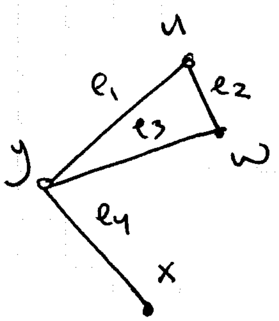
$\boxed{\text{IV}}$ Intro. To Graphs / Graph Theory

A graph G: consists of a Triple:

1. A vertex set: $V(G)$.
2. An edge set: $E(G)$.
3. A relation that associates each edge with two vertices (its endpoints).

A graph is called $\boxed{\text{simple}}$ if it contains no loops or multiple edges. When u & v are endpoints of an edge, we write "uv" for the edge or "$u \leftrightarrow v$"; we say that u & v are adjacent.

$\boxed{\text{Ex.}}$

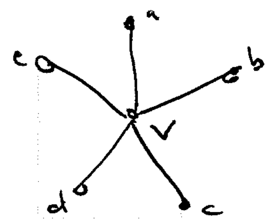


← A simple graph with
$V(G) = \{u, w, x, y\}$
$E(G) = \{e_1, e_2, e_3, e_4\}$

The $\boxed{\text{order}}$ of a graph is (generally) defined as the size of its vertex set: $|V(G)|$.

The $\boxed{\text{degree}}$ of a vertex: v, written $\boxed{\deg(v)}$ is equal to the number of edges emanating from the vertex - or equivalently, we may say $\deg(v) = \#$ edges "incident to v."

Ex.



In the previous example, $\deg(v)=5$, and $\deg(a)=\deg(b)=\cdots=\deg(e)=1$.

Many "special" or commonly-encountered graphs have particular names. The graph above (and its ilk) are called $\boxed{\text{star graphs.}}$

Other common graph types include: Paths, cycles, complete graphs, cubes, bipartite graphs & the Petersen graph (among many others!).

A $\boxed{\text{path}}$ is a simple graph whose vertices can be ordered so that two vertices are adjacent iff they are consecutive in the list.

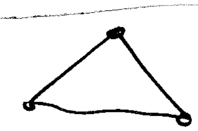
P_2 : 

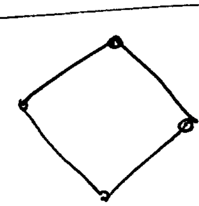
P_3 : 

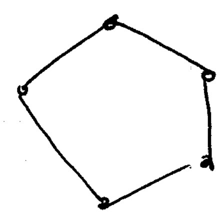
P_4 : 

$\underset{\#\text{ verts.}}{P_n}$: Path w/ n vertices.

A $\boxed{\text{cycle}}$ is a path with an equal number of vertices & edges that form a closed loop.

C_3 : 

C_4 : 

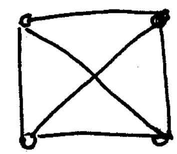
C_5 : 

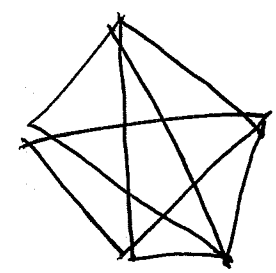
C_n : Cycle with n vertices

A **complete graph** is a graph where each vertex is adjacent to every vertex in the graph - except itself.

K_3: 

"Triangle" graph

K_4: 

K_5: 

K_n: Complete graph on n vertices

A graph is called **regular** if every vertex has equal degree.

Note that K_3 is "2-regular", since $\deg(v)=2 \ \forall v$.

K_4 is "3-regular" & K_n is "$n-1$-regular".

<u>Q</u>: What is the order of $E(K_n)$, i.e. $|E(K_n)|$?

<u>A</u>: $|E(K_n)| = \frac{n(n-1)}{2} \rightarrow \frac{(n \text{ vertices})\cdot(\deg \text{ each vertex})}{2}$ ← count each edge twice

Alternatively, $|E(K_n)| = \underline{\underline{\binom{n}{2}}} = \frac{n(n-1)}{2}$.

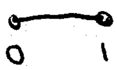
In graph theory, the **maximum degree** over all vertices in a graph is denoted: $\boxed{\Delta(G)}$; the **minimum degree** is denoted: $\boxed{\delta(G)}$.

Note that a graph is regular iff $\Delta(G)=\delta(G)$.

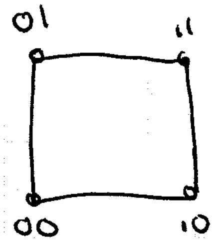
The $\boxed{\text{cube}}$ or $\boxed{k\text{-dimensional cube: } Q_k}$ is the simple graph whose vertices are k-tuples with entries $\{0,1\}$ and whose edges are the pairs of k-tuples that differ in exactly one position.

$\boxed{\text{Ex.}}$

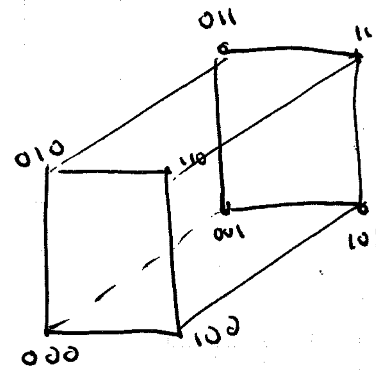
$Q_1 \rightarrow$



Q_2:



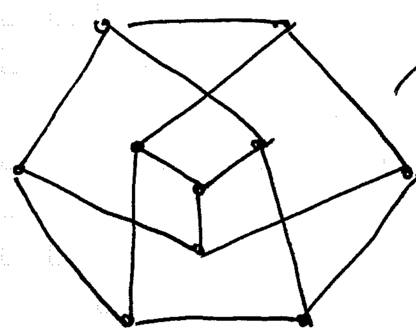
Q_3:



Note that: $|V(Q_k)| = 2^k$ & $|E(Q_k)| = k \cdot 2^{k-1}$

Also, $\Delta(Q_k) = \delta(Q_k) = k$, so Q_k is "k-regular."

The $\boxed{\text{Petersen Graph}}$ is famous graph with a number of interesting theoretical properties; it consists of 10 vertices; it is <u>3-regular</u> and contains 10 unique cycles of length six.

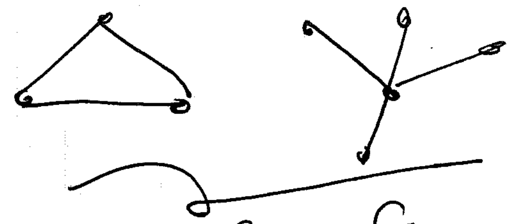


Petersen Graph: $\Delta(P) = \delta(P) = 3$.

A graph G is called $\boxed{\text{connected}}$ if for each $u, v \in V(G)$ there exists a path in G connecting these vertices.

A <u>maximally connected</u> subgraph of G is called a $\boxed{\text{component}}$.

Ex.

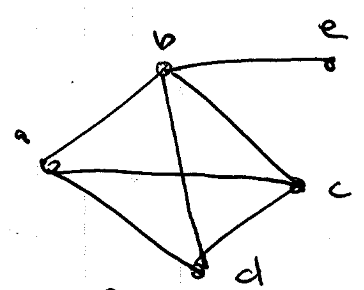


Graph G

Above, G consists of Two components: The Triangle on The left & The star on The right.

A **clique** in a graph is a set of pairwise adjacent vertices.

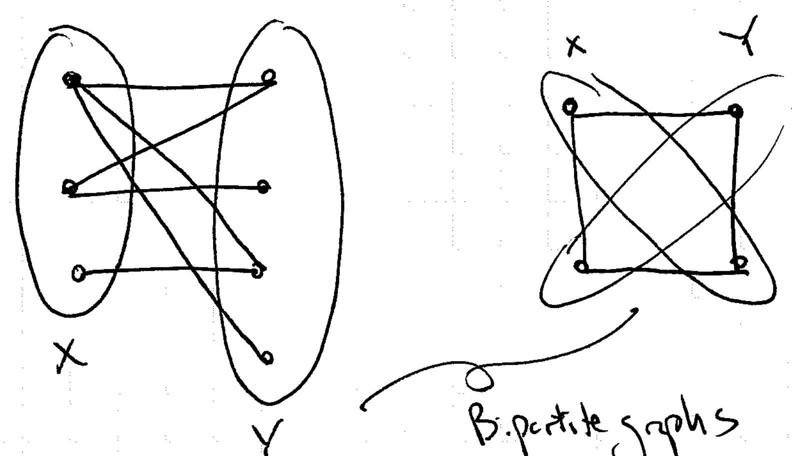
Ex.



In This graph, The set: $\{a,b,c,d\}$ forms a clique.

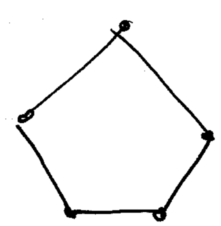
A **bipartite** graph G is a graph whose vertex set can be partitioned into Two sets: X & Y such That every edge in G has one end in X & the other in Y.

Ex.



Bipartite graphs

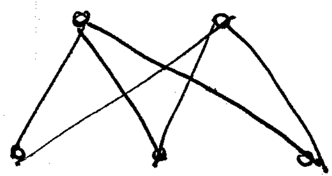
A non-bipartite graph: (why?)



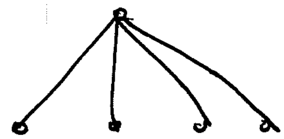
A bipartite graph with partite sets X & Y is called a $\boxed{\text{complete bipartite graph}}$ if <u>every possible</u> connection of a vertex of X with a vertex of Y is present in the graph.

Such graphs are denoted: $\boxed{K_{n,m}}$

$\boxed{\text{Ex.}}$



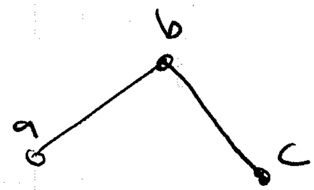
$K_{2,3}$



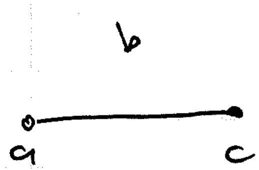
$K_{1,4}$

The $\boxed{\text{complement of a graph}}$ G, denoted $\boxed{\overline{G}}$, is the graph whose vertex set is the same as that of G & whose edge set consists of all the edges that are <u>NOT</u> present in G.

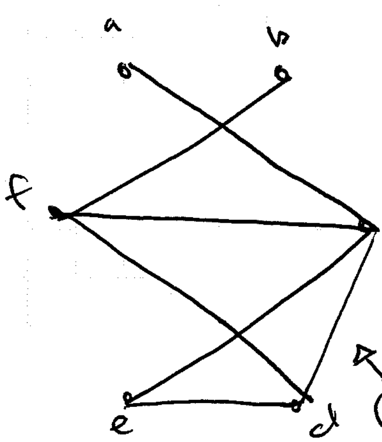
$\boxed{\text{Ex.}}$



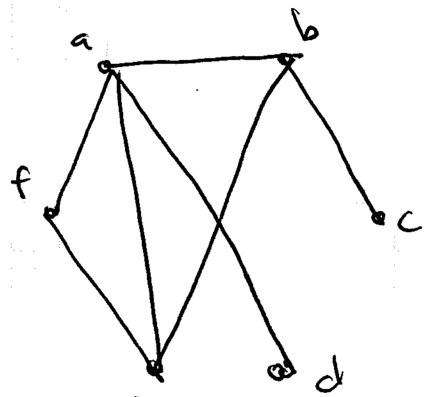
G



$\overline{G}$



G



Theorem: The "First Theorem of Graph Theory"

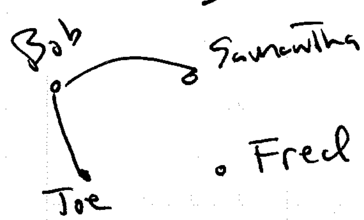
[The number of vertices with odd degree is even in any graph.]

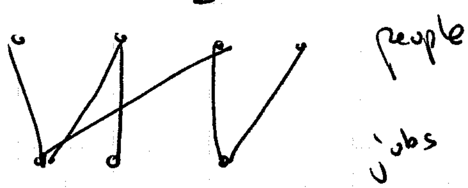
Proof: Let $S = \sum_{v \in V(G)} \deg(v)$. Notice that in counting S, we count each edge exactly twice. Hence, $S = 2|E(G)|$. Because S is even, it must be that the number of vertices of odd degree is even. $\square$

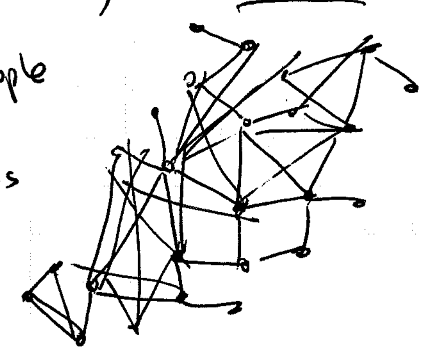
Why study graphs?

i) Graphs can be used to model a diverse array of phenomena, including (but hardly limited to): Topological relationships, geometric relationships, Networks (including "complex" networks) - incl. computer/Terminal networks, social networks, Transportation/road networks, economic networks, the brain (a network of neurons), "scheduling problems", "job assignment problems", counting/combinatorics, stochastic problems, and a multitude of other applications!

Ex. A "social network" ; a "job assignment" problem ; the internet!



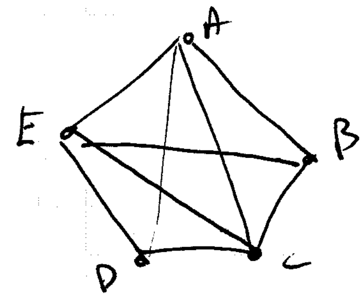




$\boxed{\text{Ex.}}$ A "Toy" example -

If five friends mutually shake hands, how many handshakes occur?
(i.e what is the "size" of this social network?)

We model the problem by representing people as vertices, edges represent handshakes. Using K_5 as our graph:



Total # handshakes $= |E(K_5)| = \frac{5 \cdot 4}{2} = \underline{10}$.

:) In the following lectures we explore some of the essential intersections between graphs & their associated matrix representations – in relation to complex networks.

:) Some of these explorations will include the further study of:

Trees, directed graphs, counting walks & paths,
solving the "shortest path problem" (i.e the Google Map problem),
"Matrix-Tree" computations, Kruskal's Algorithm,
Network centrality, community detection, PageRank
& Power Laws & Scale-Free Networks.