\}$ such that $x_i \neq y_i$, where $x_j = y_j$ for all $i \neq j$, and satisfying $\{x_i, y_i\} \cap \{x_1, \ldots x_{i-1}\} = \emptyset$. Since $x, y \in S$, we must have $i \leq k$. Hence $(x_1, \ldots, x_k) \sim (y_1, \ldots, y_k)$ in H_m^k.

Conversely, suppose $f(x) \sim f(y)$ in H_m^k. Then by Lemma 2.2, there is $i \in \{1, 2, \ldots, k\}$ such that $x_i \neq y_i$, where $x_j = y_j$ for all $i \neq j$, and satisfying $\{x_i, y_i\} \cap \{x_1, \ldots x_{i-1}\} = \emptyset$. Hence $(x_1, \ldots, x_k, l, \ldots, l) \sim (y_1, \ldots, y_k, l, \ldots, l)$ in $H_m^n[S]$.

We conclude that $x \sim y$ in $H_m^n[S]$ if and only if $f(x) \sim f(y)$ in H_m^k, thus $H_m^n[S] \cong H_m^k$. ■

3 Hamiltonicity of Hanoi Graphs

In this section we will prove the first of the main results of the paper by Hinz and Parisse. First, we present a lemma.

Lemma 3.1. Given any perfect states s_1, s_2, s_3, and s_4 in H_m^n, with $s_1 \neq s_2$ and $s_3 \neq s_4$, there exists an automorphism $f \in Aut(H_m^n)$ such that $f(s_1) = s_3$ and $f(s_2) = s_4$.

Proof. Fix any permutation π in the symmetric group $Sym(2+m)$ and define $f_\pi : H_m^n \to H_m^n$ by

$$f_\pi\big((x_1, x_2, \ldots, x_n)\big) = \big(\pi(x_1), \pi(x_2), \ldots, \pi(x_n)\big).$$

We claim that $f_\pi \in Aut(H_m^n)$. To see this, let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. We must show that $x \sim y$ if and only if $f_\pi(x) \sim f_\pi(y)$.

($\Rightarrow$) Suppose $x \sim y$. Then there is $i \in \{1, 2, \ldots, n\}$ such that $x_i \neq y_i$, where $x_j = y_j$ for all $j \neq i$, and satisfying $\{x_i, y_i\} \cap \{x_1, \ldots, x_{i-1}\} = \emptyset$. Consider $f_\pi(x) = \big(\pi(x_1), \pi(x_2), \ldots, \pi(x_n)\big)$ and $f_\pi(y) = \big(\pi(y_1), \pi(y_2), \ldots, \pi(y_n)\big)$. Since $x_i \neq y_i$, we have $\pi(x_i) \neq \pi(y_i)$. Also, since $x_i \neq x_j$ for all $j < i$, we have $\pi(x_i) \neq \pi(x_j)$ for all $j < i$. Finally, since $x_j = y_j$ for all $j \neq i$, we have $\pi(x_j) = \pi(y_j)$ for all $j \neq i$. It follows from Lemma 2.2 that $f_\pi(x) \sim f_\pi(y)$.

($\Leftarrow$) Suppose $f_\pi(x) \sim f_\pi(y)$ and consider $\pi^{-1} \in Sym(2+m)$:

$$f_{\pi^{-1}}\big(f_\pi(x)\big) = \big(\pi^{-1}(\pi(x_1)), \pi^{-1}(\pi(x_2)), \ldots, \pi^{-1}(\pi(x_n))\big) = (x_1, x_2, \ldots, x_n) = x.$$

Similarly, $f_{\pi^{-1}}\big(f_\pi(y)\big) = y$. From the argument above, we have that $f_\pi(x) \sim f_\pi(y)$ implies $f_{\pi^{-1}}\big(f_\pi(x)\big) \sim f_{\pi^{-1}}\big(f_\pi(y)\big)$. It follows that $x \sim y$.

We conclude that $x \sim y$ if and only if $f_\pi(x) \sim f_\pi(y)$, so f_π is an automorphism on H_m^n.

Finally, let $s_i = (a_i, a_i, \ldots, a_i)$ be a perfect state in H_m^n for $i = 1, 2, 3, 4$, where each $a_i \in \{0, 1, \ldots, 2+m\}$ and where $a_1 \neq a_2$ and $a_3 \neq a_4$. Let π be any permutation in

$Sym(2+m)$ such that $\pi(a_1) = a_3$ and $\pi(a_2) = a_4$. Then $f_\pi \in Aut(H_m^n)$ and we have

$$f_\pi(s_i) = \big(\pi(a_i), \pi(a_i), \ldots, \pi(a_i)\big)$$

for each i, so that $f(s_1) = s_3$ and $f(s_2) = s_4$ as desired. ■

We will use the previous lemma to aid in an inductive proof of the hamiltonicity of Hanoi graphs.

Theorem 3.2. Every Hanoi graph is hamiltonian.

Proof. Fix any $m \in \mathbb{N}$. Our proof consists of two parts. In (i) we will show by induction on $n \in \mathbb{N}$ that, for any n, there is a hamiltonian path in H_m^n starting and ending with vertices that correspond to distinct perfect states. This fact will then be used in (ii) to construct a hamiltonian cycle in H_m^{n+1}.

(i) By induction on $n \in \mathbb{N}$, we will show that, for any n, there is a hamiltonian path in H_m^n starting and ending with vertices that correspond to distinct perfect states.

Base Case. Since the case $n = 0$ is trivial, let $n = 1$. The graph H_m^1 corresponds to the Tower of Hanoi puzzle with 1 disc on $3+m$ pegs. Since there is only one disc, it can move freely among the pegs. So every state is a perfect state and any state can be reached from any other state in exactly one move of the disc. Therefore H_m^1 is the complete graph on $3+m$ vertices, K_{3+m}. The complete graph K_n is hamiltonian for any $n \in \mathbb{N}$, so H_m^1 has a hamiltonian path. Since every state is a perfect state, such a hamiltonian path begins and ends in distinct perfect states.

Induction Hypothesis. Fix any $n \geq 1$ and suppose that H_m^n has a hamiltonian path beginning and ending in distinct perfect states. Consider H_m^{n+1}, which corresponds to the puzzle with $n+1$ discs on m pegs, obtained by adding a disc with radius $n+1$ to H_m^n. Without loss of generality, suppose that all discs begin on peg 0, a perfect state. Then discs 1 through n form an n-tower sitting atop disc $n+1$ on peg 0. We can move disc $n+1$ stepwise through every peg from 0 to $m+2$, in order, in the following way. The reader may follow along with Figure 3.1. By the induction hypothesis, there is a hamiltonian path between distinct perfect states in H_m^n, and by Lemma 3.1, perfect states are isomorphic, so there is a hamiltonian path between any two distinct perfect states. Therefore, before each step moving disc $n+1$, we can perform a hamiltonian path transferring the n-tower of discs 1 through n to a peg allowing disc $n+1$ to move. Because there are always at least 3 pegs, this n-tower can be moved to a peg that is neither the same as the current peg of disc $n+1$, nor the same as where we would like to move disc $n+1$. To be concrete, let us say that each time we wish to move disc $n+1$ from peg i to peg $i+1$, we first move the n-tower to peg $i+2 (\mathrm{mod}\ 3+m)$. After the last move of disc $n+1$ to peg $2+m$, the n-tower can be transferred to peg $2+m$ as well, again on a hamiltonian path through H_m^n.

Now disc $n+1$ has been moved $2+m$ times, stepwise through each of the $3+m$

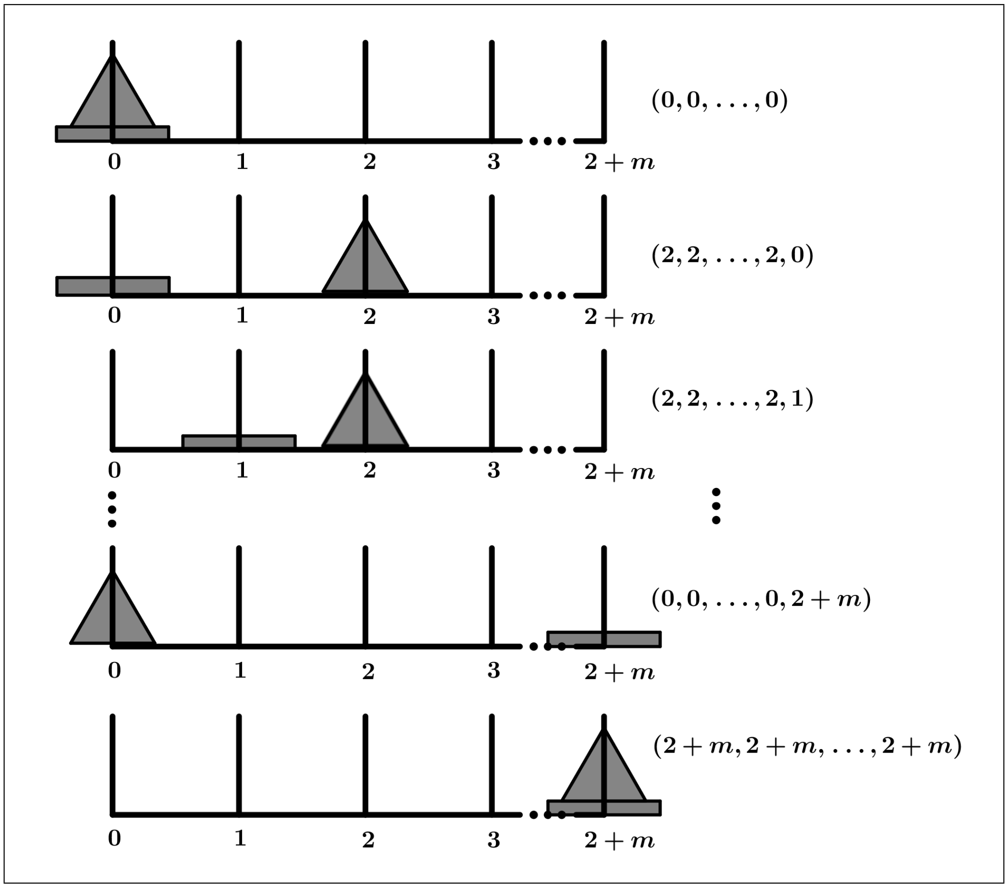

FIGURE 3.1: Part (i):Moving the n-tower and disc $n+1$ through a hamiltonian path between distinct perfect states.

pegs. Before each move of disc $n+1$, the n-tower completes a hamiltonian path between two distinct perfect states. During each such path, the n smaller discs move through every possible state for that position of disc $n+1$. In this way, every possible state of all $n+1$ discs is achieved exactly once, completing a hamiltonian path in H_m^{n+1}. Thus there is a hamiltonian path in H_m^n between vertices corresponding to distinct perfect states for any $n \in \mathbb{N}$.

(ii) We shall now construct a hamiltonian cycle on H_m^{n+1}, where $n \in \mathbb{N}$. The reader may follow along with Figure 3.2. Without loss of generality, let the initial vertex in the cycle be $(1,1,\ldots,1,0) \in V(H_m^{n+1})$, corresponding to the state where the n-tower of discs with radius at most n is on peg 1 and disc $n+1$ is on peg 0. By (i), we can transfer the n-tower of smaller discs through a hamiltonian path from peg 1 to peg 2, followed by moving disc $n+1$ to peg 1. In this step we have gone through every vertex in H_m^{n+1} that has a zero in the last entry, ending on vertex $(2,2,\ldots,2,1)$. Continuing in this way, we can transfer the n-tower of smaller discs through a hamiltonian path from peg $i+1$ to peg $i+2$ for each $i \in \{0, 1, \ldots, 2+m\}$, following each complete transfer by a single move of disc $n+1$ from peg i to peg $i+1$. In each step we go through every vertex with an i in the last entry exactly once. Consider the steps modulo $3+m$, so that when the n-tower is on peg $2+m$ the next step will transfer it to peg 0. The process terminates when we transfer the n-tower back onto peg 1, followed by moving disc $n+1$ onto peg 0. Now we have completed a path in H_m^{n+1} that goes through every vertex exactly once and ends on the initial vertex, completing a hamiltonian cycle in H_m^{n+1}.

Therefore every Hanoi graph is hamiltonian. ■

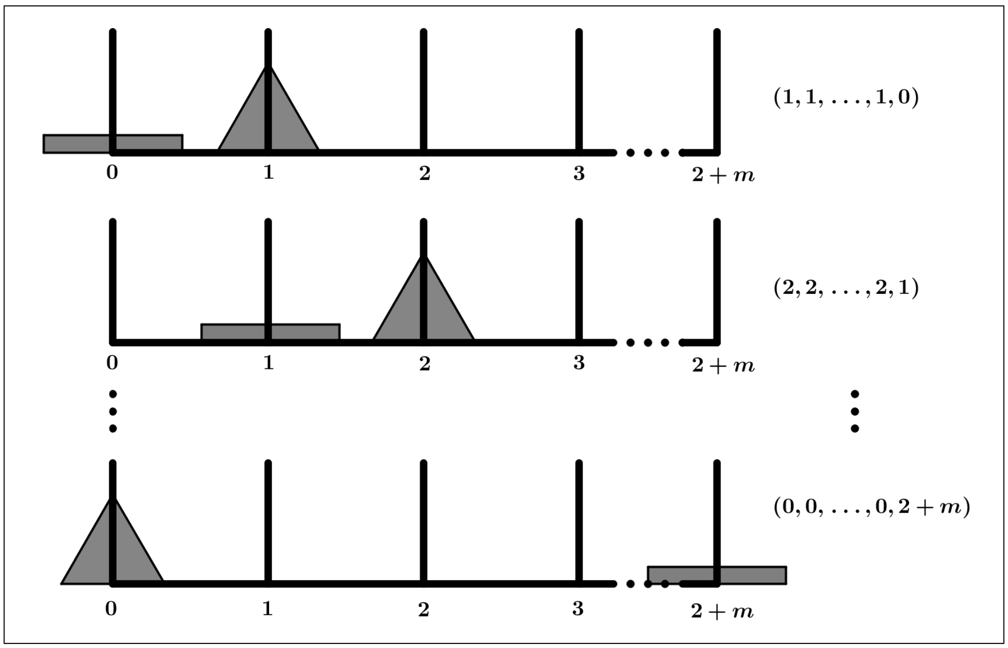

FIGURE 3.2: Part (ii): Moving the n-tower and disc $n+1$ through all possible states in H_m^{n+1} exactly once.

4 Planarity of Hanoi Graphs

Our second main result specifies for which values of m and n the Hanoi graph H_m^n is planar. As we have seen by the planar embeddings illustrated in Figure 2.5, the graphs H_0^1, H_0^2, and H_0^3 are planar. Before presenting the main result, we provide planar embeddings of two more Hanoi graphs, followed by a lemma characterizing planarity for the Hanoi graphs H_0^n.

First we consider the Hanoi graph H_1^1, which corresponds to the Tower of Hanoi puzzle with 1 disc on 4 pegs. Since there is only one disc, it can move freely between the pegs.

Since any peg is reachable from any other peg through exactly one move of the disc, every pair of vertices is adjacent in the Hanoi graph. Thus H_1^1 is isomorphic to the complete graph K_4 and is therefore planar. A planar embedding of H_1^1 is shown in Figure 4.1.

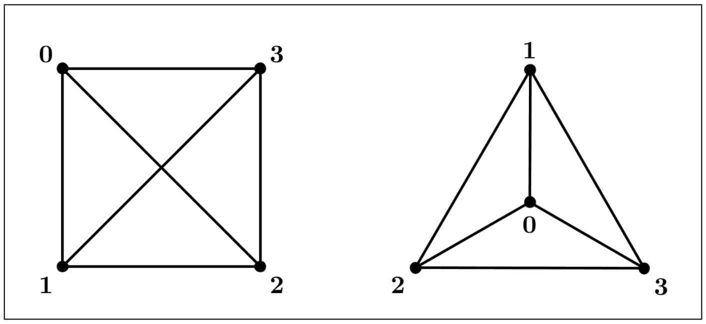

FIGURE 4.1: Two embeddings of the Hanoi graph H_1^1.

Next we consider the Hanoi graph H_1^2, which corresponds to the Tower of Hanoi puzzle with 2 discs on 4 pegs. A planar embedding of H_1^2 is shown in Figure 4.2. Note that H_1^2 is 3-connected [3, p.194], meaning that there is no pair of vertices whose deletion results in a disconnected graph. Every 3-connected planar graph has essentially only one planar embedding [6, p.376], so this planar embedding of H_1^2 is essentially unique.

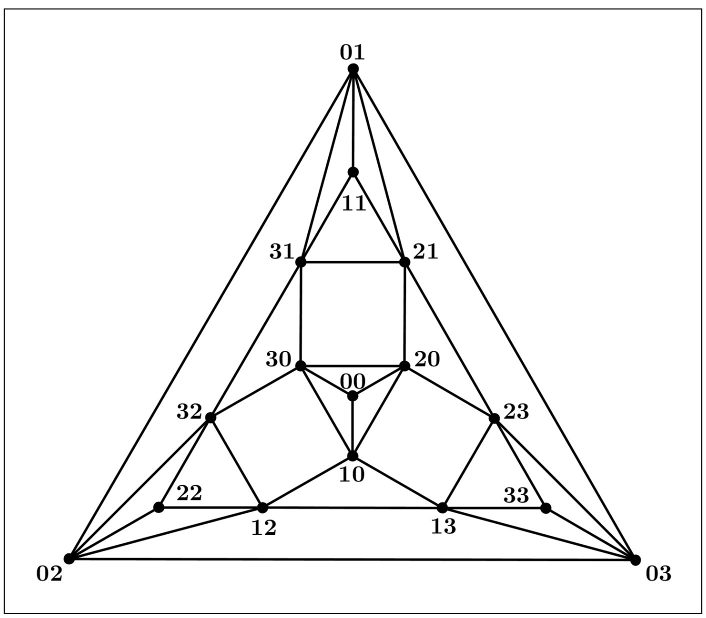

FIGURE 4.2: Planar embedding of the Hanoi graph H_1^2.

Lemma 4.1. The Hanoi graph H_0^n is planar for all $n \in \mathbb{N}$.

Proof. We will show by induction on n that H_0^n allows a planar embedding, whose infinite face is the complement of an equilateral triangle with side length $2^n - 1$ and whose corners are the perfect states.

Base Case. Since the case $n = 0$ is trivial, let $n = 1$. The graph H_0^1 corresponds to the Tower of Hanoi puzzle with 1 disc on 3 pegs. The disc can move freely between between the pegs in any order, and every state is a perfect state. So H_0^1 is isomorphic to the complete graph K_3, which can be drawn as an equilateral triangle, as shown in Figure 2.5. Thus the infinite face is the complement of an equilateral triangle and its side length is $1 = 2^1 - 1$.

Induction Hypothesis. Fix $k \in \mathbb{N}$ and suppose H_0^n has a planar embedding of the desired form for all $1 \leq n \leq k$. In particular, H_0^k can be drawn without crossings such that its infinite face is the complement of an equilateral triangle with side length $2^k - 1$ and the corners are the perfect states. We will construct a drawing of H_0^{k+1} as follows. Since the disc with radius $k+1$ is the largest disc, every state of the other k discs is possible, regardless of the position of disc $k+1$. So we take three copies of the planar graph H_0^k, one for each possible position (peg 0, 1, and 2) of the disc with radius $k+1$. The vertices of these are relabeled with $(k+1)$-tuples ending in 0, 1, and 2, respectively. Since adjacent vertices in H_m^n differ in exactly one entry of their n-tuples, we add three edges to connect the three copies of H_0^k. Namely, if we denote the perfect states in H_0^k by $([0]), ([1]), ([2])$, where $([0])$ is the k-tuple consisting of only zeros, then we add edges to form the adjacencies $([0], 1) \sim ([0], 2)$, $([1], 0) \sim ([1], 2)$, and $([2], 0) \sim ([2], 1)$. This is illustrated in Figure 4.3.

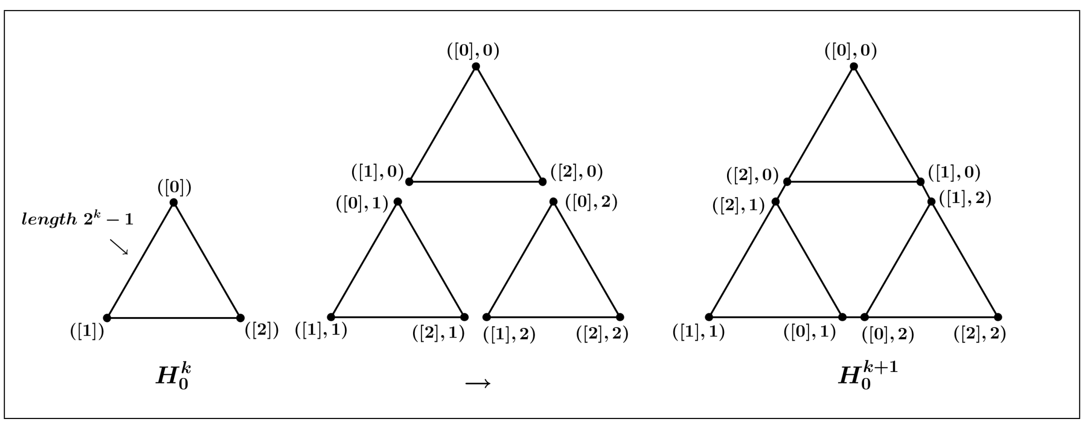

FIGURE 4.3: Construction of H_0^{k+1} from three copies of H_0^k .

We claim that no other edges are added to form H_0^{k+1}. Indeed, we cannot add an edge within a single copy of H_0^k, otherwise we would have two states that can be reached from each other in a single move in H_0^{k+1}, but not in H_0^k. This is a contradiction, since it would mean the states can be reached through a single move when the largest disc $k+1$ has fixed position, but not when disc $k+1$ is not present. We also cannot add any other edge between two copies of H_0^k. If we did, the vertices would have to have the first k entries of their $(k+1)$-tuples the same as each other, since their last entries differ. But the first k entries do not all have the same value, otherwise they would have been perfect states in H_0^k. So one of the first k entries has to have the same value as the last entry of one of the vertices, violating the adjacency relationship of Lemma 2.2.

That exactly three edges are added to connect the three copies of H_0^k and form H_0^{k+1} can also be verified using the edge count formula [4]

$$|E_m^n| = \frac{(3+m)(2+m)}{4}\Big((3+m)^n - (1+m)^n\Big),$$

which gives the number of edges in the Hanoi graph H_m^n. We get the following edge counts for H_0^k and H_0^{k+1}.

$$\begin{aligned}
|E_0^k| &= \frac{(3+0)(2+0)}{4}\Big((3+0)^k - (1+0)^k\Big) \\
&= \frac{3}{2}\big(3^k - 1\big) \\
|E_0^{k+1}| &= \frac{(3+0)(2+0)}{4}\Big((3+0)^{k+1} - (1+0)^{k+1}\Big) \\
&= \frac{3}{2}\big(3^{k+1} - 1\big) \\
&= \frac{9}{2}\cdot 3^k - \frac{3}{2} \\
&= \frac{9}{2}\big(3^k - 1\big) + \frac{6}{2} \\
&= 3\Big(\frac{3}{2}\big(3^k - 1\big)\Big) + 3 \\
&= 3|E_0^k| + 3.
\end{aligned}$$

Since each of the three copies of H_0^k is an equilateral triangle, through flips we can arrange them so that each of the three edges added are the middle edges of the sides of a new equilateral triangle with side length $2\big(2^k - 1\big) + 1 = 2^{k+1} - 1$. As each of the three copies of H_0^k is planar, and the edges added to connect them do not create any crossings, then H_0^{k+1} is planar as well.

Thus H_0^n is planar for all $n \in \mathbb{N}$. ■

We now prove the second main result of the paper.

Theorem 4.2. The only planar Hanoi graphs are H_0^n, H_1^1, and H_1^2.

Proof. We have shown that H_1^1 and H_1^2 are planar by constructing planar embeddings of their graphs, and we have shown in Lemma 4.1 that H_0^n is planar for all $n \in \mathbb{N}$, so it remains to show that all other Hanoi graphs are non-planar.

The Hanoi graph H_2^1 corresponds to the Tower of Hanoi puzzle with 1 disc on 5 pegs.

As was the case with H_1^1, since there is only one disc, it can move freely between the pegs. Since any peg is reachable from any other peg through exactly one move of the disc, every pair of vertices is adjacent in H_2^1. Thus H_2^1 is isomorphic to the complete graph K_5. Hence H_2^1 is non-planar since K_5 is non-planar.

Note that whenever $m \geq 2$ and $n \geq 1$, the Tower of Hanoi puzzle has at least 5 pegs. In any regular state of the puzzle, the smallest disc can move freely between pegs. In particular, it can move freely between any set of 5 pegs, so K_5 is a subgraph of the corresponding Hanoi graph. It follows that H_m^n is non-planar for all $m \geq 2$ and $n \geq 1$.

We summarize the planarity results we have obtained so far in a table:

$m\backslash n$	1	2	3	4	5	$\ldots$
0	Y	Y	Y	Y	Y	$\ldots$
1	Y	Y	?	?	?	$\ldots$
2	$\Downarrow$N$\Rightarrow$	N	N	N	N	$\ldots$
3	N	N	N	N	N	$\ldots$
4	N	N	N	N	N	$\ldots$
5	N	N	N	N	N	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

It remains to cover the case of H_1^n when $n \geq 3$. By Lemma 2.3, H_1^3 is a subgraph of H_1^n for all $n > 3$. So if H_1^3 is non-planar, then so is H_1^n for every $n > 3$. We will present two arguments to show that H_1^3 is non-planar; one is a direct proof using Kuratowski's Theorem, and the other is an argument by contradiction made by Hinz and Parisse.

Kuratowski's Theorem states that if a graph contains a subgraph that is a K_5 or $K_{3,3}$ subdivision, then the graph is non-planar [6, p.246]. We will demonstrate that H_1^3 does contain a K_5 subdivision. In Figure 4.4, we have constructed H_1^3 by taking four copies of H_1^2, one for each position of the largest disc, and added 24 edges corresponding to legal moves of the largest disc. We can verify that this is the correct number of edges by applying the edge count formula again [4]:

$$\begin{aligned}
|E_1^2| &= \frac{(3+1)(2+1)}{4}\big((3+1)^2 - (1+1)^2\big) \\
&= 36 \\
|E_1^3| &= \frac{(3+1)(2+1)}{4}\big((3+1)^3 - (1+1)^3\big) \\
&= 168 \\
&= 4|E_1^2| + 24.
\end{aligned}$$

In Figure 4.5 we have left as solid lines only those edges that will be used in the K_5 subdivision. In Figure 4.6, we have a subgraph of H_1^3 that is a K_5 subdivision, which has been rearranged for clarity. Thus H_1^3 is non-planar by Kuratowski's Theorem.

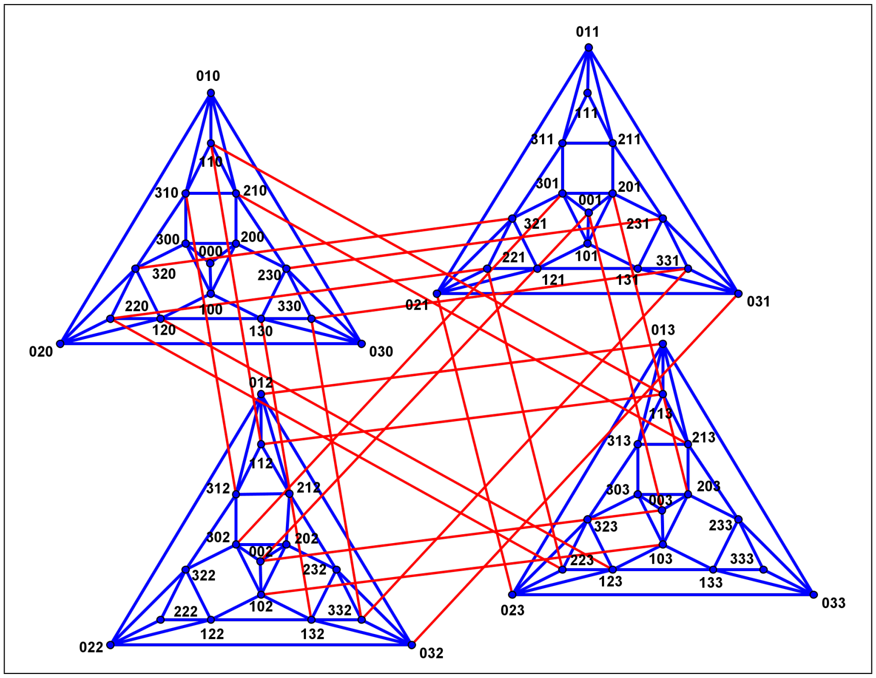

FIGURE 4.4: The Hanoi graph H_1^3.

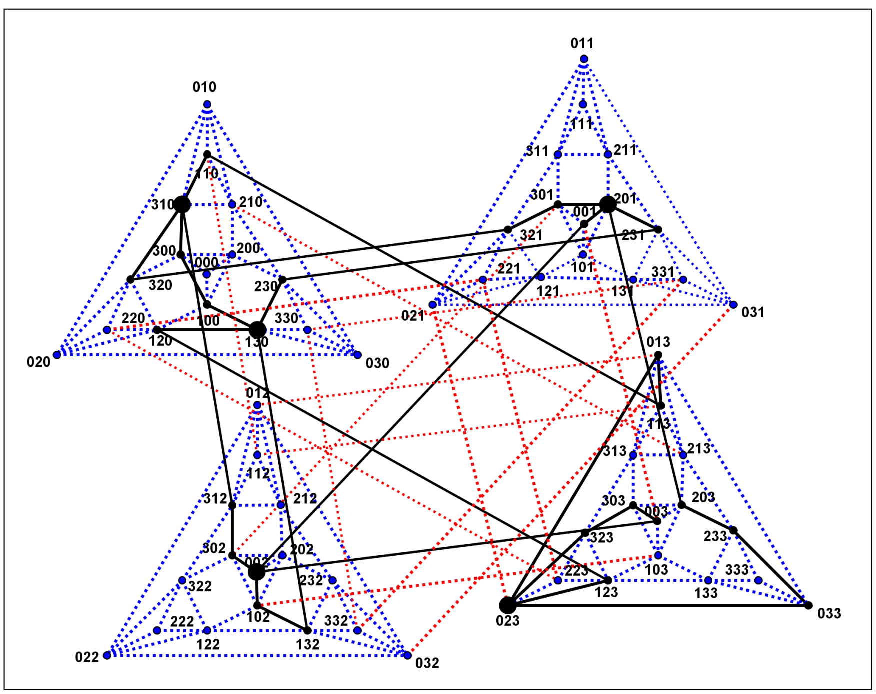

FIGURE 4.5: The Hanoi graph H_1^3, with K_5 subdivision shown by solid edges.

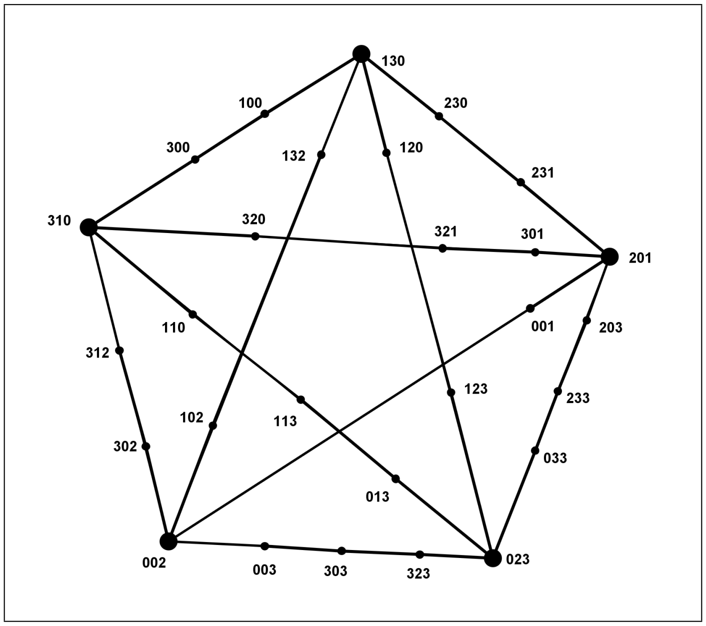

FIGURE 4.6: K_5 subdivision subgraph of H_1^3.

Now, following the argument made by Hinz and Parisse, suppose by way of contradiction that we have a planar embedding of H_1^3. As discussed previously, this consists of four copies of H_1^2, one for each position of the largest disc, interconnected by 24 additional edges. Since the planar embedding of H_1^2 in Figure 4.2 is essentially unique, we see that the faces of H_1^2 all have either three or four vertices in their boundary. In particular, the infinite face has either three or four vertices in its boundary.

Suppose the infinite face of one of the copies of H_1^2 has three vertices. Then these three vertices each have at least degree 3 in H_1^2, since the minimum degree in H_1^2 is 3. Note that there are 12 edges going out of each copy of H_1^2, as can be seen be inspection of Figure 4.4.

We can also see this by a combinatorial consideration of the possible moves of the largest disc: If the largest disc is able to move, there are either 1 or 2 empty pegs. If there is 1 empty peg, then there are $\binom{3}{2} = 3$ choices for which two pegs have the smallest two discs, and 2 ways for these smallest two discs to be positioned on those pegs, giving 6 vertices, each with one edge outward. If there are 2 empty pegs, there are 3 choices of peg for both of the smaller discs to be on, and for each there are 2 choices of peg to move the largest disc to, giving 3 vertices, each with 2 edges outward, for a total of another 6 edges.

Now, in order for the embedding of this copy of H_1^2 to remain planar, these 12 edges must be incident to the three vertices on the outer face. Then by the pigeonhole principle, at least one of these vertices gets another 4 edges added to its degree. So there is some vertex in H_1^3 with degree at least 7. In any regular state of the Tower of Hanoi puzzle with 3 discs and 4 pegs, the disc 1 is able to move to three different pegs, disc 2 is able to move to zero or two pegs, and disc 3 is able to move to zero, one, or two pegs. However, the disc 3 is able to move to two different pegs only if disc 2 cannot move at all. So from any regular state, there are at most six moves possible, so the corresponding vertex in the Hanoi graph has degree at most 6, a contradiction.

Now suppose the infinite face of one of the copies of H_1^2 has four vertices. By inspection of the planar embedding of H_1^2 in Figure 4.2, we see that every such vertex has degree 5. Similarly to the previous case, at least one of these vertices has to get at least three of the additional 12 edges. Then there is a vertex in H_1^3 with degree at least 8, a contradiction, since we have already shown that the degree of any vertex can be no more than 6.

Thus H_1^3 cannot be planar.

Now we have shown that the only planar Hanoi graphs are H_0^n, H_1^1, and H_1^2, as summarized in the table below, and the proof is complete. ■

$m \backslash n$	1	2	3	4	5	$\dots$
0	Y	Y	Y	Y	Y	$\dots$
1	Y	Y	N$\Rightarrow$	N	N	$\dots$
2	$\Downarrow$N$\Rightarrow$	N	N	N	N	$\dots$
3	N	N	N	N	N	$\dots$
4	N	N	N	N	N	$\dots$
5	N	N	N	N	N	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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