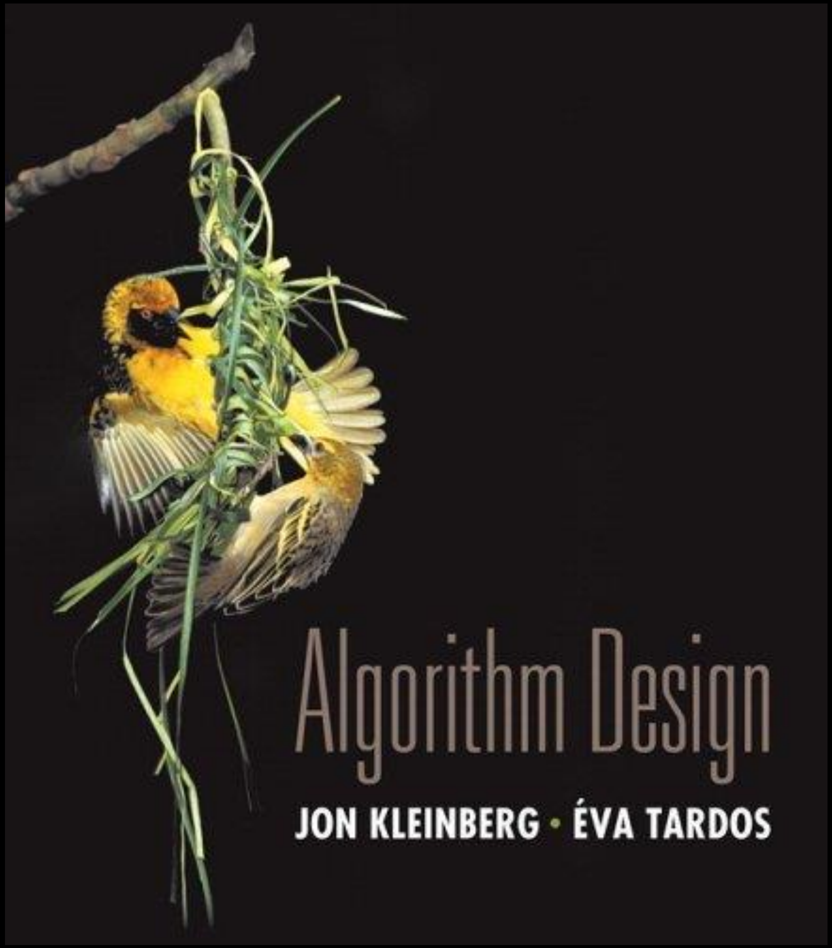


Chapter 6

Dynamic Programming

CS 350: Winter 2018

Algorithmic Paradigms

Greedy. Build up a solution incrementally, myopically optimizing some local criterion.

Divide-and-conquer. Break up a problem into sub-problems, solve each sub-problem independently, and combine solution to sub-problems to form solution to original problem.

Dynamic programming. Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems; typically each subproblem is solved just once, and the solution is stored (i.e. cached).

The next time the same subproblem occurs, instead of recomputing its solution, one simply looks up the previously computed solution, thereby saving computation time at the expense of a (hopefully) modest expenditure in storage space. Each of the subproblem solutions is indexed in some way, typically based on the values of its input parameters, so as to facilitate its lookup. The technique of storing solutions to subproblems instead of recomputing them is called memoization.

Dynamic Programming History

Bellman. [1950s] Pioneered the systematic study of dynamic programming.

Etymology.

- Dynamic programming = planning over time.
- Secretary of Defense was hostile to mathematical research.
- Bellman sought an impressive name to avoid confrontation.

"it's impossible to use dynamic in a pejorative sense"
"something not even a Congressman could object to"

Reference: Bellman, R. E. *Eye of the Hurricane, An Autobiography.*

Dynamic Programming Applications

Areas.

- Bioinformatics.
- Control theory.
- Information theory.
- Operations research.
- Computer science: theory, graphics, AI, compilers, systems, ….

Some famous dynamic programming algorithms.

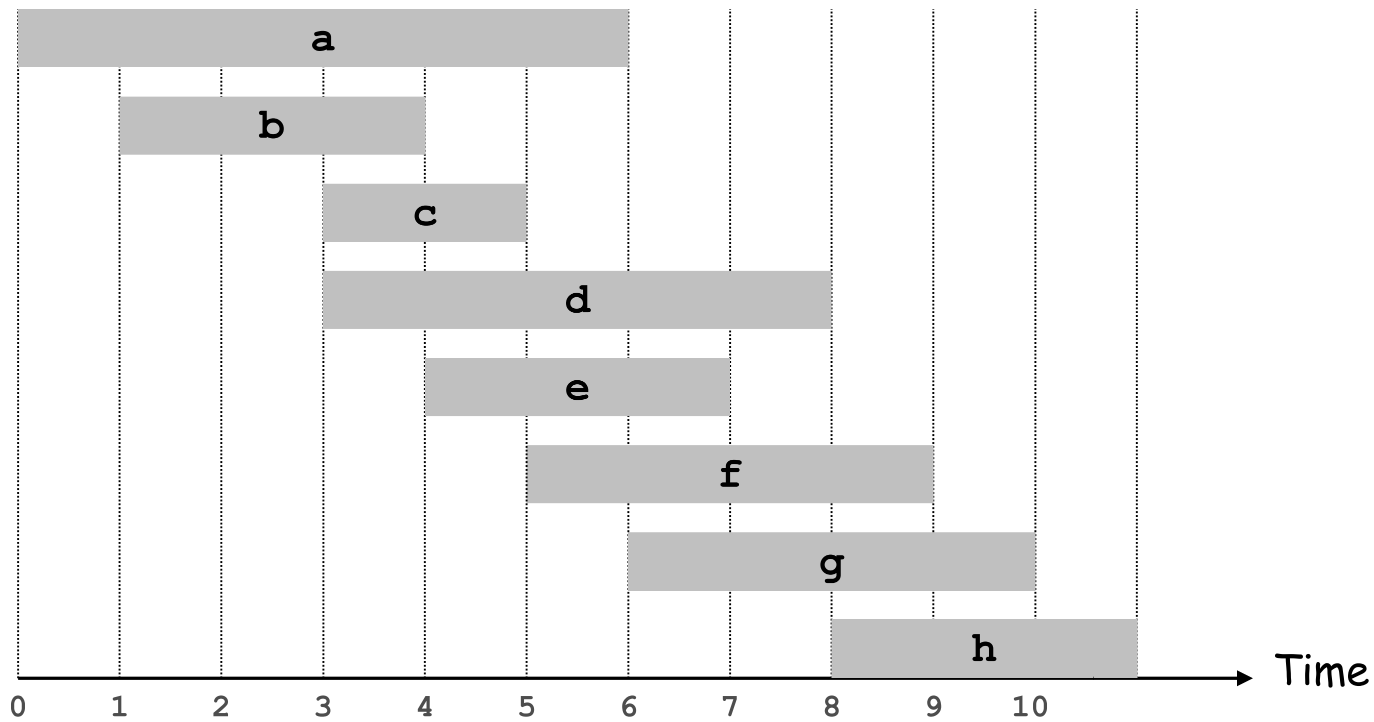
- Unix diff for comparing two files.
- Viterbi for hidden Markov models.
- Smith-Waterman for genetic sequence alignment.
- Bellman-Ford for shortest path routing in networks.
- Cocke-Kasami-Younger for parsing context free grammars.

6.1 Weighted Interval Scheduling

Weighted Interval Scheduling

Weighted interval scheduling problem.

- Job j starts at s_j, finishes at f_j, and has weight or value v_j .
- Two jobs compatible if they don't overlap.
- Goal: find maximum weight subset of mutually compatible jobs.

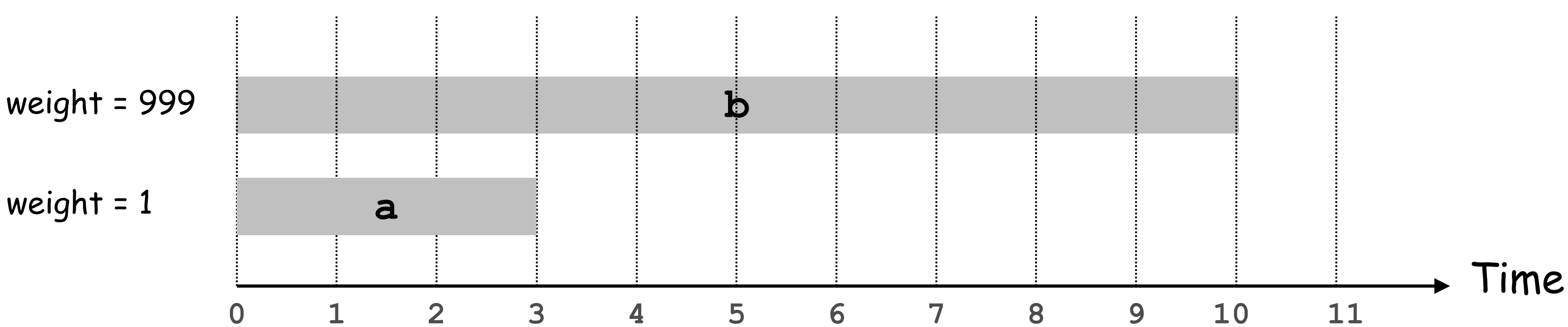


Unweighted Interval Scheduling Review

Recall. Greedy algorithm works if all weights are 1.

- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

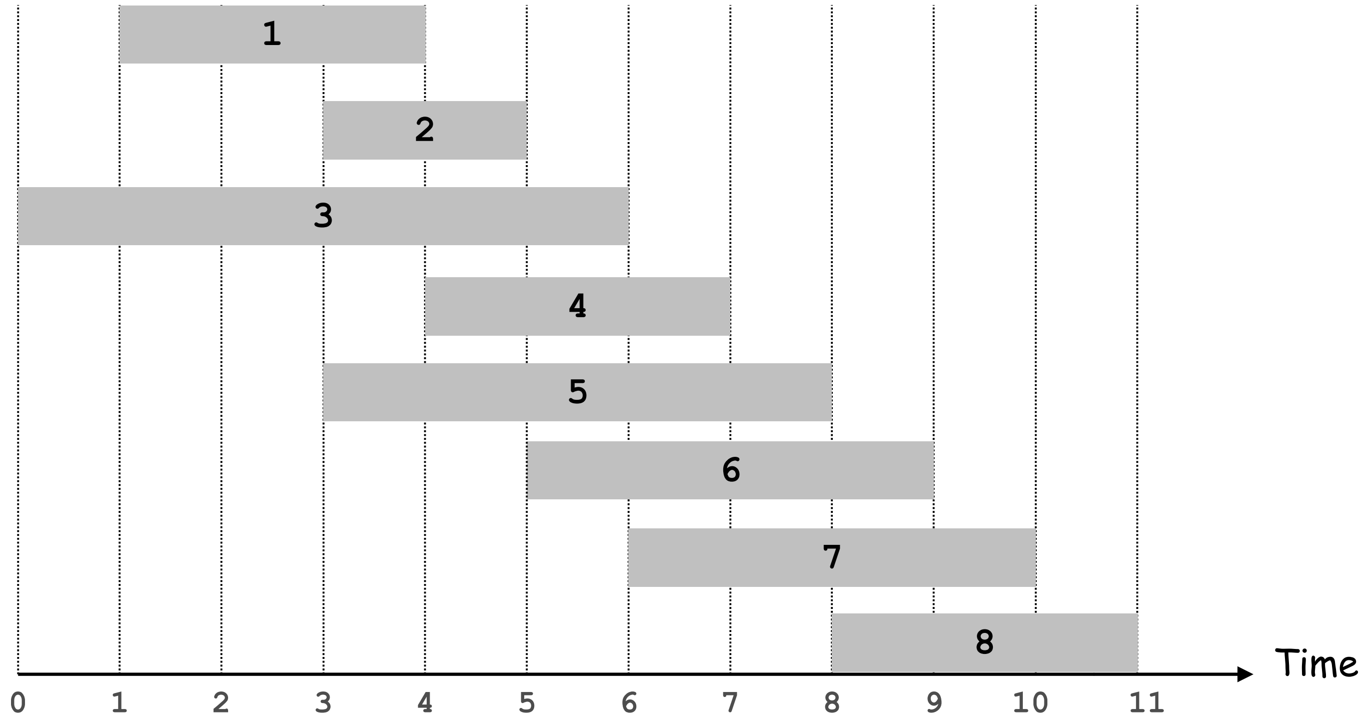
Observation. Greedy algorithm can fail spectacularly if arbitrary weights are allowed.



Weighted Interval Scheduling

Notation. Label jobs by finishing time: $f_1 \leq f_2 \leq \ldots \leq f_n$.
Def. p(j) = largest index i < j such that job i is compatible with j.

Ex: p(8) = 5, p(7) = 3, p(2) = 0.



Dynamic Programming: Binary Choice

Notation. $OPT(j)$ = value of optimal solution to the problem consisting of job requests 1, 2, ..., j.

- Case 1: OPT selects job j.
  - collect profit v_j
  - can't use incompatible jobs { p(j) + 1, p(j) + 2, ..., j - 1 }
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., p(j)

optimal substructure

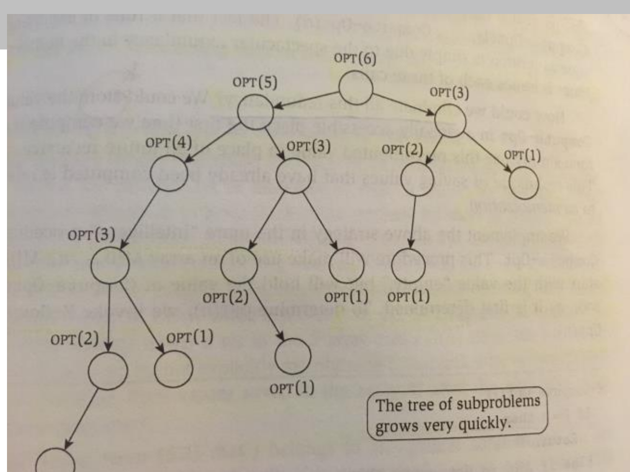
- Case 2: OPT does not select job j.
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., j-1

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max \{ v_j + OPT(p(j)),\ OPT(j-1) \} & \text{otherwise} \end{cases}$$

Weighted Interval Scheduling: Brute Force

Brute force algorithm.

```
Input: n, s1,…,sn , f1,…,fn , v1,…,vn

Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.

Compute p(1), p(2), …, p(n)

Compute-Opt(j) {
   if (j = 0)
      return 0
   else
      return max(vj + Compute-Opt(p(j)), Compute-Opt(j-1))
}
```

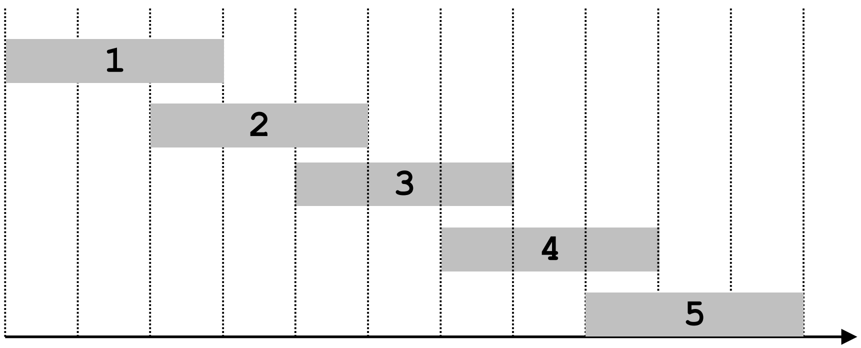


This algorithm can take <u>exponential time in the worst-case</u>, due to the potential redundancy in the recursive calls.

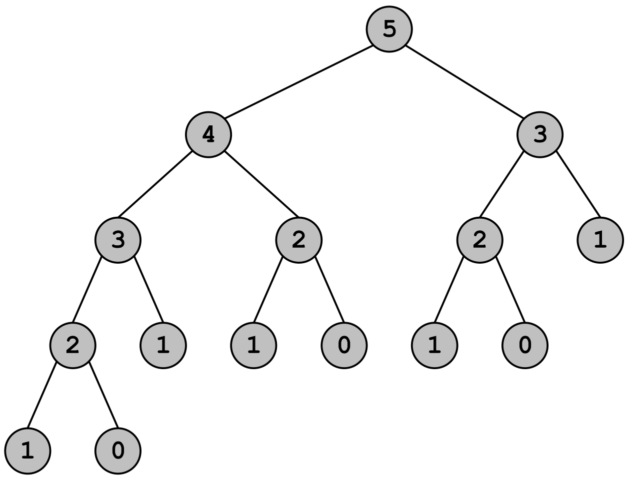
Weighted Interval Scheduling: Brute Force

Observation. Recursive algorithm fails spectacularly because of redundant sub-problems $\Rightarrow$ exponential algorithms.

Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence.



p(1) = 0, p(j) = j-2



Weighted Interval Scheduling: Memoization

Memoization. Store results of each sub-problem in a cache; lookup as needed.

```
Input: n, s_1,…,s_n , f_1,…,f_n , v_1,…,v_n

Sort jobs by finish times so that f_1 ≤ f_2 ≤ ... ≤ f_n.
Compute p(1), p(2), …, p(n)

for j = 1 to n
   M[j] = empty          global array
M[0] = 0

M-Compute-Opt(j) {
   if (M[j] is empty)
      M[j] = max(v_j + M-Compute-Opt(p(j)), M-Compute-Opt(j-1))
   return M[j]
}
```

Weighted Interval Scheduling: Running Time

Claim. Memoized version of algorithm takes $O(n \log n)$ time.

- Sort by finish time: $O(n \log n)$.
- Computing $p(\cdot)$: $O(n \log n)$ via sorting by start time.

- `M-Compute-Opt(j)`: each invocation takes $O(1)$ time and either
  - (i) returns an existing value `M[j]`
  - (ii) fills in one new entry `M[j]` and makes two recursive calls

- Progress measure Φ = # nonempty entries of `M[]`.
  - initially $\Phi = 0$, throughout $\Phi \leq n$.
  - (ii) increases Φ by 1 $\Rightarrow$ at most $2n$ recursive calls.

- Overall running time of `M-Compute-Opt(n)` is $O(n)$. ▪

Remark. $O(n)$ if jobs are pre-sorted by start and finish times.

Weighted Interval Scheduling: Finding a Solution

Q. Dynamic programming algorithms computes optimal value. What if we want the solution itself?

A. Do some post-processing.

```
Run M-Compute-Opt(n)
Run Find-Solution(n)

Find-Solution(j) {
   if (j = 0)
      output nothing
   else if (v_j + M[p(j)] > M[j-1])
      print j
      Find-Solution(p(j))
   else
      Find-Solution(j-1)
}
```

- # of recursive calls $\leq n \Rightarrow O(n)$.

Weighted Interval Scheduling: Bottom-Up

Bottom-up dynamic programming. Unwind recursion.

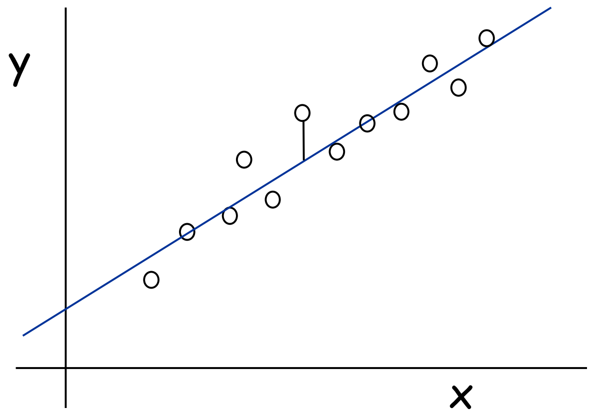
```
Input: n, s1,…,sn , f1,…,fn , v1,…,vn

Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.

Compute p(1), p(2), …, p(n)

Iterative-Compute-Opt {
   M[0] = 0
   for j = 1 to n
      M[j] = max(vj + M[p(j)], M[j-1])
}
```

6.3 Segmented Least Squares

Least squares.

- Foundational problem in statistic and numerical analysis.
- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.
- Find a line $y = ax + b$ that minimizes the sum of the squared error:

$$SSE = \sum_{i=1}^{n} (y_i - a x_i - b)^2$$



Solution. Calculus $\Rightarrow$ min error is achieved when

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2}, \quad b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

Segmented Least Squares

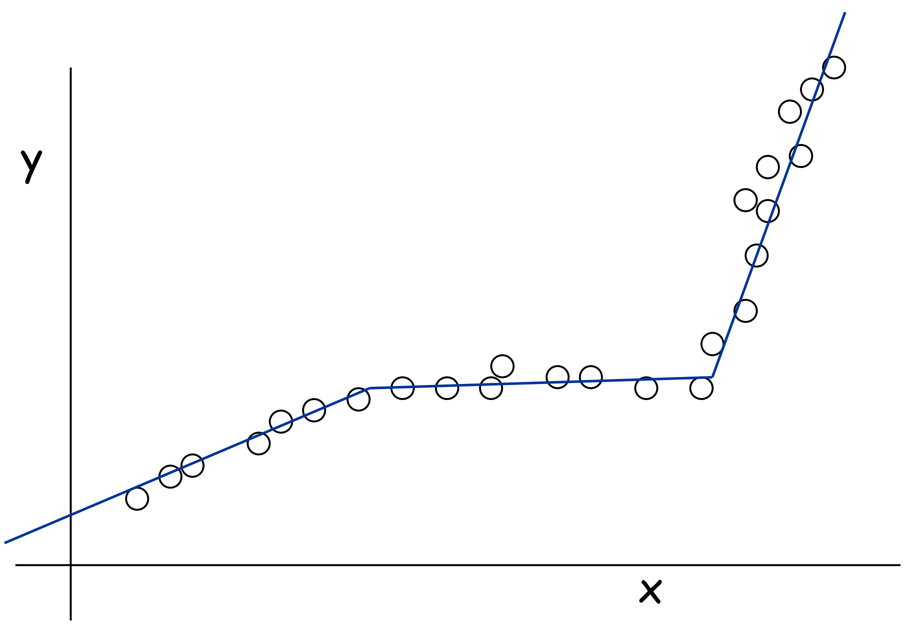
Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with
- $x_1 < x_2 < \ldots < x_n$, find a sequence of lines that minimizes $f(x)$.

Q. What's a reasonable choice for $f(x)$ to balance accuracy and parsimony?

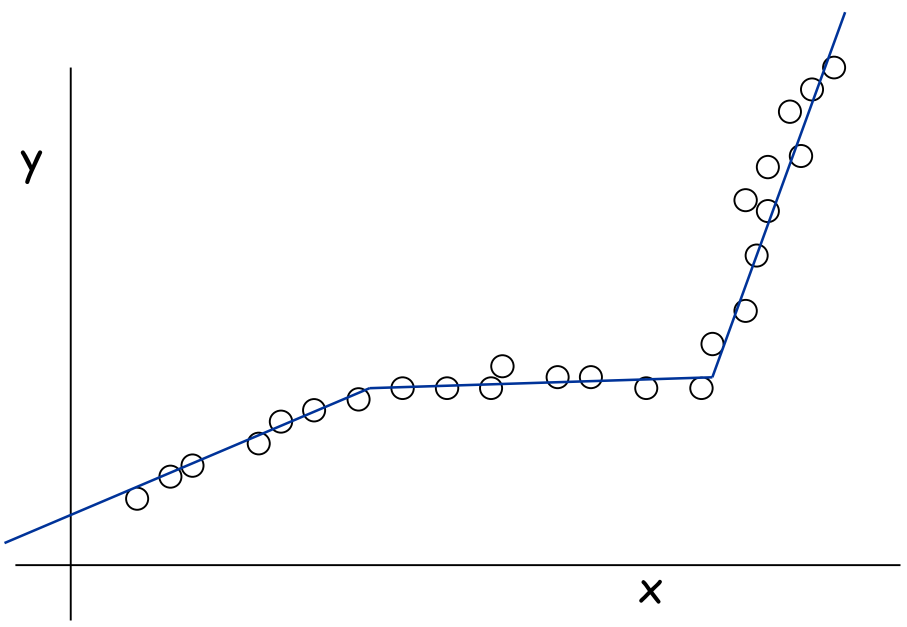
goodness of fit

number of lines



Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with
- $x_1 < x_2 < \ldots < x_n$, find a sequence of lines that minimizes:
  - the sum of the sums of the squared errors E in each segment
  - the number of lines L
- Tradeoff function: $E + c L$, for some constant $c > 0$.



Dynamic Programming: Multiway Choice

Notation.

- $OPT(j)$ = minimum cost for points $p_1, p_{i+1}, \ldots, p_j$.
- $e(i, j)$ = minimum sum of squares for points $p_i, p_{i+1}, \ldots, p_j$.

To compute OPT(j):

- Last segment uses points $p_i, p_{i+1}, \ldots, p_j$ for some i.
- Cost = $e(i, j) + c + OPT(i-1)$.

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \min\limits_{1 \le i \le j} \{ \; e(i,j) + c + OPT(i-1) \} & \text{otherwise} \end{cases}$$

Segmented Least Squares: Algorithm

```
INPUT: n, p₁,…,pN , c

Segmented-Least-Squares() {
   M[0] = 0
   for j = 1 to n
      for i = 1 to j
         compute the least square error eij for
         the segment pi,…, pj

   for j = 1 to n
      M[j] = min 1 ≤ i ≤ j (eij + c + M[i-1])

   return M[n]
}
```

Running time. $O(n^3)$. ← can be improved to $O(n^2)$ by pre-computing various statistics

- Bottleneck = computing $e(i, j)$ for $O(n^2)$ pairs, $O(n)$ per pair using previous formula.

6.4 Knapsack Problem

Knapsack Problem

Knapsack problem.

- Given n objects and a "knapsack."
- Item i weighs $w_i > 0$ kilograms and has value $v_i > 0$.
- Knapsack has capacity of W kilograms.
- Goal: fill knapsack so as to maximize total value.

Ex: { 3, 4 } has value 40.

W = 11

#	value	weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Greedy: repeatedly add item with maximum ratio v_i / w_i.

Ex: { 5, 2, 1 } achieves only value = 35 $\Rightarrow$ greedy not optimal.

Dynamic Programming: False Start

Def. OPT(i) = max profit subset of items 1, …, i.

- Case 1: OPT does not select item i.
  - OPT selects best of { 1, 2, …, i-1 }
- Case 2: OPT selects item i.
  - accepting item i does not immediately imply that we will have to reject other items
  - without knowing what other items were selected before i, we don't even know if we have enough room for i

Conclusion. Need more sub-problems!

Dynamic Programming: Adding a New Variable

Def. OPT(i, w) = max profit subset of items 1, …, i with weight limit w.

- Case 1: OPT does not select item i.
  - OPT selects best of { 1, 2, …, i-1 } using weight limit w
- Case 2: OPT selects item i.
  - new weight limit = $w - w_i$
  - OPT selects best of { 1, 2, …, i–1 } using this new weight limit

$$OPT(i,w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1,w) & \text{if } w_i > w \\ \max\{\, OPT(i-1,w),\ \ v_i + OPT(i-1, w-w_i) \,\} & \text{otherwise} \end{cases}$$

Knapsack Problem: Bottom-Up

Knapsack. Fill up an n-by-W array.

```
Input: n, W, w1,…,wN, v1,…,vN

for w = 0 to W
   M[0, w] = 0

for i = 1 to n
   for w = 1 to W
      if (wi > w)
         M[i, w] = M[i-1, w]
      else
         M[i, w] = max {M[i-1, w], vi + M[i-1, w-wi ]}

return M[n, W]
```

Knapsack Algorithm

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT: { 4, 3 }
value = 22 + 18 = 40

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Knapsack Problem: Running Time

Running time. $\Theta(n\ W)$.

- Not polynomial in input size!
- "Pseudo-polynomial."
- Decision version of Knapsack is NP-complete. [Chapter 8]

Knapsack approximation algorithm. There exists a poly-time algorithm that produces a feasible solution that has value within 0.01% of optimum. [Section 11.8]

Dynamic Programming Summary

Recipe.

- Characterize structure of problem.
- Recursively define value of optimal solution.
- Compute value of optimal solution.
- Construct optimal solution from computed information.

Dynamic programming techniques.

- Binary choice: weighted interval scheduling.
- Multi-way choice: segmented least squares.
- Adding a new variable: knapsack.
- Dynamic programming over intervals: RNA secondary structure.

Viterbi algorithm for HMM also uses DP to optimize a maximum likelihood tradeoff between parsimony and accuracy

CKY parsing algorithm for context-free grammar has similar structure

Top-down vs. bottom-up: different people have different intuitions.

6.6 Sequence Alignment

String Similarity

How similar are two strings?

- `ocurrance`
- `occurrence`

o	c	u	r	r	a	n	c	e	-
o	c	c	u	r	r	e	n	c	e

6 mismatches, 1 gap

o	c	-	u	r	r	a	n	c	e
o	c	c	u	r	r	e	n	c	e

1 mismatch, 1 gap

o	c	-	u	r	r	-	a	n	c	e
o	c	c	u	r	r	e	-	n	c	e

0 mismatches, 3 gaps

Edit Distance

Applications.

- Basis for Unix diff.
- Speech recognition.
- Computational biology.

Edit distance. [Levenshtein 1966, Needleman-Wunsch 1970]

- Gap penalty δ; mismatch penalty α_{pq}.
- Cost = sum of gap and mismatch penalties.

C	T	G	A	C	C	T	A	C	C	T
C	C	T	G	A	C	T	A	C	A	T

$\alpha_{TC} + \alpha_{GT} + \alpha_{AG} + 2\alpha_{CA}$

-	C	T	G	A	C	C	T	A	C	C	T
C	C	T	G	A	C	-	T	A	C	A	T

$2\delta + \alpha_{CA}$

Sequence Alignment

Goal: Given two strings $X = x_1\, x_2 \, . \, . \, . \, x_m$ and $Y = y_1\, y_2 \, . \, . \, . \, y_n$ find alignment of minimum cost.

Def. An alignment M is a set of ordered pairs x_i-y_j such that each item occurs in at most one pair and no crossings.

Def. The pair x_i-y_j and $x_{i'}$-$y_{j'}$ cross if $i < i'$, but $j > j'$.

$$\text{cost}(M) = \underbrace{\sum_{(x_i, y_j) \in M} \alpha_{x_i y_j}}_{\text{mismatch}} + \underbrace{\sum_{i \,:\, x_i \text{ unmatched}} \delta + \sum_{j \,:\, y_j \text{ unmatched}} \delta}_{\text{gap}}$$

Ex: CTACCG vs. TACATG.

Sol: $M = x_2\text{-}y_1,\ x_3\text{-}y_2,\ x_4\text{-}y_3,\ x_5\text{-}y_4,\ x_6\text{-}y_6$.

x_1	x_2	x_3	x_4	x_5		x_6
C	T	A	C	C	-	G
-	T	A	C	A	T	G
	y_1	y_2	y_3	y_4	y_5	y_6

Sequence Alignment: Problem Structure

Def. $OPT(i, j)$ = min cost of aligning strings $x_1\, x_2 \ldots x_i$ and $y_1\, y_2 \ldots y_j$.

- Case 1: OPT matches x_i-y_j.
  - pay mismatch for x_i-y_j + min cost of aligning two strings $x_1\, x_2 \ldots x_{i-1}$ and $y_1\, y_2 \ldots y_{j-1}$
- Case 2a: OPT leaves x_i unmatched.
  - pay gap for x_i and min cost of aligning $x_1\, x_2 \ldots x_{i-1}$ and $y_1\, y_2 \ldots y_j$
- Case 2b: OPT leaves y_j unmatched.
  - pay gap for y_j and min cost of aligning $x_1\, x_2 \ldots x_i$ and $y_1\, y_2 \ldots y_{j-1}$

$$OPT(i, j) = \begin{cases} j\delta & \text{if } i = 0 \\ \min \begin{cases} \alpha_{x_i y_j} + OPT(i-1, j-1) \\ \delta + OPT(i-1, j) \\ \delta + OPT(i, j-1) \end{cases} & \text{otherwise} \\ i\delta & \text{if } j = 0 \end{cases}$$

Sequence Alignment: Algorithm

```
Sequence-Alignment(m, n, x1x2...xm, y1y2...yn, δ, α) {
   for i = 0 to m
      M[i, 0] = iδ
   for j = 0 to n
      M[0, j] = jδ

   for i = 1 to m
      for j = 1 to n
         M[i, j] = min(α[xi, yj] + M[i-1, j-1],
                       δ + M[i-1, j],
                       δ + M[i, j-1])
   return M[m, n]
}
```

Analysis. $\Theta(mn)$ time and space.

English words or sentences: $m, n \leq 10$.

Computational biology: $m = n = 100,000$. 10 billions ops OK, but 10GB array?

6.7 Sequence Alignment in Linear Space

Sequence Alignment: Linear Space

Q. Can we avoid using quadratic space?

Easy. Optimal value in O(m + n) space and O(mn) time.

- Compute OPT(i, •) from OPT(i-1, •).
- No longer a simple way to recover alignment itself.

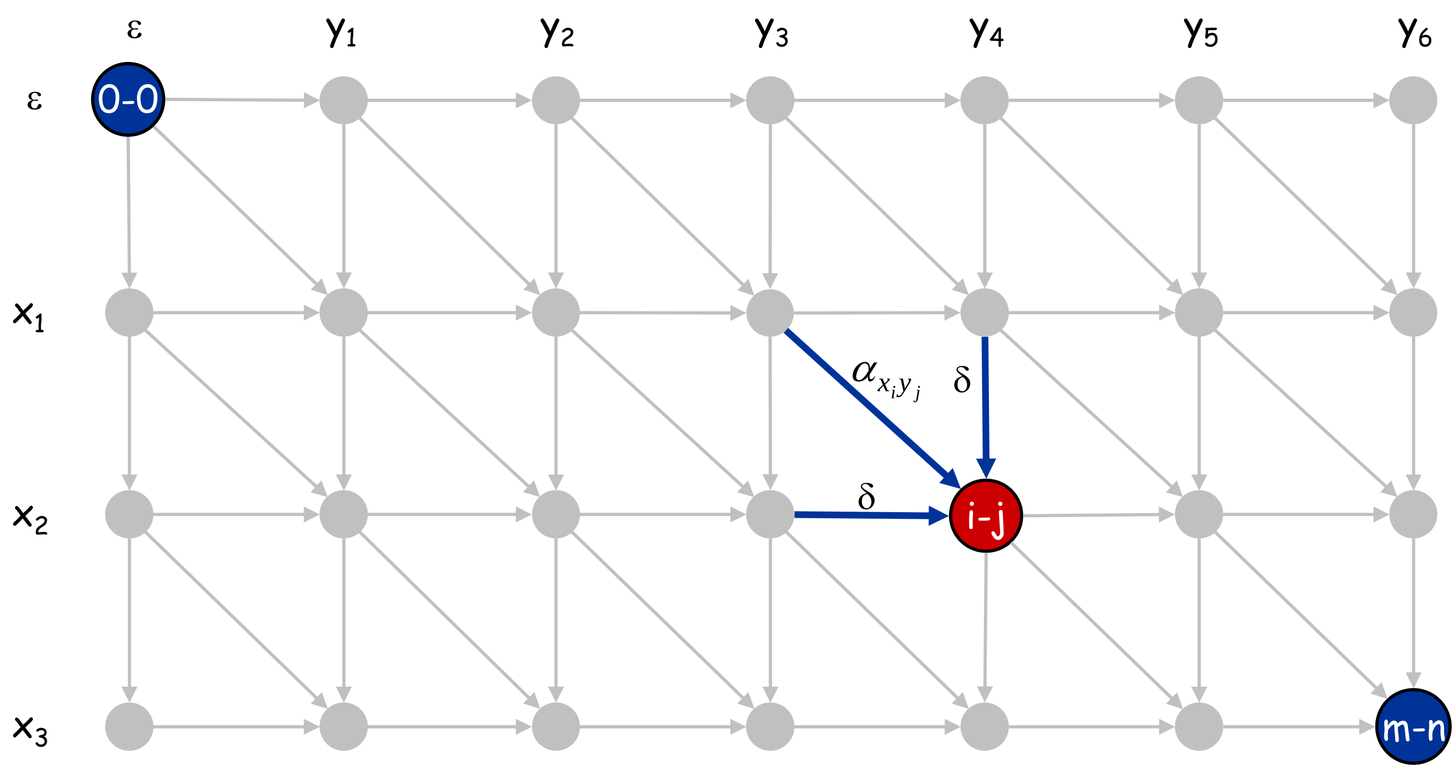
Theorem. [Hirschberg 1975] Optimal alignment in O(m + n) space and O(mn) time.

- Clever combination of divide-and-conquer and dynamic programming.
- Inspired by idea of Savitch from complexity theory.

Sequence Alignment: Linear Space

Edit distance graph.

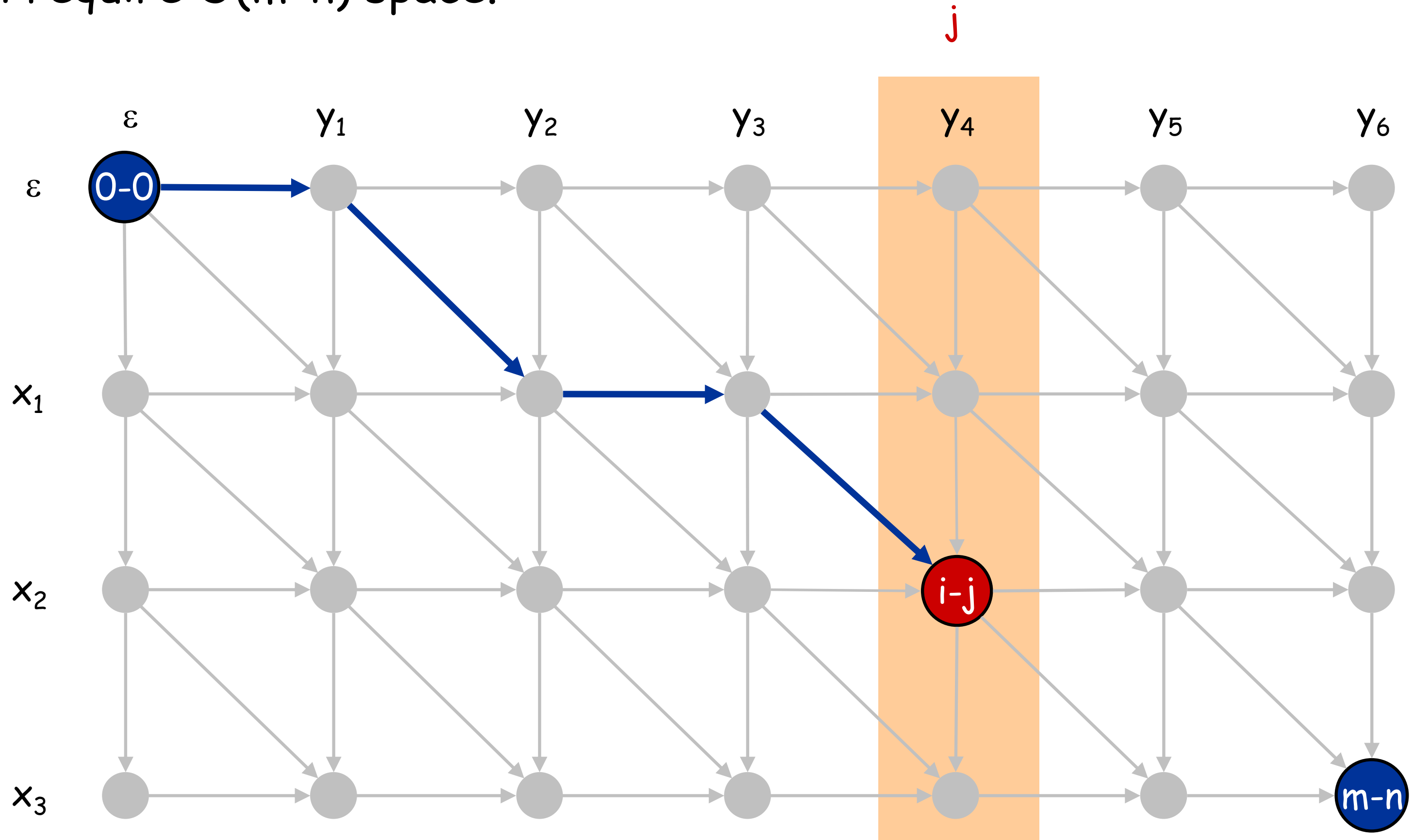
- Let $f(i, j)$ be shortest path from $(0,0)$ to (i, j).
- Observation: $f(i, j) = OPT(i, j)$ (can prove this by induction on $i+j$)



Edit distance graph.

- Let f(i, j) be shortest path from (0,0) to (i, j).
- Can compute f (•, j) for any j in O(mn) time and O(m) space.

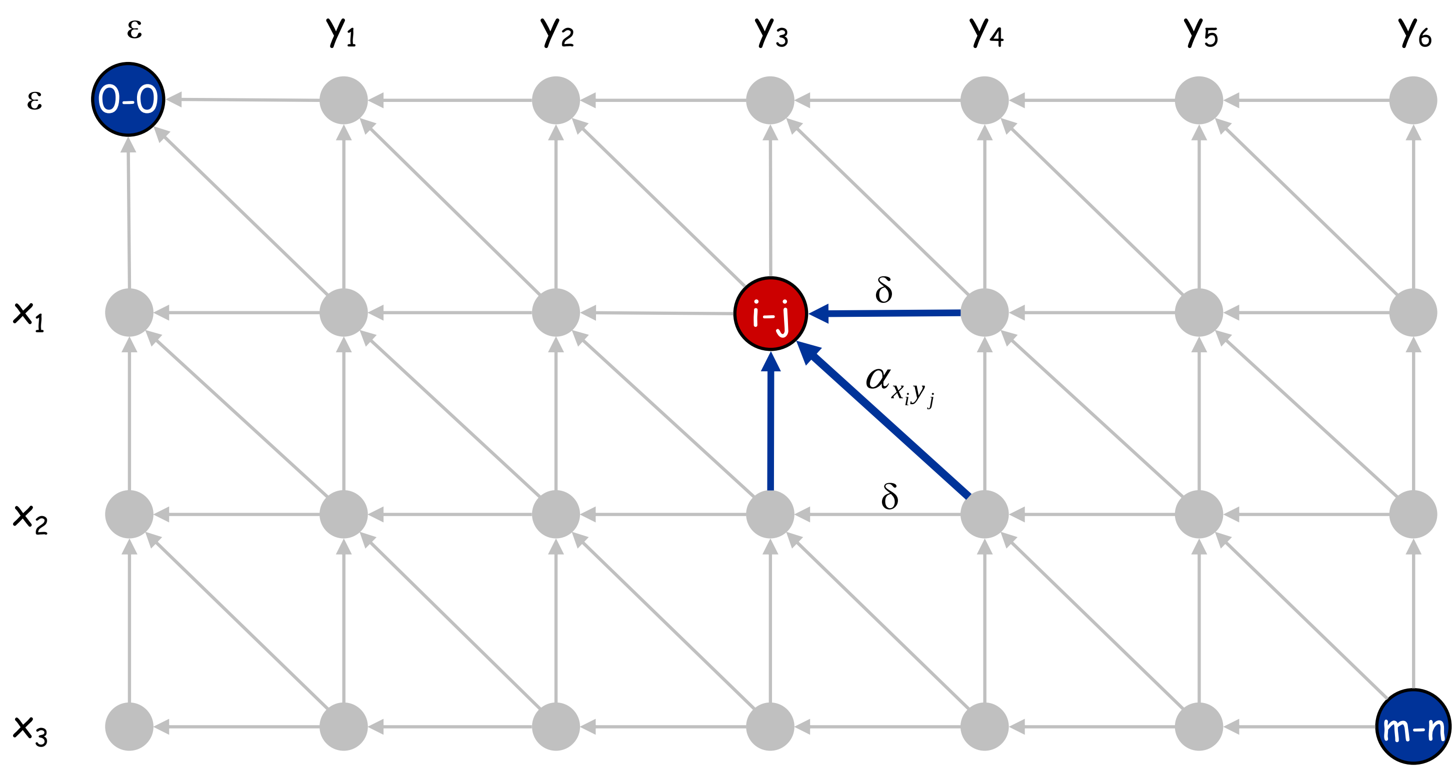
*Note: we want to recover solution for best alignment in the end; this will require O(m+n) space.



Sequence Alignment: Linear Space

Edit distance graph.

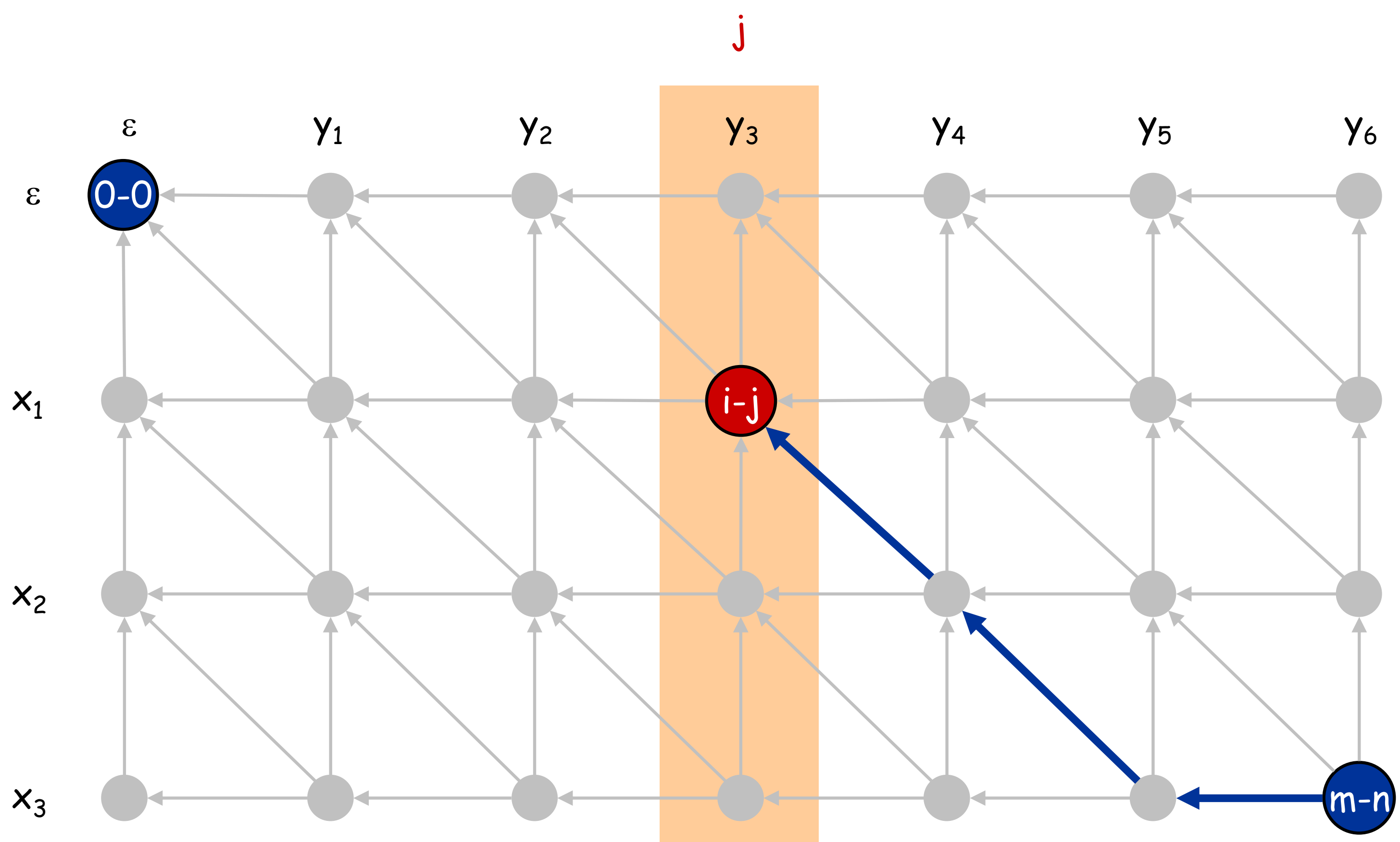
- Let $g(i, j)$ be shortest path from (i, j) to (m, n).
- Can compute by reversing the edge orientations and inverting the roles of $(0, 0)$ and (m, n)



Sequence Alignment: Linear Space

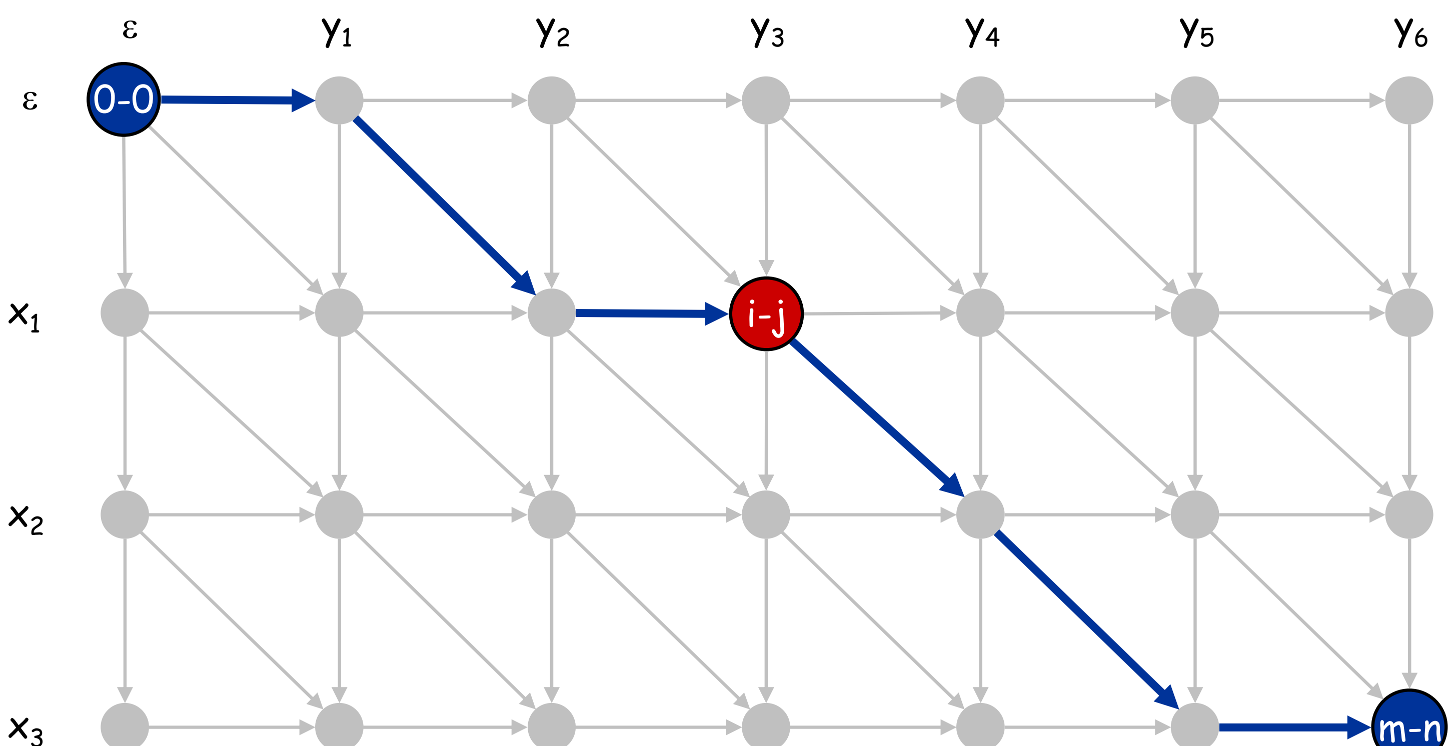
Edit distance graph.

- Let $g(i, j)$ be shortest path from (i, j) to (m, n).
- Can compute $g(\cdot, j)$ for any j in $O(mn)$ time and $O(m)$ space. (just like we did for $f()$).



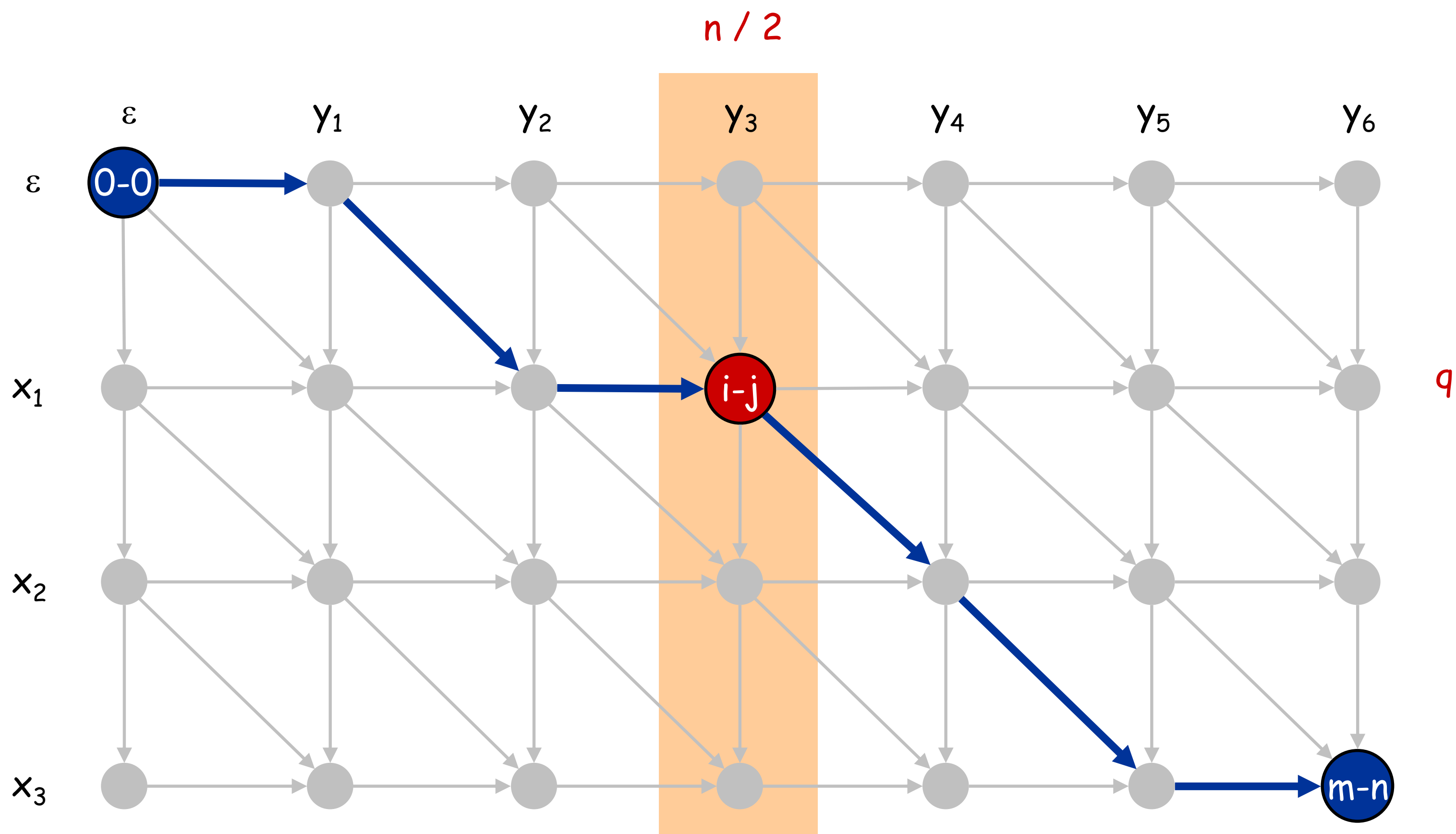
Sequence Alignment: Linear Space

Observation 1. The cost of the shortest path that uses (i, j) is $f(i, j) + g(i, j)$.



Sequence Alignment: Linear Space

Observation 2. let q be an index that minimizes f(q, n/2) + g(q, n/2). Then, the shortest path from (0, 0) to (m, n) uses (q, n/2).

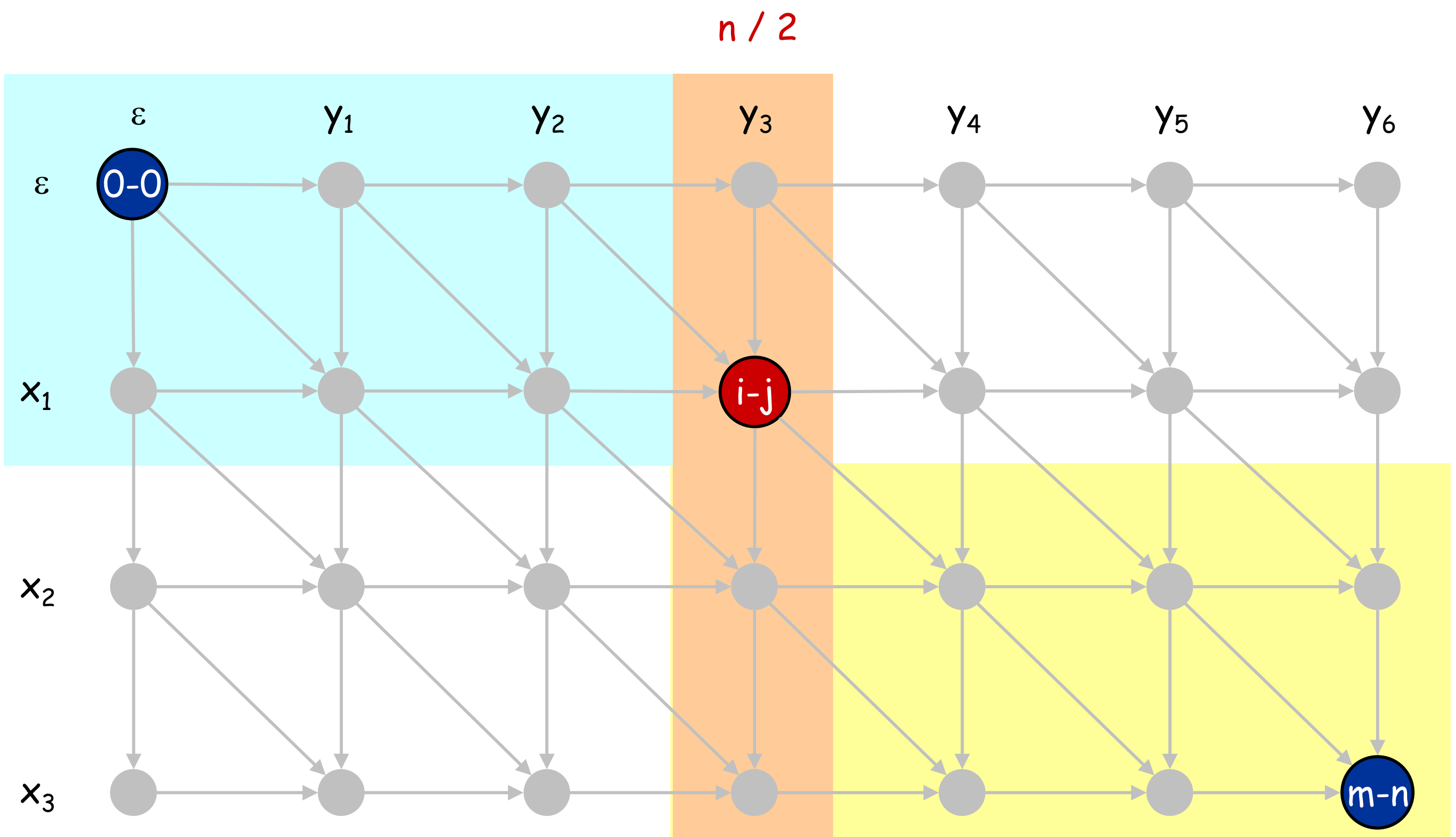


Sequence Alignment: Linear Space

Divide: find index q that minimizes $f(q, n/2) + g(q, n/2)$ using DP.

- Align x_q and $y_{n/2}$.

Conquer: recursively compute optimal alignment in each piece.



Sequence Alignment: Running Time Analysis Warmup

Theorem. Let T(m, n) = max running time of algorithm on strings of length at most m and n. T(m, n) = O(mn log n).

$$T(m,n) \le 2T(m,\ n/2) + O(mn) \quad \Rightarrow \quad T(m,n) = O(mn \log n)$$

Remark. Analysis is not tight because two sub-problems are of size (q, n/2) and (m - q, n/2). In next slide, we save log n factor.

Sequence Alignment: Running Time Analysis

Theorem. Let T(m, n) = max running time of algorithm on strings of length m and n. T(m, n) = O(mn).

Pf. (by induction on n)

- O(mn) time to compute f(•, n/2) and g (•, n/2) and find index q.
- T(q, n/2) + T(m - q, n/2) time for two recursive calls.
- Choose constant c so that:

$$\begin{aligned} T(m,\ 2) &\leq cm \\ T(2,\ n) &\leq cn \\ T(m,\ n) &\leq cmn + T(q,\ n/2) + T(m-q,\ n/2) \end{aligned}$$

- Base cases: m = 2 or n = 2.
- Inductive hypothesis: T(m, n) ≤ 2cmn.

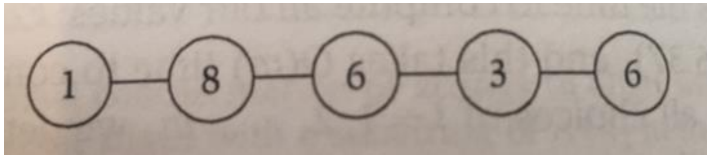
$$\begin{aligned} T(m,n) &\leq T(q,n/2) + T(m-q,n/2) + cmn \\ &\leq 2cqn/2 + 2c(m-q)n/2 + cmn \\ &= cqn + cmn - cqn + cmn \\ &= 2cmn \end{aligned}$$

Kleinberg 6.1 (HW)

#1: Let $G=(V,E)$ be an undirected graph with n nodes. Recall that a subset of the nodes is called an independent set if no two of them are joined by an edge. Finding large independent sets is difficult in general (NP-hard); but here we'll see that it can be done efficiently if the graph is "simple" enough.

Call a graph G a path if its nodes can be written as $v_1, v_2, \ldots, v_n$, with an edge between v_i and v_j iff the indices differ by one. With each node v_i, we associate a positive integer weight w_i.

Goal: Find an independent set in a path G whose total weight is as large as possible.



Kleinberg 6.1 (HW)

#1: Goal: Find an independent set in a path G whose total weight is as large as possible.

(a) Give an example to show that the following algorithm doesn't always find an independent set of max total weight.

```
Algo: "heaviest-first"
  {
     S=empty set
      while (some node remains in G)
       {
         Pick node with max weight and add it to S; delete this node and
             its neighbors
        }
    }
```

Kleinberg 6.1 (HW)

#1: Goal: Find an independent set in a path G whose total weight is as large as possible.

(c) Give an algorithm that takes an n-node path G with weights and returns an independent set of maximum total weight. The running time should be polynomial in n, independent of the values of the weights.

(Use a DP structure; start with a recursion.)

Kleinberg 6.1 (HW)

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Let S_i denote an independent set on {v1,..,vi}, and let X_i denote its weight. Define $X_0=0$ and note that $X_1=w_1$.

Now for i>1, either v_i belong to S_i or it doesn't. What is a natural recursion?

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If v_i belongs to S_i, then v_{i-1} doesn't (why?); so either $X_i=w_i+X_{i-2}$ or $X_i=X_{i-1}$ (why?).

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Hence, $X_i=\max(X_{i-1}, wi+X_{i-2})$; what's the run-time to compute S_n?

Kleinberg 6.3 (HW)

#3: Let $G=(V,E)$ be a directed graph with nodes $v_1,\ldots,v_n$. We say that G is an ordered graph if it has the following properties:

(i) Each edge goes from a node with a lower index to a node with a higher index; i.e., each directed edge has the form (v_i,v_j) with $i<j$.

(ii) Each node expect v_n has at least one edge leaving it. That is, for every node v_i, $i=1,2,\ldots,n-1$ there is at least one edge of the form (v_i,v_j).

The length of a path is the number of edges it contains. The goal is to solve:

Given an ordered graph G, find the length of the longest path that begins at v_1 and ends at v_n.

Kleinberg 6.3 (HW)

#3:The length of a path is the number of edges it contains. The goal is to solve:

Given an ordered graph G, find the length of the longest path that begins at v_1 and ends at v_n.

(b) Give an efficient algorithm that takes an ordered graph G and returns the length of the lonest path that begins at v_1 and end at v_n.

Idea: Use DP structure; consider subproblems OPT[i], the length of the longest path from v_1 to v_i in ordered graph, G. <u>One caveat</u>: not all nodes v_i necessarily have a path from v_1 to v_i; let's use the value "-inf" in this case.

Kleinberg 6.3 (HW)

#3:(b) Give an efficient algorithm that takes an ordered graph G and returns the length of the lonest path that begins at v_1 and end at v_n.

Idea: Use DP structure; consider subproblems OPT[i], the length of the longest path from v_1 to v_i in ordered graph, G. <u>One caveat</u>: not all nodes v_i necessarily have a path from v_1 to v_i; let's use the value "-inf" in this case.

Define OPT[1]=0 (base case); use for loop:

```
M[1]=0
  for i=2,..,n
   {
      M=-inf
        for all edges (j,i) in G
         {
           if M[j]=!-inf && M<M[j]+1
               then M=M[j]+1
           /endif
         } /end for
    M[i]=M
} /endfor
```

Kleinberg 6.3 (HW)

#3:(b) Give an efficient algorithm that takes an ordered graph *G* and returns the length of the lonest path that begins at v_1 and end at v_n.

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```

What's the run time?

Kleinberg 6.3 (HW)

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} /endfor
```

What's the run time? $O(n^2)$

Kleinberg 6.6 (HW)

This problem is very similar in flavor to the segmented least squares problem. We observe that the last line ends with word w_n and has to start with some word w_j; breaking off words $w_j, \ldots, w_n$ we are left with a recursive sub-problem on $w_1, \ldots, w_{j-1}$.

Thus, we define $OPT[i]$ to be the value of the optimal solution on the set of words $W_i = \{w_1, \ldots, w_i\}$. For any $i \leq j$, let $S_{i,j}$ denote the slack of a line containing the words $w_i, \ldots, w_j$; as a notational device, we define $S_{i,j} = \infty$ if these words exceed total length L. For each fixed i, we can compute all $S_{i,j}$ in $O(n)$ time by considering values of j in increasing order; thus, we can compute all $S_{i,j}$ in $O(n^2)$ time.

As noted above, the optimal solution must begin the last line somewhere (at word w_j), and solve the sub-problem on the earlier lines optimally. We thus have the recurrence

$$OPT[n] = \min_{1 \leq j \leq n} S_{i,n}^2 + OPT[j-1],$$

and the line of words $w_j, \ldots, w_n$ is used in an optimum solution if and only if the minimum is obtained using index j.

Finally, we just need a loop to build up all these values:

> `Compute all values` $S_{i,j}$ `as described above.`
> `Set` $OPT[0] = 0$
> `For` $k = 1, \ldots, n$
> $\quad OPT[k] = \min_{1 \leq j \leq k} \left(S_{j,k}^2 + OPT[j-1]\right)$
> `Endfor`
> `Return` $OPT[n]$.

As noted above, it takes $O(n^2)$ time to compute all values $S_{i,j}$. Each iteration of the loop takes time $O(n)$, and there are $O(n)$ iterations. Thus the total running time is $O(n^2)$.

By tracing back through the array OPT, we can recover the optimal sequence of line breaks that achieve the value $OPT[n]$ in $O(n)$ additional time.

Kleinberg 6.11 (HW)

#11: Suppose you're consulting for a company that manufactures PC equipment and ships it to distributors all over the country. For each of the next n weeks, they have a projected supply s_i of equipment (measured in pounds), which has to be shipped by an air freight carrier.

Each week's supply can be carried by one of two air freight companies: A or B.

(*) Company A charges a fixed rate r per pound (so it costs r*si to ship a week's supply si).

(*) Company B makes contracts for a fixed amount c per week, independent of the weight. However, contracts with company B must be made in blocks of four consecutive weeks at a time.

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A schedule, for the PC company, is a choice of air freight company (A or B) for each of the n weeks, with the restriction that company B, whenever it is chosen, must be chosen for blocks of four contiguous weeks at a time. The cost of a schedule is the total amount paid to company A and B.

Kleinberg 6.11 (HW)

#11: Give a polynomial-time algorithm that takes a sequence of supply values s_1, s_2,….,s_n and returns a schedule of minimum cost.

Example. Suppose r=1, c=10, and the sequence of values is:
11,9,9,12,12,12,12,9,9,11.

Then the optimal schedule would be to choose company A for first three weeks, then company B for a block of four consecutive weeks, and then company A for the final three weeks.

How to use DP structure to solve?

Kleinberg 6.11 (HW)

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How to use DP structure to solve?

Let OPT[i] denote the minimum cost of a solution for weeks 1 through i. In an optimal solution, we either use company A or B for the ith week.

If we use company A, we pay $r*s_i$ and behave optimally up through week i-1; else we use company B and pay 4c for this contract, and we behave optimally through week i-4.

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In summary, our recursion is as follows: $OPT[i] = \min(r*s_i+OPT(i-1), 4c+OPT(i-4))$.

What's the runtime?