32 Interpolation Error

Assuming we use the data set: \( f(x_1, y_1), \ldots, (x_n, y_n) \) to build an interpolating function: \( f(x) \).

The interpolation error \( \epsilon(x) = f(x) - P(x) \).

Then, assume \( P(x) \) is the degree \((n-1)\) or less interpolating polynomial fitted to the data set: \( f(x_1, y_1), \ldots, (x_n, y_n) \). The interpolation error is:

\[
\epsilon(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{n!} f^{(n)}(c)
\]

for \( c \) lying between the smallest and largest of the numbers: \( x_1, x_2, \ldots, x_n \).

Example: Consider again the example of polynomial interpolation for \( f(x) = \sin x \) for 4 data points equally spaced on \([0, \pi/2]\).

\[
\sin(x) - P(x) = \frac{(x-x_0)(x-x_{3/2})(x-x_{5/3})(x-x_{4/2})}{4!} f^{(4)}(c)
\]

so

\[
\left| \frac{(x-x_0)(x-x_{3/2})(x-x_{5/3})(x-x_{4/2})}{4!} \right| \leq 1 \times 10^{-11} = 0.0000000000000005348
\]

So in the "worst case", an upper bound for the error is \( 2 \times 0.0000000000000005348 \).
Theorem: The choice of real numbers \(-1 \leq x_i \leq 1\) that makes
the value of
\[
\max_{-1 \leq x \leq 1} \left| \prod_{i=1}^{n} (x-x_i) \right|
\]
as small as possible is:
\[
\chi_c = \cos \left( \frac{(2i-1)\pi}{2n} \right) \quad \text{for} \quad i = 1, \ldots, n
\]
and the minimum value is:
\[
\frac{1}{2^{n-1}}
\]
In fact, the minimum is achieved by:
\[
\prod_{i=1}^{n} (x-x_i) = \frac{1}{2^{n-1}} T_n(x)
\]
where \(T_n(x)\) denotes the degree \(n\) Chebyshev Polynomial.

Thus, the interpolation error can be minimized if the \(n\) interpolating
base points in \([-1,1]\) are chosen to be the roots of the degree \(n\)
Chebyshev interpolating polynomial \(T_n(x)\). These roots are:
\[
\chi_c = \cos \left( \frac{\text{odd } \pi}{2n-1} \right)
\]
where "odd" stands for odd numbers up to \(2n-1\).

Then we are guaranteed that the absolute value of:
\[
\prod_{i=1}^{n} (x-x_i)
\]
is less than \(\frac{1}{2^{n-1}}\) for all \(x\) in \([-1,1]\).

Choosing the Chebyshev roots as the base points for interpolation
distributes the interpolation error as evenly as possible across the interval \([-1,1]\).
We call the interpolating polynomial that uses the Chebyshev roots as base points the Chebyshev interpolating polynomial.

**Ex.** Find a worst-case error bound for the difference on \([-1,1]\) between \(f(x) - p_n(x)\) and the degree \(4\) Chebyshev interpolating polynomial:

\[
f(x) - p_n(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{5!} f^{(5)}(c)
\]

where:
- \(x_1 = \cos \frac{\pi}{10}\), \(x_2 = \cos \frac{3\pi}{10}\), \(x_3 = \cos \frac{5\pi}{10}\), \(x_4 = \cos \frac{7\pi}{10}\), \(x_5 = \cos \frac{9\pi}{10}\)

are the Chebyshev roots and where: \(-1 < c < 1\). According to the theorem above, for \(1 \leq x \leq 1\),

\[
|f(x) - p_n(x)| \leq \frac{1}{24}
\]

In addition, if \(|f'(x)| \leq \varepsilon\) on \([-1,1]\), the interpolation error is:

\[
|e^x - p_n(x)| \leq \frac{e}{245!} \approx 0.00142
\]

for all \(x \in [-1,1]\).

The error bound for Chebyshev interpolation for the entire interval is only slightly larger than the bound for a point near the center of the interval, when evenly spaced interpolation is used. Near the ends of the interval, the Chebyshev error is much smaller.
Chebyshev Polynomials

Define the \textit{Chebyshev polynomial} by \( T_n(x) = \cos(n \cos^{-1} x) \).

Despite its appearance, it is a polynomial in the variable \( x \) for every \( n \). For instance, if \( n = 0 \), we have: \( T_0 = 1 \), \( T_1(x) = \cos(x) \).

For \( n = 2 \), recall: \( \cos(2x) = \cos^2(x) - \sin^2(x) \).

Set \( y = \cos x \), so \( \cos 2y = \cos^2 y - \sin^2 y \). Thus \( T_2(x) = \cos(2x) = \cos^2 y - \sin^2 y \).

Adding, we have:
\[
T_n(x) + T_{n-1}(x) = 2 \cos ny \cos y = 2x T_n(x)
\]

Hence:
\[
T_n(x) = 2x T_n(x) - T_{n-1}(x)
\]

Fact 1: The \( T_n(x) \) are polynomials.
\[
T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x
\]
Fact 2: \( \deg(e_n) = n \) with leading coefficient: \( 2^{n-1} \). (Ref: By recursion relation.)

Fact 3: \( T_n(1) = 1 \) and \( T_n(1) = (-1)^n \). In general,
\[
T_{n+1}(1) = 2(T_n(1)) - T_{n-1}(1) = 2(-1)^{n-1} - (-1)^{n+1}
\]
\[
= (-1)^n(2-1) = (-1)^n = (-1)^{n+1}
\]

Fact 4: The maximum absolute value of \( T_n(x) \) for \(-1 \leq x \leq 1\) is 1. It follows from the fact that \( T_n(x) = \cos(\theta) \) for some \( \theta \).

Fact 5: All zeros of \( T_n(x) \) are located between 1 and 1.

In fact, the zeros are the solutions of \( 0 = \cos(n \pi \theta) \).

Since \( \cos(y) = 0 \) iff \( y \) is odd integer. \((n \pi)/2\), we find \( T_n(x) \):

\[
\begin{align*}
\cos((n \pi)/2) & = 0 \\
\Rightarrow x & = \cos\left(\frac{\text{odd } \pi}{2}ight)
\end{align*}
\]

Fact 6: \( T_n(x) \) alternates between 1 and 1 \(-\text{total of } n+1 \text{ times}\).

In fact, this happens at \( \cos(0), \cos(n\pi), \ldots, \cos((n-1)\pi/n), \cos(n\pi) \).
It follows from Fact 2 that \( \text{deg}(x) \leq \frac{1}{2^{n-1}} \).

Since, according to Fact 3, all roots of \( \text{deg}(x) \) are real, we can write \( \text{deg}(x) \) in factored form as \( (x-x_1) \ldots (x-x_n) \).

Chebyshev's Theorem follows directly from the facts.

**Proof of Theorem:** Let \( \text{deg}(x) \) be a monic polynomial with an even smaller absolute maximum on \([-1,1]\); in other words, \( |\text{deg}(x)| < \frac{1}{2n} \) for \(-1 \leq x \leq 1\). This assumption leads to a contradiction: Since \( \text{deg}(x) \) attains between \(-1 \leq x \leq 1\) a total of \( n+1 \) times at the \( n+1 \) points \( \frac{1}{n} \) are also the roots of the polynomial: \( \text{deg}(x) \) is alternately positive or negative. Therefore, \( \text{deg}(x) \) must have \( n+1 \) points that are also roots. This contradicts the fact that \( \text{deg}(x) \) is a monic polynomial.

**Change of Interval:**

We extend our discussion of Chebyshev interpolation from \([-1,1]\) to \([a,b]\).

We do this with 2 steps:
1. Scale the points by \( \frac{b-a}{2} \).
2. Translate the points by \( \frac{b+a}{2} \).
So we have the original proof:

\[
(\cos \left( \frac{\text{odd} \cdot \pi}{2n} \right)) \rightarrow \left[ \frac{b-a}{2} \right] \cdot \cos \left( \frac{\text{odd} \cdot \pi}{2n} + \frac{b-a}{2} \right)
\]

Chebyshev interpolation nodes

On the interval \([a, b]\):

\[
x_i = \frac{b-a}{2} + \frac{b-a}{2} \cos \left( \frac{2i-1) \pi}{2n} \right)
\]

for \(i = 1, \ldots, n\). The inequality:

\[
|\sin(x_1) \cdots \sin(x_n)| \leq \frac{\left( \frac{b-a}{2} \right)^n}{2^{n-1}}
\]

holds on \([a, b]/7\).

Ex. Find the four Chebyshev base points for interpolation on the interval \([0, \pi/2] \).

Find an upper bound for the Chebyshev interpolation error for \(f(x) = \sin x\).

\[
- \frac{\pi}{2} \cdot 0 \left[ \left( \frac{\text{odd} \cdot \pi}{2(n)} \right) \right]^2 \frac{\pi/2 + 0}{2} \rightarrow 8
\]

\[
x_1 = \frac{\pi}{4} + \frac{\pi}{4} \left( \frac{\pi}{8} \right), \quad x_2 = \frac{\pi}{4} + \frac{\pi}{4} \left( \frac{3\pi}{8} \right), \quad x_2 = \frac{\pi}{4} + \frac{\pi}{4} \left( \frac{5\pi}{8} \right).
\]

The worst-case interpolation error for \(0 \leq x \leq \pi/2\) is:

\[
|\sin(x) - P_3(x)| = \frac{1}{4} (x-x_1)(x-x_2)(x-x_3)(x-x_4)\left| f^{(n)}(c) \right| \leq \frac{1}{4} \frac{1}{2^3} \approx 0.00198
\]