Geometric Graph Homomorphisms and the Geochromatic Number

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Abstract

In this mathematical literature project, we study and present the results of the 2011 article by Boutin and Cockburn [2], which defines and proves some fundamental properties of geometric graph homomorphisms. The geochromatic number of a geometric graph is related to both the chromatic number and the thickness; yet we consider an infinite bipartite family of geometric graphs with thickness 2 that have arbitrarily high geochromatic number. We also examine conditions for a graph to have geochromatic number at most four.

1 Introduction

In abstract graph theory, a homomorphism is a structure-respecting map between two graphs. The structure we want to preserve is simply the set of edges of the graph. We often create drawings of abstract graphs because the visualization makes problems easier to solve, but in reality we really only need two things to define a graph: the vertex set and the edge set.

In geometric graph theory, by contrast, the particular drawing of a graph now matters. Therefore, a structure-respecting map between geometric graphs must not only preserve vertex adjacencies, but also edge crossings. This brings a unique set of tools and challenges to the question of finding homomorphisms between geometric graphs. Inspired by the 2011 article by Debra Boutin and Sally Cockburn on geometric graph homomorphisms, this mathematical literature project aims to provide a glimpse of the intricacies of geometric graphs and the maps between them. In Section 2, we present some useful background on (abstract) graph homomorphisms before introducing the notion of a geometric graph. Then in Section 3, we develop a foundation and intuition for geometric graph homomorphisms so that we have the machinery to explore the geochromatic number in Section 4.

2 Preliminaries

In this section we provide some background information to help transition smoothly to the study of geometric graph homomorphisms and the geochromatic number. First we discuss graph homomorphisms in the case of abstract graphs, keeping in mind that all necessary conditions for a function to be an abstract graph homomorphism must also hold for a geometric graph homomorphism. Then we give an overview of geometric graphs, with particular interest in edge crossings.

2.1 Graph homomorphisms

A *homomorphism* f from a graph G to a graph H is a vertex mapping that preserves adjacencies. That is, for $x, y \in V(G)$, $x \sim y$ in G implies $f(x) \sim f(y)$ in H. We write $G \to H$ (and say G is *homomorphic* to H) if there exists a homomorphism f from G to H. We will often refer to $f : G \to H$ as an *abstract* graph homomorphism to distinguish from geometric homomorphisms later on. Since homomorphisms do not need to preserve nonadjacencies, a bijective homomorphism is not necessarily a graph isomorphism. It is also worth noting that $G \to H$ does not imply $H \to G$. However, we do have transitivity: if $G_1 \to G_2$ and $G_2 \to G_3$, then $G_1 \to G_3$ (as the composition of homomorphisms is again a homomorphism).

A *proper vertex coloring* of a graph G is an assignment of colors to the vertices so that no two adjacent vertices have the same color. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors needed for a proper vertex coloring. Equivalently, $\chi(G)$ is the smallest positive integer n such that $G \to K_n$. Such a map may be viewed as a partition of $V(G)$ into independent sets. With this in mind, let $f : G \to H$ be any homomorphism. Since f preserves adjacencies, the preimage of an independent set in H is an independent set in G. Therefore, a proper vertex coloring of H corresponds to a proper vertex coloring back in G. This tells us $\chi(G) \leq \chi(H)$. As we will see, the use of this result extends nicely to geometric homomorphisms.

2.2 Geometric graphs

A *geometric graph* $\overline{G}$ is formed by drawing a simple graph G in the plane with vertices in general position, where all edges are straight line segments between vertices [2]. A *crossing* occurs in $\overline{G}$ when a pair of edges has a common interior point. We say $\overline{G}$ is a *plane* geometric graph if it has no crossings.

The following definitions are also used throughout this paper:

1. An edge e in a geometric graph $\overline{G}$ is a *crossing edge* if it crosses another edge in $\overline{G}$. The set of all crossing edges in $\overline{G}$ is denoted $E_\times$.

2. A vertex v in a geometric graph $\overline{G}$ a *crossing vertex* if it is incident to a crossing edge. The set of all crossing vertices in $\overline{G}$ is denoted $V_\times$.

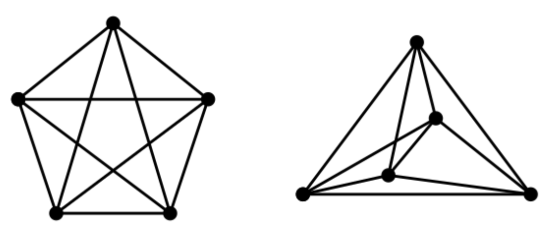

Figure 1: Two geometric graphs realizing K_5

In Figure 1 we provide two geometric realizations of K_5. As abstract graphs, they are isomorphic. As geometric graphs, they are non-isomorphic since their crossing structures differ. The first is a convex geometric graph: the vertices are in a position forming the vertex set of a convex polygon [5]. We can see that every vertex is a crossing vertex, and the inner five-cycle is made up of all the crossing edges. The second is a geometric realization with the minimum number of crossings over *all* drawings of K_5. Indeed, the crossing number of K_5 is 1. In general, this is not always possible to achieve. If G is planar, Fáry's theorem tells us there is a plane drawing of G with straight-line segments. However, if G has crossing number $n > 0$, there may or may not exist a geometric realization (which by definition has straight-line edges) of $\overline{G}$ with n crossings. In fact there is a different parameter, the rectilinear crossing number, which is the minimum number of crossings in a straight-line drawing of G. This is an active area of graph theory research; for example, [1] gives a lower bound on the rectilinear crossing number of K_n.

3 Geometric graph homomorphisms

3.1 Definitions and properties

Let $\overline{G}$ and $\overline{H}$ be geometric graphs. A *geometric homomorphism* $f : \overline{G} \to \overline{H}$ is a vertex mapping that preserves adjacencies and crossings [2]. Note that f is a homomorphism on the underlying abstract graphs G and H with the added condition that if the edges uv and xy are crossed in $\overline{G}$, then the edges $f(u)f(v)$ and $f(x)f(y)$ are crossed in $\overline{H}$.

There is also the notion of a *geometric isomorphism* $\phi : \overline{G} \to \overline{H}$, which is an isomorphism on the underlying abstract graphs G and H such that the edges uv and xy cross in $\overline{G}$ *if and only if* the edges $\phi(u)\phi(v)$ and $\phi(x)\phi(y)$ cross in $\overline{H}$. As is the case for abstract graph homomorphisms, a bijective geometric homomorphism is not necessarily a geometric isomorphism. In fact, different realizations of the same abstract graph may not even have geometric homomorphisms to each other. For example, consider the two geometric realizations of K_4 given in Figure 2. Throughout this paper we refer to the plane realization as $\overline{K}_4$ and the one-crossing realization as $\widehat{K_4}$. Any bijection from $V(\overline{K}_4)$ to $V(\widehat{K_4})$ is a geometric homomorphism since there are no crossings in $\overline{K}_4$ to be preserved. However, there is no geometric homomorphism from $\widehat{K_4}$ to $\overline{K}_4$ since there is no way to preserve the single edge crossing.

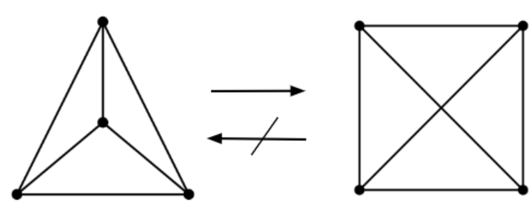

Figure 2: $\overline{K}_4 \to \widehat{K_4}$ but $\widehat{K_4} \not\to \overline{K}_4$

We can generalize the situation above. Suppose $\overline{G}$ is a plane geometric graph (with underlying abstract graph G) and H is any graph such that $G \to H$. Then $\overline{G} \to \overline{H}$ for any geometric realization $\overline{H}$ of H. We make use of this fact to simplify the work in Section 4.

3.1.1 Chromatic number and thickness

Since all geometric homomorphisms are also abstract graph homomorphisms, if $\overline{G} \to \overline{H}$, then $\chi(G) \leq \chi(H)$. The parameter we next define, the (geometric) thickness $\theta(\overline{G})$ of a geometric graph, gives us another similar result. As in [2], we define a *thickness edge coloring* of $\overline{G}$ as a coloring of the edges of $\overline{G}$ so that no two edges of the same color cross each other. Then $\theta(\overline{G})$ is the minimum number of colors required for a thickness edge coloring. The purpose of defining $\theta(\overline{G})$ in this manner is to emphasize its dependence on the particular geometric realization of a graph. By contrast, the thickness of an abstract graph is the minimum of this value over all (not necessarily geometric) realizations.

Note that a thickness edge coloring of a geometric graph is a partition of the edge set into plane subgraphs. Moreover, if $f : \overline{G} \to \overline{H}$ is a geometric homomorphism, then the preimage of any plane subgraph in $\overline{H}$ is a plane subgraph in $\overline{G}$. So any thickness edge coloring of $\overline{H}$ corresponds to a thickness edge coloring back in $\overline{G}$. Therefore, $\theta(\overline{G}) \leq \theta(\overline{H})$.

Putting these results together, we get the following lemma:

Lemma 1. *If $\overline{G} \to \overline{H}$, then*

$$\chi(G) \leq \chi(H) \text{ and } \theta(\overline{G}) \leq \theta(\overline{H}).$$

3.1.2 Lemmas: non-identifiable vertices

We end this subsection by compiling three results from [2] regarding vertices that cannot be identified by any geometric homomorphism. We will use these lemmas in Section 4.

Lemma 2. *Adjacent vertices cannot be identified by any geometric homomorphism.*

This is true because any geometric homomorphism is an abstract graph homomorphism.

Lemma 3. *Endpoints of edges that cross cannot be identified by any geometric homomorphism.*

Proof. Let $f : \overline{G} \to \overline{H}$ be a geometric homomorphism. Suppose edge uv crosses edge xy in $\overline{G}$. Since f preserves all crossings, and no pair of edges with a common endpoint can cross, each of u, v, x, w must be mapped to a distinct vertex in $\overline{H}$. □

Lemma 4. *Let $\overline{P}$ be an odd-length path in $\overline{G}$. If there is a single edge crossing all the edges in $\overline{P}$, then the endpoints of $\overline{P}$ cannot be identified by any geometric homomorphism.*

Proof. Let $\overline{P} = v_1 v_2 \ldots v_{2r}$ be such a path in $\overline{G}$, all edges of which are crossed by edge xy. Let $f : \overline{G} \to \overline{H}$ be a geometric homomorphism. For $i = 1, \ldots, 2r-1$, since edge $v_i v_{i+1}$ crosses edge xy, it follows that edge $f(v_i)f(v_{i+1})$ crosses edge $f(x)f(y)$. This means the vertices $f(v_1), f(v_3), \ldots, f(v_{2r-1})$ are on one side of the line in $\mathbb{R}^2$ containing edge $f(x)f(y)$, while vertices $f(v_2), f(v_4), \ldots, f(v_{2r})$ are on the other side. Therefore, $f(v_1) \neq f(v_{2r})$. □

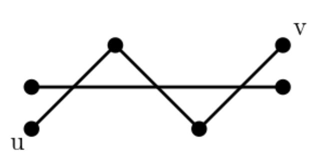

Figure 3: By Lemma 4, u and v cannot be identified.

3.2 Examples

It is interesting to restrict our attention to a single isomorphism class of abstract graphs. When we transition to viewing the various straight-line drawings of a graph as geometric graphs, these drawings suddenly represent very distinct structures. One way to highlight similarities and differences between these structures is to determine which pairs of graphs have a geometric homomorphism between them.

In Figure 4 are five geometric realizations of the Petersen graph, with the number of crossings ranging from 2 to 7. These graphs are labeled so that $\overline{P}_i$ has i crossings. In addition to the different crossing numbers, these geometric graphs also have widely varying numbers of crossing edges and crossing vertices. This information is provided in Table 1. Note that an increase in the number of crossings does not necessarily imply an increase in the number of crossing edges or crossing vertices.

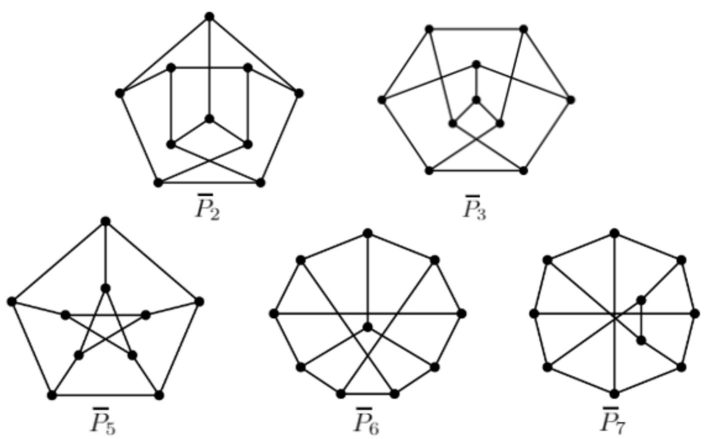

Figure 4: Five geometric realizations of the Petersen graph

	$\lvert E_\times \rvert$	$\lvert V_\times \rvert$
$\overline{P}_2$	4	8
$\overline{P}_3$	6	9
$\overline{P}_5$	5	5
$\overline{P}_6$	6	10
$\overline{P}_7$	5	8

Table 1: Counting crossing edges and crossing vertices

In the typical case, the information provided in Table 1 would not be enough for us to rule out the existence of a geometric homomorphism between graphs. However, in this case we can use the fact that the Petersen graph P is a core. That is, any endomorphism of P is actually an automorphism [4]. So if a geometric homomorphism $f : \overline{P}_i \to \overline{P}_j$ exists, then f must be a bijection that preserves all adjacencies, non-adjacencies, and edge crossings. For this reason, $\overline{P}_j$ must have at least as many crossings, crossing edges, and crossing vertices as $\overline{P}_i$ does. This observation alone allows us to rule out 14 of the 20 possible cases for a geometric homomorphism between a pair of these graphs. For the remaining 6 cases, we have to compare the crossing structures of the graphs.

For instance, each pair of crossed edges in $\overline{P}_3$ shares vertices with the other edge crossings. Thus, $\overline{P}_2 \not\to \overline{P}_3$ because it is impossible to preserve the two vertex-disjoint crossings of $\overline{P}_2$. We can determine $\overline{P}_3 \not\to \overline{P}_6$ because, while three vertices in $\overline{P}_3$ are each involved in multiple edge-disjoint crossings, there is only one vertex in $\overline{P}_6$ that is incident to multiple crossing edges. The fact that each of five crossing edges in $\overline{P}_5$ is involved in multiple crossings tells us $\overline{P}_5 \not\to \overline{P}_6$ and $\overline{P}_5 \not\to \overline{P}_7$. Now there are two cases left. For these, we can prove $\overline{P}_2 \to \overline{P}_6$ and $\overline{P}_2 \to \overline{P}_7$ by giving geometric homomorphisms as shown in Figure 5. That is, among these five geometric realizations of the Petersen graph, there are only two pairs where a geometric homomorphism exists.

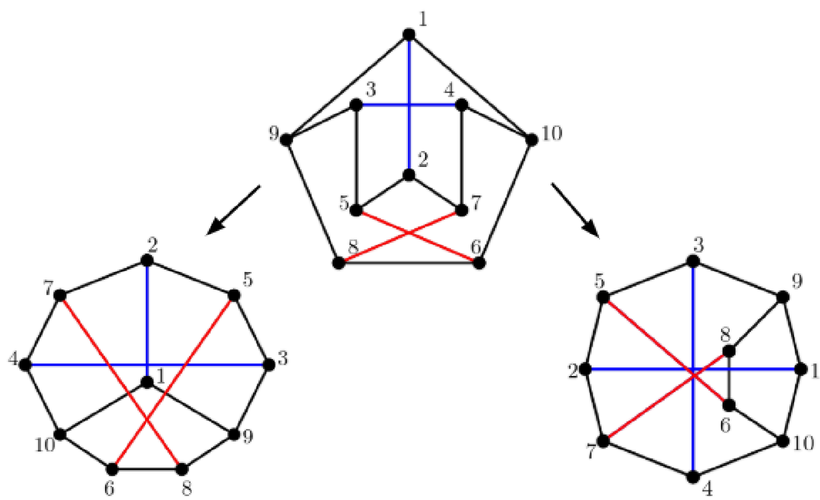

Figure 5: Geometric homomorphisms between realizations of the Petersen graph

Of course, in more general cases, a geometric homomorphism need not be injective. For example, the 8-vertex cubic graph in Figure 6 has a geometric homomorphism to $\widehat{K_4}$, despite having more crossings, crossing vertices, and crossing edges. In Section 4, we examine specific criteria for a graph to be geometrically homomorphic to $\widehat{K_4}$.

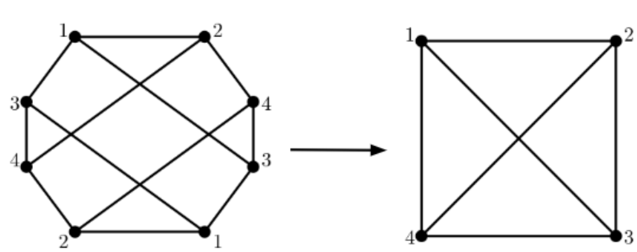

Figure 6: A geometric homomorphism to $\widehat{K_4}$

4 Geochromatic number

4.1 Definitions and properties

Now that we have developed sufficient background on geometric homomorphisms, in this section we explore the geochromatic number.

Let $\overline{G}$ be a geometric graph. As in [2], we say $\overline{G}$ is *n-geocolorable* if $\overline{G} \to \overline{K}_n$ for some geometric realization of K_n. The *geochromatic number* of $\overline{G}$, denoted $X(\overline{G})$, is the smallest positive integer n such that $\overline{G}$ is n-geocolorable.

For $n = 1, 2, 3$, there are simple conditions for a geometric graph to have geochromatic number n:

- $n = 1$: Since K_1 is just an isolated vertex, $X(\overline{G}) = 1$ if and only if $\overline{G}$ is an empty graph.
- $n = 2$: K_2 consists of a single edge, so any geometric realization is a plane graph. Thus, $X(\overline{G}) = 2$ if and only if $\overline{G}$ a plane bipartite graph with at least one edge.
- $n = 3$: Here again, the only geometric realization of K_3 is plane. Therefore, $X(\overline{G}) = 3$ if and only if $\overline{G}$ is a plane graph such that $\chi(G) = 3$.

As n increases, it quickly becomes more difficult to determine whether a given graph $\overline{G}$ is n-geocolorable. For one thing, as n gets large there are increasingly many geometric realizations of K_n to consider, and determining these realizations is a complicated problem on its own. The bounds presented in Lemma 1 give a pair of necessary conditions: if $\overline{G} \to \overline{K}_n$ for some geometric realization of K_n, then $\chi(G) \leq n$ and $\theta(\overline{G}) \leq \theta(\overline{K}_n)$.

The following definitions will be used this section:

1. The *crossing subgraph* $\overline{G}_\times$ is the geometric subgraph of $\overline{G}$ with vertex set $V_\times$ and edge set $E_\times$.
2. The *crossing components* of $\overline{G}$, denoted $\overline{C}_1, \overline{C}_2, \ldots \overline{C}_m$, are the connected components of $\overline{G}_\times$.
3. The *crossing component graph* of $\overline{G}$ is the abstract graph denoted $C_\times$ whose vertices correspond to the crossing components of $\overline{G}$, with an edge between vertices $\overline{C}_i$ and $\overline{C}_j$ if an edge of $\overline{C}_i$ crosses an edge of $\overline{C}_j$ in $\overline{G}$.
4. The *induced crossing subgraph* $\overline{G}[V_\times]$ is the geometric subgraph of $\overline{G}$ induced by $V_\times$.

Examples of these definitions are provided in Figures 7 and 8. Note that $\overline{G}, \overline{G}_\times$, and $\overline{G}[V_\times]$ may not all be distinct for a given geometric graph. For example, the induced crossing subgraph of $\overline{P}_6$ is the same as $\overline{P}_6$ itself since all vertices are crossing vertices. $\overline{P}_5$ has the same crossing subgraph and induced crossing subgraph, since the only edges between crossing vertices are crossing edges. In most of the examples in this paper, the crossing component graph is a simple graph. However, $C_\times$ has a loop if $\overline{G}$ has a crossing component that is not plane. This the case for $\overline{P}_5$, which has a single crossing component which is not plane. Therefore, its crossing component graph consists of a single vertex with a loop.

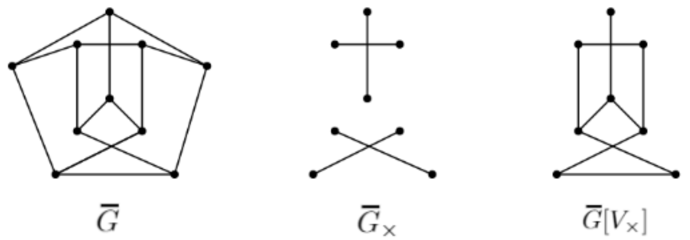

Figure 7: Crossing subgraph and induced crossing subgraph

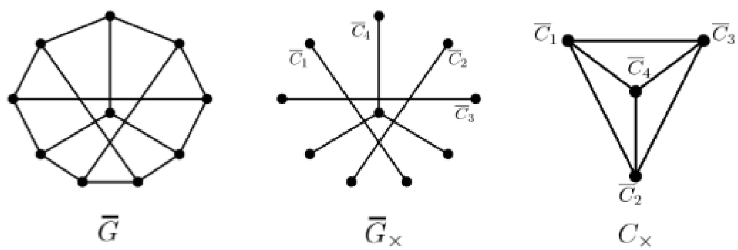

Figure 8: Crossing subgraph and crossing component graph

Using these definitions, we can provide another set of necessary conditions for $\overline{G} \to \overline{H}$.

Lemma 5. *If $\overline{G} \to \overline{H}$, then*

1. $\overline{G}_\times \to \overline{H}_\times$,
2. $\overline{G}[V_\times] \to \overline{H}[V_\times]$,
3. $C_\times(\overline{G}) \to C_\times(\overline{H})$. *That is, there is an (abstract) homomorphism from the crossing component graph of $\overline{G}$ to the crossing component graph of $\overline{H}$.*

Proof. Let $f : \overline{G} \to \overline{H}$ be a geometric homomorphism. Since f preserves crossing edges and crossing vertices, we can obtain homomorphisms for (1) and (2) by appropriate restrictions of f. For (3), we define a homomorphism by mapping each crossing component $\overline{C}_i$ of $\overline{G}$ to its image $f(\overline{C}_i)$ in $\overline{H}$. First we show this map is well-defined. Let $x, y \in \overline{C}_i$ be vertices in the same crossing component of $\overline{G}$. This means there is a path from x to y in $\overline{G}$ consisting only of crossing edges. Since f preserves adjacencies and crossing edges, there is a path of equal or shorter length from $f(x)$ to $f(y)$ in $\overline{H}$ consisting only of crossing edges. Therefore, $f(x)$ and $f(y)$ are in the same crossing component of $\overline{H}$. Now we verify this map is indeed a homomorphism. Suppose $\overline{C}_i \sim \overline{C}_j$ in $C_\times(\overline{G})$. This means an edge of $\overline{C}_i$ crosses an edge of $\overline{C}_j$ in $\overline{G}$. Since f preserves crossings, it follows that an edge of $f(\overline{C}_i)$ crosses an edge of $f(\overline{C}_j)$ in $\overline{H}$. But this means $f(\overline{C}_i) \sim f(\overline{C}_j)$ in $C_\times(\overline{H})$, as needed. $\square$

Figure 9 gives an example to show that the conditions in Theorem 5 are not sufficient for $\overline{G} \to \overline{H}$. Here $\overline{G}$ is the Petersen graph, which we have noted is a core, and so it does not retract to a proper subgraph. Even though $\overline{G}$ and $\overline{H}$ have the same crossing subgraph and induced crossing subgraph, we see $\overline{G} \not\to \overline{H}$.

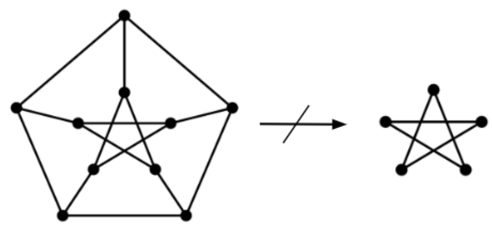

Figure 9: $\overline{G} \not\to \overline{H}$ even though $\overline{G}_\times \to \overline{H}_\times$.

4.2 Geochromatic number at most four

In this section we examine conditions for a geometric graph to be 4-geocolorable. First we note that any geometric graph homomorphic to $\overline{K}_4$ (the plane realization) is also homomorphic to $\widehat{K_4}$ (the one-crossing realization). Therefore, we only need to study criteria for $\overline{G} \to \widehat{K_4}$. As in Figure 6, $\widehat{K_4}$ has vertex set $\{1, 2, 3, 4\}$ with edge 13 crossing edge 24.

First we provide a definition:

For any subset of vertices $Y = \{\overline{C}_{i_1}, \ldots, \overline{C}_{i_r}\}$ of the crossing component graph $C_\times$, Let $\overline{G}_Y$ denote the subgraph of $\overline{G}$ induced by the vertices in $\overline{C}_{i_1} \cup \ldots \cup \overline{C}_{i_r}$. Note that $\overline{G}_Y$ contains any non-crossing edges among the vertices in $\overline{C}_{i_1} \cup \ldots \cup \overline{C}_{i_r}$, so $\overline{G}_Y$ is a subgraph of $\overline{G}[V_\times]$.

Now we present a set of necessary conditions for $\overline{G}$ to have geochromatic number at most four.

Theorem 1. *[2] If $\overline{G}$ is 4-geocolorable, then*

1. *Each crossing component $\overline{C}_i$ is a bipartite plane subgraph of $\overline{G}$.*
2. *There is a proper bipartition (U, V) of $V(C_\times)$ so that $\overline{G}_U$ and $\overline{G}_V$ are bipartite plane subgraphs of $\overline{G}$.*

Proof. If $\overline{G}$ is 4-geocolorable, then there exists a geometric homomorphism $f : \overline{G} \to \widehat{K_4}$. By Lemma 5, we can restrict f to get a homomorphism from the crossing subgraph $\overline{G}_\times$ to the crossing subgraph $(\widehat{K_4})_\times$. Note that $(\widehat{K_4})_\times$ consists of the disjoint edges 13 and 24. Therefore, each crossing component $\overline{C}_i$ of $\overline{G}_\times$ is mapped to either edge 13 or edge 24. This proves each $\overline{C}_i$ is bipartite. Further, the preimage of a plane graph under a geometric homomorphism is a plane graph; this proves each $\overline{C}_i$ is plane.

To prove (2), partition $V(C_\times)$ by setting $U = \{\overline{C}_i | f(\overline{C}_i) = 13\}$ and $V = \{\overline{C}_i | f(\overline{C}_i) = 24\}$. Without loss of generality, if $e_i \in \overline{C}_i$ crosses $e_j \in \overline{C}_j$ in $\overline{G}_\times$, then $f(e_i) = 13$ and $f(e_j) = 24$. In $C_\times$, this shows that $\overline{C}_i \sim \overline{C}_j$ if and only if $\overline{C}_i$ and $\overline{C}_j$ are in different cells of the bipartition. This proves $C_\times$ is bipartite. Now consider the subgraph $\overline{G}_U$ of G. Since $\overline{G}_U$ is induced by the set of vertices in the crossing components that all get mapped to edge 13, we see that $\overline{G}_U$ is homomorphic to the edge 13. The same argument shows $\overline{G}_V$ is homomorphic to the edge 24. Therefore, $\overline{G}_U$ and $\overline{G}_V$ are bipartite and plane. $\square$

The geometric graph $\overline{G}$ in Figure 10 shows that the conditions in Theorem 1 are not sufficient for the existence of a 4-geocoloring. Since each of two crossing components in $\overline{G}_\times$ is a single edge, conditions 1 and 2 are satisfied. Now observe that one of the crossing edges needs to be mapped to 13 while the other crossing edge is mapped to 24. But then the fifth vertex would need a fifth color since it is adjacent to each of the other four vertices. This proves $X(\overline{G}) > 4$. Since there is only one crossing, it is easy to create a homomorphism from $\overline{G}$ to a geometric realization of K_5. Thus, $X(\overline{G}) = 5$.

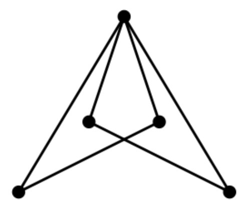

Figure 10: [2] The conditions in Theorem 1 are not sufficient for $\overline{G} \to \widehat{K_4}$.

Although the conditions in Theorem 1 are not sufficient for $X(\overline{G}) = 4$, the next theorem shows that any geometric graph satisfying those conditions has geochromatic number at most 8.

Theorem 2. *If $\overline{G}$ satisfies conditions 1 and 2 in Theorem 1, then $X(\overline{G}) \leq 8$.*

Proof. First we show that $\overline{G}[V_\times]$ is 4-geocolorable. Let U, V be the cells of the bipartition of the crossing component graph of $\overline{G}$ as in condition 2. Since $\overline{G}_U$ is biparite, we can properly color it using colors 1 and 3. Similarly, we can properly color $\overline{G}_V$ using colors 2 and 4. Since $\overline{G}_U$ and $\overline{G}_V$ are plane, any pair of crossed edges in $\overline{G}$ must have one edge in $\overline{G}_U$ and the other edge in $\overline{G}_V$ (thus, they are colored 13 and 24). Note that there may be edges in $\overline{G}[V_\times]$ that are not in $\overline{G}_U$ or $\overline{G}_V$. However, any such edge is a non-crossing edge with one vertex in each subgraph, and so its endpoints are different colors. Therefore, we have defined a 4-geocoloring of $\overline{G}[V_\times]$. Now, the subgraph induced by the non-crossing vertices is plane, so by the Four color theorem we can properly color it using colors 5,6,7,8. This gives a proper 8-coloring of $\overline{G}$ where all pairs of crossed edges are colored 13 and 24. Thus, $\overline{G} \to \widehat{K_8}$, where $\widehat{K_8}$ is the convex realization shown in Figure 11. This proves $X(\overline{G}) \leq 8$. □

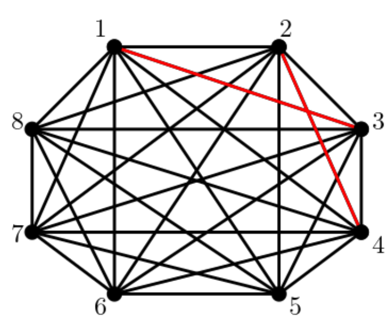

Figure 11: Convex realization of K_8

It turns out the two conditions in Theorem 1 are both necessary and sufficient for a 4-geocoloring if $\overline{G}$ is bipartite. This brings us to the next theorem.

Theorem 3. *Let $\overline{G}$ be a **bipartite** geometric graph. Then $\overline{G}$ is 4-geocolorable if and only if both conditions of Theorem 1 are satisfied.*

Proof. If $\overline{G}$ is 4-geocolorable, then Theorem 1 completes this direction of the proof.

Now assume $\overline{G}$ meets both conditions of Theorem 1. Since $\overline{G}$ is bipartite, we can properly 2-color the vertices of $\overline{G}$ using colors 1 and 2. Now in $\overline{G}_U$, re-color all vertices colored 1 with color 4; in $\overline{G}_V$, re-color all vertices colored 2 with color 3. Note that all vertices in $\overline{G}_U$ are colored 2 or 4, all vertices in $\overline{G}_V$ are colored 1 or 3, and all non-crossing vertices are still colored 1 or 2. Since $\overline{G}_U$ and $\overline{G}_V$ are both plane subgraphs of $\overline{G}$, each pair of crossing edges has one edge in $\overline{G}_U$ and the other in $\overline{G}_V$. This means every pair of crossing edges in $\overline{G}$ gets mapped to the pair of crossing edges in $\widehat{K}_4$. So, this proper 4-coloring preserves crossings and is in fact a 4-geocoloring of $\overline{G}$. □

Figure 6 provides an example of a 4-geocolorable bipartite graph. In this case, each of the four crossing components is a single edge that crosses two other crossing components. Thus, the crossing component graph $C_\times$ is a 4-cycle. The two graphs $\overline{G}_U$ and $\overline{G}_V$ induced by a proper bipartition of $C_\times$ are both plane 4-cycles in $\overline{G}$.

4.3 Unbounded geochromatic number

We end this section by presenting a family of bipartite, thickness-2 geometric graphs with arbitrarily high geochromatic number. This highlights the fact that we really need more information than just the chromatic number and thickness of a geometric graph in order to determine its geochromatic number. As we have seen in previous examples, it really comes down to the adjacency and crossing structures.

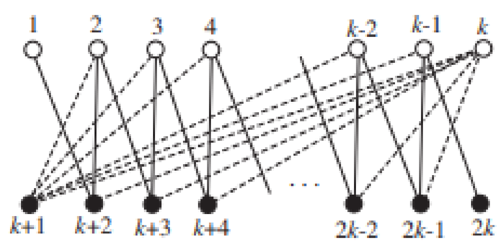

Figure 12: [2] A bipartite, thickness 2 geometric graph with $X(\overline{G}) = 2k$.

Theorem 4. *The geochromatic number of a bipartite, thickness-2 geometric graph is arbitrarily large. In particular, the construction of $\overline{G}$ given in Figure 12 yields a geometric graph with geochromatic number $2k$.*

Proof. We construct $\overline{G}$ on $2k$ vertices as in Figure 12. Position k white vertices labeled $1, \ldots, k$ in a row, and k black vertices labeled $k+1, \ldots, 2k$ in another row below them. If the vertices are to be in general position, we cannot have three vertices on the same line. However, we can skew the rows slightly without affecting the construction or the proof.

First connect 1 to $k+2$ with a solid edge. Then for $i = 2, \ldots, k-1$, connect i to $k+i$ and $k+i+1$ with a solid edge. Observe that none of the solid edges cross each other. Now connect $k+1$ to each of $2, \ldots, k$ with a dashed edge and connect k to each of $k+2, \ldots, 2k-1$ with a dashed edge. Observe that none of the dashed edges cross each other. Therefore, we have two plane layers and so $\overline{G}$ is a thickness-2 bipartite graph.

Next we prove that $X(\overline{G}) \geq 2k$. We can do this by showing that no two vertices in $\overline{G}$ can be identified by any geometric homomorphism. This implies that any geometric homomorphism $f : \overline{G} \to \overline{K}_n$ must be injective, and so $n \geq 2k$. We consider two cases:

Vertex i and vertex j are in the same cell of the bipartition: First suppose that i and j are both white vertices. Without loss of generality, assume $i < j$. Note that i and j are involved in a common crossing since the dashed edge from j to $k+1$ crosses the solid edge from i to $k+i+1$. By Lemma 3, i and j cannot be identified by any geometric homomorphism. By the symmetry of the construction, a similar argument holds if i and j are both black vertices.

Vertex i and vertex j are in opposite cells of the bipartition: Suppose i is a white vertex and j is a black vertex. First consider $i = k$ and $j = k+1$. Then i and j are adjacent and cannot be identified by Lemma 2. Next consider the case where $i \neq k$ and $j \neq k+1$. Then i and j are joined by an odd-length path of solid edges, all of which are crossed by the edge $k(k+1)$. By Lemma 4, i and j cannot be identified. Now let $j = k+1$ with $i \neq k$. Then edge $k(k+1)$ crosses edge $i(i+k+1)$ and so by Lemma 3, i and $j = k+1$ cannot be identified. Similarly, $i = k$ cannot be identified with $j \neq k+1$.

We have exhausted all possibilities, so we get $X(\overline{G}) \geq 2k$. But by adding all the missing edges to $\overline{G}$ until we have a complete graph, we get a geometric realization $\overline{K}_{2k}$ such that $\overline{G} \to \overline{K}_{2k}$. Therefore, $X(\overline{G}) = 2k$. If we want an odd geochromatic number, we can delete vertex $2k$ and all arguments are the same.

□

5 Further study

A natural continuation of this study of the geochromatic number might be to examine conditions for $X(\overline{G}) \leq 5$. To begin, we first should find all geometric realizations of K_5. Then to simplify the work as we did in Section 4, it is useful to find which of these realizations are homomorphic to other realizations. This type of problem is addressed in another paper by Boutin and Cockburn [3], where they use geometric homomorphisms to define a partial order on the set of geometric realizations of a given abstract graph. To ensure the relation is antisymmetric, the following definition is used: $\overline{G} \preceq \overline{H}$ if there is a geometric homomorphism $f : \overline{G} \to \overline{H}$ that induces an isomorphism on the underlying abstract graphs. For example, the poset for K_4 is a single chain of two elements as we have seen in Figure 2. The poset for K_5 consists of a single chain of three elements as shown in Figure 13 below. Therefore, to check if a given geometric graph $\overline{G}$ is 5-geocolorable, it is sufficient to check if there is a geometric homomorphism from $\overline{G}$ to the five-crossing realization of K_5 (the maximal element of the chain).

Things become considerably more complicated for K_6. In [3], the Hasse diagram for the poset of K_6 realizations is provided. There are a total of 15 non-isomorphic geometric realizations, with the number of edge crossings ranging from 3 to 15 (but no realizations with 13 or 14 crossings). The poset has three maximal elements. Therefore, checking if $\overline{G}$ is 6-geocolorable requires checking if $\overline{G}$ has a geometric homomorphism to at least one of these three maximal realizations of K_6.

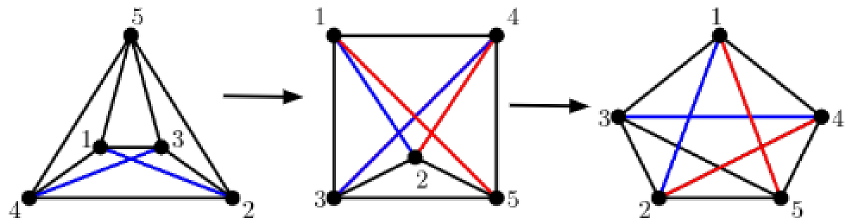

Figure 13: Poset of geometric realizations of K_5

Other questions for continued study of geocolorings are presented in [2]. For example, we know that any geometric graph satisfying the conditions in Theorem 1 has geochromatic number at most 8. But what is the largest geochromatic number possible in a geometric graph that meets those conditions? Figure 10 gives such an example where $X(\overline{G}) = 5$.

Geometric graphs have not been studied nearly as extensively as abstract graphs, so there is still plenty of territory that is yet to be explored. Boutin and Cockburn's 2011 article was the first work in the literature to extend the theory of graph homomorphisms to geometric graphs. Any question that is interesting for abstract graphs is bound to be interesting for geometric graphs, most likely with additional intricacies to consider.

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