CHAPTER THREE

Theoretical Orientation and Conceptual Framework

Introduction

Having now introduced and motivated the topic, defined the research questions, and looked at some of the research germane to the proposed study, this chapter is devoted to establishing the theoretical orientation and conceptual framework within which the study will take place. Some in the mathematics education community may talk about theoretical or conceptual frameworks as if they were interchangeable phrases, while others make a distinction (Eisenhart, 1991; Lester, 1991). Particularly for novices, Lester (1991) comments about the "very good doctoral students" he has worked with, saying that

> "as they begin their doctoral dissertations almost all of them have very little, if any, understanding of what it means to have a conceptual or theoretical framework for their research" (p. 194).

Romberg (1992) affirms this by noting that novice researchers often jump from the identification of an interesting problem straight to the design of a study without "situating their study with the work of others" (p. 56). More is needed than just a review of research to inform and guide the study: An explicit description of the theoretical orientation as well as a conceptual framework for the study is required. To define the aforementioned phrases it is first useful to expound the distinctions between theoretical and conceptual frameworks that others have delineated; it will then be made explicit what is meant by adopting a theoretical orientation as well as a conceptual framework for this study.

Eisenhart (1991) described a theoretical framework as “a structure that guides research by relying on a formal theory…constructed by using an established, coherent explanation of certain phenomena and relationships” (p. 205). She cites as an example Vygotsky’s theory of socio-historical constructivism. Marshall and Rossman (1989) explain how a researcher will locate a research problem in a body of theory; the location is chosen on the basis of the researcher’s own underlying assumptions, and these assumptions must be explicitly stated. Thus, theoretical frameworks can be expected to invoke a host of values and beliefs, not unique to the researcher, but shared in a common paradigm with other scholars. This is why researchers generally do not locate their work in some completely new theoretical framework, creating something that sounds compelling to them alone, but will seek to identify the perspectives that align their work with other researchers.

On the other hand, a conceptual framework is described by Eisenhart (1991) as “a skeletal structure of justification, rather than a skeletal structure of explanation” (p. 209); this structure is based on either formal logic or experience. As such, it consists of an argument which can incorporate differing points of view, and which culminates in the articulation of a rationale for the adoption of some ideas or concepts in favor of others. The chosen ideas or concepts serve to guide the data collection and analysis:

> “Crucially, a conceptual framework is an argument that the concepts chosen for investigation or interpretation, and any anticipated relationships among them, will be appropriate and useful, given the research problem under investigation” (p. 209).

Lester (1991) finds the distinction between justification and explanation useful for mathematics education, noting that the most persistent needs in research are to justify “why a particular question is proposed to be studied in a particular way and why certain factors (e.g., concepts, behaviors, attitudes, societal forces) are more important than others" (p. 195). Thus, conceptual frameworks, like theoretical frameworks, also embody certain values and beliefs; the key difference that I see is that the latter are more rigidly tied to the body of theory they adhere to, while the former are built to be more flexible. That is, a drawback in relying too heavily on formal theory is that such an approach may minimize potentially important information (Becker, 1991). On the other hand, "conceptual framework facilitate more comprehensive ways of investigating a research problem" (Eisenhart, 1991, p. 211).

What I have chosen to do is organize some main assumptions underlying the study in what I am calling the theoretical orientation of social constructivism. These assumptions are intended to broadly cover the domains of knowledge, learning, and the implications for teacher training, and are articulated in the following sections of this chapter. Moreover, in establishing the perspective of the study, it is shown how these views are derived from or tied to existing bodies of theory in the literature regarding the nature of mathematical knowledge, how one comes to learn and what this means for prospective teachers.

While the theoretical orientation lays out a more general attitude, what I am calling the conceptual framework more specifically looks at the matter of understanding variation. As such, a working model for studying conceptions of variation is established, and this model reflects a kind of synthesis of earlier research on this topic. This framework guides the study by helping to inform not only the statement of the research questions, but the design of the data collection and subsequent analysis to help answer the questions. It also helps suggest what concepts of variation should be examined, and in what context.

Knowledge

The various forms of constructivism come with several different names, and a review of theory makes it clear that any author should not rely solely on a particular label to convey the perspective taken. Rather, it seems better to be explicit about the terms being used and what they mean. For example, Begg (1995) claims that "constructivism is a theory of knowing and learning, rather than a theory of teaching" (p. 71). Ernest (1996), on the other hand, claims that the forms of constructivism comprise an educational paradigm, represented by theories of ontology, epistemology, methodology, and a theory of teaching (pedagogy). This section, about the nature of mathematical knowledge, addresses the epistemological aspects of social constructivism; that is, the intention is not to portray the entire gamut of views on mathematical knowledge in detail, but to articulate the main components which this study assumes.

In order to explicate the essential features of social constructivism as an epistemological stance, it is useful to distinguish between personal and public philosophies about mathematics. The former are implicit and held more privately, while the latter "are explicitly stated and subject to critical analysis" (Neyland, 1995, p. 142). Ernest (1991) uses the Perry theory to give examples of three personal beliefs: Dualism, for example, views mathematics as "concerned with facts, rules, correct procedures and simple truths determined by absolute authority" (p. 113). In Multiplistic views, a plurality of perspectives in epistemology is admitted, but there is no rational basis for any particular choice (Ernest, 1998). Neyland (1995) suggests that this position is "illustrated by children who see mathematics as a mass of rules from which they make random selections in an effort to achieve the required answer" (p. 142); such a child is profiled in the research of Erlwanger (1973). This study more closely reflects the Relativistic views of mathematics, which not only acknowledges multiple approaches to problems, but holds that the key to evaluating the different approaches depends on the features of the overall context (Ernest, 1998; Ernest 1991).

For the public philosophies, Ernest offers the two categories of Absolutism and Fallibilism; the former, he claims, is "the dominant epistemological perspective of mathematics" (1991, p. 3). Absolutism holds that mathematics contains infallible, unquestionable, and unchangeable truths. The position of this study is more in the direction of Fallibilism; as the name implies, this view sees mathematical truth as fallible. What this means is that

> "Mathematical concepts and proofs can never be regarded as beyond revision and correction; they may require renogiation as standards of rigour change or new meanings emerge…Mathematics is a dialogue between people exploring mathematical problems, and it must be viewed in its historical and social context" (Neyland, 1995, p. 143).

Taken together, the more personal belief of Relativism along with the public Fallibilist perspective combine in what Ernest (1991, 1998) calls the of social constructivist philosophy of mathematical knowledge. It is important to understand that this aspect plays more to the epistemological interpretation of constructivism; the subsequent section of this chapter discusses aspects of constructivism in terms of building knowledge, or learning . As Janvier (1996) put it, "For me, constructivism is primarily a philosophical theory or position about knowledge and knowledge acquisition" (p. 449).

In assuming a social constructivist stance towards mathematical knowledge, an example pertinent to this study is illustrative. Consider the notion of randomness, and then consider what it means to know if some outcome is or is not random. Somewhat surprisingly, not only is randomness "a complicated concept that is describable in many ways" (May, 1997, p. 222), but there remains a "fundamental question of the real nature of randomness" (Rial, 1998, p. 482). In fact, the exact wording "What is random?" is the title of more than one article, at least one book, and the question is addressed by several other authors as well (Kac, 1983; May, 1997; Beltrami, 1999; Pagels, 1982; Kolata, 1986; Bennett, 1998). Thus, knowledge of what is random can seem tenuous and very much dependent on context; therefore, some advocate leaving the notion of randomness to the realm of the undefined, much the same way that the notions of point or line are left undefined in geometry. Pagels (1982) not only commented that "mathematicians have never succeeded in giving a precise definition of randomness" (p. 85), but even went so far as to exclaim: "Mathematicians don't know what random is!" (p. 87). While a concise definition seems intractable, facets of the concept such as uncertainty, unpredictability, and lack of pattern or causes do provide useful lenses with which to examine randomness (Canada, 1999). However, in terms of epistemology, determining what counts as random proves to be more of a human construct and less of an absolute, objective knowledge. For instance, in considering the randomness of a coin toss, Ford (1983) claims that "coin tosses are universally presumed completely random despite their obvious underlying determinism" (p. 40). As another example, which sequence looks more random: Sequence A (01101100) or Sequence B (01010101) ? Seife (1997) claims that any mathematician could tell you that Sequence A is more random, "but none could tell you just how much more random" (p. 532). Seife discusses recent tests which have been designed to measure randomness, and points to conclusions that let theorists claim that $\sqrt{2}$ is more random than **e**, but both are less random than π. However, Batanero and Serrano (1999) admit that "in a test of randomness there is always a small probability that the sequence is not random in spite of having passed the randomness tests" (pp. 560-561). Especially for finite sequences, whether or not they are in fact random can never be definitively answered (Steinbring, 1990), a perspective which leads Kac (1983) to state that, from an operational point of view, "the concept of randomness is so elusive as to cease to be viable" (p. 406). Beltrami (1999) affirms that "in response to the persistent question 'Is it random?' the answer must now be 'probably, but I'm not sure'" (p. 109).

The point of the above example is that the knowledge of what truly constitutes randomness has proven to be more of a human construct rather than an absolute and infallible truth. What we know as random as today, we may not know as random tomorrow (Wild & Pfannkuch, 1999). If there ever was a mathematical concept which defies Absolutist thinking and lends itself to Relativist / Fallibilist perspective of social constructivism, surely that concept is randomness. Moreover, this example shows the kind of theoretical orientation to mathematical knowledge which is taken by this study. In assuming social constructivist aspects of epistemology, the view is that many concepts in mathematics, like randomness, fit a cycle of subjective and objective knowledge, whereby the social negotiation process interacts between private beliefs and publicly accepted knowledge (Ernest, 1991). Batanero, Green, and Serrano (1998) share that

> "Epistemological problems play a fundamental role for mathematics educators, because analysing the obstacles that have historically emerged in the formation of concepts can help to understand students' difficulties in learning mathematics" (p. 113).

While this section concentrated more on epistemology, another aspect of constructivism is discussed in terms of a theory of learning, and this is presented in the following section.

Learning

While it may be questionable that "mathematics educators almost universally accept that learning is a constructive process", it is easier to agree with Ernest's observation that "the term *constructivism* itself covers a panoply of theoretical positions" (Cobb, Yackel, & Wood, 1992, p. 3). A review of any amount of literature will show that while some use talk of constructivism in quite broad terms, others make fine gradations of distinction. Nor is there clear consensus of which specific terms have what meaning, or of how many different types of constructivism there are. For example, Cobb and Yackel (1996) distinguish between psychological, sociocultural, and emergent perspectives of constructivism; Ernest (1996) discusses radical, social, and weak constructivism. In regards to Ernest's typology, Herscovics (1996) declares: "Although I consider myself a constructivist, I fail to identify with any of these types" (p. 352). Fox (1997) notes that

> "constructivism is a metaphor, and also an umbrella term, covering a family of theories of perception, memory, learning and teaching, which have in common a shared concern with human knowledge as the product of an active, constructive process" (p. 14).

With a similar intent shown in the previous section with regard to an epistemological stance, this section does not aim to delve into every detail of the many forms of constructivism. Rather, the intention is to explicate some key aspects comprising different forms of constructivism, highlighting those perspectives which are assumed by this study as elements of a theoretical orientation of learning.

Von Glasersfeld (1995) points to one of Jean Piaget's (1937) major works as introducing the notion of constructivism and making the term popular. Rooted in the work of Piaget, this notion focused on the individual construction of knowledge, whereby "we construct our knowledge of our world from our perceptions and experiences, which are themselves mediated through our previous knowledge" (Simon, 1995, p. 114). According to von Glasersfeld (1987), "to have 'learned' means to have drawn conclusions from experience and to act accordingly" (p. 8). Piaget's contention was that "learning is subordinated to development" (1964, p. 184), and he posited four hierarchical stages of cognitive development; the transition through these stages depended on the four main factors of maturation, experience, social transmission, and equilibration. Maturation, for example, was linked to embryogenesis, which "concerns the development of the body, but it concerns as well the development of the nervous system and the development of mental functions" (Piaget, 1964, p. 176). Equilibration was described as a self-balancing of two intrinsic polar yet complementary processes or behaviors, assimilation and accommodation (Fosnot, 1996; Herscovics, 1996). Assimilation enables the learner to "fit new knowledge into his or her existing cognitive structure", while accommodation comprises the learner's "reorganization and expansion of such a cognitive structure" (Herscovics, 1996, p. 351). Mental structures develop via a cycle of disequibrilation, accommodation, assimilation, and equilibration within the individual learner (Janvier, 1996).

On the one hand I do agree with Herscovics (1996) that Piaget's stage theory, especially with its emphasis on age progression through the hierarchy, can be a poor guide to what people can learn in regard to specific topics. For example, reliance on this

Comment: Move this to the end, when you can have a discussion about learning theory was what seemed to be influential in Piaget and Inhelder's (1957) determination that many young children were unable to distinguish between deterministic and chance events, while other researchers were able to set the age threshold lower (Fischbein, Pampu, & Minzat, 1975; Kuzmak & Gelman, 1986). On the other hand, I also agree that "one cannot ignore the very major intellectual changes occurring at the ages of 5 to 7 and 11 to 13 years among most children (Herscovics, 1996, p. 352). Moreover, the process of having disequilibration resolve itself in favor of equilibration seems to be an adequate description of a way of constructing individual knowledge.

Piagetian constructivism, to use Janvier's (1996) term, "is seen as an individual cognitive activity that involves the internal reorganization of mental schema" (Teppo, 1997, p. 3). Others called this merely constructivism, but concerned that some people were missing the epistemological implications behind Piaget's ideas, von Glasersfeld was prompted to "add the qualifier *radical* to constructivism" (von Glasersfeld, 1998, p. 23). Ernest (1996) affirms that radical constructivism is based on two key principles of von Glasersfeld (1989). Paraphrased by Begg (1995), these principles are:

> "(i) knowledge is not passively received, but actively built up, by a thinking learner; and
> (ii) thinking is an adaptive process that helps the learner to organize his or her experiential world, rather than to know absolute reality" (p. 71).

Additional labels associated with this perspective are cognitive or psychological constructivism . As Simon (1995) wrote, "the radical constructivist position focuses on the individual's construction, thus taking a cognitive or psychological perspective" (p. 16). The label I'll choose to use for a perspective of learning which is characterized by the two principles of von Glasersfeld's is cognitive constructivism, since for me that evokes the focus on individuals who construct their own knowledge. It should be noted, however, that other authors prefer the label of psychological constructivism (Cobb, 2000).

In addition to the individual aspects of constructing knowledge, there is also a contribution from social interactions as reflected through language and culture or environment: Piaget himself claimed that the social transmission factor in developing knowledge was "fundamental" but "insufficient" (1964, p. 180). Fosnot (1996) comments,

> "Although the main body of Piaget's work centered on illuminating the progressive cognitive structuring of individuals, he did not overlook the effect of social interaction on learning" (p. 18).

Thus, while cognitive constructivism does affirm social interaction as important for learning, the focus is on the reorganization of an individual's cognition (Simon, 1995; Cobb, 2000). A sociocultural perspective towards constructivism, on the other hand, "views learning as the enculturation of an individual into a community of practice, and the focus of inquiry is placed on the individual's participation in social practice" (Teppo, 1997, p. 3). The sociocultural perspective owes much to the work of Vygotsky, whose emphasis was on the language and the social world (Cobb, 1994; Vygotsky, 1986). Vygotsky claimed that cognitive development resulted from social interaction and education, by means of language: "Knowledge is socially constructed. Vygotsky emphasises the part played by social activity and cultural practices as sources of thinking," observes Harvard (1997, p. 40). A central idea expressed in Vygotsky's writings is a person's increasing capacity for self-regulation. His notion of the zone of proximal development was used to describe "the place at which a child's empirically rich but disorganized spontaneous concepts 'meet' the systematicity and logic of adult reasoning" (Kozulin, 1986, p. xxxv). Thus, learners realize their capacity for self-regulation by being guided in their participation by others who are more capable; the result is a level of performance which the learners would otherwise have been unable to reach on their own (Harvard, 1997). Cobb (2000) notes that sociocultural constructivism, the label I'll use for those perspectives following a Vygotskian tradition, tends to "elevate the social process above the psychological" (p. 309).

Thus, a dichotomy of sorts is presented between the cognitive constructivism perspective (focusing more on individual thinking processes, and the sociocultural perspective (focusing more on the sociological influences on learning). Cobb, Yackel and Wood (1992) invoked the label of *social constructivism* to emphasize both the role of the individual and the social environment in learning, and they used this theoretical orientation in their research. In their opinion,

> "the question of whether mathematics is essentially cognitive or whether it is essentially sociological or cultural in nature is considered to be irrelevant. We have instead suggested that it is useful to see mathematics as both a cognitive activity constrained by social and cultural processes and as a social and cultural phenomenon that is constituted by a community of actively cognizing individuals" (p. 32).

In reviewing the literature, it should be noted that Cobb and Yackel subsequently referred to this very same position under the label *emergent perspective* (Cobb, 1994; Cobb & Yackel, 1996; Cobb, 2000). Referring to these authors' work, Teppo wrote that "their *social constructivist* or 'emergent' framework" implies that the development of individual and the development social meaning cannot exist independently of each other (1997, pp. 3-4). Others continued the tradition of this paper in simply referring to this perspective under its original label of social constructivism.

The explication given thus far is important because, as noted earlier, to simply invoke claim that this study takes a constructivist orientation towards learning leaves the door open to a host of interpretations. Is the orientation more focused on individual cognition, or the sociological aspects of learning? This study does ascribe to the essential spirit of von Glasersfeld's (1989) principles , and thus acknowledges the value of a cognitive constructivist perspective. Also, the study values social context in learning, thus validating the sociocultural perspective. Simon joins Cobb, Yackel, and Wood in proclaiming that "we refer to this coordination of psychological and sociological analyses as 'social constructivism'" (Simon, 1995, p. 117). Thus, as a learning theory, social constructivism suggests that individuals internally construct their own knowledge via the effects of social mediation, and language plays an important part in this overall process (Fosnot, 1996; Telese, 1999). I assume this perspective in my study.

In summary, my philosophical orientation towards knowledge and learning comprises elements of established theory. Regarding the nature of knowledge, I do see mathematics as an evolving construct, something to which humans ascribe meaning. To say that mathematical knowledge is relative and fallible is for me a reflection of the way that people decide upon accepted meaning in the discipline. This reflects what Ernest (1991) calls social constructivism as epistemology. As far as building or acquiring knowledge, individuals create their own understandings, but are assisted in this endeavor by language and mediation within a social environment: The individual and social aspects of learning are co-dependent. This reflects what many call a social constructivist orientation towards learning.

Application to Teacher Training

Now that a theoretical orientation towards knowledge and learning has been discussed, another theoretical perspective is articulated concerning the knowledge that teachers need to have, and how they can develop this knowledge. This is an important issue for university programs, which research suggests play a key influence on the preservice teacher's knowledge of and beliefs about mathematics (Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992). Perhaps because teacher knowledge an incredibly complex issue (Lehrer & Franke, 1992), it has been studied in terms of different types or components of knowledge. However, while many different components of teachers' knowledge have been identified, only a few of these components have gained the attention of many researchers (Fennema & Franke, 1992). Leinhardt and Smith (1985) saw the cognitive aspects of teaching arise out of two main types of knowledge - Lesson structure knowledge and subject matter knowledge. Shulman (1986) casts a historical net over the definition of teacher knowledge and concludes that subject matter knowledge is a definite prerequisite to teaching. However, he proposed a distinction among "three categories of content knowledge: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge" (Shulman, 1986, p.9). In terms of mathematics, Cooney (1994) cites the framework of Theule-Lubienski in portraying three overlapping domains of knowledge: That of the subject, the pedagogy, and the students of mathematics. Ball and McDiarmid (1988a) acknowledged the underdevelopment and incompleteness of researchers' understanding of different kinds of knowledge such as subject matter and pedagogical knowledge, and there has been a call for increasing research into these components of teachers' knowledge (Shulman, 1988; Ball & McDiarmid, 1988a; Leinhardt & Smith, 1985). That call has been increasingly answered in recent years, and two components which research has delineated are subject matter knowledge (SMK) and pedagogical content knowledge (PCK).

Although naming subject matter as a key component of teacher knowledge "is neither a new nor controversial assertion" (Ball & McDiarmid, 1988b, p. 437), the authors pointed out that research on the issue of SMK was a relatively recent domain of inquiry, heavily influenced by Shulman. Shulman (1988) gave a definition of SMK as "that comprehension of the subject appropriate to a content specialist in that domain" (p.26), a comprehension which includes "the key facts, concepts, principles, and explanatory framework of a discipline" (Borko et. al., 1992, p.195). Regarding mathematics, the SMK can be further delineated into procedural knowledge (for example, knowing how to use the symbols, rules, algorithms and syntax of mathematics) and conceptual knowledge (for example, knowing the underlying structure and interconnections of mathematics). This delineation can be likened to Ball's distinction in SMK between knowledge of mathematics versus knowledge about mathematics (Eisenhart, Borko, Underhill, Jones, & Agard., 1993; Borko et. al., 1992; Ball, 1990). Some have extended SMK to include "knowledge of curriculum activities, effective methods of presentation, and assessment procedures" (Fennema & Franke, 1992, p. 158), but this extension seems to point more to the overlaps between SMK, PCK, knowledge of the student, and Shulman's aforementioned curricular knowledge (Leinhardt & Smith, 1985).

While it may be true that "teachers' subject-matter knowledge and its interrelations with pedagogical content knowledge are still very much unknown" (Even, 1993, p. 94), these components can be distinguished. Grossman, Wilson, and Shulman (1989) point out "our growing awareness that there may be fundamental differences between the subject matter knowledge necessary for teaching and subject matter knowledge per se" (p.24). The *Professional Standards for Teaching Mathematics* (NCTM, 1991) includes a section addressing the professional development of teachers; Standards 2, entitled "Knowing Mathematics and School Mathematics," states that "teachers of mathematics should develop their knowledge of the content and discourse of mathematics, including mathematical concepts, procedures, and the connections among them..." (p. 132). This description seems very much in line with the idea that SMK is about knowledge of mathematics and knowledge about mathematics (Simon, 1993).

Standard 4, however, "Knowing Mathematics Pedagogy," speaks about "teachers' knowledge and ability to use and evaluate...ways to represent mathematics concepts and procedures..." (NCTM, 1991, p. 151), and this relates to PCK. This idea of representing mathematics is mentioned by Shulman (1986), who includes in his definition of PCK the use of analogies, examples, illustrations, and demonstrations - "in a word, the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). Knowing how to accomplish this representation - how to select the tasks, ask the questions, and assess what students understand - is seen as at the heart of PCK (McDiarmid, Ball, & Anderson, 1989; Borko et. al., 1992; Tirosh, 2000). Ball (1993) writes that "pedagogical content knowledge includes understandings about what students find interesting and difficult as well as a repertoire of representations, tasks, and ways of engaging students in the content" (p. 160). Beginning teachers find one of their first challenges to be the transformation of disciplinary knowledge (SMK) into a form appropriate to teaching their pupils, and this transformation requires (among other things) a subject-specific pedagogical knowledge (PCK) (Grossman et. al, 1989). Cooney (1994) notes how appealing Shulman's emphasis on PCK has been to the field of mathematics education, and relates this knowledge to "how specific content can be interpreted in teaching situations" (p.611).

Thus, a main theme of SMK can be portrayed as asking what the teacher knows about the subject: How deep is their understanding, and how well connected is their knowledge (Ball, 1990; Simon, 1993) ? In the case of mathematics, do they only know basic facts, algorithms, and rules? A main theme of PCK concerns knowing how to teach the subject: Are they aware of common representations for the subject, and of the sources of misconceptions commonly held by learners (Tirosh, 2000) ? Because PCK involves transforming SMK into ways that allow pupils to learn, it seems natural that “teachers' pedagogical content knowledge is influenced by their subject matter knowledge” (Even, 1993, p. 97). Further overlap exists among not only SMK and PCK, but also among knowledge of learning, the learner and curricular knowledge (McDiarmid et. al., 1989; Porter & Brophy, 1988). A university program aimed at equipping students to teach elementary mathematics for understanding should have as its primary goals the fostering of these types of knowledge.

Regarding the manner in which teachers can develop their SMK or PCK, Standard 1 for the professional development of teachers – “Experiencing Good Mathematics Teaching” (NCTM, 1991, p. 127) - implies that preservice teachers need a learning environment which reflects the very tenets of pedagogy they themselves will want to transfer to their own classrooms. That is, "the pedagogy must be tied to the mathematics taught" (Franke, 2000, p. 9). Individual reflection, combined with work in small groups with an emphasis on discussion, can create an environment wherein preservice teachers can "explore, develop mathematical arguments, conjecture, validate possible solutions, and identify connections among mathematical ideas" (NCTM, 1991, p. 128). This environment should reflect the social constructivist aspects of learning described earlier.

This study is primarily concerned with the SMK of teachers in regards to the topic of statistical variation. While the component of PCK is also validated as important, for the topic of variation the research on how children learn is still sparse: Hence, it is difficult to know just how to help teachers work with their pupils when research is still painting the picture of how students think about this topic. It is also seems problematic to address PCK when even the teachers' own SMK is unclear: That is, at least in the case of elementary preservice teachers, it is unknown what conceptions of variation they have. Thus, in the last main section of this paper, a conceptual framework is explicated to help guide the study in researching the understanding of variation.

Understanding Variation

Just as variation is at the heart of a statistical investigation, so too is the understanding of variation at the heart of this study. In looking at elementary preservice teachers' conceptions, it is helpful to have a framework which helps organize the aspects of an understanding of variation. Eisenhart (1991) gave a useful description of the purpose of conceptual frameworks in guiding research, and Miles and Huberman (1994) support Eisenhart's view by saying that

> "A conceptual framework explains, either graphically or in narrative form, the main things to be studied – the key factors, constructs, or variables – and the presumed relationships among them. Frameworks can be rudimentary or elaborate, theory-driven or commonsensical, descriptive or casual" (p. 18).

The conceptual framework for this study is a synthesis of the ideas promoted by other studies that have looked at variation along with my own ideas based on experiences studying and teaching stochastics. There are two main themes in the framework: One theme pertains to four different aspects of understanding variation, and the other theme has to do with three contexts in which variation can be exposed and studied. By nesting the aspects of understanding within the contextual theme, a coherent framework is postulated as useful in guiding the data collection; conjoining this framework to my overall theoretical orientation toward knowledge and learning suggests the method for analysis of the data.

Before explaining the elements of the conceptual framework developed for this study, it is useful to critique another framework which has been relied on in other research: The neo-Piagetian model of cognitive development known as the Structure of Observed Leaning Outcomes (SOLO). This model of multimodal functioning has been used to study, among other topics, understanding of sampling, averages, and even geometric thinking (Watson & Moritz, 2000a; Watson & Moritz, 2000c; Olive, 1991). The SOLO model posits five distinct, hierarchical modes of thinking: sensorimotor, ikonic, concrete symbolic, formal and post formal. The first three modes are progressed through by most people with the help of others, while "progression into formal and post formal modes depends on interest, motivation, ability, and instruction" (Watson, Collis, & Moritz, 1995). As an exemplar of the kinds of knowledge exhibited in a mode, intuitive thinking would be characterized as ikonic, while quantitative symbolic thinking would be concrete symbolic. The deductive reasoning of abstract concepts would be in the formal mode. Because a transition into later modes does not preclude continued development in earlier modes, the model is referred to as multimodal. The three main cycles within each mode, also hierarchical, are summarized by Watson et. al. (1995) as follows:

> "(U) *Unistructural responses*, which represent the use of only one relevant aspect of the mode;
> (M) *Multistructural responses*, in which several disjoint relevant aspects are processed, usually in sequence; and
> (R) *Relational responses*, in which an integrated understanding of the relationships between the different aspects is exhibited" (p. 4).

By fitting student responses into a U-M-R cycle within the various modes, researchers have found the SOLO taxonomy useful in characterizing knowledge of the concepts under study.

Some examples of responses and their subsequent classification according to the SOLO taxonomy show the nature of this framework in practice. Watson and Moritz (2000a) asked students the question: "If you were given a 'sample', what would you have?" (p. 52). According to these researchers, the use of SOLO as a framework allows the following distinctions among responses: One fifth grader's response of "A sample of work" was "classified as having ikonic support for a unistructural idea in the concrete symbolic mode", which the researchers notate as "U1+IK" (2000a, pp. 12-13). This is less sophisticated than the "U1" classification, denoting a fully unistructural response in the concrete symbolic mode, given to another fifth grader who said "Something the same as something else" (p. 13). Although these may seem to be fine distinctions, they do seem to represent a lower level of understanding than the multistructural response of the ninth grader who said: “Only part of the whole thing” (p. 13). This “M1” response was, in turn, lower than the relational “R1” response of an eleventh grader: “A random selection of data from a wider group, meant to represent the entire group” (p. 14). While some of the distinctions in progressing through the U-M-R cycle in a given mode do seem understandable and justifiable, it can be less clear how some distinctions are made. For example, an eighth grader said “A small portion” (p. 13); this was classified as unistructural because there was no explicit mention to the ‘whole’ from which the sample came from. The sixth grader who said “A small piece of something” (p. 14) did receive the multistructural classification. Yet, for many students I’ve worked with, invoking the term ‘portion’ implies the existence of a larger unit, so my interpretation might have been that the eighth grader was in fact implicitly assuming the ‘whole.’ The point is that, by only using written responses and not allowing the opportunity for reflective explanations, it can be problematic to administer rigid coding schemes that adhere to a predetermined taxonomy.

Another difficulty in using the SOLO model is in falling prey to seeing the data only through those theoretical lenses; this means that the data is going to fall into a number of levels whether it wants to or not, and any interpretation must be made in light of the underlying theory. An example of a potentially unwarranted reliance on the SOLO taxonomy is demonstrated in the work to characterize children’s statistical thinking along the four constructs of describing, organizing, representing, and analytically interpreting data (Jones, Thornton, Langrall, Mooney, Perry, & Putt, 2000). While I do find the four constructs very helpful in striating the aspects of statistical thinking for the purposes of analysis, what these researchers did is seek to translate the SOLO modes and cycles into “levels of statistical descriptors that were stable with respect to students’ thinking across constructs” (Jones et. al., 2000, p. 288). By stable, they meant that if a student is at a certain level in one construct, they should be at that same level in all other constructs. They write that in considering their framework,

> “we claim that stability across all four constructs is important because it enhances the coherence of the framework, and greater coherence is seen to make the framework more viable to curriculum builders and teachers when developing instructional programs in data exploration for the elementary school” (p. 288).

Notice that their primary reason in adopting a certain stance does not seem to be because it is a faithful representation of the data, but because it supports the framework. This does seem understandable for this particular piece of research, whereby the admitted goal was to establish a viable framework. However, for the purposes of this study, the framework is meant to support the data and not the other way around.

The general appeal of a model like SOLO for a study like this is that it offers a way of organizing the questions and responses, but the main drawback is in the rigidity of the approach. Since a goal of this study is to find out what elementary preservice teachers know about variation, it seems appropriate to be open and accommodating of whatever the data reveals. Still, organizing principles can be helpful, and in this case the themes of data analysis and statistical investigation offer guidance on how an understanding of variation can be characterized in a structured but flexible manner. As mentioned in the previous chapter, and adopted by Jones et. al. (2000) in their research, a working definition of data analysis includes the organization, description, representation, and analysis of data (Shaughnessy, Garfield, & Greer, 1996): These are facets deemed important when dealing with data. Also, these facets are complementary to the overall aspects of a statistical investigation, which include formulating a question, collecting, describing, and analyzing the data (Gal and Garfield, 1997; Friel, Bright, Frierson, & Kader, 1997). The aspect of formulating a question fits nicely with the professional statistician's mindset of noticing variation and wondering why it exists (Wild & Pfannkuch, 1999). Combining these various elements of statistical inquiry, along with my own appreciation of the ways in which other stochastical research fits together (Watson, 2000b; Watson, Kelly, Callingham, & Shaughnessy, 2002), lets me propose a four-faceted model for looking at an understanding of variation. The four aspects of understanding variation which are postulated as useful in organizing this study are the expectation of variation, the presentation of variation, the description of variation, and the interpretation of variation: These aspects are discussed further in the following paragraphs.

The expectation of variation is suggested as a way for students to demonstrate their intuitive, experientially or mathematically based reasoning in situations for which variation is inherent. For example, when drawing repeated samples of the same size from a population, it is useful to think ahead of time what kinds of variation are reasonable to expect. Particularly illuminating are the ends of the spectrum: On one end, a person expects their samples to look very much alike due to a small amount of variation from sample to sample. On the other end, a person may huge disparities between samples owing to large amounts of expected variation. In preparing for this study, a class of elementary preservice teachers was led into an investigation using bags of M&Ms: Some students, who had no apparent knowledge of the nature of the way these candies are packaged, anticipated no variation. The bags should not only have the same number of candies, but in the same color distribution. In a sense, experience with some other candies justifies this expectation: For example, rolls of Life Savers do in fact all have the same number of candies per package. Once it became apparent that the bags of M&Ms did in fact differ from one another, the posing of other questions was facilitated. While I used standard-size bags in my class, my officemate taught a similar class and used smaller "fun-size" bags. The latter are appropriately named if the idea of fun is to get none of a certain color, because the variation among the smaller bags is so extreme that sometimes an entire color has a frequency of zero (Steve Blair, personal communication, March 4, 2002). For situations involving chance, whereby the data is a compilation of the outcomes of repeated events, the expectation of variation is tied to the research on probabilistic thinking. A reliance on the outcome approach or on proportional reasoning, for example, could result in a minimization of variation. A spinner which is three-fourths black and only one-fourth white might be expected to produce mostly black outcomes, but what variation inheres in the term 'mostly' ? That is, in 20 spins would they expect all the outcomes to be black? Expecting variation to occur seems to be a fundamental part of an overall understanding of the concept, and in this study it is one aspect that will be researched. Before conducting any experiments or attempting a task, and prior to looking at data relevant to the situation, it should be asked what is expected.

The identification of variation, as a framework element for this study, falls into the area of representing the data that is a part of other researchers' frameworks. That is, when faced with variation in the data, do the subjects notice? How do they want to display data so as to draw attention to this variation? In previous work with elementary preservice teachers, it is not clear what distinctions are made between, for examples,

Comment: Identification = Presentation?

different bar graphs (for the same data set) which emphasise or diminish the variation present. In a task involving different displays of data, some students felt box plots gave a good sense of variation within the data; a few others thought that only the range was highlighted while variation within the data was obscured. Some students have shown that they notice variation even when raw data is presented, before attempts at data reduction are made; other find that different displays are useful for the purposes of not only summarizing data, but drawing attention to the variation they notice. It has also happened whereby a student conducted an experiment and was unable to discuss the variability in the data: It wasn't immediately clear, however, if the student simply didn't notice the variation or lacked the means with which to talk about what they had noticed. This element of presenting the variation not only has strong tie-ins to the research on graphicacy, but also in the small amount of research done on looking at the distributions of data sets. Identifying the variation in data as it is presented using different displays is an important aspect of an overall understanding of variation, and thus is incorporated as an element in the conceptual framework for this study.

Comment: Notifying = Identifying

The description of variation encompasses the research which points out how measures of central tendency can often be integral in the way students measure of talk about the variation they notice. For example, in a study about drawing repeated samples with replacement from a known population, one component of the students' responses was to give an indication of spread that was tied to their idea of the center (Shaughnessy, Watson, Moritz, & Reading, 1[illegible]; Reading & Shaughnessy, 2000; Shaughnessy & Ciancetta, 2001). What language do people use when trying to describe variation? It should be noted that, in quantifying variation, more than just traditional measures of standard deviation and variance can be used; Torok and Watson (2000), in reflecting on their own research, comment that "this study successfully explored students' understanding of variation without ever employing the phrase 'standard deviation'" (p. 166). Other research shows the use of broad descriptors to classify responses about variation, such as the NARROW-REASONABLE-HIGH coding scheme (Shaughnessy et. al., 1999; Reading & Shaughnessy, 2000; Shaughnessy & Ciancetta, 2001). Thus, it is not clear what level of description will be given, nor what language – verbal, symbolic, pictorial, or numeric – would be used by elementary preservice teachers in describing the variation they have noticed. It seems reasonable to assume a certain amount of interdependence between the way the variation is presented, and the way it is described; the framework allows for this relationship. One direction of thinking may be that people will engage in the process of noticing variation, followed by a description of what they are attending to in the display. This element in the framework affords a window for viewing the way that elementary preservice teachers describe the variation they notice.

The interpretation of variation goes beyond description in the sense that there is on the one hand an inferential component: What does the variation tell us about other data sets based on the same investigation? For example, how would another sample or batch of samples be similar to the current one? On the other hand there is also a component of justification, in that the causes of variation can be discussed. That is, students can reason about the sources of the variation, such as common or special causes (Pfannkuch, 1997); in the case of probability situations, the cause is random, but research proves it is erroneous to assume that people know this. Another way that variation gets interpreted is in terms of what it means: Suppose the variation has been expected, noticed, and described – what then does it imply about the data? How confident are we in the trustworthiness of data from other sources when the variation is not what we would have expected? For example, some of the tasks used in previous research on chance situations have asked students whether the purported results were actually obtained from doing the experiment or simply fabricated (Watson, Kelly, Callingham, & Shaughnessy, 2002). It would be interesting to hear elementary preservice teachers' reactions to similar tasks, and find how they interpret variation that is unreasonable. The interpretation of variation seems related to the expectation of variation, and it may be that a person's expectation influences their interpretation and vice versa.

The fours aspects of variation described above are offered as potential areas in which an understanding of variation can be organized; moreover, there are three contexts for looking at variation which this study will use. These contexts are based on the kinds of tasks that other researchers used for the studies discussed in the literature review: They include variation in sampling, chance situations, and data sets. It should be acknowledged that these are not suggested as all the environments in which variation reveals itself, but rather these are some areas which have been looked at by others and have proved to be useful for mining students' understanding. A sampling situation has different dimensions for exploring variation: There is, for example, the effect of sample size on the variation of the mean, and also there is the matter of taking repeated samples of the same size. There are also possibilities for variation to be explored in the context of creating survey techniques. In chance situations, the context provides opportunities to move away from answers that rely on an expected value for one outcome, but move towards the anticipated results of many outcomes in which random variation is sure to play a part. In data sets, of course the other contexts do allow for the generation of data, but the essential feature of this context is that the data is either already provided or a matter of straightforward measurement. The latter happens, for example, in media contexts, or other forums where some data is presented and it is left to reader to make sense of what the data is trying to say. The former occurs in cases such as when a class may simply want to gather some quick data, such as measuring heights or armspans, where the purpose was is to get the set of data so that sense can be made of it.

By taking the four aspects of understanding of variation, and embedding them within the three contexts for looking at the concept, the following graphical representation helps define the conceptual framework (see Fig. 6):

UNDERSTANDING VARIATION
A Conceptual Framework

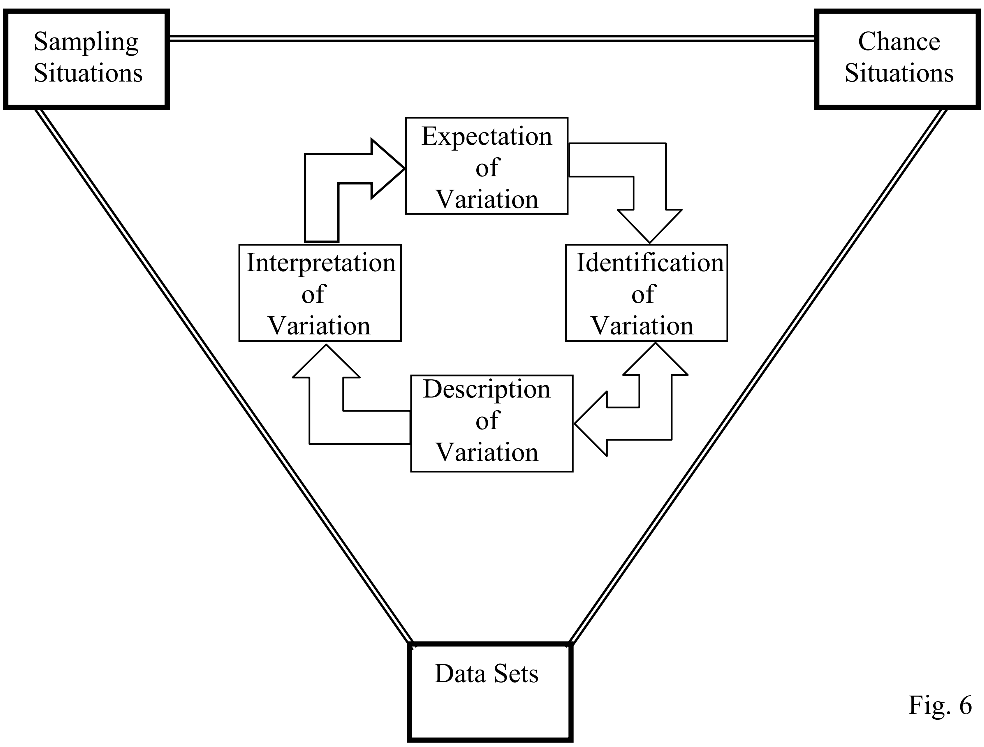

Fig. 6

Within the contexts of sampling, chance situations, or data sets, a persons' understanding can be seen in the aspects of expecting, identifying, describing, and interpreting variation. The expectation of variation may be seen as a natural starting place, with the other aspects following in the course of a statistical inquiry. The identification and description of variation may influence one another, and both inform the interpretation, along with possible influence of initial expectations. The interpretation of variation may influence expectations for future stochastical situations.

Comment: Identifying= presenting

Conclusion

The conceptual framework depicted above is meant to function much in the spirit of Eisenhart's (1991) or Miles and Huberman's (1994) definitions: It portrays the elements of the concept under study and suggests the means for inquiry. In this case, an understanding of variation can be examined by looking at the way that subjects expect, present, describe, and interpret variation in different situations. The three contexts for this study, chosen because of their roots in previous research, are sampling, chance situations, and data sets. The conceptual framework is suggested as a way to help organize the data collection and analysis, by aiming questions at these particular aspects in the chosen contexts. It is a deliberately loose framework in the sense that it allows for a variety of types of responses, yet adheres to the ideas generated in past research, so it does add a certain degree of direction and structure.

Although it does not depend on a specific model of cognitive development like the SOLO taxonomy, the conceptual framework does derive from my overall theoretical orientation, which takes a social constructivist position not only in terms of epistemology but also in the methodology of knowledge acquisition. That is, knowledge is, in general, seen as a human construct - fallible and relative as opposed to infallible and absolute. Learning occurs both as individuals make internal sense of situations (a process which often involves the unsettling of prior notions into a state of disequilibrium), and also in the course of social engagement with others (a process heavily dependent on the communication of ideas and articulation of knowledge between learners). Components of teachers' knowledge include subject matter knowledge and pedagogical content knowledge, and this study is more concerned with the former. The learning environment for preservice teachers should reflect the overall social constructivist values espoused earlier.

Taking together, the elements of the theoretical orientation and the components of the conceptual framework pave the way for the methodology to be explained in the next chapter. In discussing the methodology, the issues of data collection and analysis will be described, showing how to carry out the study.