IV: Section (iii) Eigenvalues & Path-counts in a Network

In graph/Network Theory, it is commonly the case that we would like to measure the "centrality" (i.e. "importance") of a vertex or subgraph. One conventional way to assess the centrality of a vertex is by measuring the number, say, of networks paths with which it "contributes." Put another way, we can define centrality as a function of the number of unique closed walks of a specified length K from any vertex back to itself.

Below we develop a technique to compute the total number of closed walks in a graph for a fixed length K from any vertex back to itself.

Recall, from previous discussions, that a non-zero vector $\vec{v}$ is called an eigenvector of the matrix $A_{n \times n}$ with associated eigenvalue λ if the following equation is satisfied.

$$\boxed{A\vec{v} = \lambda\vec{v}}$$

Furthermore, an $n \times n$ matrix is diagonalizable if it has $\underline{n}$ (linearly independent) eigenvectors.

Remember, in addition, That if a matrix is <u>diagonalizable</u>, Then it factors in The following way:

$$A = PDP^{-1}$$ where D is The diagonal matrix consisting of The eigenvalues of A; $D = \begin{bmatrix} \lambda_1 & & \emptyset \\ & \lambda_2 & \\ \emptyset & & \ddots \lambda_n \end{bmatrix}$

and P is an (invertible) matrix whose columns are The respective eigenvectors of A.

One application of diagonalization already encountered is That of <u>matrix exponentiation</u>. To wit:

$$A^k = \underbrace{(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})}_{k\text{ Times}} = PD^kP^{-1}$$

$\boxed{\text{Thm.}}$ If A is a <u>symmetric</u> $n \times n$ real-valued matrix, Then <u>A has n linearly independent eigenvectors</u>.

Observe That The adjacency matrix $A(G)$ on a graph/Network G is <u>symmetric</u> $(A^T = A)$ & <u>real-valued</u>.
The Theorem above indicates That $A(G)$ is Therefore <u>diagonalizable</u>!

We are almost ready To count The Total number of closed walks in a given graph.
First, a little more machinery.

Def. The **Trace** of a matrix $(Tr(A))$ is defined as the sum of its diagonal entries.

Ex. $Tr\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = 1+4 = 5.$

A nice property of any diagonal matrix D is that successive powers of D (D^k) are still diagonal matrices. In fact,

$$D^k = \begin{bmatrix} \lambda_1^k & & & \phi \\ & \lambda_2^k & & \\ & & \ddots & \\ \phi & & & \lambda_n^k \end{bmatrix},$$

so the diagonal entries of powers of D are just the diagonal entries of D raised to that power.

Ergo, $Tr(D^k) = \sum_{i=1}^{n} \lambda_i^k = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k.$

One last theorem from Linear Algebra is needed as a prerequisite for our "big" theorem.

Theorem: The trace of the matrix D equals the trace of PDP^{-1}.

This tells us that $Tr(A(G)^k) = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k.$

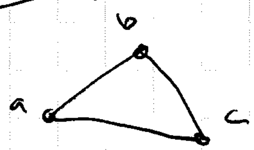
Putting this together, we are ready for our key result.

Theorem The total number of closed walks of length k in a graph G from any vertex back to itself is given by:

$$\boxed{\lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k}$$
$(k \geq 1)$

where the λ_i's are the eigenvalues of the adjacency matrix, $A(G)$ of G.

Ex. Let $G = K_3$



Then $A(G) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Using earlier techniques, we solve for the eigenvalues: $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = -1$.

Thus the total # of closed walks of length k is given by the formula: $2^k + (-1)^k + (-1)^k$.

When $k = 1$, we get, Total $= 2^1 - 1 - 1 = \underline{0}$, which is easily verified.
When $k = 2$, we have, Total $= 2^2 + (-1)^2 + (-1)^2 = \underline{\underline{6}}$, also easily verified.
More concisely, we can easily solve an otherwise challenging combinatorics problem with the general formula:

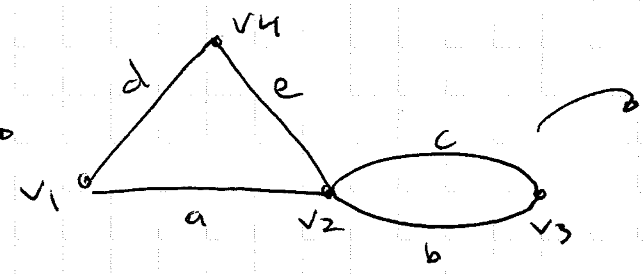
$$\text{Total \# closed walks in } K_3 \text{ (of length } k) = \begin{cases} 2^k + 2 & \text{for } k \text{ even, } k \geq 2 \\ 2^k - 2 & \text{for } k \text{ odd, } k \geq 1 \end{cases}$$

So for, $k = 10$, we have, for instance,

Total $= 2^{10} + 2 = 1026$ closed walks!

Ex. Returning to a previous example.

Given: G →



$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}}_{A(G)}$$

The eigenvalues of $A(G)$ are as follows:

$$\lambda_1 \approx 2.68133,\ \lambda_2 \approx -2.3234,\ \lambda_3 \approx -1,\ \lambda_4 \approx .642074$$

By our main Theorem then, the total # of closed walks of length k in G is $\approx (2.68133)^k + (-2.3234)^k + (-1)^k + (.642074)^k$ $(\forall k \geq 1)$

Let $k=1$, then Total $= 2.68133 - 2.3234 - 1 + .642074 \approx 0$, which is easily confirmed by the graph.

In addition, if $\underline{k=2}$, the formula yields: 14 total closed walks.

We verify this result.

1. $v_1 \to v_4 \to v_1$
2. $v_1 \to v_2 \to v_1$
3. $v_4 \to v_1 \to v_4$
4. $v_4 \to v_2 \to v_4$
5. $v_2 \to v_1 \to v_2$
6. $v_2 \to v_4 \to v_2$
7. $v_2 \to v_4 \to v_2$
8. $v_2 \to v_1 \to v_2$
9. $v_2 \xrightarrow{c} v_3 \xrightarrow{c} v_2$
10. $v_2 \xrightarrow{b} v_3 \xrightarrow{b} v_2$
11. $v_2 \xrightarrow{c} v_3 \xrightarrow{b} v_2$
12. $v_3 \xrightarrow{b} v_2 \xrightarrow{c} v_3$
13. $v_3 \xrightarrow{b} v_2 \xrightarrow{b} v_3$
14. $v_3 \xrightarrow{c} v_2 \xrightarrow{c} v_3$ ✓