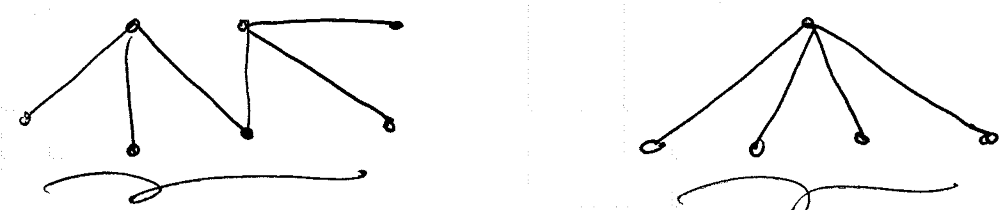
IV: Section (ii) Trees, Directed Graphs & Isomorphisms

Def. A **tree** is a connected, acyclic (i.e. contains no cycles) graph.

Ex.



Tree w/ 7 vertices

Tree w/ 5 vertices

i) Essential facts about Trees

Trees are often characterized according to the following (4) equivalences.

(1) G is connected & contains no cycles, n vertices

(2) G is connected & contains $n-1$ edges

(3) G has $n-1$ edges & no cycles

(4) For each $u, v \in V(G)$, there exists a unique path from u to v contained in G.

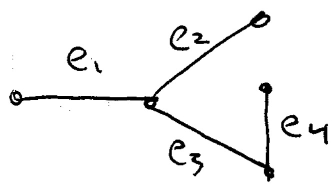
Note that by equivalences we mean that, say, if property (1) holds for a graph with n vertices, then so does (2), (3), (4), etc.

Observe that in the previous example, the tree on the left has 7 vertices & 6 edges; the tree on the right consists of 5 vertices & 4 edges - each tree is acyclic & connected - confirming the above-noted equivalences.

Additional pertinent facts about Trees:

(a) Every edge of a tree is a so-called $\boxed{\text{cut-edge}}$, meaning that its deletion increases the number of components in the graph.

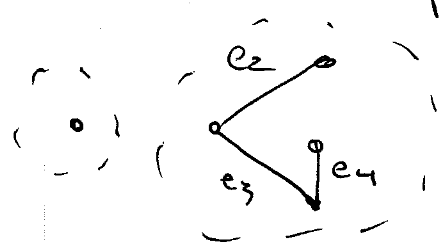
$\boxed{\text{Ex.}}$



↙ A tree with 5 vertices, 4 edges; note graph is connected & acyclic.

Observe that the deletion/removal of any edge of this tree increases the number of components of the graph.

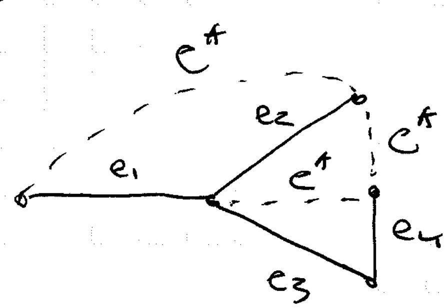
deletion of e_1 →



2 components

(b) Adding one edge to a tree forms exactly one cycle.

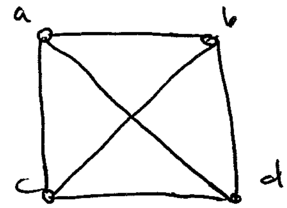
$\boxed{\text{Ex.}}$



Note that adding any of the edges labeled e^* forms exactly one cycle in the graph.

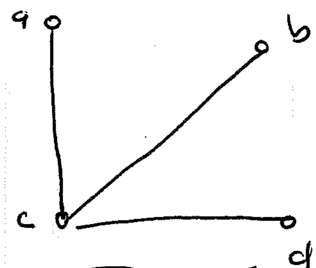
$\boxed{\text{def.}}$ A $\boxed{\text{spanning tree}}$ of a graph G is a subgraph of G with vertex set: $V(G)$ satisfying the tree criteria (i.e. it is connected & acyclic).

$\boxed{\text{Ex.}}$ Consider: $G = K_4 \rightarrow$



A spanning tree of G is given by:

Note that this choice is not unique, and that a graph in general possesses many spanning trees.

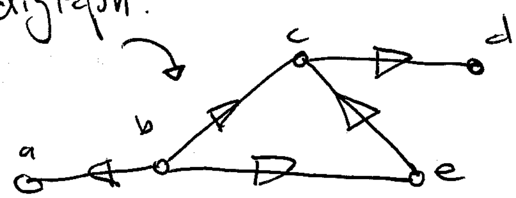


spanning tree of G

(c) Every connected graph contains a spanning tree. (See the example above).

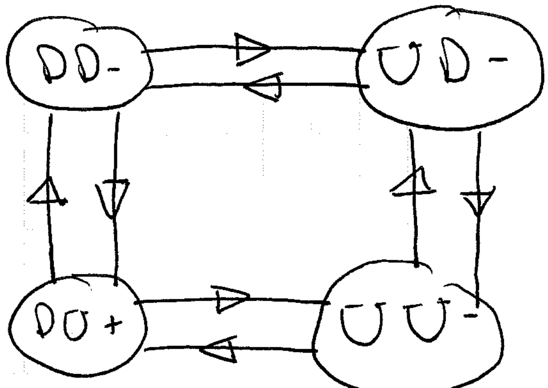
$\boxed{\text{Def.}}$ A $\boxed{\text{directed graph}}$ (or "digraph" for short) is a graph whose edges are directed/oriented from one incident vertex to the other. Conventionally, for a directed edge: $u \rightarrow v$, we say that u is the tail & v is the head, so that an edge is from its tail to its head. Frequently graphs are presented as $\boxed{\text{weighted}}$ if each edge is attributed a real-valued weight. Such weights often represent "flow" (e.g. traffic/current modeling), "cost", "distance", or some such real-world measure.

$\boxed{\text{Ex.}}$ A digraph.

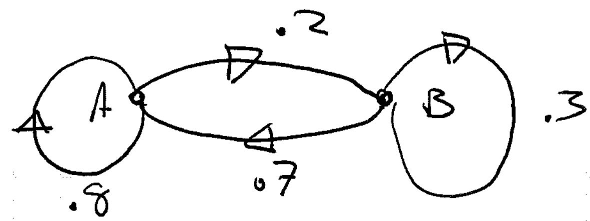


Here the arrows indicated the orientation of each edge in the graph.

Ex. Digraphs & weighted digraphs.



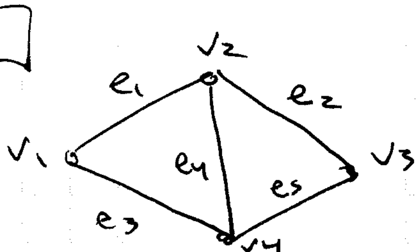
A finite automaton for two light switches: D - Down, U - Up



A "Markov chain" digraph

Def. The $\boxed{\text{incidence matrix}}$, $M(G)$ of an <u>(undirected)</u> graph G is an <u>$n \times m$ matrix</u> where $|V(G)| = n$ & $|E(G)| = m$, respectively, such that $m_{ij} = 1$ if vertex v_i & edge e_j are incident, & $m_{ij} = 0$ otherwise.

Ex.



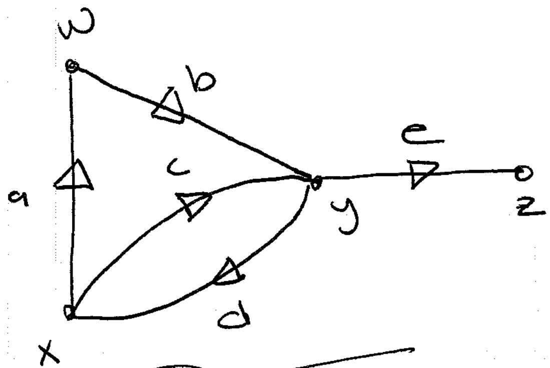
G

$$
\begin{array}{c}
 \\ v_1 \\ v_2 \\ v_3 \\ v_4
\end{array}
\begin{array}{c}
\begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1
\end{bmatrix}
\end{array}
\begin{array}{l}
 \\ \to \text{row sum} = \deg(v_1) = 2 \\ \to = 3 \\ \to = 2 \\ \to = 3
\end{array}
$$

$M(G)$

Def. For a <u>directed graph</u> G, the incidence matrix is defined analogously; we define <u>$m_{ij} = +1$</u> if v_i is the <u>tail</u> of e_j; similarly, we define <u>$m_{ij} = -1$</u> if v_i is the <u>head of</u> e_j.

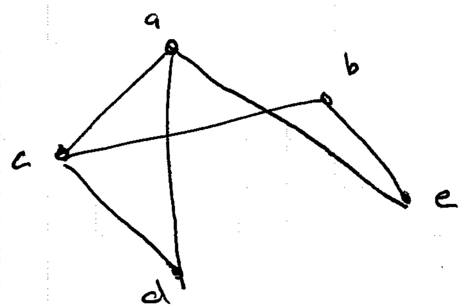
Ex.



$$\underbrace{G}_{} \longrightarrow \underbrace{\begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} w \\ x \\ y \\ z \end{matrix} & \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}}_{M(G)}$$

Def. The $\boxed{\text{adjacency matrix}}$, $A(G)$ of an undirected/directed graph G is an <u>$n \times n$ matrix</u> where the entry a_{ij} equals the number of edges from v_i to v_j in G.

Ex.



$$\underbrace{G}_{} \longrightarrow \underbrace{\begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}}_{A(G)}$$

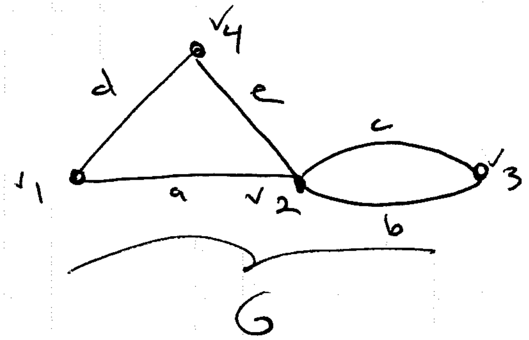
(Note: $A^T = A$)

Def. A $\boxed{\text{walk}}$ in a graph is a sequence $v_0, e_1, v_1, e_2, v_2, \ldots, v_k$ of graph vertices & edges such that edge e_i has endpoints v_{i-1} & v_i.

Next we show how the adjacency matrix of a graph can be used to count all the walks in a graph of a specified length.

$\boxed{\text{Theorem}}$ Let $\boxed{A(G)}$ be the adjacency matrix of a graph having vertices: $v_1, \ldots, v_n$. The number of distinct $v_i - v_j$ walks of length k $(k \geq 1)$ is equal to the (i,j) element of A^k.

$\boxed{\text{Ex.}}$ Consider the graph:



$$\underbrace{\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 0 & 1 \\ v_2 & 1 & 0 & 2 & 1 \\ v_3 & 0 & 2 & 0 & 0 \\ v_4 & 1 & 1 & 0 & 0 \end{array}}_{A(G)}$$

$\cdot$) Note that for walks of length 1 (i.e. $k=1$), say, the number of walks from $v_2 \to v_3$ is $(a_{23})^1 = 2$; similarly no walks of length 1 exist for $v_3 \to v_4$ and we see that $(a_{34})^1 = 0$, etc.

Now consider walks of length 2 $(k=2)$.

$$A^2 = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 6 & 0 & 1 \\ 2 & 0 & 4 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix}$$

So, for instance this shows that there are two unique walks of length 2 from $v_1 - v_1$

walk 1: $v_1 \to v_4 \to v_1$ walk 2: $v_1 \to v_2 \to v_1$

In addition, we see from A^2 that there are no walks of length 2 from $v_2 \to v_3$, etc.

A^2 also reveals, for instance, the presence of six unique walks from $v_2 \to v_2$; we enumerate this list to verify.

Walk 1: $v_2 \xrightarrow{c} v_3 \xrightarrow{c} v_2$ **Walk 2:** $v_2 \xrightarrow{b} v_3 \xrightarrow{b} v_2$

Walk 3: $v_2 \xrightarrow{c} v_3 \xrightarrow{b} v_2$ **Walk 4:** $v_2 \xrightarrow{b} v_3 \xrightarrow{c} v_2$

Walk 5: $v_2 \to v_4 \to v_2$ **Walk 6:** $v_2 \to v_1 \to v_2$.

These examples are relatively trivial due to the small length size. However, suppose we wanted to count the number of unique walks between two vertices in G for length 10 ($k=10$).

Exponentiating A yields:
$$A^{10} = \begin{bmatrix} 3218 & 4329 & 4382 & 3217 \\ 4329 & 10{,}870 & 4276 & 4329 \\ 4382 & 4276 & 6488 & 4382 \\ 3217 & 4329 & 4382 & 3218 \end{bmatrix}$$

Incredibly, this shows, for instance, that there are 4,276 unique walks in G between $v_2 \to v_3$! Clearly it would be tremendously onerous (if not impossible) to count all of these walks by hand - or even with the aid of a computer, for that matter.

N.B.: In statistics & applied mathematics, the notion of a "random walk" in a graph has become something of a sine qua non for estimating complex probabilities. In this vein, if we wanted

To determine how many random walks of length 10 emanating from v_1 exist in G, we would simply add the components from row 1 of A^{10}.

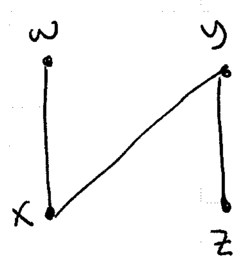
This gives: $3218 + 4329 + 4382 + 3217 = 15,146$.

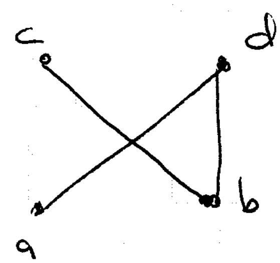
In summary, there are a total of 15,146 random walks of length 10 (beginning @ v_1) in G – we note again that such a computation would, naturally, be extremely difficult to achieve by hand.

Graph Isomorphisms

Informally, two graphs are said to be isomorphic if they are "structurally identical" – in other words, if a re-labeling of the vertices of graph G yields graph H, the graphs are isomorphic & we write: $\boxed{G \simeq H}$.

Ex.

G: 

H: 

Note that if we impose the following vertex re-labeling on

$$G: \left.\begin{array}{l} w \to a \\ x \to d \\ y \to b \\ z \to c \end{array}\right\}$$

we get the graph H.

Thus, $G \simeq H$.

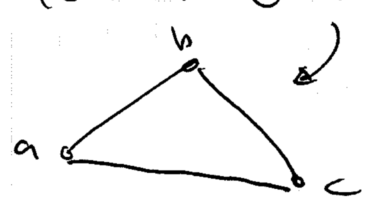
Def. More formally, we say G is **isomorphic** to H (i.e. $G \cong H$) if there exists a bijection f (one-to-one, onto map) from $V(G)$ to $V(H)$ such that $uv \in E(G)$ iff $f(u)f(v) \in E(H)$. Put another way, f "preserves" edge incidences in H.

From the previous example, then, let $f: V(G) \to V(H)$ with: $f(w)=a$, $f(x)=d$, $f(y)=b$ & $f(z)=c$. This defines an isomorphism between G & H.

Def. If a graph is <u>isomorphic to itself</u>, we say that there exists an **automorphism** on G.

Note that the (trivial) <u>identity map</u> is an automorphism on G, always. Such an automorphism is consequently said to be <u>Trivial</u>.

Ex. Consider $G = K_3$



Note that any <u>permutation</u> of the vertices of K_3 yields an automorphism.

For instance: $f(a)=b$, $f(b)=c$, $f(c)=a$ is an automorphism of K_3.

In total there are <u>3</u> such automorphisms.

In general, $G = K_n$ has <u>n</u> total automorphisms. We say the "automorphism class" of K_n is of <u>size n.</u>

Lastly, lets explore a deep connection between isomorphic graphs and their adjacency matrices.

$\boxed{\text{Thm}}$: Two simple graphs <u>G & H are isomorphic</u> iff <u>there exists a permutation matrix P</u> such that: <u>$A(G) = P \cdot A(H) P^T$</u>, where $A(G)$ is the adjacency matrix of G, and $A(H)$ is the adjacency matrix of H.

<u>Q</u>: why does this work?

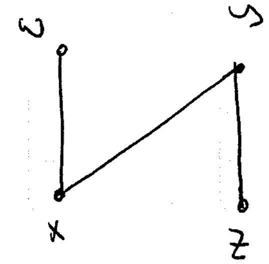
If $G \cong H$, then some re-labeling of the vertices of G yields H. Equivalently, relabeling the rows/cols. of $A(G)$ appropriately gives $A(H)$. (or vice versa)

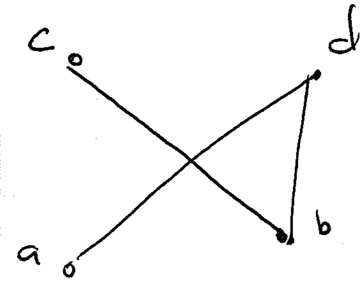
Note that $\underbrace{P}_{\text{perm. matrix}} \cdot A(H)$ effectively performs <u>row swaps</u> of $A(H)$, while $A(H)\underbrace{P^T}_{\text{Transpose of perm matrix}}$ performs the equivalent <u>column swaps</u> on $A(H)$.

Consequently, $A(G) = P \cdot A(H) P^T$ means that we can effectuate a relabeling of row vertices (& equivalent column vertices) on $A(G)$ to yield $A(H)$ (or vice versa).

We demonstrate with an earlier example.

Recall the example:

G: 

H: 

Where $G \cong H$ with the explicit isomorphism:
$w \to a$
$x \to d$
$y \to b$
$z \to c$

Consider the respective adjacency matrices:

$$A(G) = \begin{array}{c} w \\ x \\ y \\ z \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \& \quad A(H) = \begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

As indicated, the isomorphism is achieved thusly:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad P^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\to \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A(G)} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{P^T} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}}_{A(H)}.$$