3.2 Interpolation Error

Assuming we use the data set: $\{(x_1,y_1),\dots,(x_n,y_n)\}$ to build an interpolating function: $P(x)$.

The interpolation error @ x is $f(x)-P(x)$

Thm Assume $P(x)$ is the degree ($n-1$ or less) interpolating polynomial fitting the data set: $\{(x_1,y_1)\dots(x_n,y_n)\}$. The interpolation error is:

$$\boxed{f(x)-P(x) = \frac{(x-x_1)(x-x_2)(\cdots)(x-x_n)}{n!} f^{(n)}(c)}$$

for some c lying between the smallest & largest of the numbers: $x_1,x_2,\dots,x_n$.

Ex. Consider, again, the example of polynomial interpolation for $f(x)=\sin x$, for 4 data pts equally spaced on $[0,\pi/2]$.

$$\sin(x)-P(x) = \frac{(x-0)\left(x-\frac{\pi}{6}\right)\left(x-\frac{\pi}{3}\right)\left(x-\frac{\pi}{2}\right)}{4!} f''''(c)$$

$$|\sin x - P(x)| \le \frac{\left|(x-0)(x-\pi/6)(x-\pi/3)(x-\pi/2)\right|}{4!}\cdot|1| \approx \boxed{.0005348}$$

So in the "worst case", an upper bound for the error is $\approx .0005348$.

$\boxed{\text{Theorem}}$ The choice of real numbers $-1 \le x_1, \dots, x_n \le 1$ That makes

The value of: $\max\limits_{-1 \le x \le 1} \left| (x-x_1) \cdots (x-x_n) \right|$ as small as possible is:

$$x_i = \cos\left[\frac{(2i-1)\pi}{2n}\right] \quad \text{for } i=1,\dots,n$$

& The minimum value is: $\boxed{\frac{1}{2^{n-1}}}$. In fact, The minimum is

achieved by: $\boxed{(x-x_1)\cdots(x-x_n) = \frac{1}{2^{n-1}} T_n(x)}$

where $\boxed{T_n(x)}$ denotes The $\boxed{\text{degree } n \text{ Chebyshev Polynomial.}}$

Thus, The interpolation error can be minimized if The n interpolating base points in $[-1,1]$ are chosen To be The roots of The degree n Chebyshev interpolating polynomial $T_n(x)$. These roots are:

$$x_i = \cos\left(\frac{\text{odd }\pi}{2n}\right)$$ where "odd" stands for odd numbers 1 To $2n-1$

Then we are guaranteed That The absolute value of:

$(x-x_1)\cdots(x-x_n)$ is less Than $\frac{1}{2^{n-1}}$ for all x in $[-1,1]$.

Choosing The Chebyshev roots as The base points for interpolation distributes The interpolation error as evenly as possible across The interval $[-1,1]$.

We call the interpolating polynomial that uses the Chebyshev roots as base points the $\boxed{\text{Chebyshev Interpolating Polynomial.}}$

$\boxed{\text{Ex.}}$ Find a worst-case error bound for the difference on $[-1,1]$ between $f(x)=e^x$ & the degree 4 Chebyshev interpolating polynomial.

$$f(x)-P_4(x)=\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{5!}f^{(5)}(c)$$

where: $x_1=\cos\frac{\pi}{10}$, $x_2=\cos\frac{3\pi}{10}$, $x_3=\cos\frac{5\pi}{10}$, $x_4=\cos\left(\frac{7\pi}{10}\right)$, $x_5=\cos\frac{9\pi}{10}$

are the Chebyshev roots & where: $-1<c<1$. According to the theorem above, for $-1\le x\le 1$,

$$|(x-x_1)\cdots(x-x_5)|\le\frac{1}{2^4}$$

In addition, $|f^{(5)}|\le e^1$ on $[-1,1]$. The interpolation error is:

$$|e^x-P_4(x)|\le\frac{e}{2^4 5!}\approx \underline{.00142}$$

for <u>all</u> $x\in[-1,1]$.

The error bound for Chebyshev interpolation for the entire interval is only <u>slightly larger</u> than the bound for a point near the center of the interval when evenly spaced interpolation is used. Near the ends of the interval, the Chebyshev error is much smaller.

Chebyshev Polynomials

Define the nth Chebyshev polynomial by $\boxed{T_n(x) = \cos(n \cos^{-1} x)}$.

Despite its appearance, it is a polynomial in the variable x for each n. For instance, if $n=0$, we have: $T_0 = 1$, $n=1 \to T_1(x) = \cos(\cos^{-1} x) = x$.

For $n=2$, recall $\boxed{\cos(a+b) = \cos a \cdot \cos b - \sin a \cdot \sin b}$

Set $y = \cos^{-1} x$, so $\cos y = x$. Then $T_2(x) = \cos(2y) = \cos^2 y - \sin^2 y$

$= 2\cos^2 y - 1 = \boxed{2x^2 - 1}$.

In general, note:

$$T_{n+1}(x) = \cos(n+1)y = \cos(ny+y) = \boxed{\cos ny \cos y - \sin ny \sin y}$$

$$T_{n-1}(x) = \cos(n-1)y = \cos(ny-y) = \boxed{\cos ny \cos y - \sin ny \sin(-y)}$$

$(\sin(-y) = -\sin y)$

Adding, we have:

$$T_{n+1}(x) + T_{n-1}(x) = 2\cos ny \cos y = 2x T_n(x)$$

Hence, $\boxed{T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)}$ {Recursion Relation}

Fact 1 The T_n's are polynomials.

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1$

$T_3(x) = 4x^3 - 3x$ ...

Fact 2 $\deg(T_n) = n$ with leading coefficient: $\boxed{2^{n-1}}$.

(Pf: By recursion relation).

Fact 3 $T_n(1) = 1$ & $T_n(-1) = (-1)^n$. In general,

$$T_{n+1}(1) = 2(1)T_n(1) - T_{n-1}(1) = 2 \cdot 1 - 1 = \boxed{1}$$

&

$$T_{n+1}(-1) = 2(-1)T_n(-1) - T_{n-1}(-1) = 2(-1)^n - (-1)^{n-1}$$
$$= (-1)^{n-1}(2-1) = (-1)^{n-1} = \boxed{(-1)^{n+1}}$$

Fact 4 The maximum absolute value of $T_n(x)$ for $-1 \le x \le 1$ is $\boxed{1}$.

This follows from the fact that $T_n(x) = \cos y$ for some y

Fact 5 All zeros of $T_n(x)$ are located between -1 & 1.

In fact, the zeros are the solution of $0 = \cos(n \cos^{-1}(x))$.

Since ~~y~~ $\cos y = 0$ iff $y = \text{odd integer} \cdot (\pi/2)$, we find that:

$$n \cos^{-1}(\cos x) = \text{odd} \cdot \pi/2 \longrightarrow \boxed{x = \cos\left(\frac{\text{odd} \cdot \pi}{2n}\right)}$$

Fact 6 $T_n(x)$ alternates between -1 & 1 a total of $n+1$ times.

In fact, this happens @ $\cos(0), \cos(\pi/n), \ldots, \cos((n-1)\pi/n), \cos \pi$.

It follows from fact 3 that $\frac{T_n(x)}{2^{n-1}}$ is $\boxed{\text{monic}}$.

Since, according to fact 5, all roots of $T_n(x)$ are real we can write $\frac{T_n(x)}{2^{n-1}}$ in factored form as: $(x-x_1)\cdots(x-x_n)$.

Chebyshev's Theorem follows directly from these facts.

$\boxed{\text{Proof of Theorem}}$: Let $P_n(x)$ be a monic polynomial with an even smaller absolute maximum on $[-1,1]$; in other words, $|P_n(x)| < \frac{1}{2^{n-1}}$ for $-1 \leq x \leq 1$. This assumption leads to a contradiction: since $T_n(x)$ alternates between -1 & 1 a total of $n+1$ times at these $n+1$ points the difference $P_n - T_n/2^{n-1}$ is alternately positive & negative. Therefore, $P_n - T_n/2^{n-1}$ must cross zero at least n times; that is, it must have @ least n roots. This contradicts the fact that, because P_n & $T_n/2^{n-1}$ are monic, their difference is of degree $\leq n-1$.

$\boxed{\text{Change of Interval}}$

We extend our discussion of Chebyshev interpolation from $[-1,1]$ to $[a,b]$.

We do this with (2) steps:

(1) Stretch the points by factor: $\left(\frac{b-a}{2}\right)$

(2) Translate the points by $\left(\frac{b+a}{2}\right)$

So, we move the original points:

$$\cos\left(\frac{odd \cdot \pi}{2n}\right) \to \left[\frac{b-a}{2}\right] \cdot \cos\left(\frac{odd \cdot \pi}{2n} + \frac{b+a}{2}\right)$$

Chebyshev interpolation nodes

On the interval $[a,b]$:

$$\boxed{x_i = \frac{b+a}{2} + \frac{b-a}{2}\cos\left(\frac{(2i-1)\pi}{2n}\right)}$$

for $i = 1, \ldots, n$. The inequality:

$$|(x-x_1)\cdots(x-x_n)| \le \frac{\left(\frac{b-a}{2}\right)^n}{2^{n-1}}$$

holds on $[a,b]$.

Ex. Find the four Chebyshev base points for interpolation on the interval $[0, \pi/2]$ & find an upper bound for the Chebyshev interpolation error for $\boxed{f(x) = \sin x}$.

$$\to \frac{\frac{\pi}{2} - 0}{2} \cos\left[\left(\frac{odd\, \pi}{2(4)}\right) + \frac{\pi/2 + 0}{2}\right] \to$$

$x_1 = \frac{\pi}{4} + \frac{\pi}{4}\cos\left(\frac{\pi}{8}\right)$, $x_2 = \frac{\pi}{4} + \frac{\pi}{4}\cos\left(\frac{3\pi}{8}\right), \ldots, x_4 = \frac{\pi}{4} + \frac{\pi}{4}\cos\left(\frac{7\pi}{8}\right)$

The worst-case interpolation error for $0 \le x \le \pi/2$ is:

$$|\sin(x) - P_3(x)| = \frac{|(x-x_1)(x-x_2)(x-x_3)(x-x_4)|}{4!}|f''''(c)| \le \frac{\left(\frac{\pi - 0}{2}\right)^4}{4!\, 2^3} \cdot 1 \approx \boxed{.00198}$$