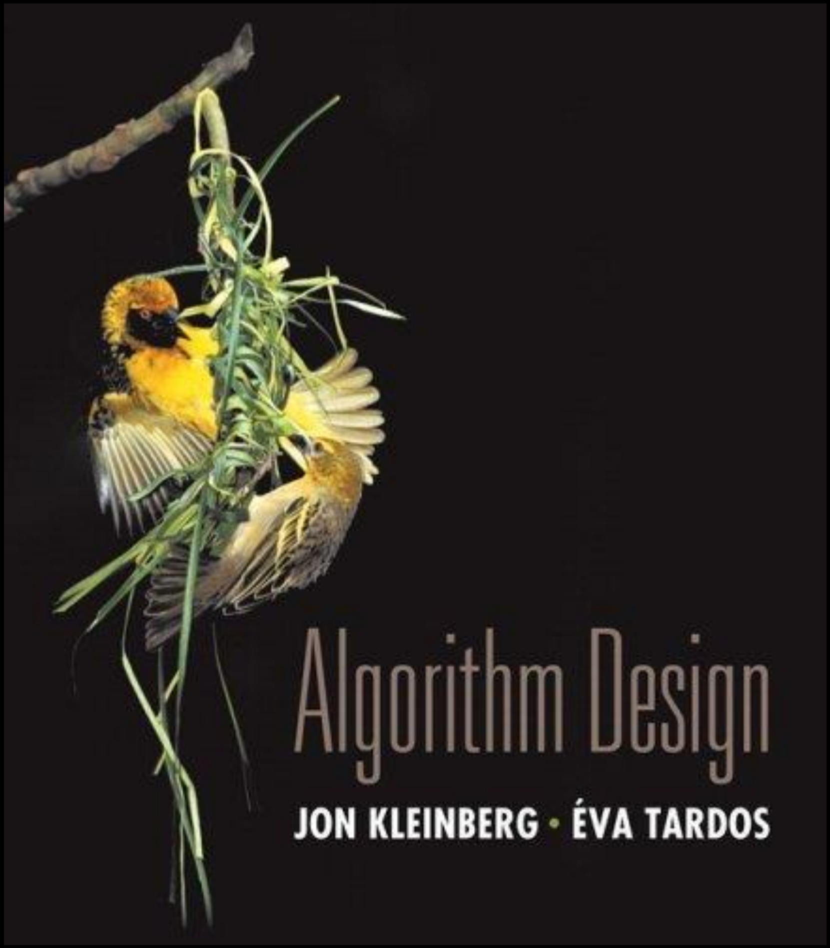


Chapter 3

Graphs

CS 350 Winter 2018

3.0 Outline

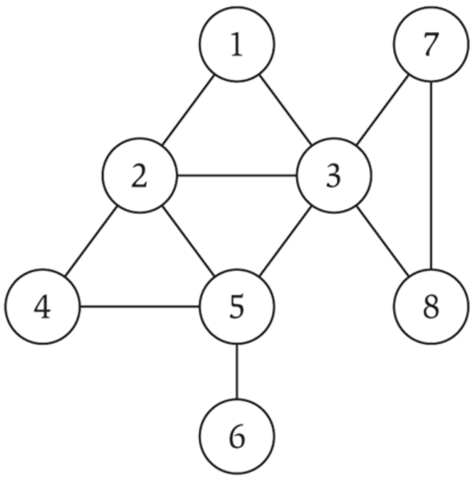
(i) Graphs
(ii) BFS & DFS
(iii) Connectivity and Graph Traversals
(iv) Testing Bipartiteness
(v) DAGS

3.1 Basic Definitions and Applications

Undirected Graphs

Undirected graph. G = (V, E)

- V = nodes.
- E = edges between pairs of nodes.
- Captures pairwise relationship between objects.
- Graph size parameters: n = |V|, m = |E|.



V = { 1, 2, 3, 4, 5, 6, 7, 8 }

E = { 1-2, 1-3, 2-3, 2-4, 2-5, 3-5, 3-7, 3-8, 4-5, 5-6 }

n = 8

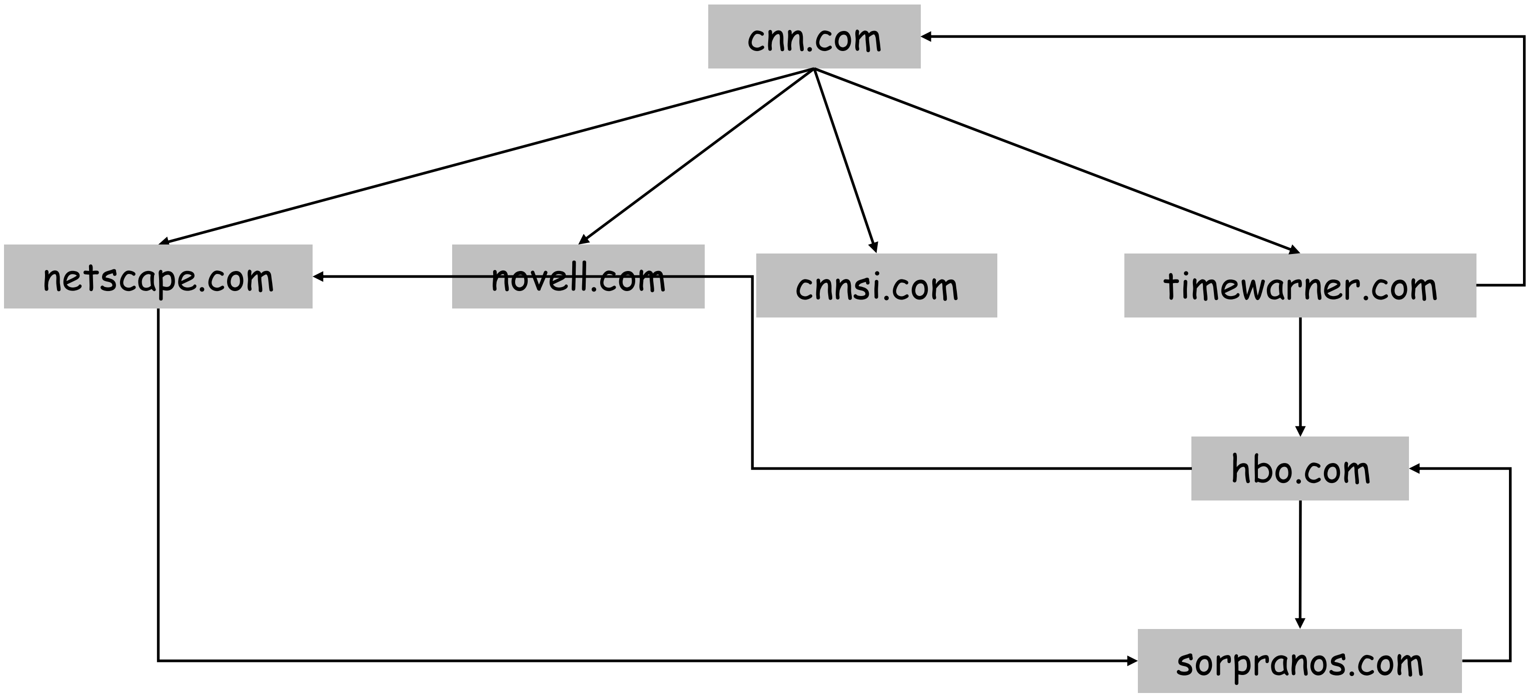
m = 11

Some Graph Applications

Graph	*Nodes*	*Edges*
transportation	street intersections	highways
communication	computers	fiber optic cables
World Wide Web	web pages	hyperlinks
social	people	relationships
food web	species	predator-prey
software systems	functions	function calls
scheduling	tasks	precedence constraints
circuits	gates	wires

World Wide Web

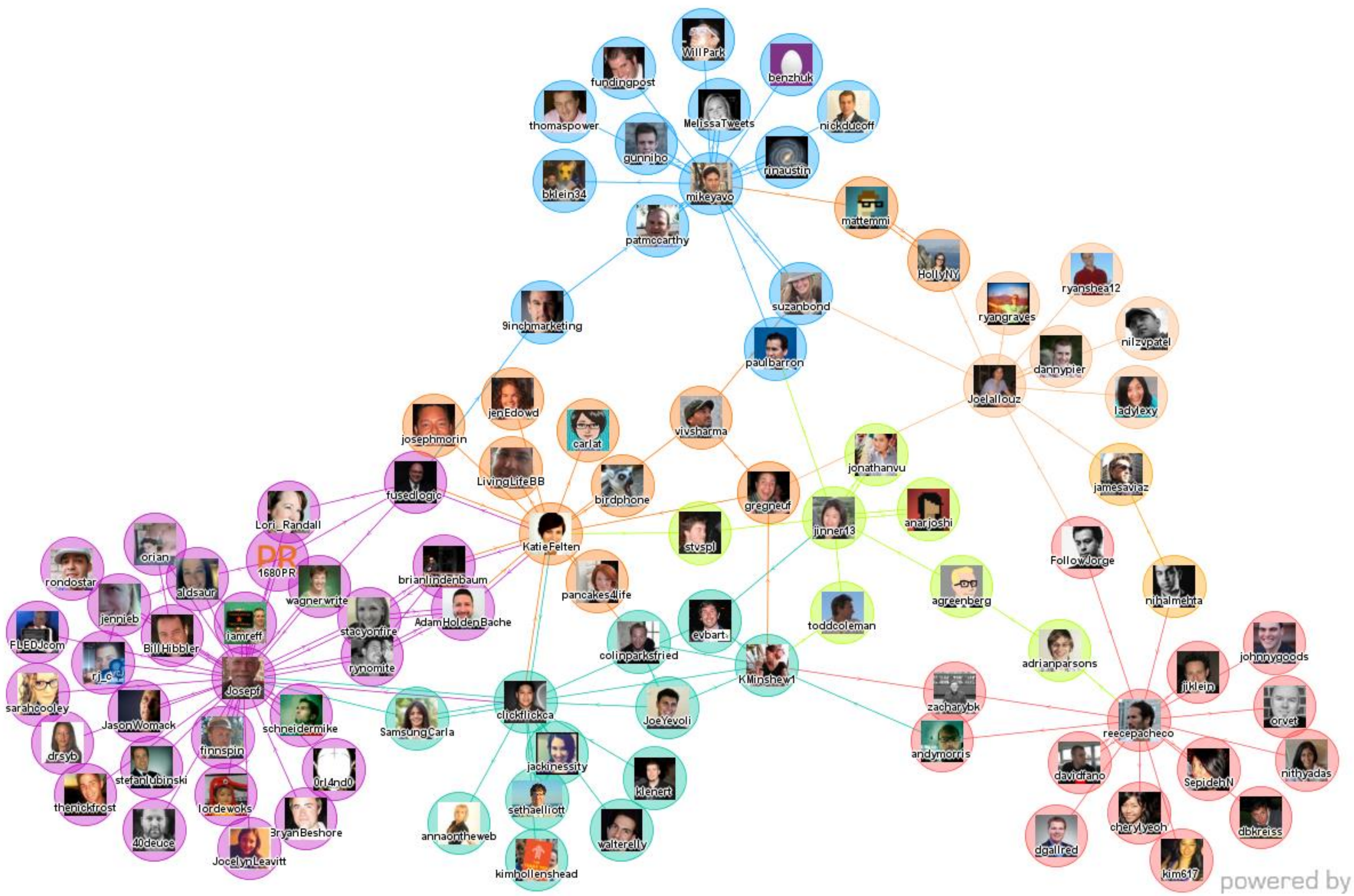
Web graph.
- Node: web page.
- Edge: hyperlink from one page to another.



Social Network

Social network graph.

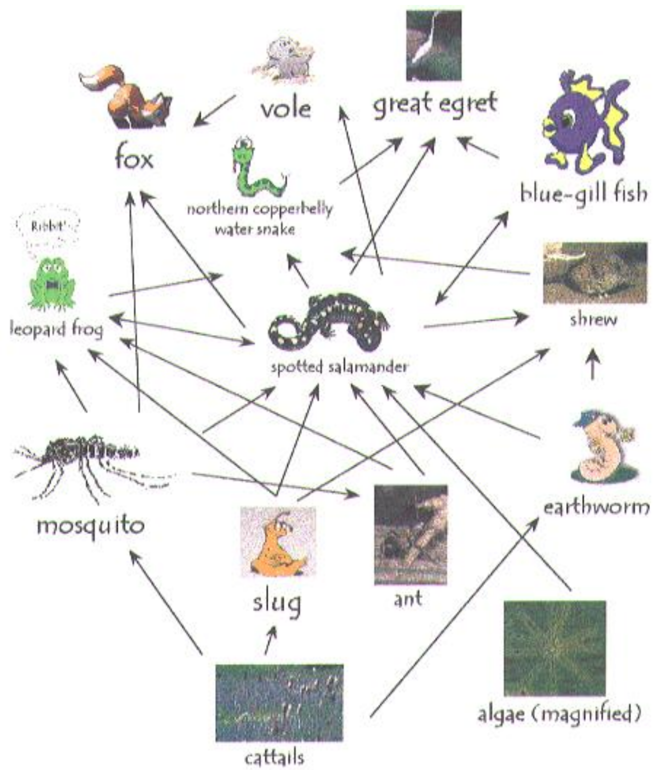
- Node: people.
- Edge: relationship between two people.



Ecological Food Web

Food web graph.

- Node = species.
- Edge = from prey to predator.

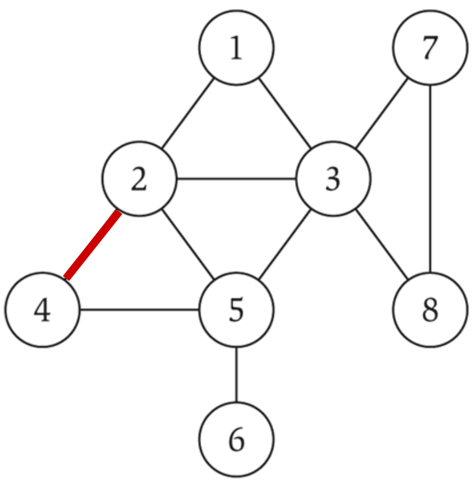


Reference: http://www.twingroves.district96.k12.il.us/Wetlands/Salamander/SalGraphics/salfoodweb.giff

Graph Representation: Adjacency Matrix

Adjacency matrix. n-by-n matrix with $A_{uv} = 1$ if (u, v) is an edge.

- Two representations of each edge.
- Space proportional to n^2.
- Checking if (u, v) is an edge takes $\Theta(1)$ time.
- Identifying all edges takes $\Theta(n^2)$ time.



	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	1	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

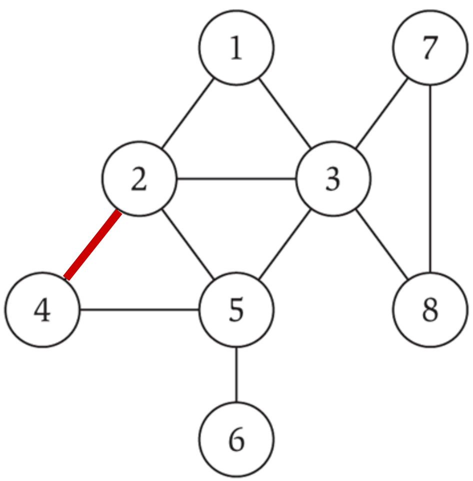
Graph Representation: Adjacency List

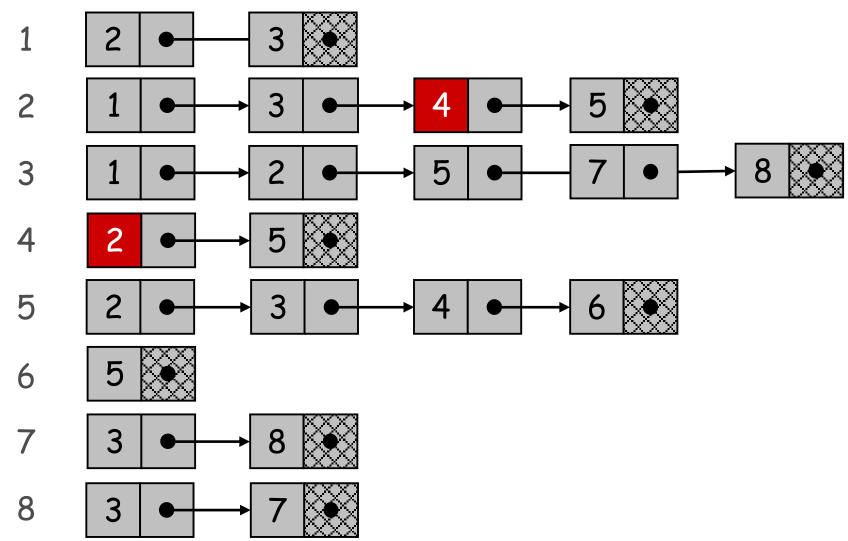
Adjacency list. Node indexed array of lists.

- Two representations of each edge.
- Space proportional to m + n.

degree = number of neighbors of u

- Checking if (u, v) is an edge takes O(deg(u)) time.
- Identifying all edges takes Θ(m + n) time.





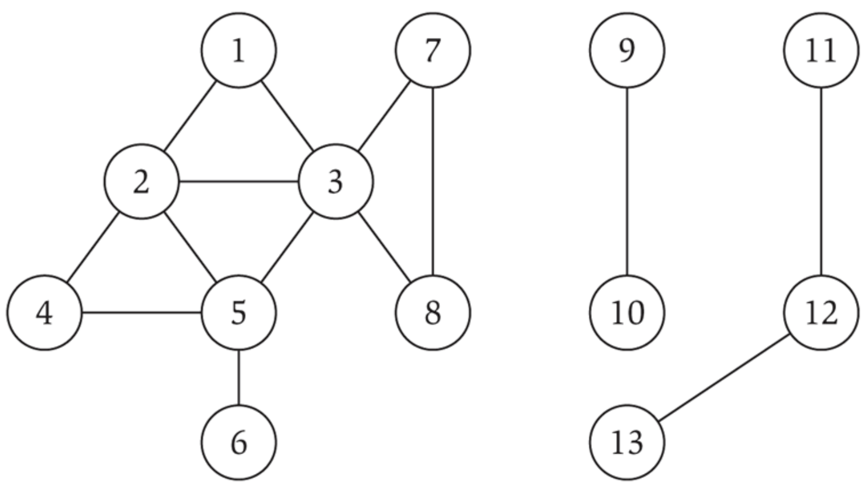
Paths and Connectivity

Def. A path in an undirected graph $G = (V, E)$ is a sequence P of nodes $v_1, v_2, \ldots, v_{k-1}, v_k$ with the property that each consecutive pair v_i, v_{i+1} is joined by an edge in E.

Def. A path is simple if no multi-edges or loops.

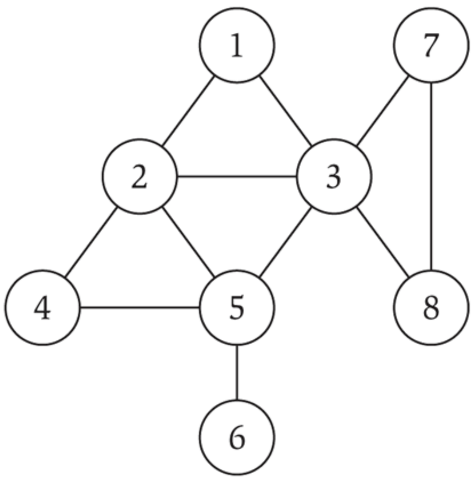
Q: What is the maximum number of edges possible in a simple graph?

Def. An undirected graph is connected if for every pair of nodes u and v, there is a path between u and v.



Cycles

Def. A cycle is a path $v_1, v_2, \ldots, v_{k-1}, v_k$ in which $v_1 = v_k$, $k > 2$, and the first $k-1$ nodes are all distinct.



cycle C = 1-2-4-5-3-1

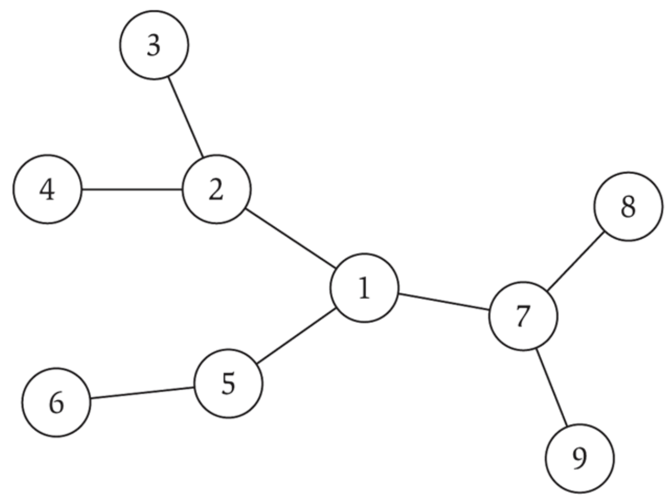
Trees

Def. An undirected graph is a tree if it is connected and does not contain a cycle.

Theorem. Let G be an undirected graph on n nodes. Any two of the following statements imply the third.

- G is connected.
- G does not contain a cycle.
- G has n-1 edges.

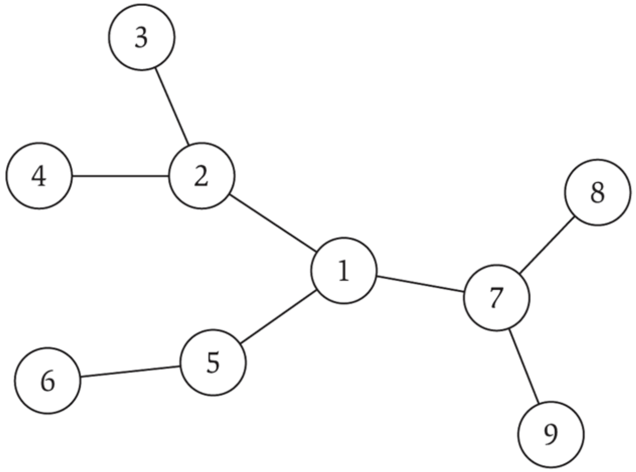
- Q: How would we prove this Theorem?



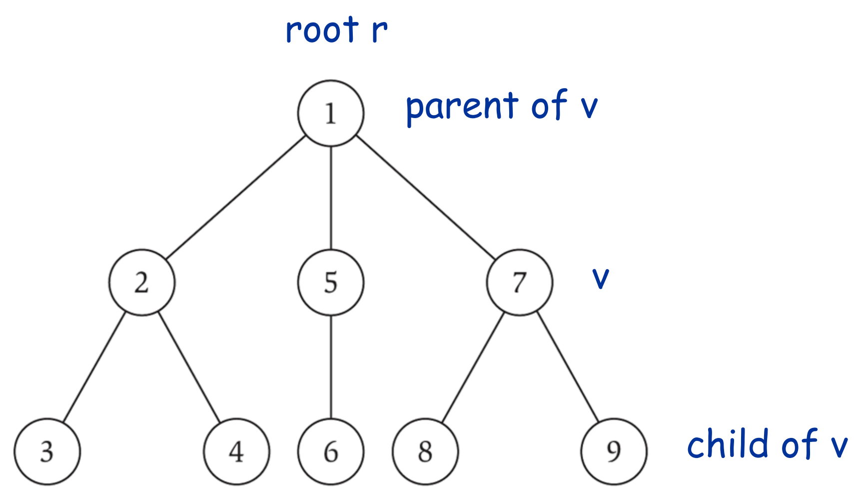
Rooted Trees

Rooted tree. Given a tree T, choose a root node r and orient each edge away from r.

Importance. Models hierarchical structure.



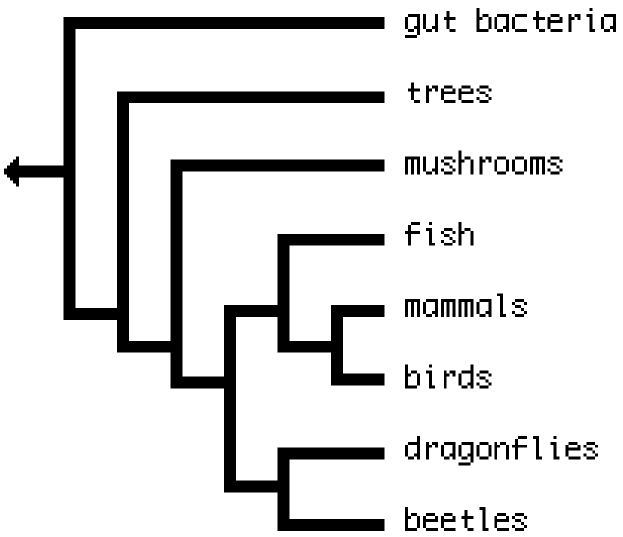
a tree



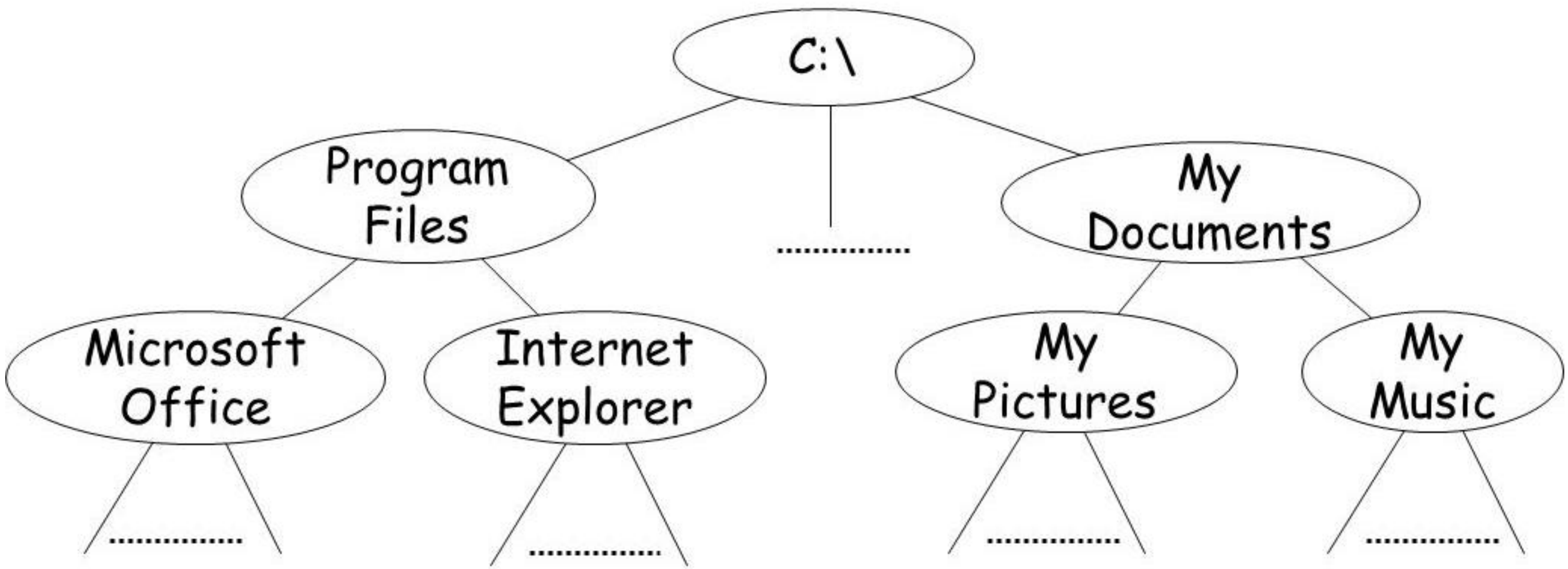
the same tree, rooted at 1

Phylogeny Trees

Phylogeny trees. Describe evolutionary history of species.



Rooted Tree



C:\
Program Files
....
My Documents
Microsoft Office
Internet Explorer
My Pictures
My Music
....
....
....
....

3.2 Graph Traversal

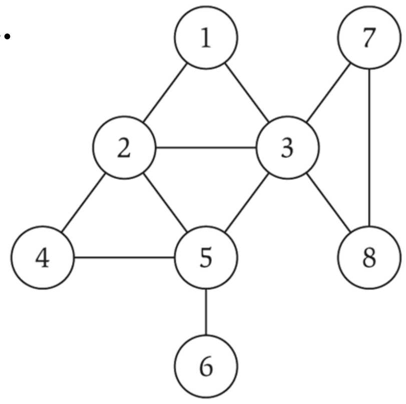
Connectivity

s-t connectivity problem. Given two node s and t, is there a path between s and t?

s-t shortest path problem. Given two node s and t, what is the length of the shortest path between s and t?

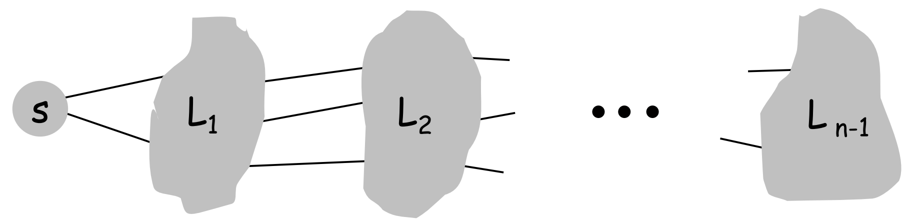
Applications.

- Google Maps.
- Maze traversal.
- Kevin Bacon number.
- Fewest number of hops in a communication network.



Breadth First Search (BFS)

BFS intuition. Explore outward from s in all possible directions, adding nodes one "layer" at a time.



BFS algorithm.

- $L_0 = \{ s \}$.
- L_1 = all neighbors of L_0.
- L_2 = all nodes that do not belong to L_0 or L_1, and that have an edge to a node in L_1.
- L_{i+1} = all nodes that do not belong to an earlier layer, and that have an edge to a node in L_i.

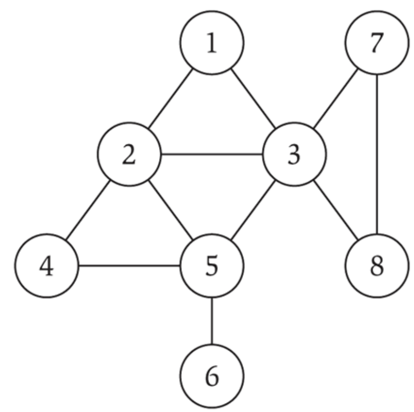
Theorem. For each i, L_i consists of all nodes at distance exactly i from s. There is a path from s to t iff t appears in some layer.

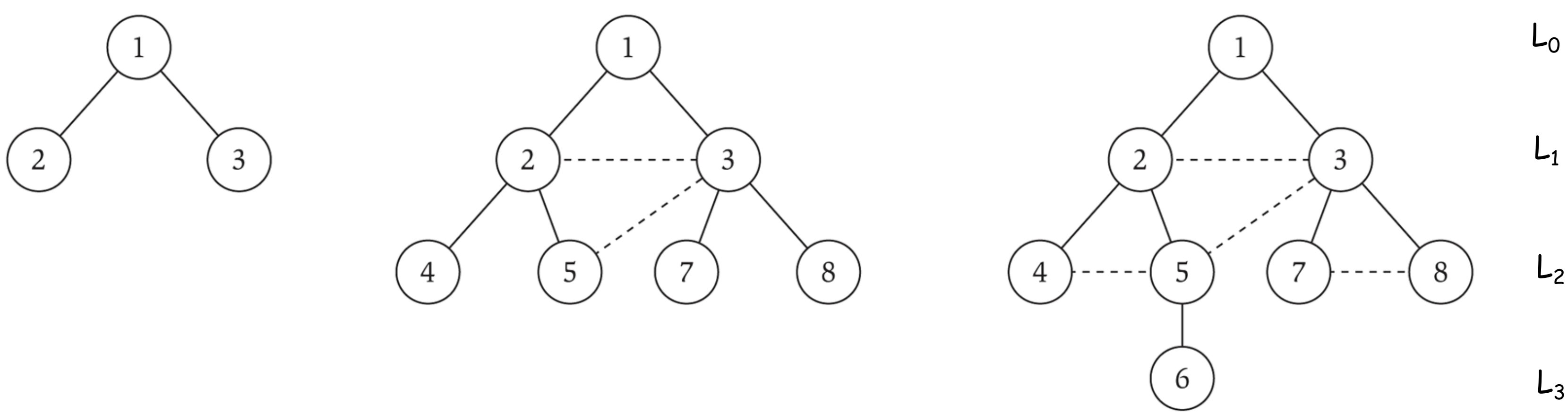
Q: How would we prove this?

Breadth First Search

Property. Let T be a BFS tree of $G = (V, E)$, and let (x, y) be an edge of G. Then the level of x and y differ by at most 1.

(Proof by contradiction)





(a) (b) (c)

Breadth First Search: Analysis

Theorem. The above implementation of BFS runs in $O(m + n)$ time if the graph is given by its adjacency representation. (NB: the data structure/graph representation matters for algorithm efficiency!)

Pf.

- Easy to prove $O(n^2)$ running time:
  - at most n lists L[i]
  - each node occurs once at most on each list; for loop runs $\leq n$ times
  - when we consider node u, there are $\leq n$ incident edges (u, v), and we spend $O(1)$ processing each edge

- Actually runs in $O(m + n)$ time:
  - when we consider node u, there are deg(u) incident edges (u, v)
  - total time processing edges is $\Sigma_{u \in V} \deg(u) = 2m$

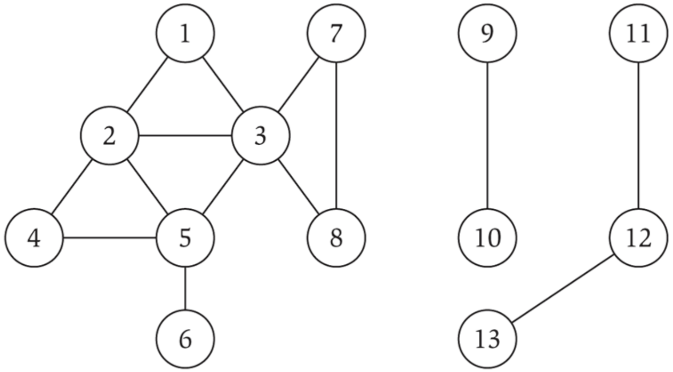
each edge (u, v) is counted exactly twice in sum: once in deg(u) and once in deg(v)

"First Theorem of Graph Theory":

$\Sigma_{u \in V} \deg(u) = 2m$

Connected Component

Connected component. Find all nodes reachable from s.

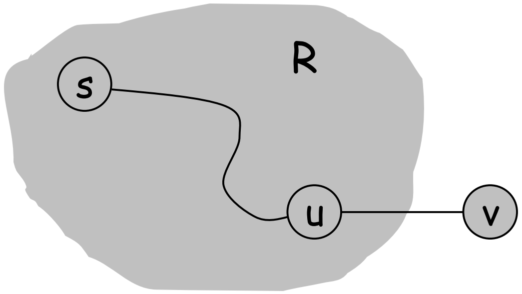


Connected component containing node 1 = { 1, 2, 3, 4, 5, 6, 7, 8 }.

Connected Component

Connected component. Find all nodes reachable from s.

```
R will consist of nodes to which s has a path
Initially R = {s}
While there is an edge (u, v) where u ∈ R and v ∉ R
  Add v to R
Endwhile
```



it's safe to add v

Theorem. Upon termination, R is the connected component containing s.
- BFS = explore in order of distance from s. (use stack, LIFO)
- DFS = explore in a different way: explore until reaching dead-end, then backtrack. (use queue, FIFO)

Breadth-first search

BFS is a simple strategy in which the root node is expanded first, then all the successors of the root node are expanded next, then their successors, etc.

BFS is an instance of the general graph-search algorithm in which the shallowest unexpanded node is chosen for expansion.

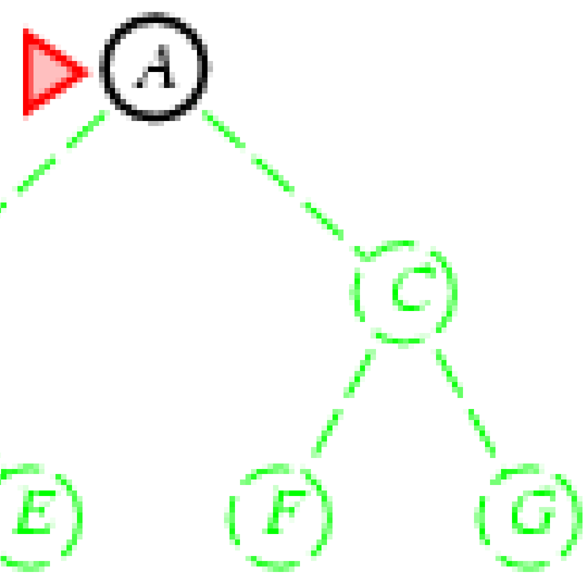
This is achieved by using a **FIFO queue** for the frontier. Accordingly, new nodes go to the back of the queue, and **old nodes, which are shallower than the new nodes are expanded first**.
NB: The *goal test* is applied to each node when it is *generated*.

Breadth-first search

Expand shallowest unexpanded node

Implementation:

- *frontier* is a FIFO queue, i.e., new successors go at end

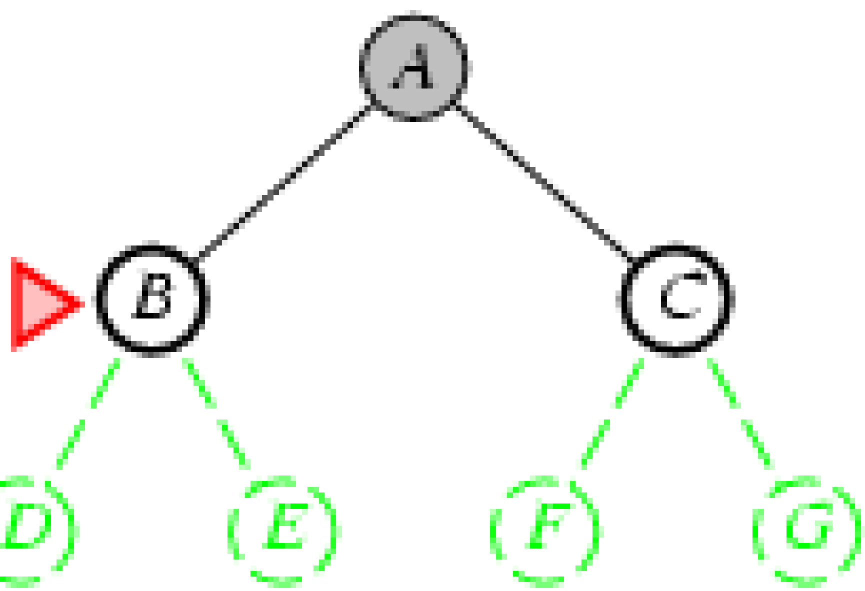


Breadth-first search

Expand shallowest unexpanded node

Implementation:

- *frontier* is a FIFO queue, i.e., new successors go at end

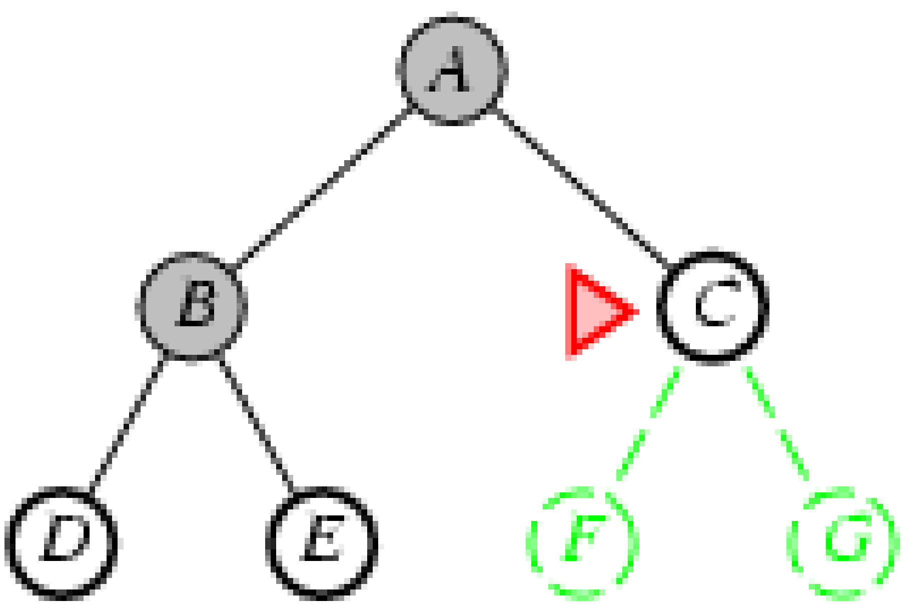


Breadth-first search

Expand shallowest unexpanded node

Implementation:

- *fringe* is a FIFO queue, i.e., new successors go at end



Breadth-first search

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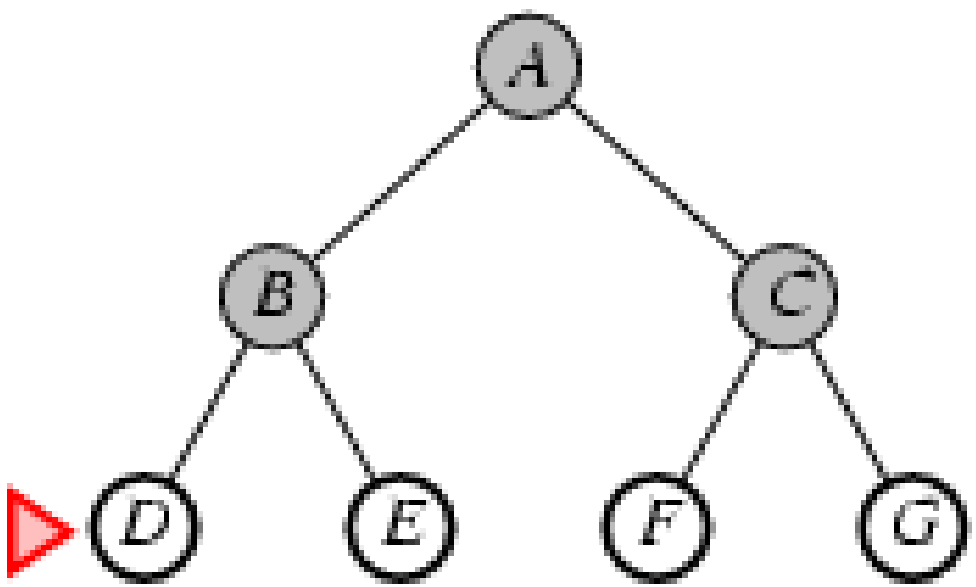
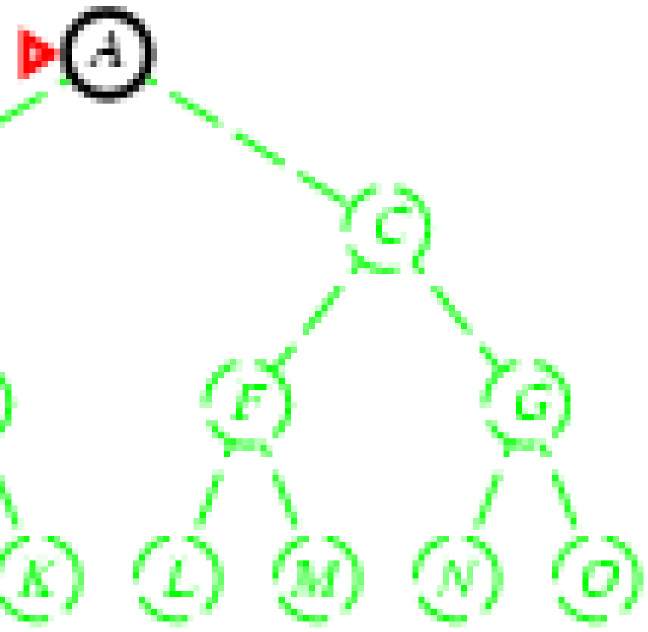


Expand deepest unexpanded node

Implementation:

- *fringe* = LIFO queue, i.e., put successors at front

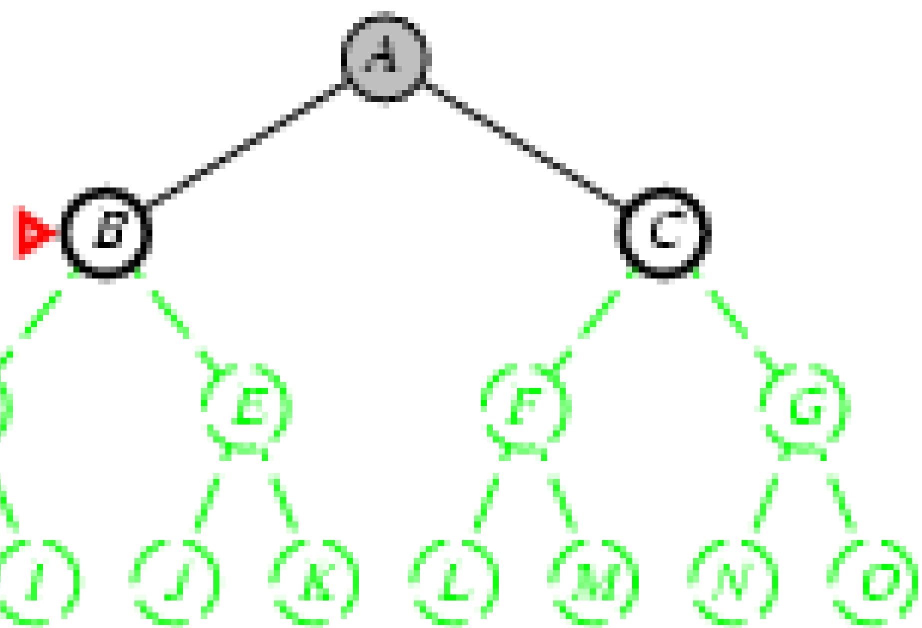


Depth-first search

Expand deepest unexpanded node

Implementation:

- *fringe* = LIFO queue, i.e., put successors at front
-

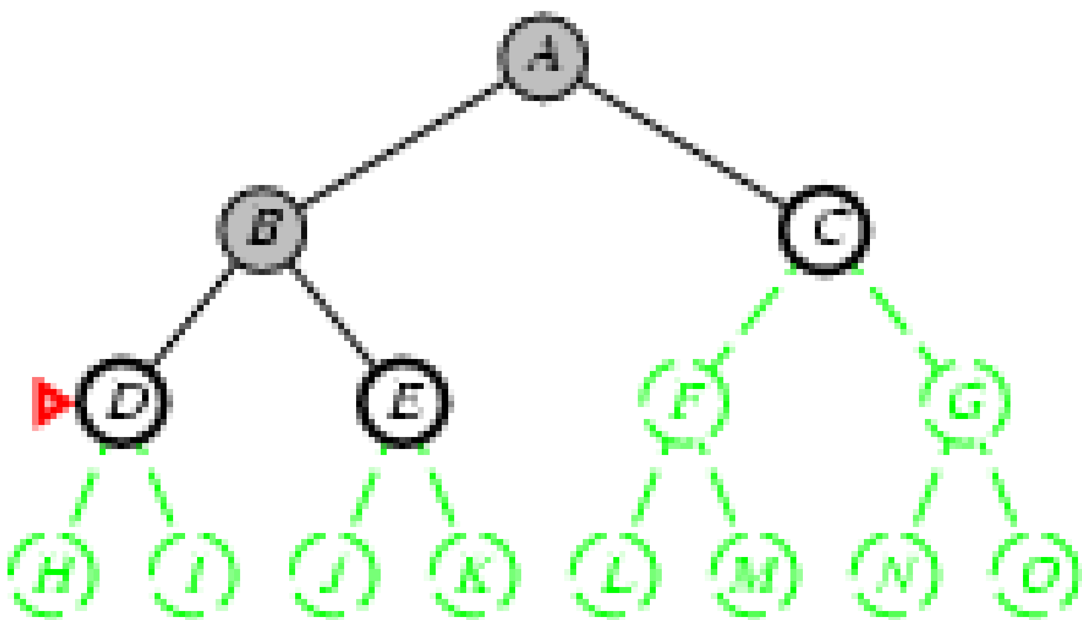


Depth-first search

Expand deepest unexpanded node

Implementation:

- *fringe* = LIFO queue, i.e., put successors at front
-

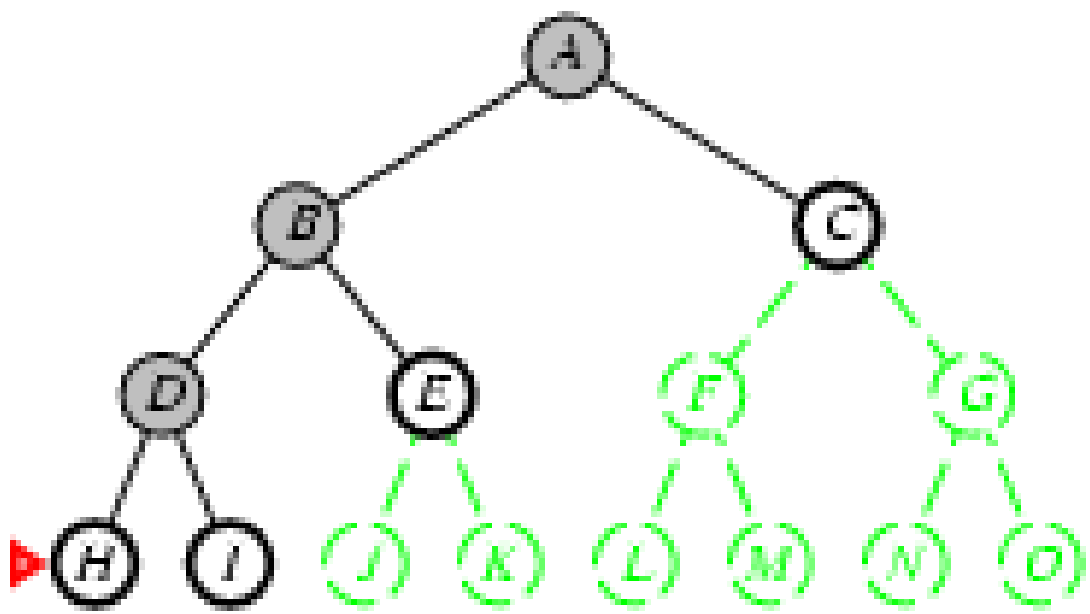


Depth-first search

Expand deepest unexpanded node

Implementation:

- *fringe* = LIFO queue, i.e., put successors at front
-

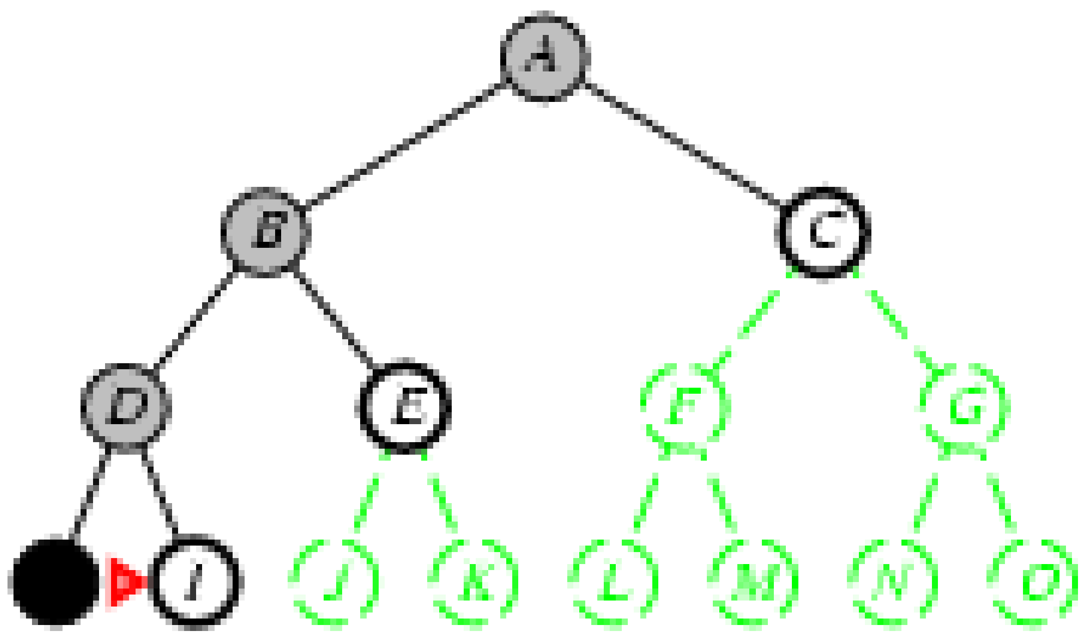


Depth-first search

Expand deepest unexpanded node

Implementation:

- *fringe* = LIFO queue, i.e., put successors at front
-

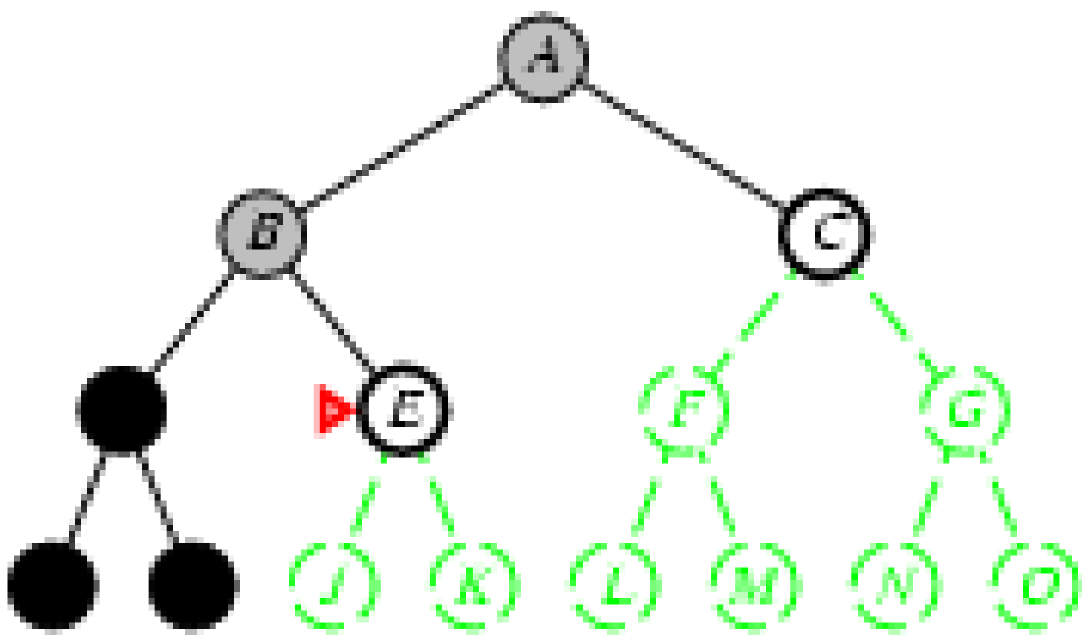


Depth-first search

Expand deepest unexpanded node

Implementation:

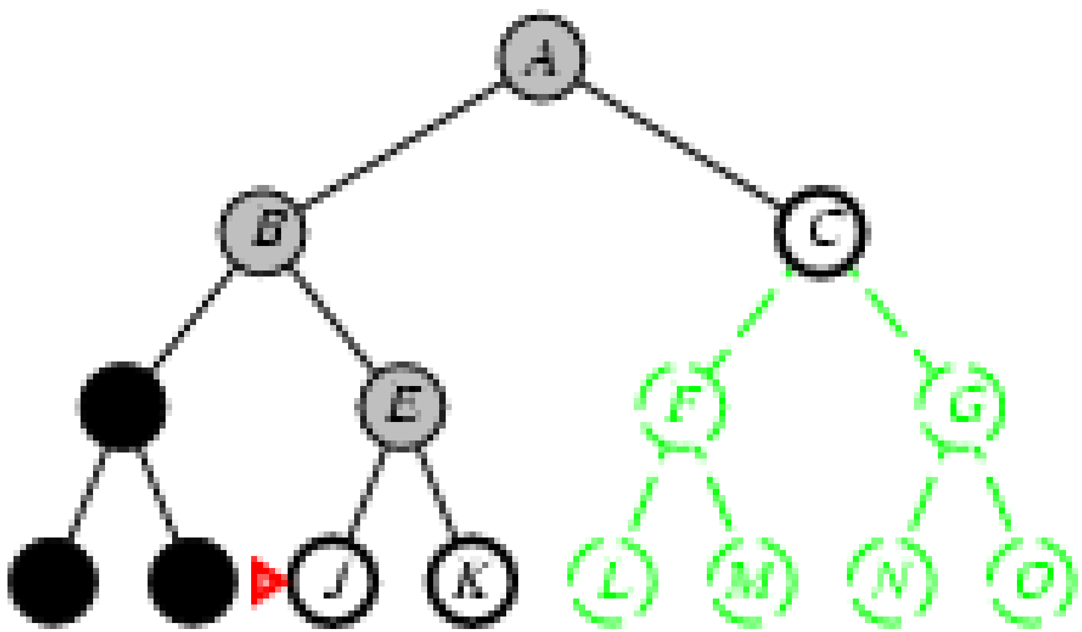
- *fringe* = LIFO queue, i.e., put successors at front
-



Depth-first search

Expand deepest unexpanded node

Implementation:
- *fringe* = LIFO queue, i.e., put successors at front
-

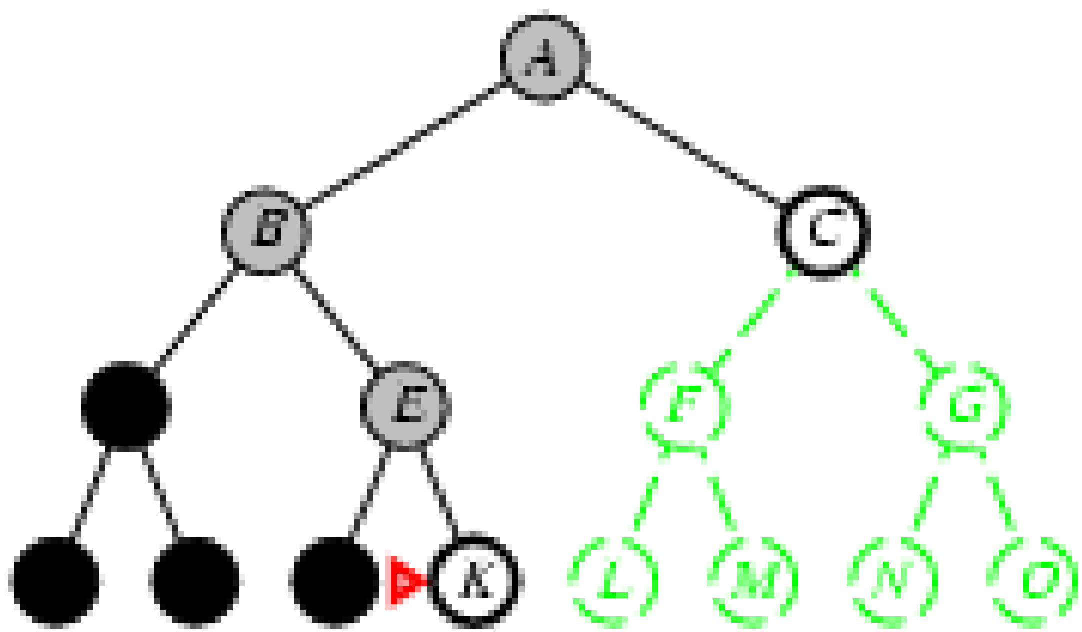


Depth-first search

Expand deepest unexpanded node

Implementation:

- *fringe* = LIFO queue, i.e., put successors at front
-

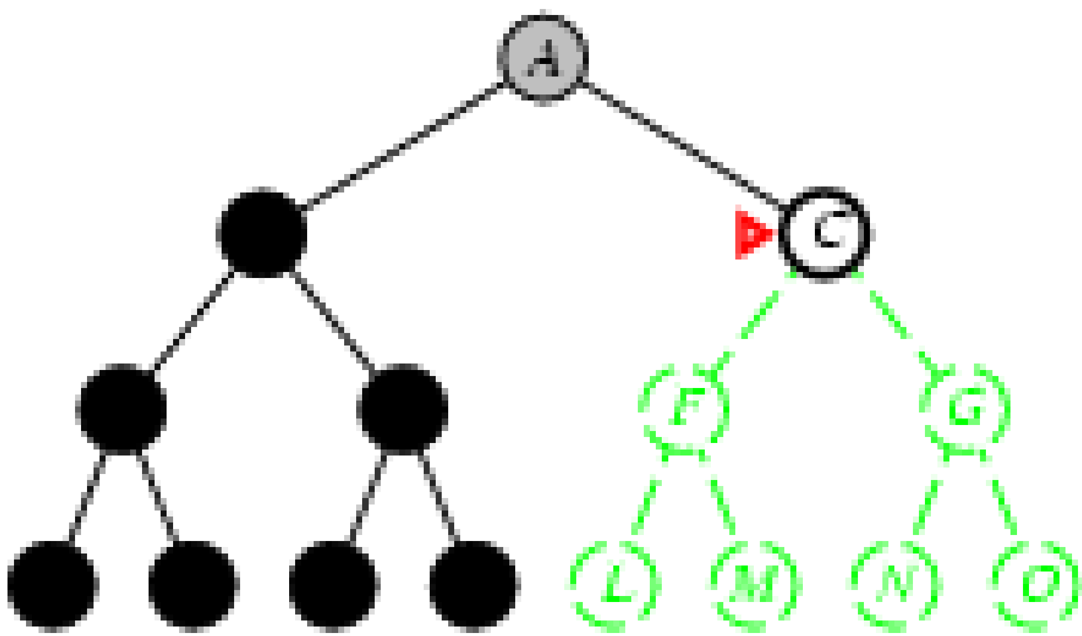


Depth-first search

Expand deepest unexpanded node

Implementation:

- *fringe* = LIFO queue, i.e., put successors at front
-

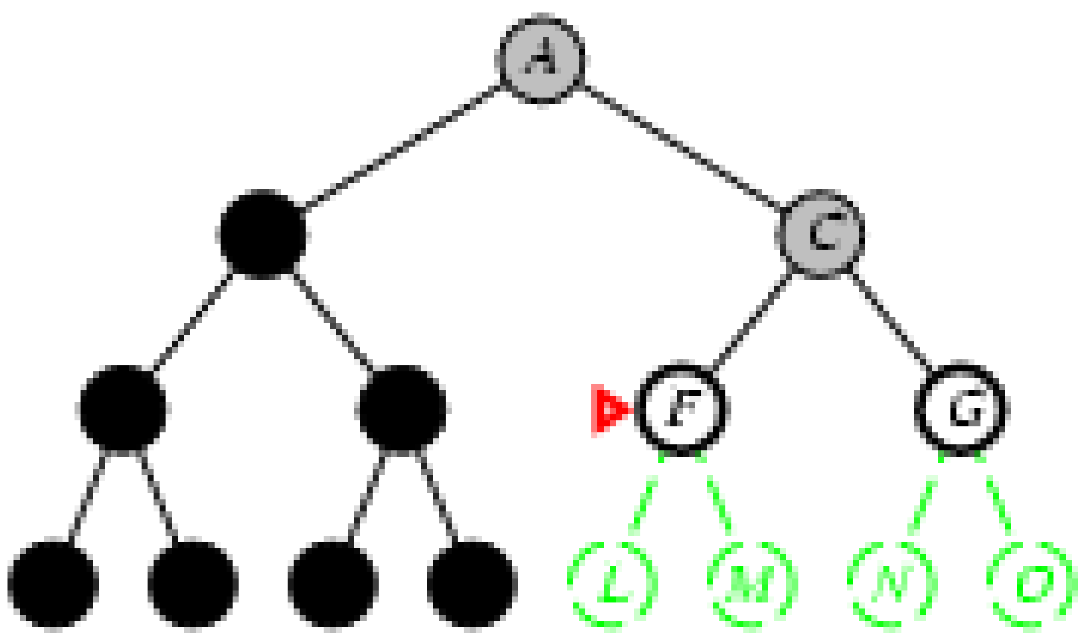


Depth-first search

Expand deepest unexpanded node

Implementation:

- *fringe* = LIFO queue, i.e., put successors at front
-

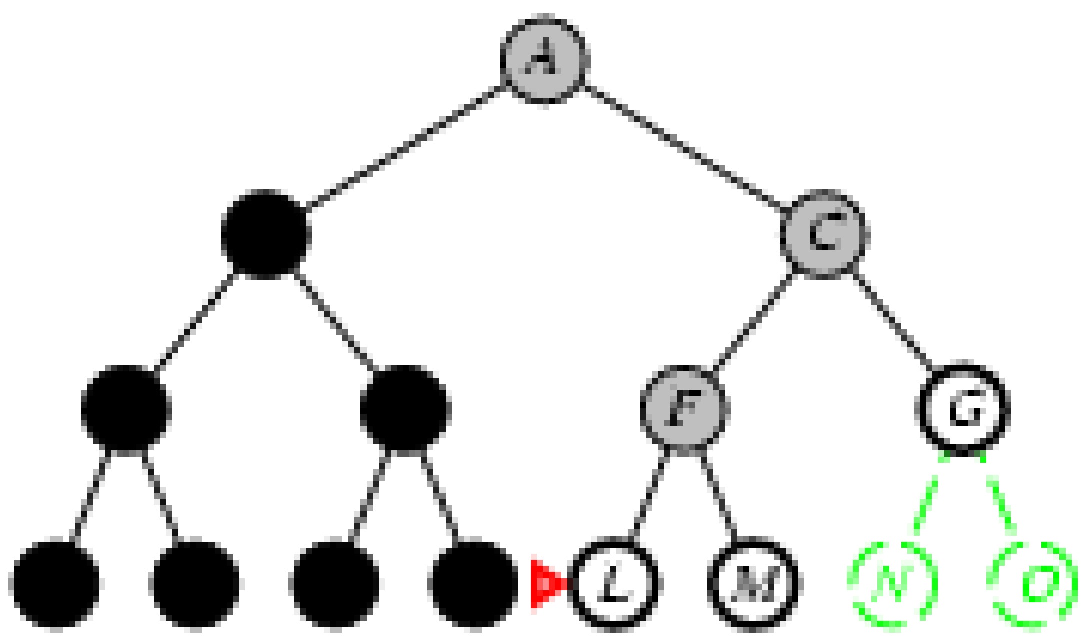


Depth-first search

Expand deepest unexpanded node

Implementation:

- *fringe* = LIFO queue, i.e., put successors at front
-

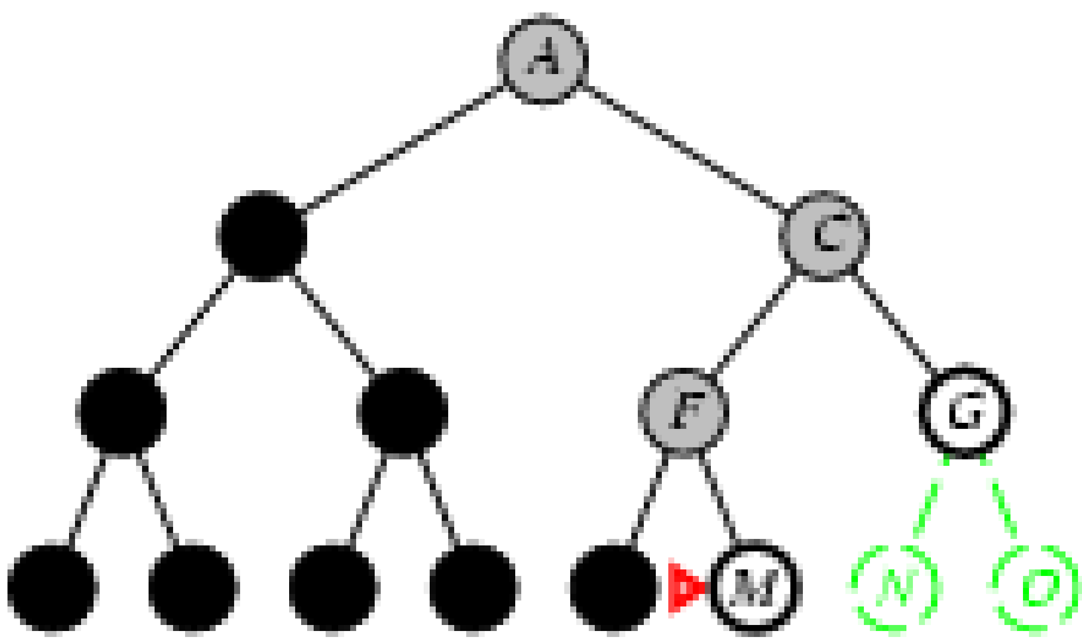


Depth-first search

Expand deepest unexpanded node

Implementation:

- *fringe* = LIFO queue, i.e., put successors at front
-



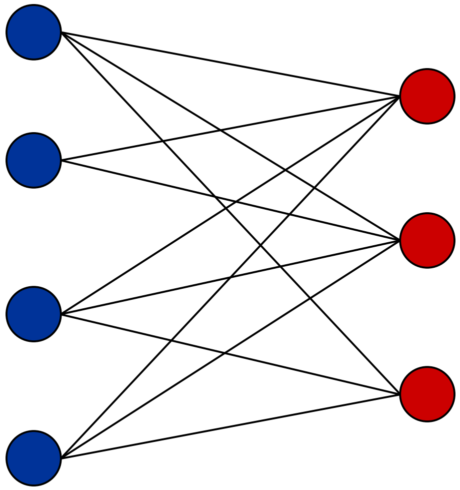
3.4 Testing Bipartiteness

Bipartite Graphs

Def. An undirected graph G = (V, E) is bipartite if the nodes can be colored red or blue such that every edge has one red and one blue end.

Applications.

- Stable marriage: men = red, women = blue.
- Scheduling: machines = red, jobs = blue.

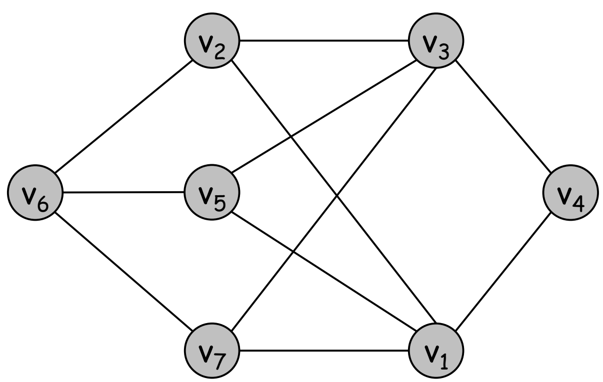


a bipartite graph

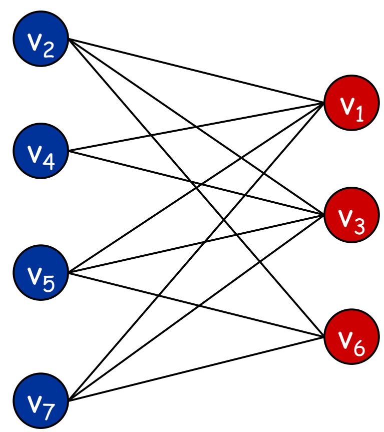
Testing Bipartiteness

Testing bipartiteness. Given a graph G, is it bipartite?

- Many graph problems become:
  - easier if the underlying graph is bipartite (matching)
  - tractable if the underlying graph is bipartite (independent set)
- Before attempting to design an algorithm, we need to understand structure of bipartite graphs.



a bipartite graph G



another drawing of G

A Structural Obstruction to Bipartiteness

Lemma. If a graph G is bipartite, it cannot contain an odd length cycle.

Pf. Not possible to 2-color the odd cycle, let alone G.

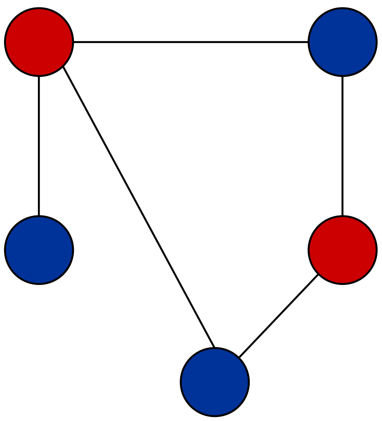
(Max number of color classes for a proper coloring of a graph is called the chromatic number of the graph).

bipartite
(2-colorable)

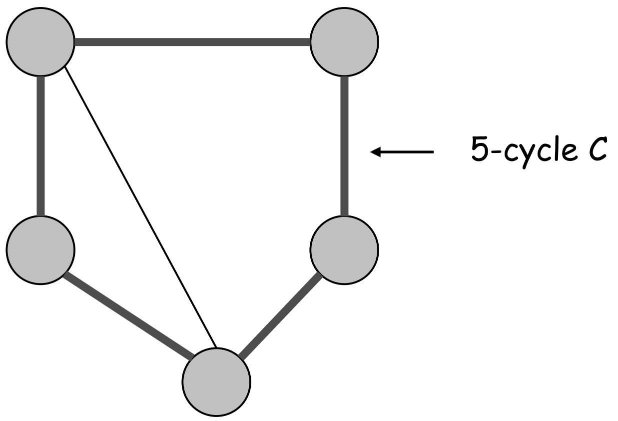
not bipartite
(not 2-colorable)

Obstruction to Bipartiteness

Corollary. A graph G is bipartite iff it contain no odd length cycle.



bipartite
(2-colorable)



not bipartite
(not 2-colorable)

A Structural Obstruction to Bipartiteness

In fact, the previous condition is <u>even stronger than previously stated</u>.

A graph G is bipartite iff it cannot contain an odd length cycle.

Pf. We already showed that if G is bipartite, then it cannot contain an odd cycle.

Q: How do we show the converse?

A Structural Obstruction to Bipartiteness

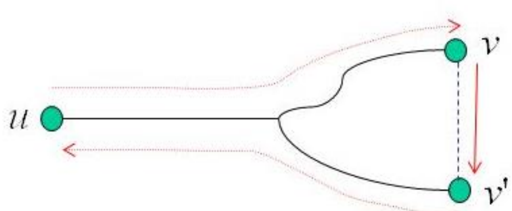
In fact, the previous condition is <u>even stronger than previously stated</u>.

A graph G is bipartite iff it cannot contain an odd length cycle.

Pf. We already showed that if G is bipartite, then it cannot contain an odd cycle.

Q: How do we show the converse?

- Let $X=\{v \in V(H)\colon f(v) \text{ is even}\}$ and $Y=\{v \in V(H)\colon f(v) \text{ is odd}\}$
- An edge v,v' within X (or Y) would create a closed odd walk using a shortest u, v-path, the edge v, v' within X (or Y) and the reverse of a shortest u, v'-path.



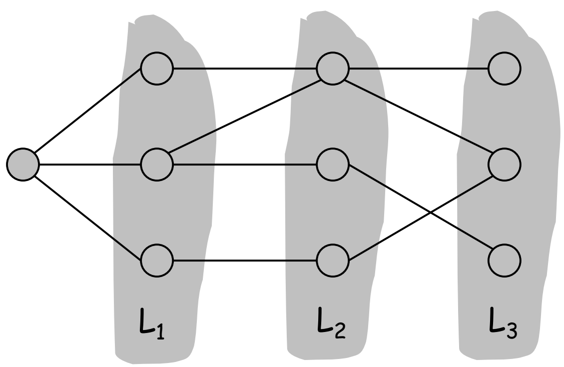
A closed odd walk using
1) a shortest u, v-path,
2) the edge v, v' within X (or Y), and
3) the reverse of a shortest u, v'-path.

Bipartite Graphs

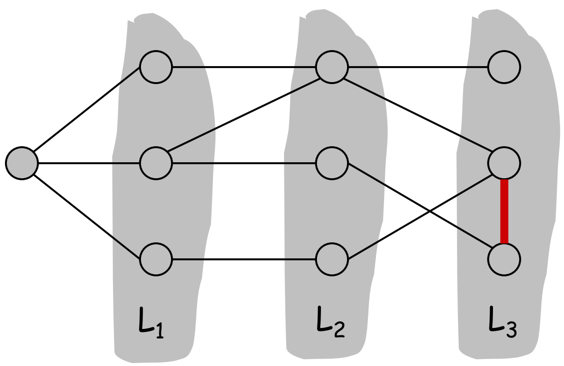
Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.

(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).



Case (i)



Case (ii)

Bipartite Graphs

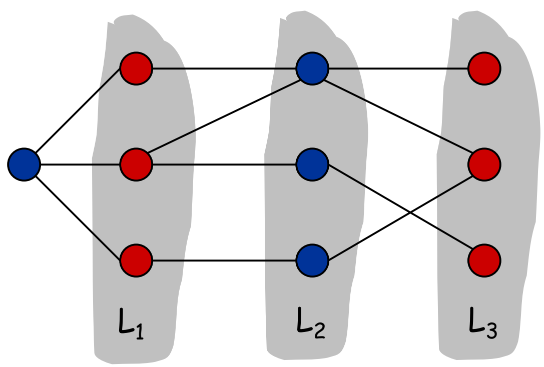
Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.

(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

Pf. (i)

- Suppose no edge joins two nodes in same layer.
- By previous lemma, this implies all edges join nodes in adjacent layers.
- Bipartition: red = nodes on odd levels, blue = nodes on even levels.



Case (i)

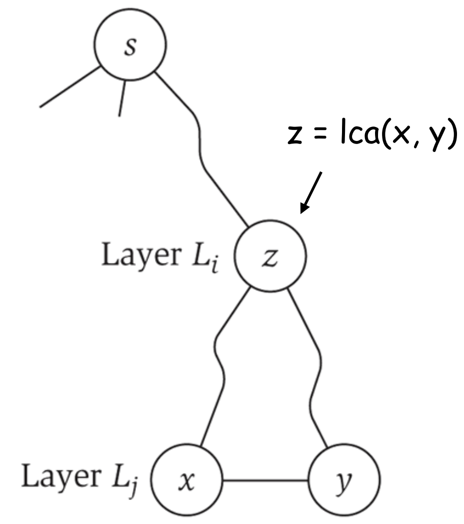
Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.

(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

Pf. (ii)

- Suppose (x, y) is an edge with x, y in same level L_j.
- Let $z = \text{lca}(x, y)$ = lowest common ancestor.
- Let L_i be level containing z.
- Consider cycle that takes edge from x to y, then path from y to z, then path from z to x.
- Its length is $\underbrace{1}_{(x,\,y)} + \underbrace{(j-i)}_{\text{path from } y \text{ to } z} + \underbrace{(j-i)}_{\text{path from } z \text{ to } x}$, which is odd. ▪



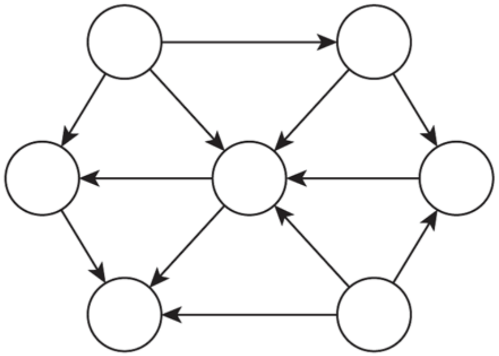
- In Summary: Using BFS yields $O(m+n)$ algorithm to check for Bipartiteness.

3.5 Connectivity in Directed Graphs

Directed Graphs

Directed graph. $G = (V, E)$

- Edge (u, v) goes from node u to node v.



Ex. Web graph - hyperlink points from one web page to another.

- Directedness of graph is crucial.
- Modern web search engines exploit hyperlink structure to rank web pages by importance.

Graph Search

Directed reachability. Given a node s, find all nodes reachable from s.

Directed s-t shortest path problem. Given two node s and t, what is the length of the shortest path between s and t?

Graph search. BFS extends naturally to directed graphs.

Web crawler. Start from web page s. Find all web pages linked from s, either directly or indirectly.

Strong Connectivity

Def. Node u and v are **mutually reachable** if there is a path from u to v and also a path from v to u.

Def. A graph is **strongly connected** if every pair of nodes is mutually reachable. (note that if the "underlying" graph is connected will call the graph merely "connected").

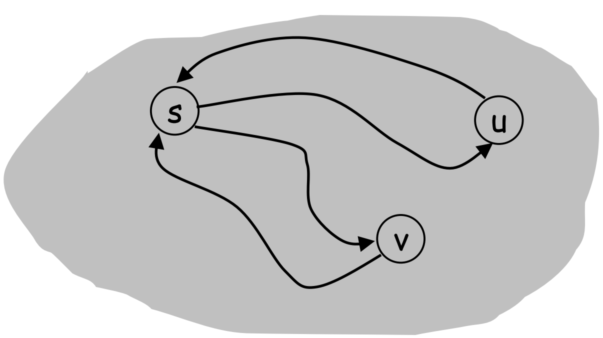
Lemma. Let s be any node. G is strongly connected iff every node is reachable from s, and s is reachable from every node.

Pf. $\Rightarrow$ Follows from definition.

Pf. $\Leftarrow$ Path from u to v: concatenate u-s path with s-v path.

Path from v to u: concatenate v-s path with s-u path. ▪

ok if paths overlap

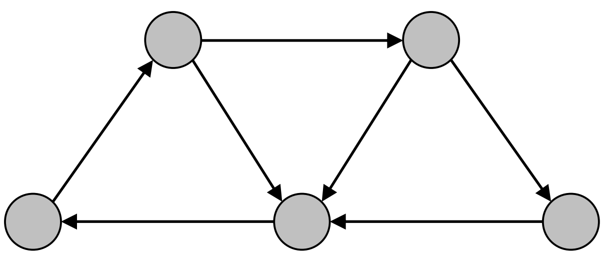


Strong Connectivity: Algorithm

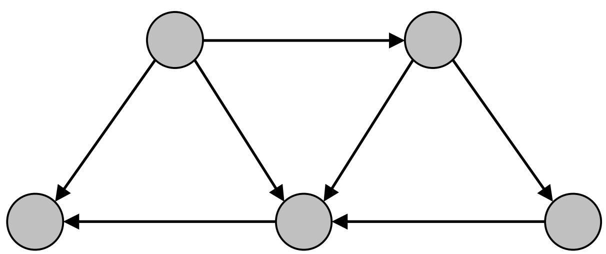
Theorem. Can determine if G is strongly connected in $O(m + n)$ time.

Pf.

- Pick any node s.
- Run BFS from s in G.
- Run BFS from s in G^{rev}.
  reverse orientation of every edge in G
- Return true iff all nodes reached in both BFS executions.
- Correctness follows immediately from previous lemma. ▪



strongly connected



not strongly connected

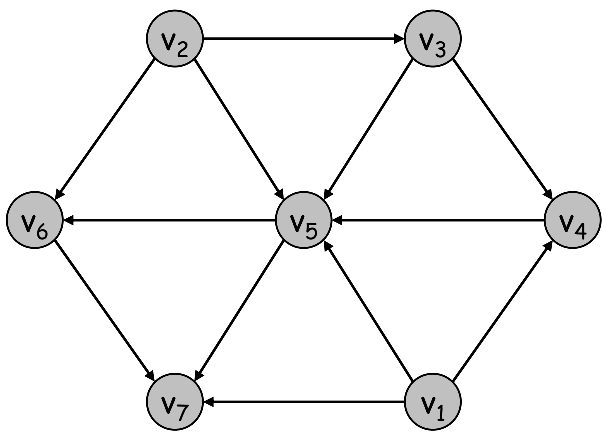
3.6 DAGs and Topological Ordering

Directed Acyclic Graphs

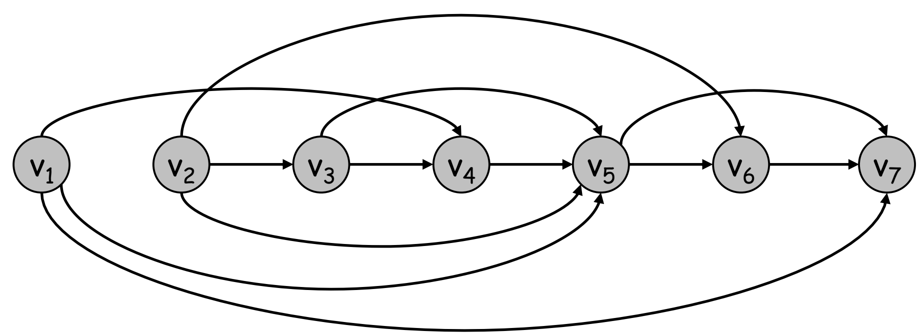
Def. An DAG is a directed graph that contains no directed cycles.

Ex. Precedence constraints: edge (v_i, v_j) means v_i must precede v_j.

Def. A topological ordering of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge (v_i, v_j) we have $i < j$.



a DAG



a topological ordering

Precedence Constraints

Precedence constraints. Edge (v_i, v_j) means task v_i must occur before v_j.

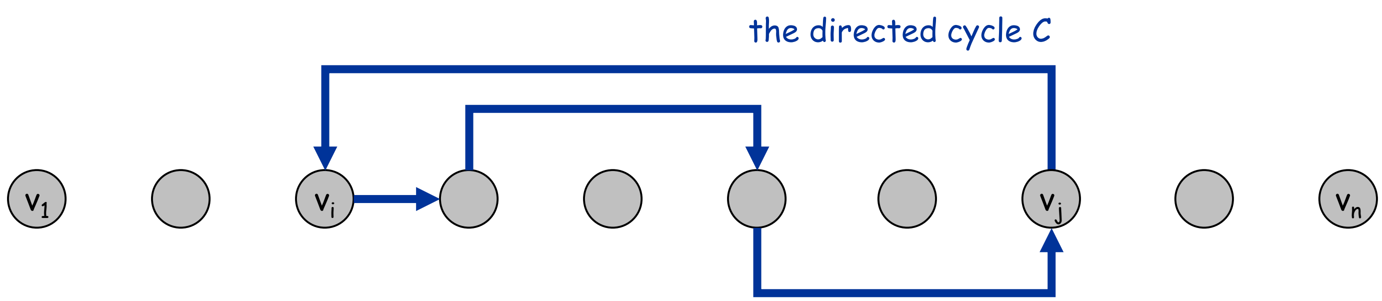
Applications.

- Course prerequisite graph: course v_i must be taken before v_j.
- Compilation: module v_i must be compiled before v_j. Pipeline of computing jobs: output of job v_i needed to determine input of job v_j.
- Markov Chains.

Lemma. If G has a topological order, then G is a DAG.

Pf. (by contradiction)

- Suppose that G has a topological order $v_1, \ldots, v_n$ and that G also has a directed cycle C. Let's see what happens.
- Let v_i be the lowest-indexed node in C, and let v_j be the node just before v_i; thus (v_j, v_i) is an edge.
- By our choice of i, we have $i < j$.
- On the other hand, since (v_j, v_i) is an edge and $v_1, \ldots, v_n$ is a topological order, we must have $j < i$, a contradiction. ▪



the supposed topological order: $v_1, \ldots, v_n$

Lemma. If G has a topological order, then G is a DAG.

Q. Does every DAG have a topological ordering?

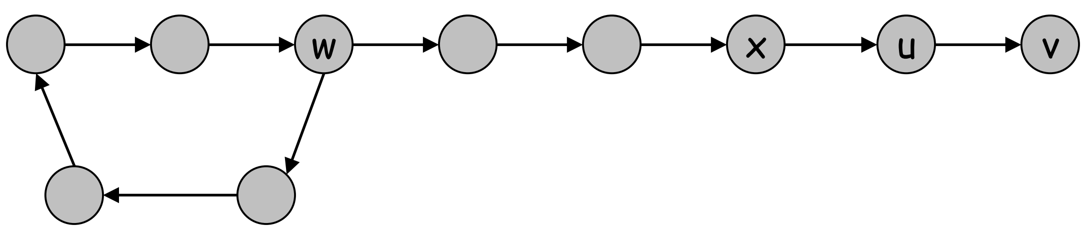
Q. If so, how do we compute one?

Directed Acyclic Graphs

Lemma. If G is a DAG, then G has a node with no incoming edges.

Pf. (by contradiction)

- Suppose that G is a DAG and every node has at least one incoming edge. Let's see what happens.
- Pick any node v, and begin following edges backward from v. Since v has at least one incoming edge (u, v) we can walk backward to u.
- Then, since u has at least one incoming edge (x, u), we can walk backward to x.
- Repeat until we visit a node, say w, twice.
- Let C denote the sequence of nodes encountered between successive visits to w. C is a cycle. ▪

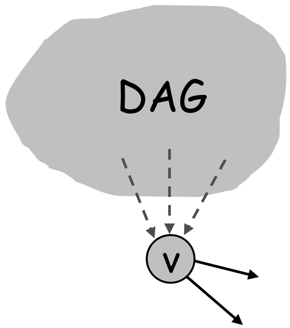


Lemma. If G is a DAG, then G has a topological ordering.

Pf. (by induction on n)

- Base case: true if n = 1.
- Given DAG on n > 1 nodes, find a node v with no incoming edges.
- G - { v } is a DAG, since deleting v cannot create cycles.
- By inductive hypothesis, G - { v } has a topological ordering.
- Place v first in topological ordering; then append nodes of G - { v }
- in topological order. This is valid since v has no incoming edges. ▪

To compute a topological ordering of G:
Find a node v with no incoming edges and order it first
Delete v from G
Recursively compute a topological ordering of $G-\{v\}$
  and append this order after v



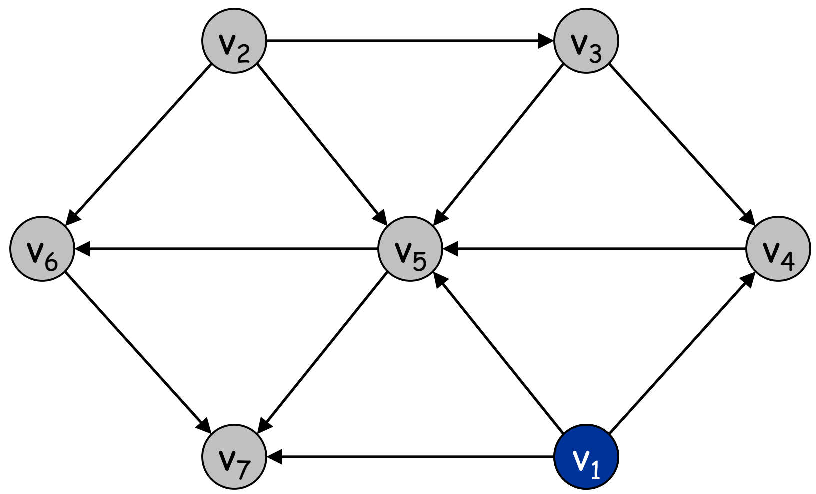
Topological Sorting Algorithm: Running Time

Theorem. Algorithm finds a topological order in $O(m + n)$ time.

Pf.

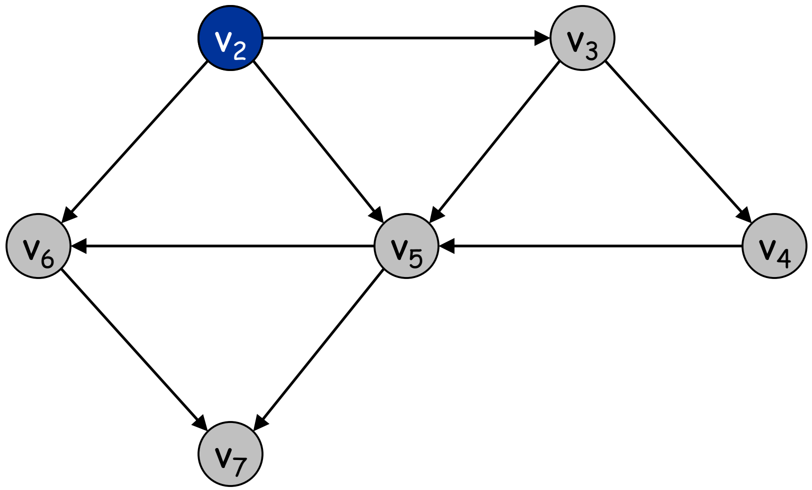
- Maintain the following information:
  - `count[w]` = remaining number of incoming edges
  - S = set of remaining nodes with no incoming edges
- Initialization: $O(m + n)$ via single scan through graph.
- Update: to delete v
  - remove v from S
  - decrement `count[w]` for all edges from v to w, and add w to S if c `count[w]` hits 0
  - this is $O(1)$ per edge ▪

Topological Ordering Algorithm: Example



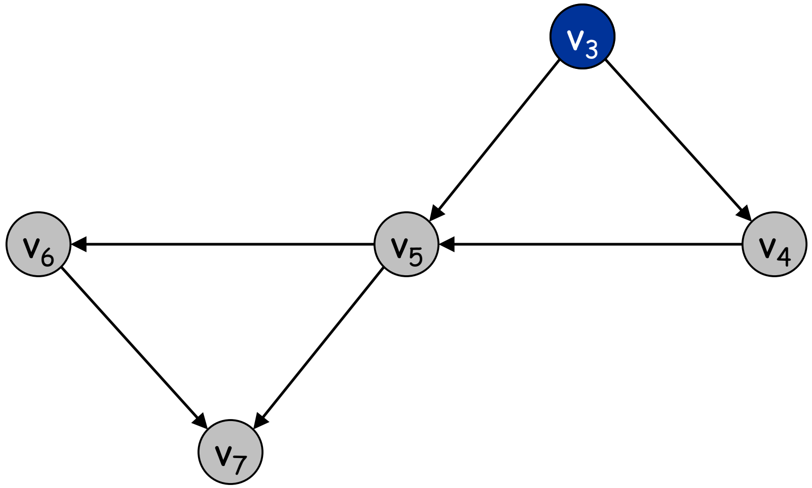
Topological order:

Topological Ordering Algorithm: Example



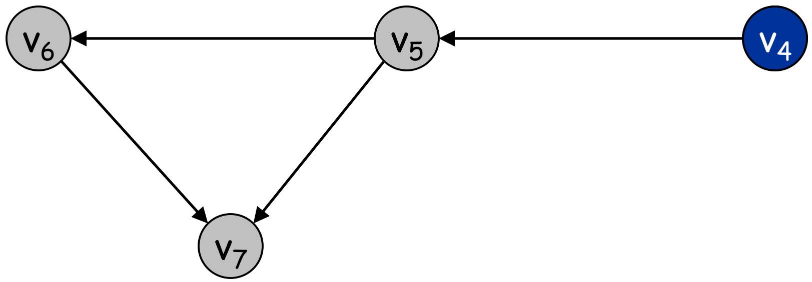
Topological order: v_1

Topological Ordering Algorithm: Example



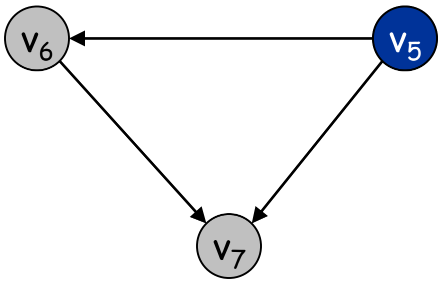
Topological order: v_1, v_2

Topological Ordering Algorithm: Example



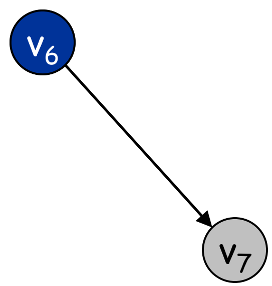
Topological order: v_1, v_2, v_3

Topological Ordering Algorithm: Example



Topological order: v_1, v_2, v_3, v_4

Topological Ordering Algorithm: Example

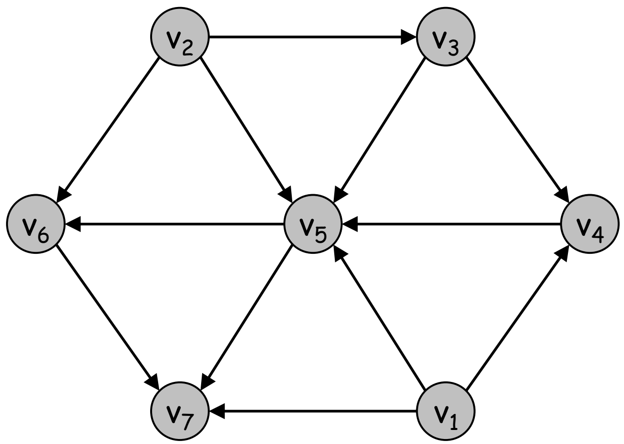


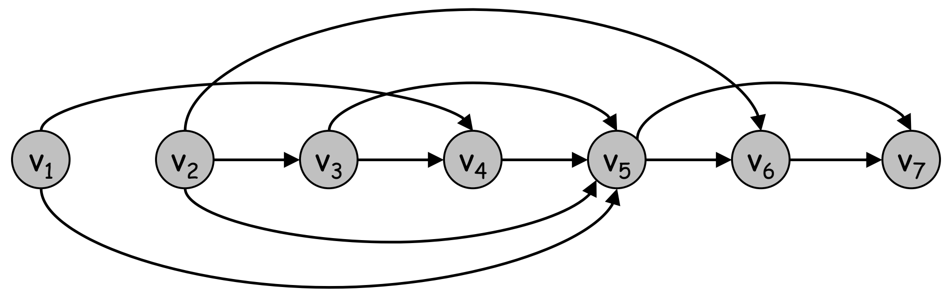
Topological order: v_1, v_2, v_3, v_4, v_5

Topological Ordering Algorithm: Example

Topological order: $v_1, v_2, v_3, v_4, v_5, v_6$

Topological Ordering Algorithm: Example





Topological order: $v_1, v_2, v_3, v_4, v_5, v_6, v_7$.

Example: HW #3.1

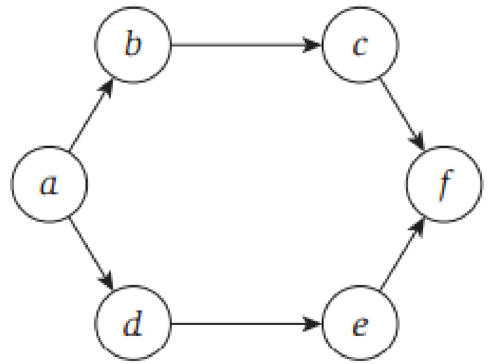


Figure 3.10 How many topological orderings does this graph have?

Example: HW #3.1

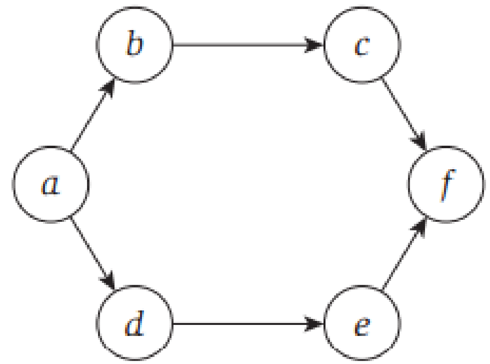


Figure 3.10 How many topological orderings does this graph have?

Consider the fact that a topological ordering must start with a and end with f. Why?

Example: HW #3.1

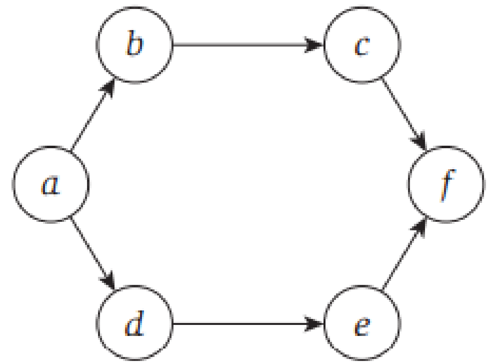


Figure 3.10 How many topological orderings does this graph have?

So, we have: a __ __ __ __ f

What can go in the middle?

Example: HW #3.1

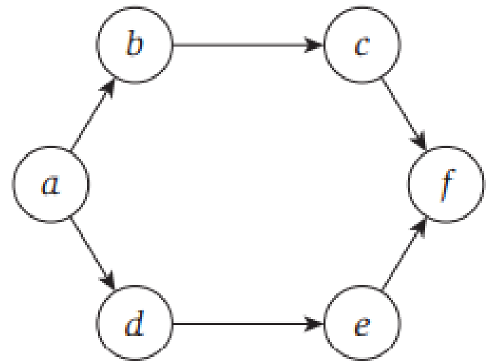


Figure 3.10 How many topological orderings does this graph have?

So, we have: a __ __ __ __ f

What can go in the middle? Note that b must precede c and d must precede e. Why?

Example: HW #3.1

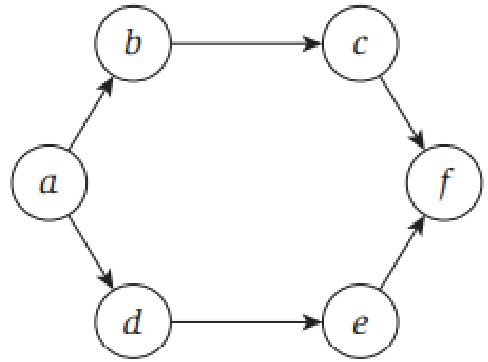


Figure 3.10 How many topological orderings does this graph have?

So, we have (2) cases:

a b __ __ __ f

And

a d __ __ __ f

How many valid orderings exist for each case?

Example: HW #3.1

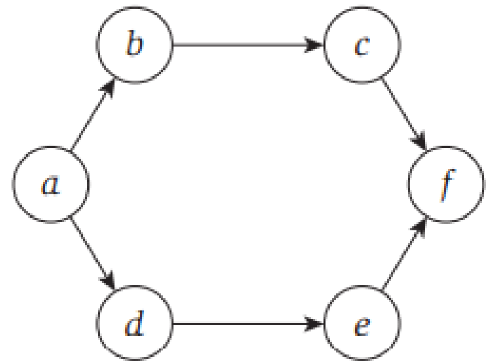


Figure 3.10 How many topological orderings does this graph have?

So, we have (2) cases:

a b __ __ __ f

And

a d __ __ __ f

How many valid orderings exist for each case?

Three. Why?

Example: HW #3.1

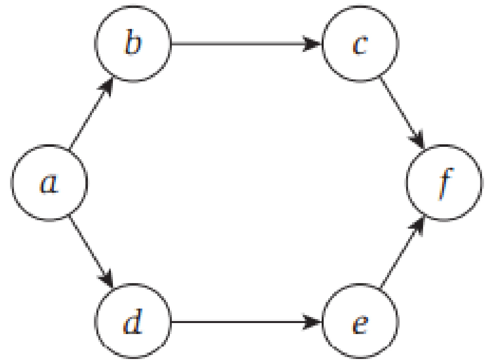


Figure 3.10 How many topological orderings does this graph have?

Final answer: 6 topological orderings exist.

Example: HW #3.2

Q: (Cycle Detection) Give an $O(m+n)$ algorithm to detect whether a given undirected graph contains a cycle.

Where should we start?

Example: HW #3.2

Q: (Cycle Detection) Give an $O(m+n)$ algorithm to detect whether a given undirected graph contains a cycle.

Where should we start?

Start with BFS from any node, say, s in G. This process produces a tree (specifically: a tree rooted at s).

If $G=T$ then we return false. Why?

Example: HW #3.2

Q: (Cycle Detection) Give an $O(m+n)$ algorithm to detect whether a given undirected graph contains a cycle.

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Otherwise, we locate an edge in G that is not in T. How would this step be implemented efficiently?

Example: HW #3.2

Q: (Cycle Detection) Give an $O(m+n)$ algorithm to detect whether a given undirected graph contains a cycle.

Where should we start?

Start with BFS from any node, say, s in G. This process produces a tree (specifically: a tree rooted at s).

If $G=T$ then we return false. Why?

Otherwise, we locate an edge in G that is not in T. How would this step be implemented efficiently?

Adding this edge to T produces a cycle (why?), and thus we return true in this case.

Example: HW #3.5

Q: A binary tree is a rooted tree in which each node has at most two children.

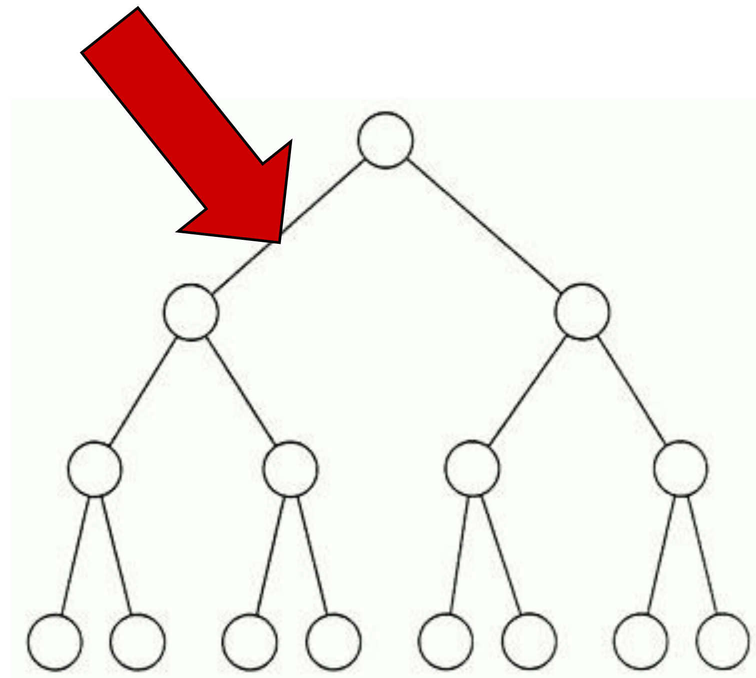
Show by induction that in any binary tree, the number of nodes with two children is exactly one less than the number of leaves.

Example: HW #3.5

Q: A binary tree is a rooted tree in which each node has at most two children.

Show by induction that in any binary tree, the number of nodes with two children is exactly one less than the number of leaves.

First, convince yourself that this seems intuitively correct.



Example: HW #3.5

Q: A binary tree is a rooted tree in which each node has at most two children.

Show by induction that in any binary tree, the number of nodes with two children is exactly one less than the number of leaves.

Pf. Base case: n=1 (trivial).

Inductive Hypothesis: Suppose that $n_{2\_children}(T)=n_{leaves}(T)-1$ for all trees with n=k nodes.

Consider a tree T with n=k+1 nodes.

Let v be a leaf in T (guaranteed to exist – why?)

Example: HW #3.5

Q: A binary tree is a rooted tree in which each node has at most two children.

Show by induction that in any binary tree, the number of nodes with two children is exactly one less than the number of leaves.

Pf. Base case: n=1 (trivial).

Inductive Hypothesis: Suppose that $n_{2\_children}(T)=n_{leaves}(T)-1$ for all trees with n=k nodes.

Consider a tree T with n=k+1 nodes.

Let v be a leaf in T denote u as the parent of v. Call T* the tree: T-v.

Example: HW #3.5

Q: A binary tree is a rooted tree in which each node has at most two children.

Show by induction that in any binary tree, the number of nodes with two children is exactly one less than the number of leaves.

Pf. Base case: $n=1$ (trivial).

Inductive Hypothesis: Suppose that $n_{2\_children}(T)=n_{leaves}(T)-1$ for all trees with $n=k$ nodes.

Consider a tree T with $n=k+1$ nodes.

Let v be a leaf in T denote u as the parent of v. Call T* the tree: T-v.

Case 1: If node u had no other children, then u is now a leaf in T*, and $n_{leaves}(T^*) = n_{leaves}(T)$, and $n_{2\_children}(T^*)= n_{2\_children}(T)$.

By the inductive hypothesis, $n_{2\_children}(T)=n_{leaves}(T)-1$, as was to be shown.

Now you finish the proof by providing case 2.