3.3 Chebyshev Interpolation

The motivation for chebyshev interpolation is to improve control of the maximum value of the interpolation error:

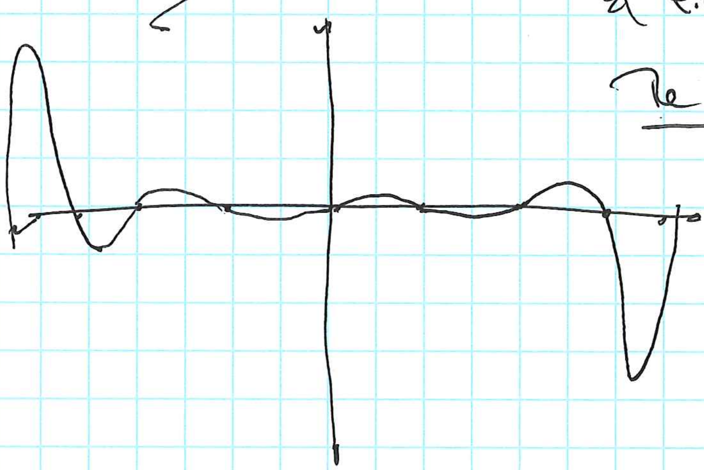
$$\boxed{\frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{n!} f^{(n)}(c)}$$

on the interpolation interval. We fix the interval to $[-1,1]$.

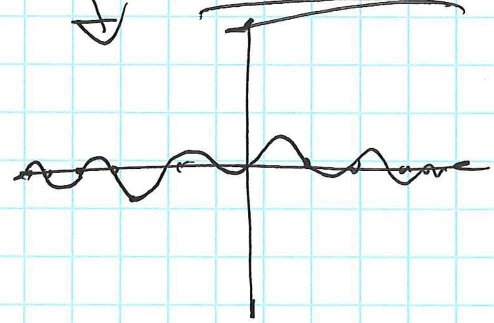
The numerator: $(x-x_1)\cdot(x-x_2)\cdots(x-x_n) = \prod_{i=1}^{n}(x-x_i)$

of the interpolation error formula is itself a degree n polynomial in x & has some maximum value on $[-1,1]$. Is it possible to find particular $x_1,\ldots,x_n$ in $[-1,1]$ that cause this numerator to be as small as possible?

Recall (from 3.2) the "Runge Phenomenon"; we would like to avoid this & find points: $x_1,\ldots,x_n$ that equalizes the polynomial throughout $[-1,1]$



"Ideal" choice



Let $x=.2$

$$|\sin(.2)-P(.2)| \le \frac{|(.2-0)(.2-\pi/6)(.2-\pi/3)(.2-\pi/2)|}{24}\,|1| \approx \underline{.00313}$$

Six Times larger Than The worst case.

The actual error @ $x=.2$ is:

$$|\sin(.2)-P(.2)| = |.19867 - .20056| = \underline{.00189}$$

Ex. Find an upper bound for The differe @ $x=.25$ & $x=.75$ between $f(x)=e^x$ & The Polynomial interpolating e^x @ $\boxed{x=-1,-.5,0,.5,1}$ $(n=5)$

$$f(x)-P_4(x) = \frac{(x+1)(x+.5)x(x-\frac{1}{2})(x-1)}{5!}f^{(5)}(c)$$

where $c\in(-1,1)$ & $f^{(5)}(x)=e^x$. Note e^x is $\boxed{\text{monotonic}}$ so $|f^{(5)}(c)|\le e^1$ on $(-1,1)$. Thus for $x\in(-1,1)$

$$|e^x-P_4(x)| \le \frac{|(x+1)(x+\frac{1}{2})x(x-\frac{1}{2})(x-1)|}{5!}\cdot e^1 \to @ x=.25$$

$$\to |e^{.25}-P_4(.25)| \le \frac{(1.25)(.75)(.25)(-.25)(-.75)}{120}\cdot e^1 \approx \underline{.000995}$$

@ $x=.75$

$$\to |e^{.75}-P_4(.75)| \le \frac{(1.75)(1.25)(.75)(.25)(.25)}{120}\cdot e^1 \approx \underline{.002323}$$

(*) Note That The interpolating error Tends To be smaller close To The center on The interpolation interval.

Proof of the Newton form & error formula

Recall that if $x_1, \dots, x_n$ are $\underline{n}$ distinct points on the real line & $y_1, \dots, y_n$ are arbitrary, we know from a Theorem (in 3.1) that there is $\boxed{\text{exactly one}}$ (degree @ most $n-1$) interpolating polynomial $P_{n-1}(x)$ for those points. We also know that the <u>Lagrange interpolating formula</u> gives such a polynomial.

Next we prove that $\boxed{\text{Newton's divided difference}}$ formula also gives an <u>interpolating polynomial</u>.

Let $P(x)$ denote the (unique) polynomial that interpolates $\{(x_1, f(x_1)), \dots, (x_n, f(x_n))\}$. Denote by $\boxed{f[x_1, \dots, x_n]}$ the <u>degree $n-1$ coefficient</u> of $P(x)$.

Thus, $P(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1}$, where $\boxed{a_{n-1} = f[x_1, \dots, x_n]}$.

Observe <u>two facts</u>:

$\boxed{1}$ $f[x_1, \dots, x_n] = f[\sigma(x_1), \dots, \sigma(x_n)]$ for any $\boxed{\text{permutation } \sigma}$ of the $\{x_i\}$.

<u>Pf:</u> Proved by the uniqueness of the interpolating polynomial.

$\boxed{2}$ $P(x)$ can be written in the form:

$$P(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2) + \dots + c_{n-1}(x - x_1) \cdots (x - x_{n-1}).$$

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... could choose $c_{n-1} = a_{n-1}$. Then the remaining $c_{n-2}, \dots, c_0$ are defined recursively by setting c_k to be the degree k coeff of the (degree @ most k) polynomial:

$$P(x) - c_{n-1}(x-x_1)\cdots(x-x_{n-1}) - c_{n-2}(x-x_1)\cdots(x-x_{n-2})$$
$$- \cdots - c_{k+1}(x-x_1)\cdots(x-x_{k+1}).$$

(This is a <u>degree @ most k</u> polynomial due to the choice of c_{k+1}).

(*)

Theorem Let $P(x)$ be the interpolating polynomial of $\{(x_1, f(x_1)), \dots, (x_n, f(x_n))\}$ where the x_i are distinct.

<u>Then:</u> **(a)** $P(x) = f[x_1] + f[x_1, x_2](x-x_1) + f[x_1, x_2, x_3](x-x_1)(x-x_2)\cancel{(x-x_3)} + \cdots + f[x_1, x_2 \dots x_n](x-x_1)\cdots(x-x_{n-1})$ &

(b) for $k > 1$, $f[x_1 \dots x_k] = \dfrac{f[x_2 \dots x_k] - f[x_1 \dots x_{k-1}]}{x_k - x_1}$

Pf (a) We must prove that $c_{k-1} = f[x_1 \dots x_k]$ for $k = 1, \dots, n$.

It is already clear for $k = n$ by definition.

In general, successively substitute $x_1, \dots, x_k$ into the form of $P(x)$ in <u>Fact (2)</u> (above).

Only the <u>first k terms</u> are <u>non-zero</u>. We conclude that the polynomial consisting of the first k terms of $P(x)$ suffices to interpolate $x_1, \dots, x_k$, and so, by the <u>uniqueness</u> of the interpolating polynomial, $c_{k-1} = f[x_1 \dots x_k]$.

$\boxed{\text{Proof (b)}}$: According to part (a), the interpolating polynomial of $x_2, \dots, x_{k-1}, x_1, x_k$ is:

$$P_1(x) = f[x_2] + f[x_2 x_3](x-x_2) + \dots + f[x_2 x_3 \cdots x_{k-1} x_1](x-x_2)\cdots(x-x_{k-1})$$
$$+ f[x_2 x_3 \cdots x_{k-1} x_1 x_k](x-x_2)\cdots(x-x_{k-1})(x-x_1)$$

& the interpolating polynomial of $x_2, x_3, \dots, x_{k-1}, x_k, x_1$ is:

$$P_2(x) = f[x_2] + f[x_2 x_3](x-x_2) + \dots + f[x_2 x_3 \cdots x_{k-1} x_k](x-x_2)\cdots(x-x_{k-1})$$
$$+ f[x_2 x_3 \cdots x_{k-1} x_k x_1](x-x_2)\cdots(x-x_{k-1})(x-x_k).$$

By uniqueness, $P_1 = P_2$. Setting $P_1(x_k) = P(x_k)$ & cancelling terms yields:

$$f[x_2 \cdots x_{k-1} x_1](x_k - x_2)\cdots(x_k - x_{k-1}) + f[x_2 \cdots x_{k-1} x_k](x_k - x_2)$$
$$\cdots(x_k - x_{k-1})(x_k - x_1) = f[x_2 \cdots x_k](x_k - x_2)\cdots(x_k - x_{k-1})$$ or,

$$f[x_2 \cdots x_{k-1} x_1] + f[x_2 \cdots x_{k-1} x_1 x_k](x_k - x_1) = f[x_2 \dots x_k]$$

Using <u>fact ①</u> this can be rearranged to:

$$f[x_1 \dots x_k] = \frac{f[x_2 \dots x_k] - f[x_1 \dots x_{k-1}]}{x_k - x_1}.$$

Next, we prove the $\boxed{\text{interpolation error formula}}$ provided earlier.

$\boxed{\text{Pf}}$: Consider adding <u>one more point</u> x to the set of interpolation points. The new interpolating polynomial would be:

$$P_n(t) = P_{n-1}(t) + f[x_1 \dots x_n x](t - x_1)\cdots(t - x_n).$$

Evaluated @ the extra point x, $P_n(x) = f(x)$, so:

$$f(x) = P_{n-1}(x) + f[x_1, \ldots x_n\, x](x-x_1)\cdots(x-x_n).$$

This formula is true for all x. Now define:

$$\boxed{h(t) = f(t) - P_{n-1}(t) - f[x_1 \ldots x_n\, x](t-x_1)\cdots(t-x_n).}$$

Notice: $h(x) = 0$ & $h(x_1) = \cdots = h(x_n) = 0$, because P_{n-1} interpolates f @ these points.

By $\boxed{\text{Rolle's Theorem}}$ between each neighboring pair of the $n+1$ points, $x, x_1, \ldots, x_n$, there must be a new point where $\boxed{h' = 0}$. There are n of these points. Between each pair of these, there must be a new point where $h'' = 0$; there are $n-1$ of these. Continuing in this fashion, there must be one point c for which $h^{(n)}(c) = 0$, for c between the smallest & largest of: $x, x_1, \ldots, x_n$. Notice that:

$$\boxed{h^{(n)}(t) = f^{(n)}(t) - n!\, f[x_1 \ldots x_n\, x]} \quad \left(\text{since } P_{n-1}^{(n)}(t) = 0\right).$$

Substituting c gives: $\quad f[x_1 \ldots x_n\, x] = \dfrac{f^{(n)}(c)}{n!}$, which leads to:

$$f(x) = P_{n-1}(x) + \frac{f^{(n)}(c)}{n!}(x-x_1)\cdots(x-x_n). \quad \square$$

The Runge Phenomenon

Polynomials can fit any set of data points (FACT).

$\boxed{\text{Ex.}}$ If we try to interpolate equally-spaced data points:

$x = -3, -2.5, -2, \ldots, 0, \ldots, 2.5, 3$ @ zero except for $\boxed{x=0}$, where we set: $(0, \underline{\underline{1}})$. The interpolating function exhibits a "bump" @ zero, and possibly, other points.

Although the data is relatively "well-behaved" ie $y_i \in [0,1]$, the interpolating polynomial yields a larger range/variance, & a "wiggle" is noticeable.

The "polynomial wiggle" associated w/ high-degree interpolation is illustrated by the Runge Phenomenon.

$\boxed{\text{Ex.}}$ Interpolate $f(x) = \dfrac{1}{1+12x^2}$ @ evenly-spaced points in $[-1,1]$.

The "Runge Example"