1.5 Root Finding Without Derivatives

- Recall that in general, Newton's Method is locally quadratically convergent for simple roots. However NM's requires computations involving a derivative (namely: $f'(x_n)$) - which is often computationally "expensive" or worse yet - intractable/undefined.
- As a natural alternative, we introduce the secant Method (and related algorithms) which replaces $f'(x_n)$ in NM's with a secant line approximation.

Secant Method

- As indicated we replicate the formula for NM's -

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

replacing $f'(x_n)$ with the difference quotient : $\underbrace{\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}}_{\text{slope of secant line on } [x_{i-1}, x_i]}$.

This yields the following:

Secant Method Formula

(*) Initial guess: $\boxed{x_0, x_1}$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)\,(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}} \quad i = 1, 2, 3 \ldots$$

Note that unlike NM's, the Secant Method requires <u>two initial starting guesses</u> (e.g. x_0, x_1).

It can be shown under the assumption that the Secant Method <u>converges to r</u> (with $f'(r) \neq 0$) the approximate <u>error relationship</u>:

$$\boxed{e_{i+1} \approx \left| \frac{f''(r)}{2f'(r)} \right| e_i e_{i-1}}$$

<u>holds</u>, implying:

$$\boxed{e_{i+1} \approx \left| \frac{f''(r)}{2f'(r)} \right|^{\alpha-1} e_i^{\alpha}}$$

where: $\boxed{\alpha = \frac{1+\sqrt{5}}{2}} = \phi$

i.e. $\frac{e_{i+1}}{e_i^{\alpha}} \approx \left| \frac{f''(r)}{2f'(r)} \right|^{\alpha-1}$

The convergence of the Secant Method to <u>simple roots</u> is said to be $\boxed{\text{superlinear}}$. $(1 < \alpha < 2)$.

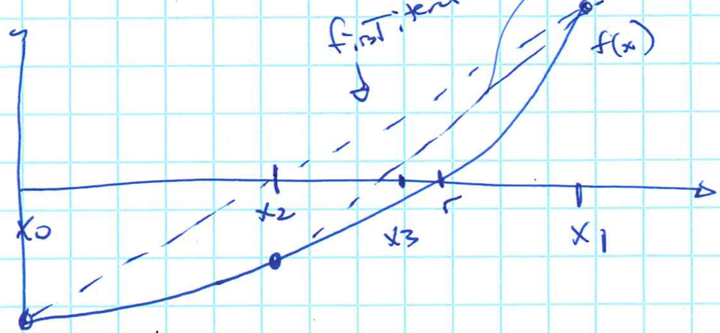
$\boxed{\text{Ex.}}$ Apply SM with $x_0 = 0$, $x_1 = 1$, $f(x) = x^3 + x - 1$.

$$x_{i+1} = x_i - \frac{(x_i^3 + x_i - 1)(x_i - x_{i-1})}{x_i^3 + x_i - (x_{i-1}^3 + x_{i-1})}$$

Rapid convergence!

$\rightarrow x_2 = \frac{1}{2}$, $x_3 = \frac{7}{11}$, ... $\boxed{x_9 = .682327803}$.

$\boxed{\text{Visualization of SM}}$



A well-known generalization of the Secant Method called the **Method of False Position** (also: Regula Falsi) works like the Bisection Method (so it is a "bracketing" algorithm) where midpoints are replaced with the point at which the secant line crosses the x-axis. (This can speed up convergence over Bisection significantly)

Given an interval $[a,b]$ that brackets a root, define the next point, c, as:

$$c = a - \frac{f(a)(a-b)}{f(a)-f(b)} = \boxed{\frac{b\,f(a) - a\,f(b)}{f(a)-f(b)}}$$

simplified form.

Unlike the Secant Method, this new point is guaranteed to lie in $[a,b]$ & one of the new intervals: $[a,c]$ or $[c,b]$ necessarily contains the root r.

Method of False Position - Pseudocode

Given interval $[a,b]$ such that $f(a)f(b) < 0$

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for i = 1, 2, 3, ...
    c = (b f(a) - a f(b)) / (f(a) - f(b))
    if f(c) = 0 END        (found root!)
    if f(a)f(c) < 0        ([a,c] contains root)
        b = c
    else  a = c            ([c,b] contains root)
end
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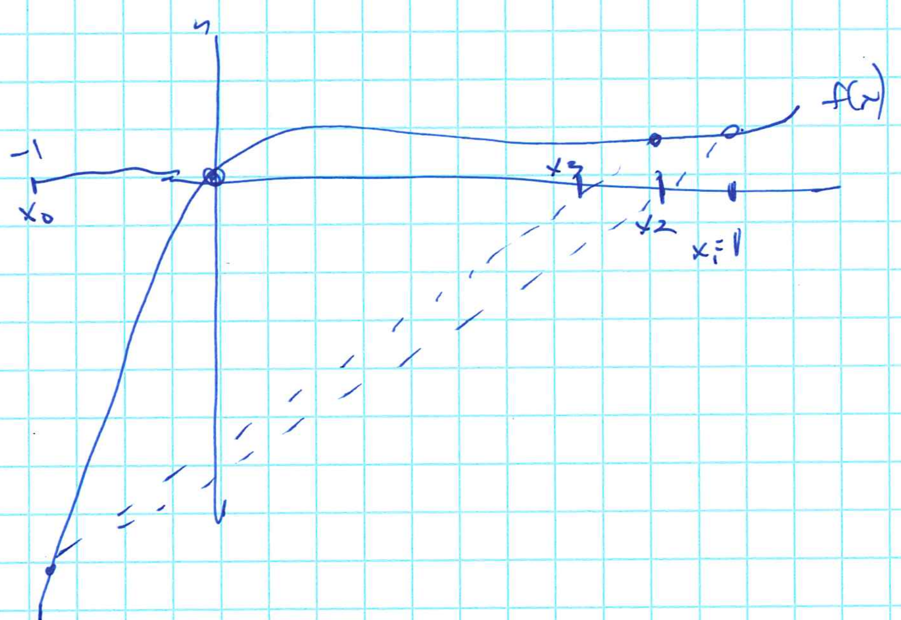
Note That FP can offer an improvement over The Bisection Method in many cases, it nonetheless <u>does not</u> always guarantee an improvement. While BM cuts uncertainty of The interval containing a root by 1/2 each step, FP can make no such guarantee; sometimes FP converges very slowly, as The next example shows.

$\boxed{\text{Ex.}}$ Use FP on $[-1,1]$ To find root of $f(x) = x^3 - 2x^2 + \frac{3}{2}x$.

$$x_0 = -1,\ x_1 = 1,\ x_2 = \frac{1(-9/2) - (-1)\,1/2}{-9/2 - 1/2} = 4/5.$$ New interval: $[-1, .8]$



slow convergence of FP

∙) Other generalizations of The Secant Method include: <u>Muller's Method</u> & <u>Inverse Quadratic Interpolation</u>, each of which use <u>a parabola</u> in place of a <u>secant line</u>; importantly, Muller's Method can identify <u>complex roots</u>.

∙) Lastly, <u>Brent's Method</u> (1973) is a contemporary method That <u>combines</u> <u>Bisection</u> with <u>Quadratic Interpolation</u> (a hybrid method.)