MATH300

Brief Review of Calculus:

Major Theorems: FTC, EVT, IMVT, MVT

Fund. Theorem of Calc.

FTC I

$$\int_a^b f(x)\,dx = \underbrace{F(b) - F(a)}_{\text{Evaluation of a definite integral}} \qquad (F' = f)$$

FTC II

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

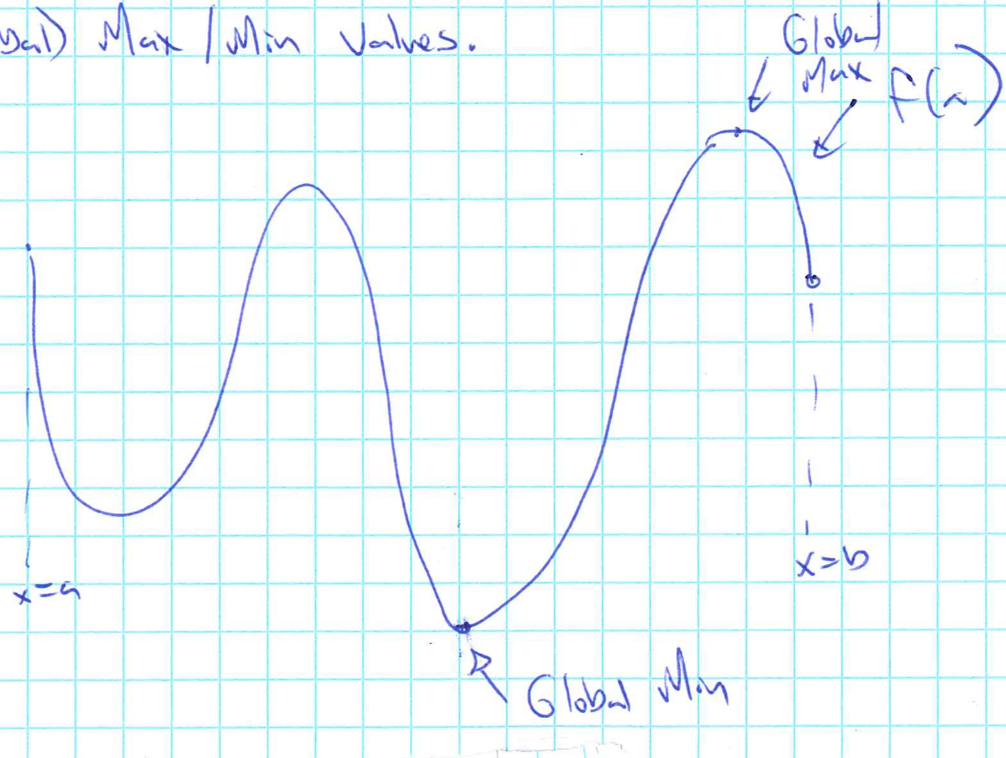
$\Rightarrow$ Shows that derivative & integral are inverse "operators".

Extreme Value Theorem

EVT Posits existence of extreme values under requisite conditions.

If $f(x)$ is <u>continuous</u> on $[a,b]$, then $f(x)$ attains its (global) Max / Min values.

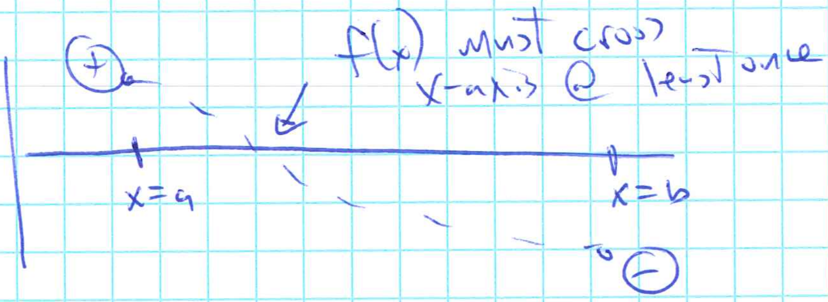
Ex.



IMVT: Intermediate Value Theorem (simplified version)

If $f(x)$ is continuous on $[a,b]$ & $f(a)$, $f(b)$ differ in sign, then there exists (at least one) value $c \in (a,b)$ where $f(c) = 0$.

Picture →



Q: Why is IMVT useful for numerical analysis?

A: We can use it to guarantee the existence of a root/zero of a function - or equivalently - we can use it to guarantee the solution to an equation exists.

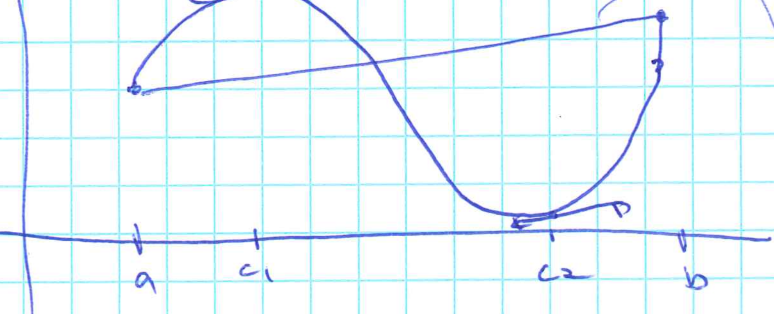
MVT: Mean Value Theorem

If $f(x)$ is differentiable on (a,b) then there exists a $c \in (a,b)$ such that:

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

← Avg Rate of Chge of $f(x)$ on $[a,b]$, i.e. slope of secant line.

Picture



slope $= \dfrac{f(b)-f(a)}{b-a}$

$$\boxed{f'(c_1) = \frac{f(b)-f(a)}{b-a}}$$

Taylor Series Suppose that $f(x)$ is infinitely differentiable at the point $x=a$. Then the Taylor Series of $f(x)$ about the point $x=a$ is defined:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \dots$$

If $a=0$, this expansion is known as a Maclaurin Series.

Some common Taylor Series from beginning Calculus are as follows:

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$
$(a=0)$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$
$(a=0)$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x|<1.$$
$(a=0)$.

Taylor series & Taylor polynomials (their finite brethren) provide a useful tool for approximating functions locally, i.e. near to the "center" $(x=a)$.

In the following example we approximate the irrational number e using a third degree Taylor polynomial/Maclaurin Polynomial.

Ex. $f(x) = e^x$, $a=0$

$\boxed{e = 2.7182818\ldots}$

$$f'(x) = f''(x) = f'''(x) = e^x$$

$$e^x = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \cdots$$

$$e^x \approx \underbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{6}}_{T_3}$$

$$e^1 \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} = \boxed{2.\overline{6}}$$

Note That a higher order Taylor expansion yields, quite naturally, a better result.

$$T_4 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \rightarrow \boxed{e^1 \approx 2.708\overline{3}}$$

$$T_5 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} \rightarrow \boxed{e^1 \approx 2.71\overline{6}}$$

The "Remainder Term" for a Maclaurin series is expressed:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1},$$

which is often used To construct upper-bounds for error with Taylor Polynomial approximation.

Note That for $f(x)$ infinitely differentiable, $\lim_{n \to \infty} R_n = 0$, as expected.

The Bisection Method (1700 BC)

The **Bisection Method** is a root-finding procedure that "brackets off" smaller & smaller intervals that contain the root/zero of a function. (using IMVT)

Ex. Use several iterations of the Bisection Method to find the root of: x^3+x-1 lying in $[0,1]$.

Let $f(x) = x^3+x-1$, Note that $f(x)$ is a polynomial function & therefore continuous on $\mathbb{R}$, so IMVT applies.

Furthermore, $f(0)<0$ & $f(1)>0$, so we know $f(x)$ has at least one root in $[0,1]$.

I $\left[\underset{\ominus}{0} \quad x=\tfrac{1}{2} \quad \underset{\oplus}{1}\right]$

Bisect interval

Check: $f\left(\frac{1}{2}\right)<0$, so II $\left[\overset{\ominus}{\tfrac{1}{2}}, \overset{\oplus}{1}\right]$ contains the root.

Bisect again → III $f\left(\frac{3}{4}\right)>0$, so $\left[\overset{\ominus}{\tfrac{1}{2}}, \overset{\oplus}{\tfrac{3}{4}}\right]$ contains the root.

IV $\left[\overset{\ominus}{.625}, \overset{\oplus}{.75}\right]$

V $\left[\overset{\ominus}{.625}, \overset{\oplus}{.6875}\right]$

⋮ After 10 iterations

X $\left[\overset{\ominus}{.6816}, \overset{\oplus}{.6836}\right]$

Actual Answer: $\boxed{x \simeq .68233}$

Pseudo Code for Bisection Method

Given: initial interval $[a,b]$ such that $f(a)f(b)<0$.

TOL: fixed Tolerance

```
While (b-a)/2 < TOL          (stopping condition)
    c = (a+b)/2              (c = midpoint)
    if f(c)=0 end            (we found root!)
    if f(a)f(c)<0            (f(a) & f(c) differ in sign)
          b = c
    else                     (f(b) & f(c) differ in sign)
          a = c
    end
end
```

Accuracy & Efficiency of Bisection Method

After n steps: $\left| \underset{\text{midpoint}}{x_c} - \underset{\text{actual value}}{r} \right| < \dfrac{b-a}{2^{n+1}}$; (Error $= |x_c - r|$)

for instance, for $n=0$, $|x_c - r| < \dfrac{b-a}{2}$ i.e. the midpoint of $[a,b]$ differs by no more than $\dfrac{b-a}{2}$ from the actual value r.

After one iteration $(n=1)$ of Bisection, $|x_c - r| < \dfrac{b-a}{4}$, etc.

$\boxed{\text{Definition}}$ A solution is $\boxed{\text{correct within } p \text{ decimal places}}$ if the error $< 0.5 \times 10^{-p}$.

$\boxed{\text{Ex.}}$ How many iterations of the Bisection Method are required to find a root of $f(x) = \cos x - x$ in $[0,1]$, accurate to 6 decimal places?

As before, $|x_c - r| < \frac{b-a}{2^{n+1}} = \frac{1-0}{2^{n+1}}$,

so we need: $\frac{1}{2^{n+1}} < .5 * \underline{\underline{10^{-6}}}$

$\longrightarrow 2^{-(n+1)} = .5 * 10^{-6} \longrightarrow -(n+1)\ln 2 = \ln(.5 * 10^{-6})$

$\longrightarrow n \approx 19.9$, so as long as $\boxed{n \geq 20}$, the condition is met.

After 20 iterations... $x_c = .739085$ & $r = .739085$.

Computational Efficiency

What is the best way to evaluate

$$P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1 \quad @\, x = 1/2 ?$$

Direct Method:

$$P\left(\tfrac{1}{2}\right) = 2*\tfrac{1}{2}*\tfrac{1}{2}*\tfrac{1}{2}*\tfrac{1}{2} + 3*\tfrac{1}{2}*\tfrac{1}{2}*\tfrac{1}{2} - 3*\tfrac{1}{2}*\tfrac{1}{2} + 5*\tfrac{1}{2} - 1 = \tfrac{5}{4}$$

"flops": $\underbrace{10}_{\text{multiplications}} + \underbrace{4}_{\text{additions}} = \boxed{14}$

Method 2: Store repeated values in memory.

$$P\left(\tfrac{1}{2}\right) = 2*\left(\tfrac{1}{2}\right)^4 + 3\left(\tfrac{1}{2}\right)^3 - 3\left(\tfrac{1}{2}\right)^2 + 5*\tfrac{1}{2} - 1 = \tfrac{5}{4}$$

$$\tfrac{1}{2}*\tfrac{1}{2} = \left(\tfrac{1}{2}\right)^2, \quad \tfrac{1}{2}*\tfrac{1}{2}*\tfrac{1}{2} = \left(\tfrac{1}{2}\right)^3, \text{ etc.}$$

"flops": 7 mult. + 4 additions = $\boxed{11}$

Method 3: Nested Multiplication: "Horner's Method"

$$\begin{aligned}
P(x) &= -1 + x\left(5 - 3x + 3x^2 + 2x^3\right) \\
&= -1 + x\left(5 + x\left(-3 + 3x + 2x^2\right)\right) \\
&= -1 + x\left(5 + x\left(-3 + x\left(3 + 2x\right)\right)\right) \\
&= -1 + x\left(5 + x\left(-3 + x\left(3 + x*2\right)\right)\right)
\end{aligned}$$

Now an evaluation of $P(x)$ requires only: 4 mult. + 4 additions $= \boxed{8 \text{ operations!}}$

In general, then, a fourth degree polynomial can be written in nested form as such:

$$\boxed{c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 = c_1 + x(c_2 + x(c_3 + x(c_4 + x \cdot c_5)))}.$$

Ex. Find a computationally efficient factored form for: $P(x) = 4x^5 + 7x^8 - 3x^{11} + 2x^{14}$

$$P(x) = x^5(4 + 7x^3 - 3x^6 + 2x^9) = \ldots =$$
$$x^5(4 + x^3(7 + x^3(-3 + x^3 \cdot 2))).$$

Evaluation of the nested form of $P(x)$ require:

7 mults + 3 additions = $\boxed{10 \text{ ops.}}$

Binary Numbers

Base 10: $123 \rightarrow 1 * 10^2 + 2 * 10^1 + 3 * 10^0$

Base 2: $(1101)_2 \rightarrow 1 * 2^3 + 1 * 2^2 + 0 * 2^1 + 1 * 2^0$
$= (13)_{10}$

Converting: Decimal → Binary

Repeatedly divide $\boxed{\text{integer part}}$ by 2 successively & record the remainders.

Ex.

$52 \div 2 = 26 \text{ R } \underline{0}$

$26 \div 2 = 13 \text{ R } \underline{0}$

$13 \div 2 = 6 \text{ R } \underline{1}$

$6 \div 2 = 3 \text{ R } \underline{0}$

$3 \div 2 = 1 \text{ R } \underline{1}$

$1 \div 2 = 0 \text{ R } \underline{1}$

Thus, $(52)_{10} = (110100)_2$

check: $2^5 + 2^4 + 2^2 = 32 + 16 + 4 = \boxed{52}$

To convert the $\boxed{\text{fractional part}}$ of a decimal number to binary, we reverse the previous steps – by ~~dividing~~ multiplying by 2 – and recording the remaining integer part.

Ex.

$.625 * 2 = .25 + \underline{1}$

$.25 * 2 = .5 + \underline{0}$

$.5 * 2 = 0 + \underline{1}$

end

Thus, $(.625)_{10} = (.101)_2$

check: $1 * 2^{-1} + 0 * 2^{-2} + 1 * 2^{-3}$

$= \frac{1}{2} + \frac{1}{8} = \boxed{.625}$

Infinite Binary Fractions

Ex. Convert $\left(\frac{1}{10}\right)_{10} \to$ Binary

$.1 * 2 = .2 + \underline{0}$

$.2 * 2 = .4 + \underline{0}$

$.4 * 2 = .8 + \underline{0}$

$.8 * 2 = .6 + \underline{1}$

$.6 * 2 = .2 + \underline{1}$

$.2 * 2 = .4 + \underline{0}$

$.4 * 2 = .8 + 0$, etc.

Thus, $(.1)_{10} = (.0\overline{0011})_2$.

Converting from Binary To Decimal

Converting a Binary number to decimal is trivial so long as the fractional part of the binary number is finite. In the case were the fractional part is infinite, more work is required.

Ex. Convert $x = (.\overline{101})_2 \to$ Decimal

Multiply by $2^3 \to$ $2^3 x = 101.\overline{101}$

$- \quad x = .\overline{101}$

Subtract $\to$ $7x = (101)_2 \to$ $7x = 5$

$\boxed{x = \frac{5}{7}}$

$\boxed{\text{Ex.}}$ Convert $(.10\overline{101})_2 \to$ Decimal.

Let $\boxed{x = .10\overline{101}}$

$\underbrace{}_{\text{call This } y}$ & let $z = .\overline{101}$

Multiplying by $2^2 \to$ $2^2 x = 10.\overline{101}$

Multiply z by $2^3 \to$

$$2^3 z = 101.\overline{101}$$
$$-\ z = \ \ .\overline{101}$$

As before

$$7z = (101)_2 \to \boxed{z = \frac{5}{7}}$$

Recall, $y = (10.\overline{101})_2 = (10)_2 + (.\overline{101})_2$

$\to$ $y = 2 + \frac{5}{7} = \frac{19}{7}$

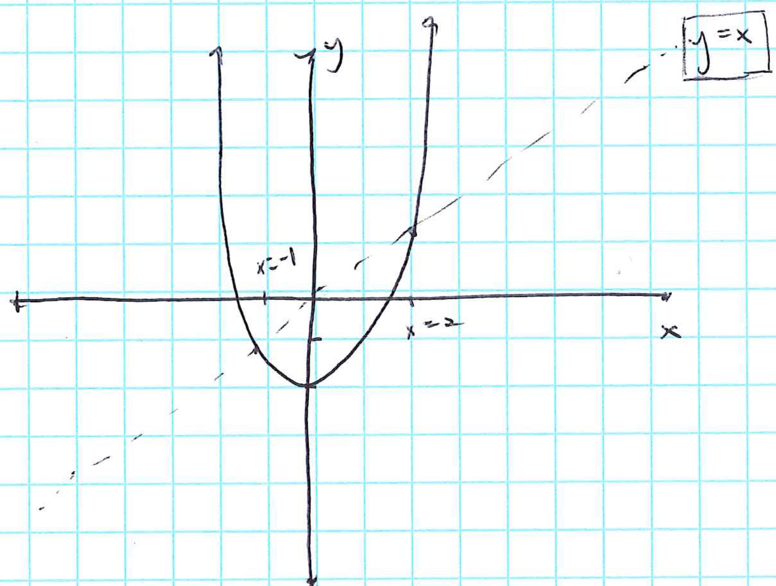
Solve for $x \to 2^2 x = y = \frac{19}{7}$

Thus, $x = \frac{19}{7} \cdot \frac{1}{4} = \boxed{\frac{19}{28}}$

1.2 Fixed-Point Iteration

Definition The real number $(r \in \mathbb{R})$ is called a <u>fixed-point</u> of the function g if $\boxed{g(r) = r}$.

Ex. The function $g(x) = x^2 - 2$ has fixed points @ $x = -1$ & $x = 2$ since $g(-1) = -1$ and $g(2) = 2$.



<u>Note</u> That the fixed points of $g(x) = x^2 - 2$ coincide with the intersection of the graphs of $g(x)$ & $y = x$.

<u>Note:</u> As was the case with the <u>Bisection Method</u> (1.1), we can freely transition between "equation-solving" and "root-finding", e.g. $\underbrace{x^2 - 2}_{g(x)} = x \longrightarrow \boxed{x^2 - 2 - x = 0}$ (or vice-versa).

Observe that the equation $x^2 - 2 - x = 0$ could equally be solved using the Bisection Method.

Fixed-Point Iteration Pseudo-code

x_0: initial guess
$x_{i+1} = g(x_i)$ for $i = 0, 1, 2, \ldots$

(Algorithm is very sensitive to initial guess)

Thus,
$$x_1 = g(x_0)$$
$$x_2 = g(x_1)$$
$$x_3 = g(x_2)$$
$$\vdots$$

<u>NB</u>: The sequence $\{x_i\}$ may or may not converge as $n \to \infty$ (n: # of steps). However, if g is continuous & $\{x_i\}$ converges, say to the number r, then r is a <u>fixed point</u>.

$$\lim_{i\to\infty} g(x_i) = g\left(\lim_{i\to\infty} x_i\right) = \lim_{i\to\infty} x_{i+1}$$

$$\boxed{g(r) = r}$$

since g is continuous

Again, we note that <u>every</u> equation $f(x)=0$ can be re-written as a fixed-point problem: $g(x) = x$; The "catch" is that there are <u>many possible</u> candidate $g(x)$ functions – and that for some choices of $g(x)$ the FPI algorithm <u>will not converge</u>. We must accordingly be <u>careful</u> with our choice of $g(x)$ function.

Ex. Consider the root-finding equation from (1.1),

$$\boxed{x^3 + x - 1 = 0}$$

Case 1 Solve for x: $\boxed{x = 1 - x^3}$ $\left(g_1(x) = 1 - x^3\right)$.

Case 2 Solve for x by first isolating x^3.

$$x^3 = 1 - x \to \boxed{x = \sqrt[3]{1-x}} \quad \left(g_2(x) = \sqrt[3]{1-x}\right)$$

Case 3 (Less obvious) Add $2x^3$ to both sides...

$$x^3 + x + \underline{2x^3} = 1 + \underline{2x^3}$$

$$\to x\left(3x^2 + 1\right) = 1 + 2x^3$$

$$\to \boxed{x = \frac{1+2x^3}{1+3x^2}} \quad \left(g_3(x) = \frac{1+2x^3}{1+3x^2}\right)$$

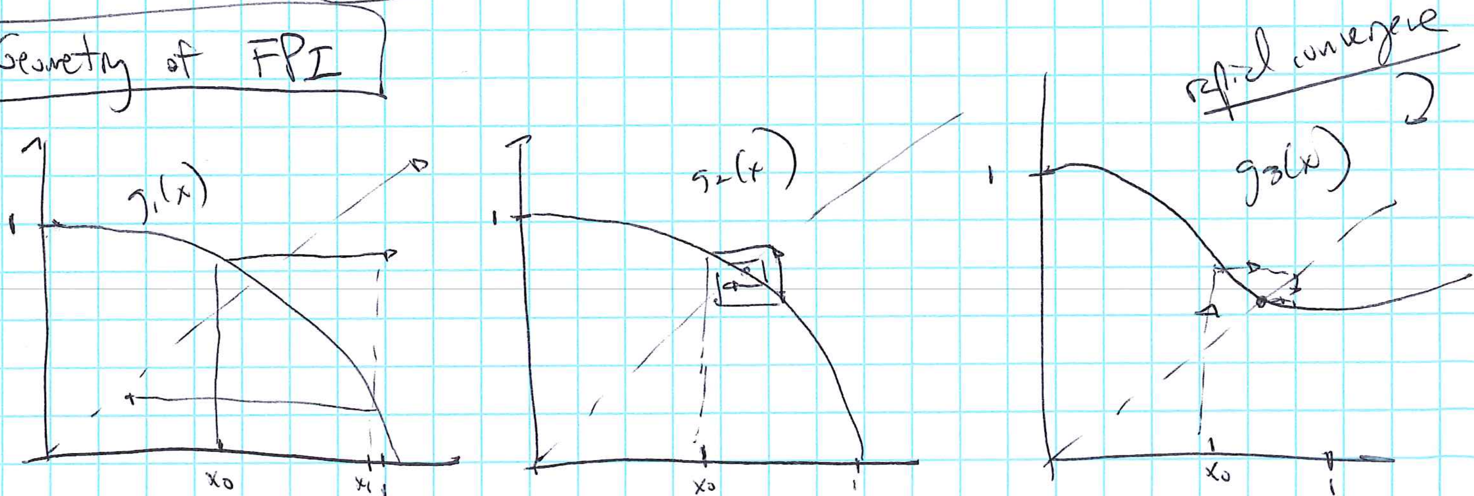
FPI After 12 iterations; $x_0 = .5$

$g_1(x) \to \boxed{x_{12} = 0}$

$g_2(x) \to \boxed{x_{12} = .6792234}$, $\boxed{x_{25} = .68236807}$

$g_3(x) \to \boxed{x_7 = .68232780}$ ← superior result!

Geometry of FPI



Q: Why did FPI fail to converge for $g_1(x)$ & converge for $g_2(x)$ & $g_3(x)$?

A: The tangent slope (i.e. the derivative) of g_2 & g_3 are sufficiently "small", which forces the algorithm to converge (see cobweb diagrams) for a "good" initial guess, x_0.

Definition An iterative method is said to obey **linear convergence** with rate S if $\boxed{\lim_{i\to\infty} \frac{e_{i+1}}{e_i} = S < 1}$. $\boxed{e_{i+1} \approx S e_i}$

(where e_i denotes the error @ step "i").

Theorem Assume that g is continuously differentiable & $g(r) = r$ (so r is a fixed point of g). Then FPI converges linearly with rate S to r for x_0 sufficiently close to r if $|g'(r)| < 1$ $\quad (S = |g'(r)|)$.

Pf: By $\boxed{\text{MVT}}$, there exists a $c_i \in (x_i, r)$ such that:

$$\boxed{x_{i+1} - r = g'(c_i)(x_i - r)} \qquad (x_{i+1} = g(x_i))$$

Let $e_i = |x_i - r|$, then $e_{i+1} = |g'(c_i)| e_i$

So $\lim_{i\to\infty} \frac{e_{i+1}}{e_i} \approx |g'(c_i)| = S < 1$ (for values sufficiently close to r).

$\boxed{\text{Def}}$. An iterative method is called locally convergent to r if the method converges to r for initial guesses sufficiently close to r.

Returning to the previous example.

$g_1(x) = 1-x^3 \rightarrow g_1'(x) = -3x^2 \quad S = |g_1'(r)| = |-3(.6823)^2| \approx 1.4 > 1$

$g_2(x) = \sqrt[3]{1-x} \rightarrow g_2'(x) = \frac{1}{3}(1-x)^{-2/3}(-1)$

$S = |(1-.6823)^{-2/3}/3| \approx .716 < 1.$

$g_3(x) = \dfrac{1+2x^3}{1+3x^2} \rightarrow g_3'(x) = \dfrac{6x(x^3+x-1)}{(1+3x^2)^2}$

$S = |g'(r)| = 0. \rightarrow$ fast convergence!

$\boxed{\text{Ex.}}$ Explore $\boxed{\text{FPI}}$ for $g(x) = \cos x$.

$|g'(r)| = |-\sin r| \approx |-\sin(.74)| \approx |-.67| < 1$

FPI will converge for $x_0 \approx r$.

$\boxed{\text{Ex.}}$ Explore $\boxed{\text{FPI}}$ for $\cos x = \sin x \rightarrow \underbrace{x + \cos x - \sin x}_{g(x)} = x$ $\quad (+x)$

$x_0 = 0 \rightarrow \boxed{x_{19} \approx .7853982}$

$S = |g'(r)| = |1 - \sin r - \cos r| = |1 - \sqrt{2}| < 1.$ $\square$

Stopping Criteria

$|x_{i+1} - x_i|$ Absolute error

For a set tolerance $\boxed{TOL}$ we may ask for an absolute error stopping condition:

$$\boxed{|x_{i+1} - x_i| < TOL}$$

or an absolute relative error stopping condition:

$$\boxed{\frac{|x_{i+1} - x_i|}{|x_{i+1}|} < TOL}$$

HW#1 Find all fixed pts. for given $g(x)$.

(a) $\frac{3}{x} \longrightarrow \frac{3}{x} = x \longrightarrow x^2 = 3 \quad \boxed{x = \pm\sqrt{3}}$

(b) $x^2 - 2x + 2 \longrightarrow x^2 - 2x + 2 = x \longrightarrow x^2 - 3x + 2 = 0$

$(x-1)(x-2) = 0 \longrightarrow \boxed{x = 1, 2}$

HW#11 (a) Express each equation as a fixed-point problem $x = g(x)$ in (3) ways. $\boxed{x^3 - x + e^x = 0}$

① $\longrightarrow \boxed{x = x^3 + e^x}$

② $\longrightarrow x^3 = x - e^x \longrightarrow \boxed{x = \sqrt[3]{x - e^x}}$

③ $\longrightarrow e^x = x - x^3 \longrightarrow \boxed{x = \ln(x - x^3)}$