10.5 Root Finding without Derivatives

} Recall that in general, Newton's Method is locally quadratically convergent for simple roots. However, NMS requires computation involving a derivative (namely \( f'(x_n) \)) — which is often computationally "expensive" or worse yet — intractable/undefined.

} As a natural alternative, we introduce the secant method (and related algorithms) which replaces \( f'(x_n) \) in NMS's with a secant line approximation.

**Secant Method**

} As indicated, we replicate the formula for NMS's:

\[
X_{i+1} = X_i - \frac{f(x_i)}{f'(x_i)}
\]

replacing \( f'(x_n) \) with the difference quotient:

\[
\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}
\]

This yields the following:

**Secant Method Formula**

\[
X_{i+1} = X_i - \frac{f(x_i)}{f(x_i) - f(x_{i-1})}
\]

initial guess \( x_0, x_1 \)

\[
X_{i+1} = X_i - \frac{f(x_i)}{f(x_i) - f(x_{i-1})}
\]
Note that unlike NMF, the Secant Method requires two initial starting guesses (e.g., \(x_0, x_1\)).

It can be shown under the assumption that the Secant Method converges to \(c\) (with \(f''(c) \neq 0\)) the approximate error relationship:

\[
|e_i| \leq \left| \frac{f''(c)}{2f'(c)} \right| |e_{i-1}|^2
\]

holds, implying:

\[
e_i \leq \left| \frac{f''(c)}{2f'(c)} \right| |e_{i-1}|^2
\]

The convergence of the Secant Method to simple roots is said to be superlinear (\(O(x^{-2})\)).

Ex. Apply SM with \(x_0 = 0, x_1 = 1\), \(f(x) = x^3 + x - 1\):

\[
x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{(x_i^3 + x_i - 1)(x_i^3 - x_{i-1})}{x_i^3 + x_i - (x_{i-1}^3 + x_{i-1})}
\]

\[
\rightarrow x_2 = \frac{1}{2}, \quad x_3 = \frac{2}{11}, \quad \ldots, \quad x_8 = 0.682527803
\]

Visualization of SM

[Diagram of function and iteration process]
A well-known generalization of the Secant Method, called the Method of False Position (also: Regular Falsi), works like the Bisection Method (so it is a 'bisection' algorithm) with endpoints replaced with the point at which the secant line crosses the x-axis. (This can speed up convergence over Bisection significantly.)

Given an interval \([a, b]\) that brackets a root, define the next point \(c\), as:

\[
c = a - \frac{f(a)(b-a)}{f(a)-f(b)} = \frac{bf(a)-af(b)}{f(a)-f(b)}
\]

Unlike the Secant Method, this new point is guaranteed to lie in \([a, b]\) and one of the new intervals \([a, c]\) or \([c, b]\) necessarily contains the root \(c\).

**Method of False Position - Pseudocode**

Given interval \([a, b]\) such that \(f(a)f(b) < 0\) for \(c = 1, 2, 3, \ldots\)

- \[c = \frac{bf(a)-af(b)}{f(a)-f(b)}\]

- if \(f(c) = 0\) END (final root found!)
- if \(f(a)f(c) < 0\) then \(b = c\)
- else \(a = c\)

([a, c] contains root)

([c, b] contains root)
Note that FP can offer an improvement over the Bisection Method in many cases, it nonetheless does not always guarantee an improvement. While BM cuts uncertainty of the interval in half by 1/2 each step, FP can make no guarantee; sometimes FP converges very slowly, as the next example shows.

Ex: Use FP on [-1, 1] to find root of \( f(x) = x^3 - 2x^2 + \frac{3}{2} x \).

\( x_0 = -1, x_1 = 1, x_2 = \frac{1(-3) - (-1)^{1/2}}{-3/2 - 1/2} = 4/3. \) New interval: 

\([-1, 4/3]\)

Other generalizations of the Secant Method include: Muller’s Method and Inverse Quadratic Interpolation, each of which use a parabola in place of a linear line; importantly, Muller’s Method can identify multiple roots.

Lastly, Brent’s Method (1973) is a contemporary method that combines Bisection with Quadratic Interpolation (a hybrid method).