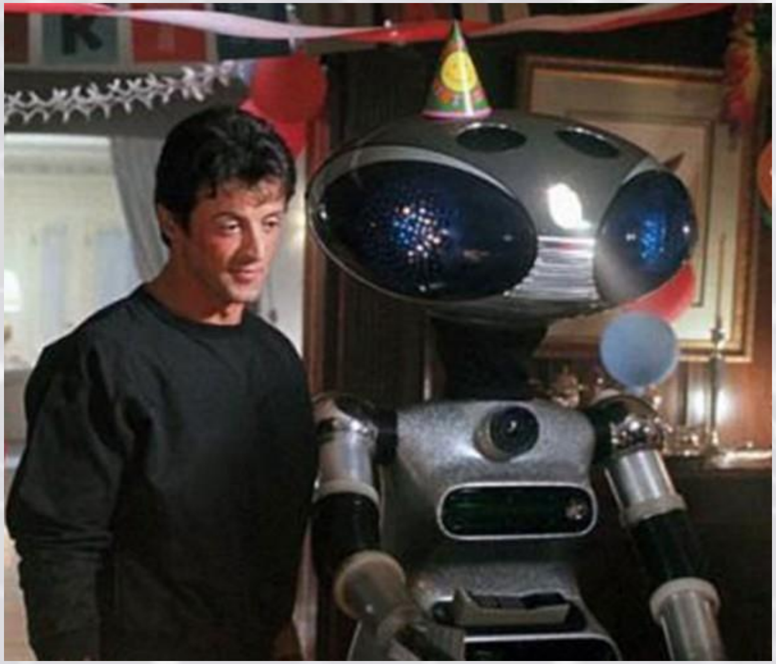
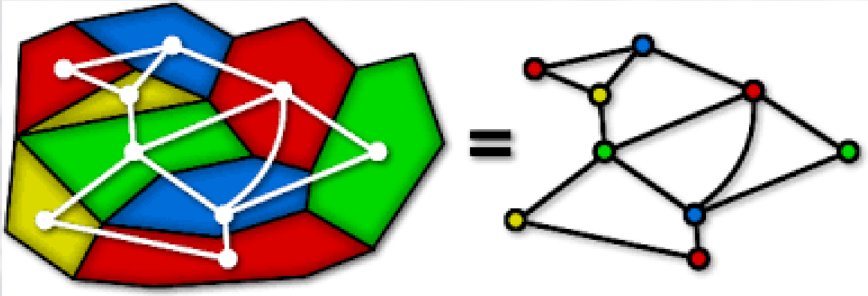
Artificial Intelligence
Chapter 6: Constraint Satisfaction Problems



Digression: The Four Color Theorem



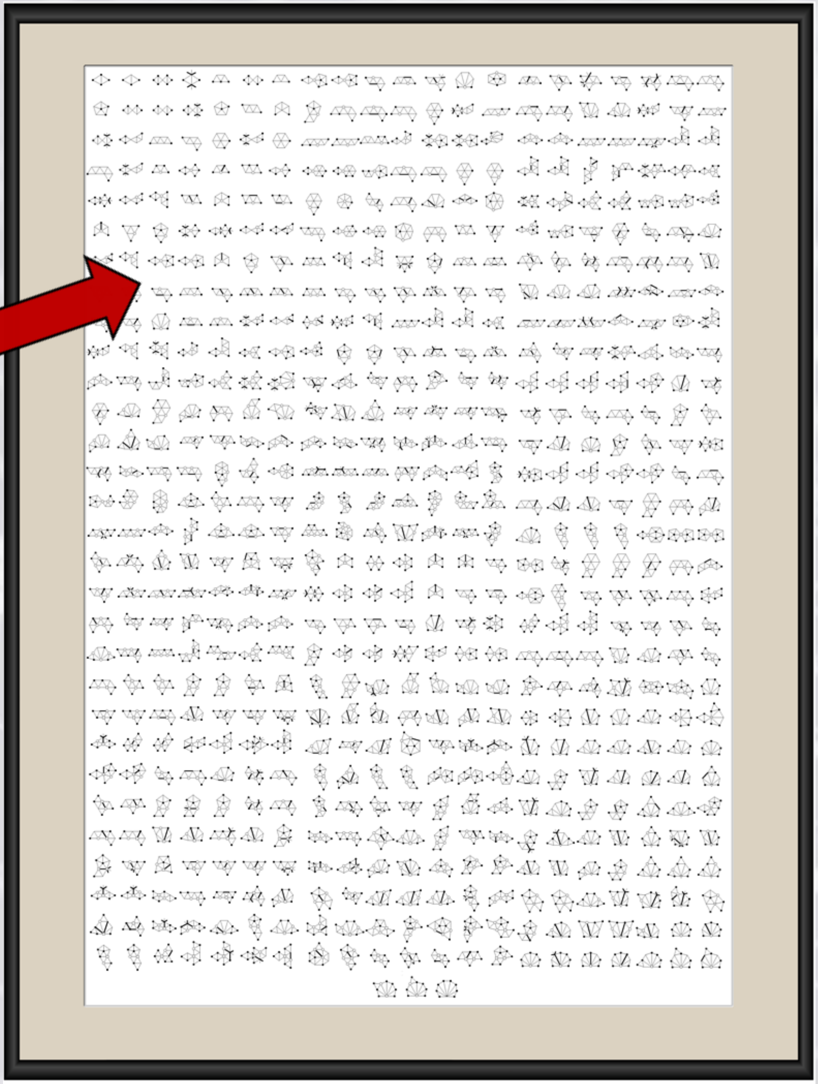
- One of the most famous results in the history of mathematics.

- It was first conjectured in 1852, but only finally proven in 1976. Notably it was the first math proof to rely crucially on computers (for a large set of configuration/case checks) – and for this reason was considered controversial.

Digression: The Four Color Theorem

- Why was the proof so difficult?

Because the best-known technique relied on (originally) 1936 unavoidable configurations -- like these.

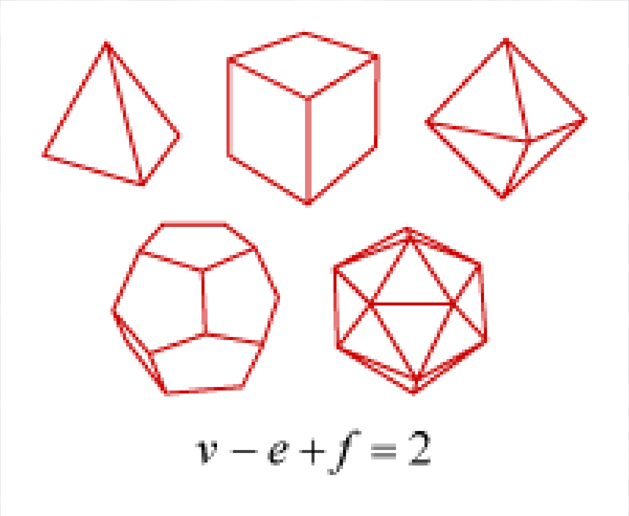


Digression: The Four Color Theorem

- While we can't prove the FCT in lecture (!), we can prove its baby brother, *the 6 color theorem*.
- Let's do this…
- (1) Prove Euler's Polyhedron Formula.

Digression: The Four Color Theorem

- (1) Prove Euler's Polyhedron Formula. (we use induction on E(G)).



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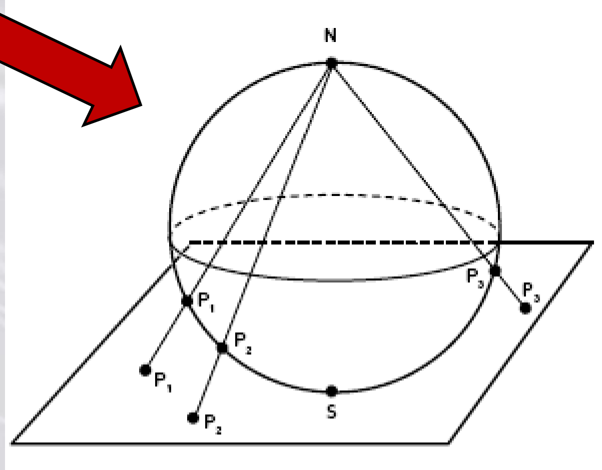
This is called a “stereographic projection”



Digression: The Four Color Theorem

- (1) Prove Euler's Polyhedron Formula.
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- (3) Prove that if G is planar (with $n \geq 3$), then $|E(G)| \leq 3n-6$.

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- (4) Claim: Every planar graph contains a vertex of degree 5 or less.

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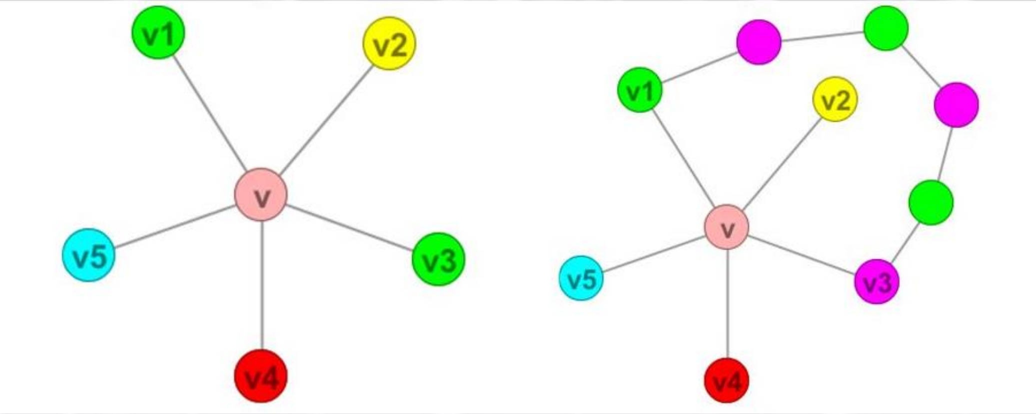
- (3) Prove that if G is planar (with $n \geq 3$), then $|E(G)| \leq 3n-6$.
- (4) Claim: Every planar graph contains a vertex of degree 5 or less. (Hint: use the fact that the sum of the degrees of vertices in a graph equals twice the number of edges).

Digression: The Four Color Theorem

- (1) Prove Euler's Polyhedron Formula.
- (2) Convince yourself that embedding on the sphere is equivalent to planarity.
- (3) Prove that if G is planar (with $n \geq 3$), then $|E(G)| \leq 3n-6$.
- (4) Claim: Every planar graph contains a vertex of degree 5 or less.
- Now put it all together and we have the 6 Color Theorem!

Digression: The Four Color Theorem

- So how about a Five Color Theorem?
- Actually, it's not *that* bad. We use the Six Color Theorem plus an additional clever argument utilizing a structure called a "Kempe Chain" (1890); the result follows by contradiction.



Digression: The Four Color Theorem

- Unfortunately, no one to date has found a simple way to reduce Kempe Chains and similar structures to efficiently solve 4CT (aside from exhaustive case-checking.

- Nevertheless, the 4CT and its proof serve evidence that humankind and machines can work together productively and harmoniously!

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SKYNET

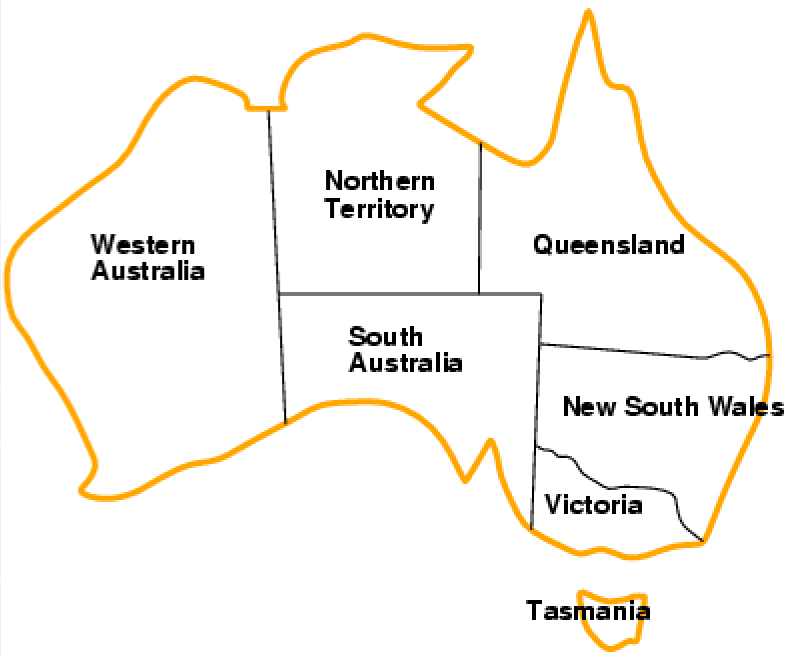
NEURAL NET-BASED ARTIFICIAL INTELLIGENCE

Constraint satisfaction problems (CSPs)

- Standard search problem: state is a "black box“ – any data structure that supports successor function and goal test
- CSP:
  - state is defined by variables X_i with values from domain D_i
  - goal test is a set of constraints specifying allowable combinations of values for subsets of variables

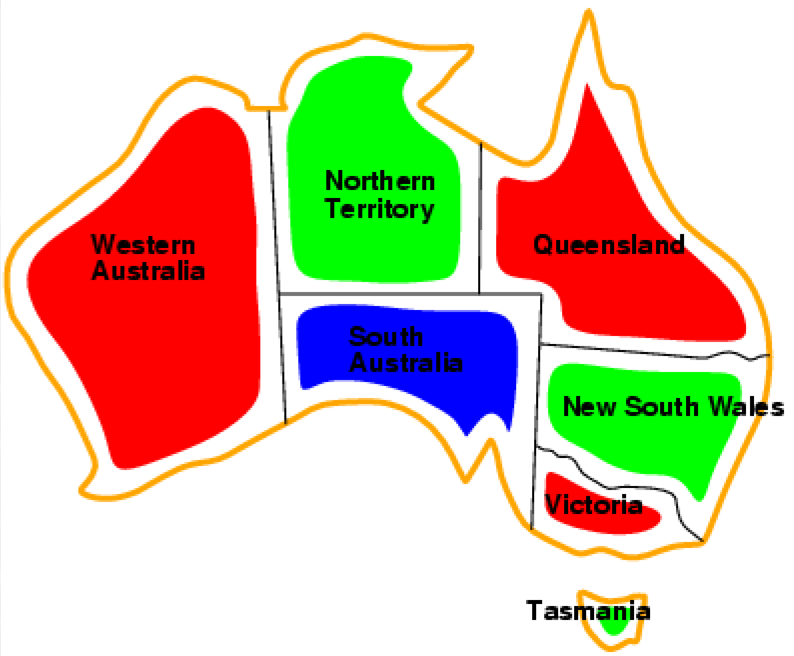
- Allows useful general-purpose algorithms with more power than standard search algorithms.

Example: Map-Coloring



- Variables *WA, NT, Q, NSW, V, SA, T*
- Domains D_i = {red,green,blue}
- Constraints: adjacent regions must have different colors
- e.g., WA ≠ NT, or (WA,NT) in {(red,green),(red,blue),(green,red),(green,blue),(blue,red),(blue,green)}

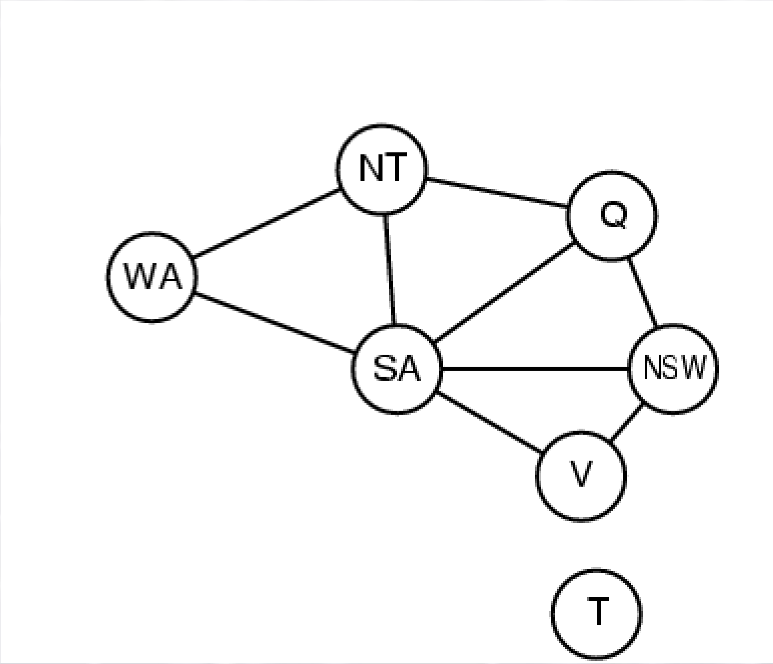
Example: Map-Coloring



- Solutions are complete and consistent assignments
- e.g., WA = red, NT = green, Q = red, NSW = green,V = red,SA = blue,T = green

Constraint graph

- Binary CSP: each constraint relates two variables
- Constraint graph: nodes are variables, arcs are constraints



Varieties of CSPs

- Discrete variables
  - finite domains:
    - n variables, domain size $d \rightarrow O(d^n)$ complete assignments
    - e.g., Boolean CSPs, incl. Boolean satisfiability (NP-complete)
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., $StartJob_1 + 5 \le StartJob_3$

- Continuous variables
  - e.g., start/end times for Hubble Space Telescope observations
  - linear constraints solvable in polynomial time by LP

Varieties of constraints

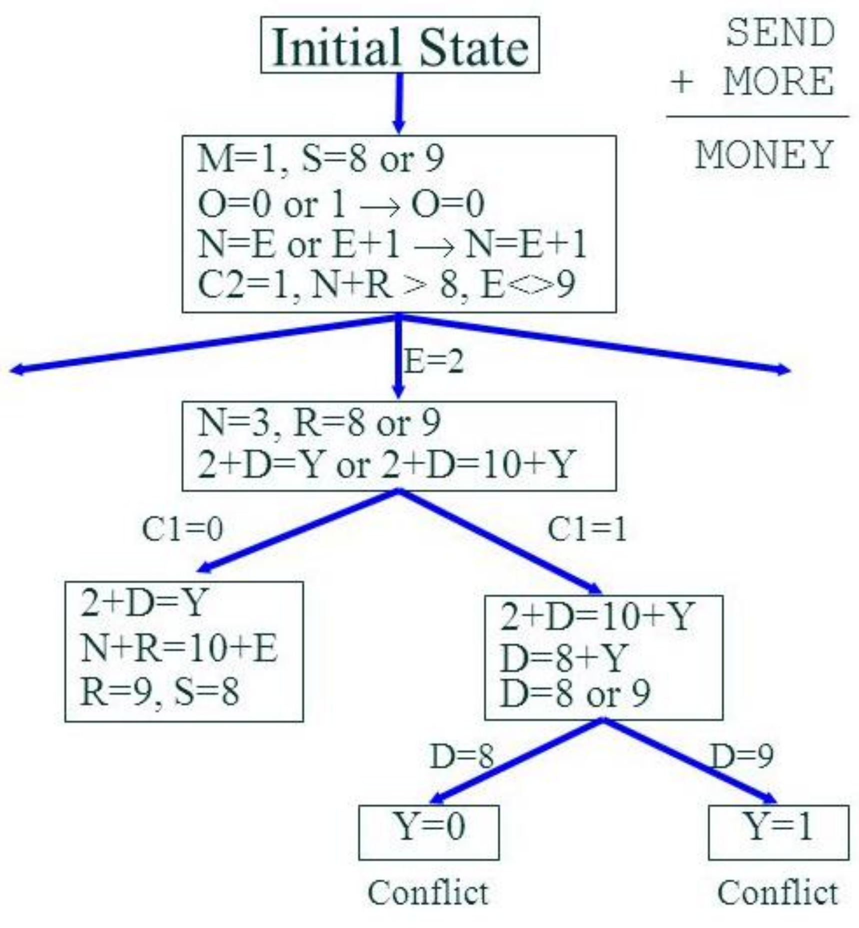
- Unary constraints involve a single variable,
  - e.g., SA ≠ green
- Binary constraints involve pairs of variables,
  - e.g., SA ≠ WA
- Higher-order constraints involve 3 or more variables,
  - e.g., cryptarithmetic column constraints

Cryptarithmetic Problem

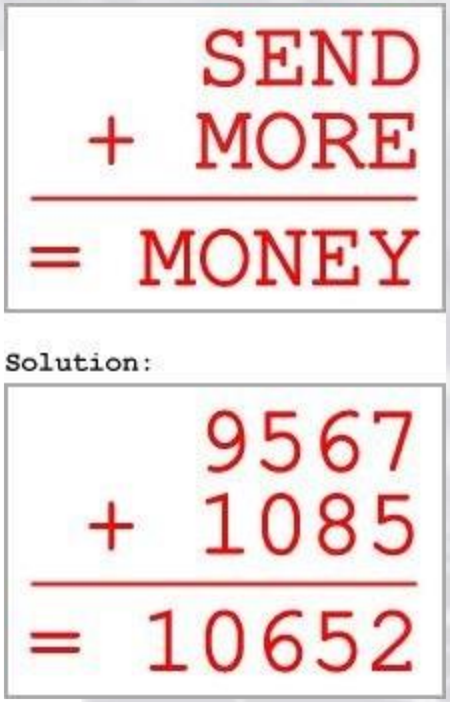
- Try this one:

```
  SEND
+MORE
------
 MONEY
```

Cryptarithmetic Problem



- A useful heuristic can help to select the **best guess** to try first.
- If there is a letter that participate in many constraints, then it is a good idea to prefer it to a letter that participates in a few.



Backtracking search

- Backtracking search is used for a **depth-first search** that chooses values for one variable at a time and backtracks when a variable has no legal values left to assign.

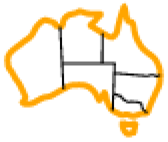
- It repeatedly chooses an unassigned variable, and then tries all values in the domain of that variable in turn, trying to find a solution.

- If an inconsistency is detected, then BACKTRACK returns failure, causing the previous call to try another value.

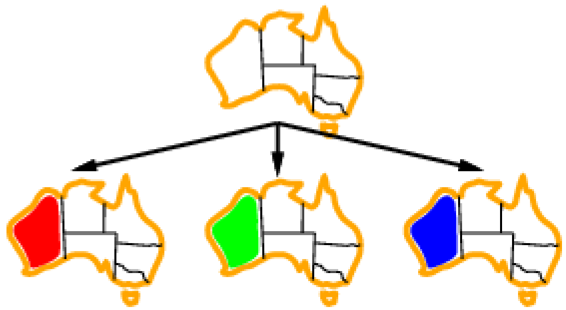
Backtracking Search

- Depth-first search for CSPs with single-variable assignments is called backtracking search.
- Variable assignments are commutative, i.e.,
  [WA = red then NT = green] same as [NT = green then WA = red].
- => Only need to consider assignments to a single variable at each node, so the algorithm keeps only a single representation of a state and alters that representation rather than creating new ones.
- Can solve n-queens for $n \approx 25$.

Backtracking example



Backtracking example



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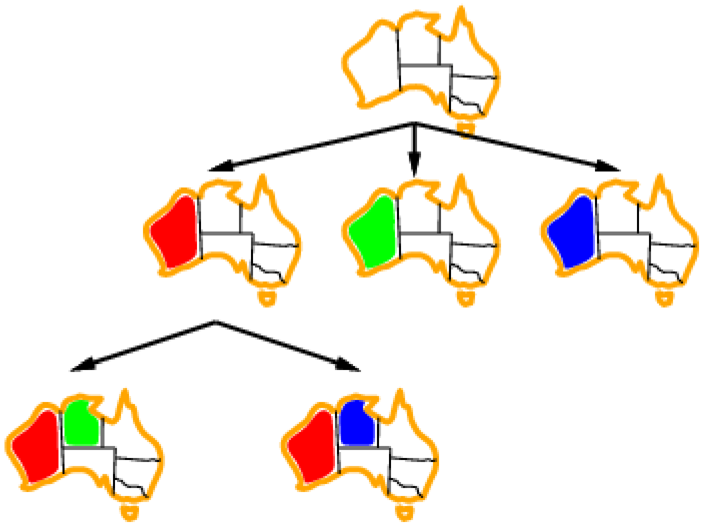
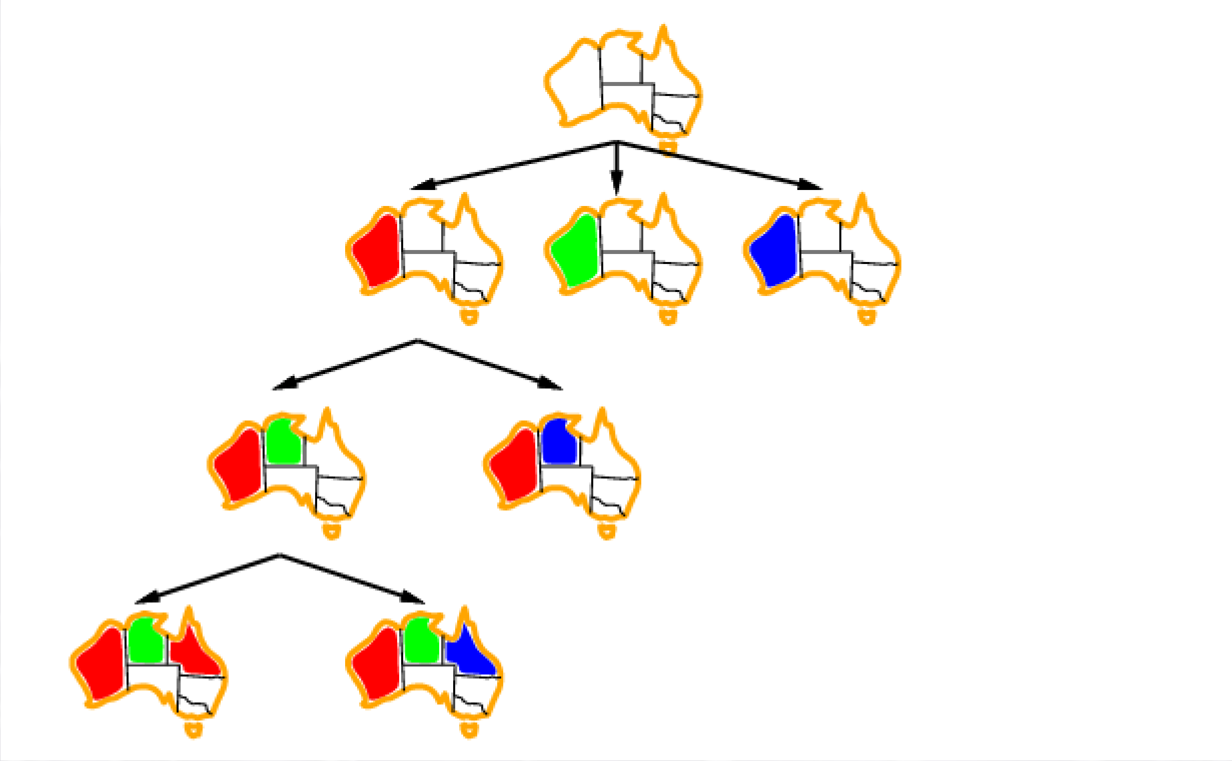


Backtracking example



Improving backtracking efficiency

- (3) General-purpose methods can give huge gains in speed:
  - Which variable should be assigned next? (e.g. MRV)
  - In what order should its values be tried? (e.g. inference/forward checking)
  - Can we detect inevitable failure early? (e.g. constraint learning)

Constraint Propagation

- In regular state-space search, an algorithm can do only one thing: *search*.

- In CSP there is a choice: (1) an algorithm can *search* (i.e. choose a new variable assignment from several possibilities); (2) perform a specific type of *inference* called **constraint propagation**.

Constraint Propagation

- Constraint propagation uses the constraints to reduce the number of legal values for a variable, which in turn can reduce the legal values for another variable, and so on.

- Constraint propagation may be interleaved with search, or it can be done as a preprocessing step.

- The key idea is: **local consistency**.

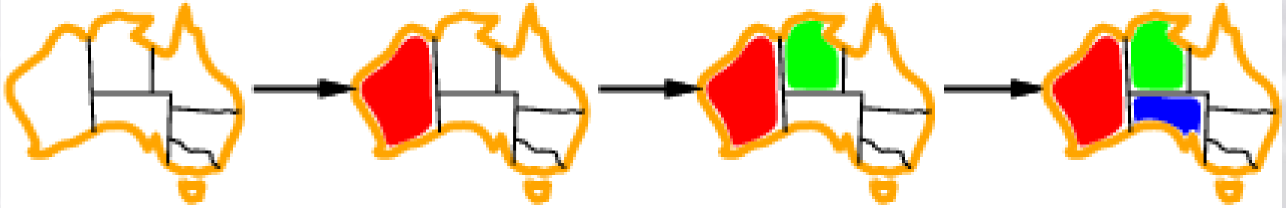
Constraint Propagation

- The key idea is: **local consistency** (e.g. node consistency, arc consistency, etc.).

- If we treat each variable as a **node** in the graph, and each binary constraint as an **arc**, then the process of enforcing local consistency in each part of the graph causes inconsistent values to be eliminated throughout the graph.

Most constrained variable

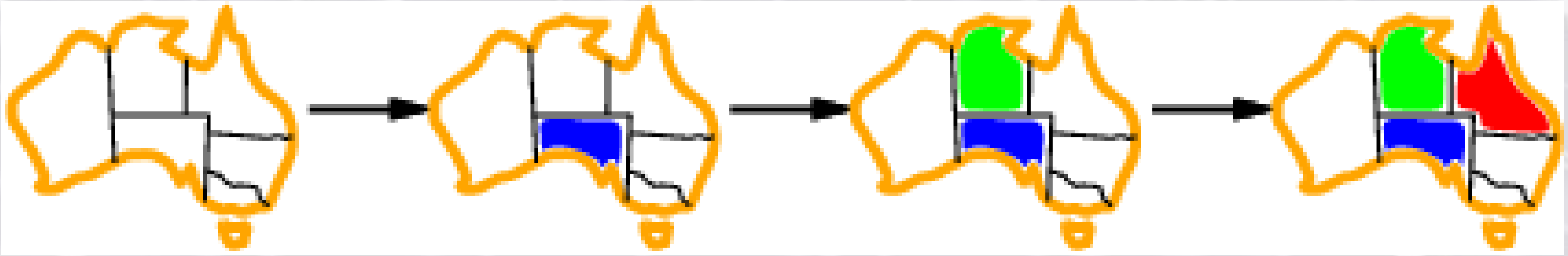
- Most constrained variable:
  choose the variable with the fewest legal values



- a.k.a. minimum remaining values (MRV) heuristic

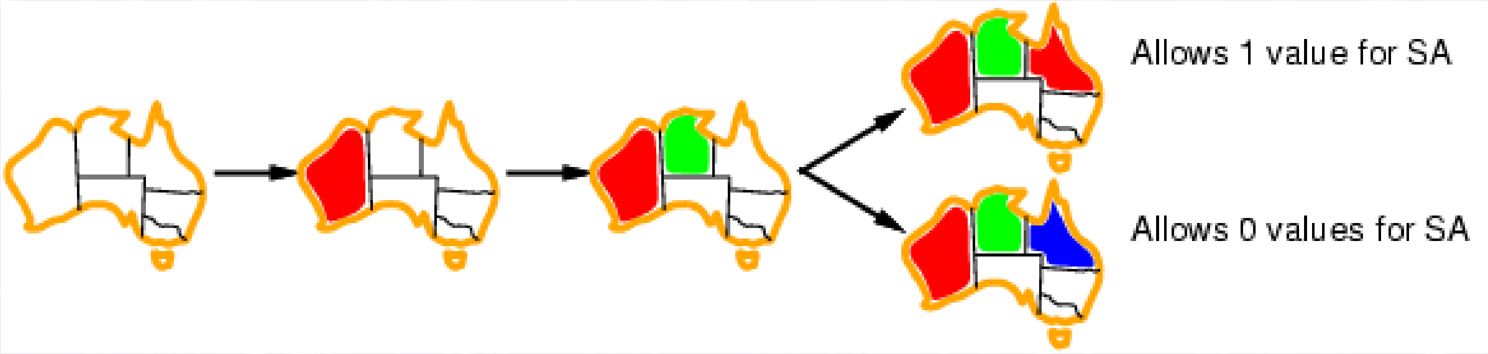
Most constraining variable

- A good idea is to use it as a tie-breaker among most constrained variables
- Most constraining variable:
  – choose the variable with the most constraints on remaining variables



Least constraining value

- Given a variable to assign, choose the least constraining value:
  - the one that rules out the fewest values in the remaining variables



- Combining these heuristics (MRV + LCV) makes 1000 queens feasible!

Forward checking

- MRV and LCV and related approaches can infer reduction in the domain variables *before* we begin the search.

- But <u>inference can be even more powerful in the course of a search</u>: every time we make a choice of a value for a variable, we have a brand-new opportunity to infer new domain reductions on neighboring values.

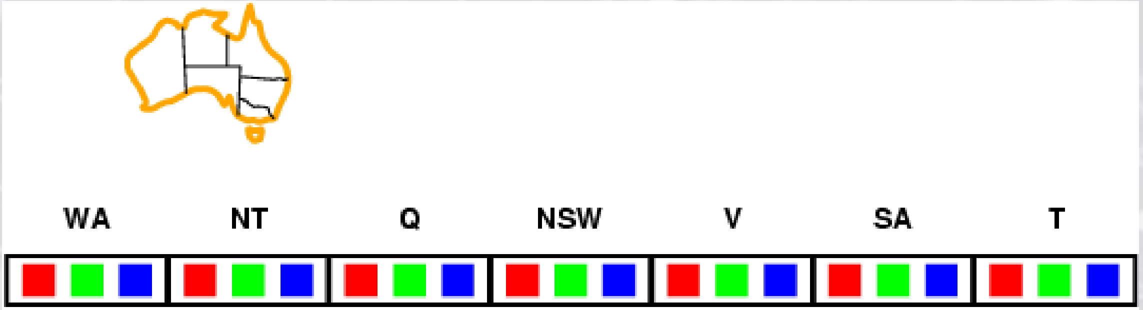
Forward checking

- Forward checking is one of the simplest forms of inference.

- Whenever a variable X is assigned, the forward-checking process establishes **arc consistency** for it: for each unassigned variable Y that is connected to X by a constraint, delete from Y's domain any value that is inconsistent with the value chosen for X.

Forward checking

- Idea:
  - Keep track of remaining legal values for unassigned variables.
  - Terminate search when any variable has no legal values.



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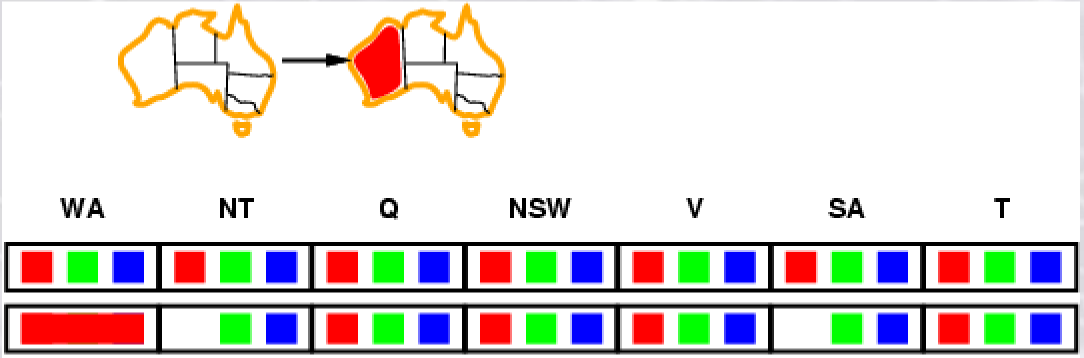

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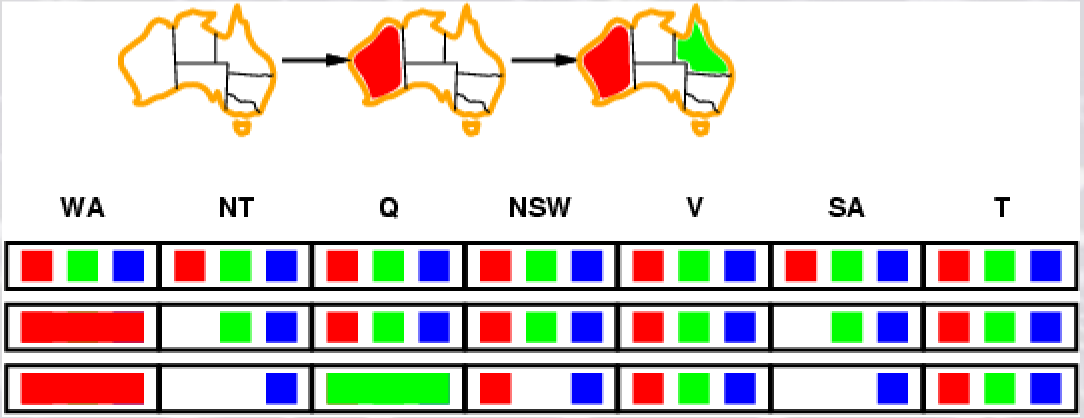

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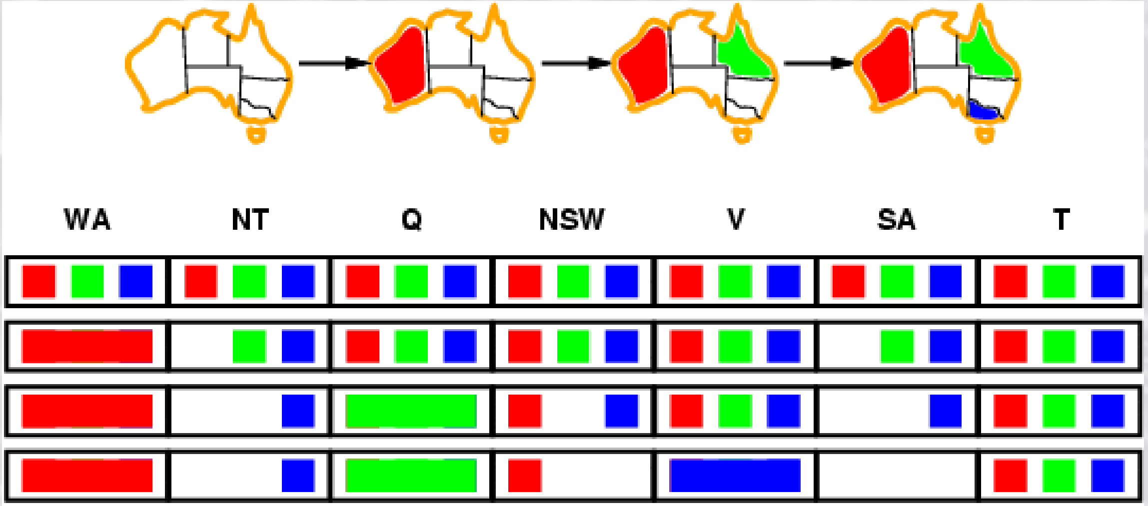
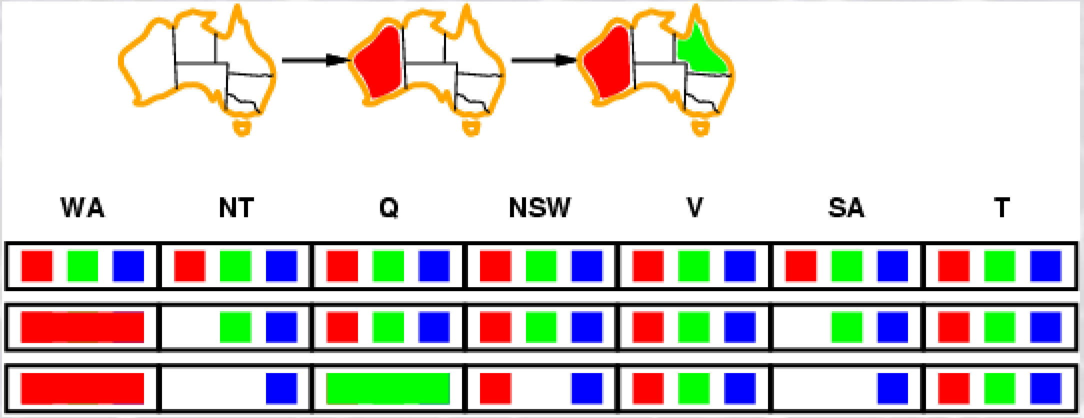


Constraint propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:
- 

- NT and SA cannot both be blue!
- Constraint propagation algorithms repeatedly enforce constraints locally…

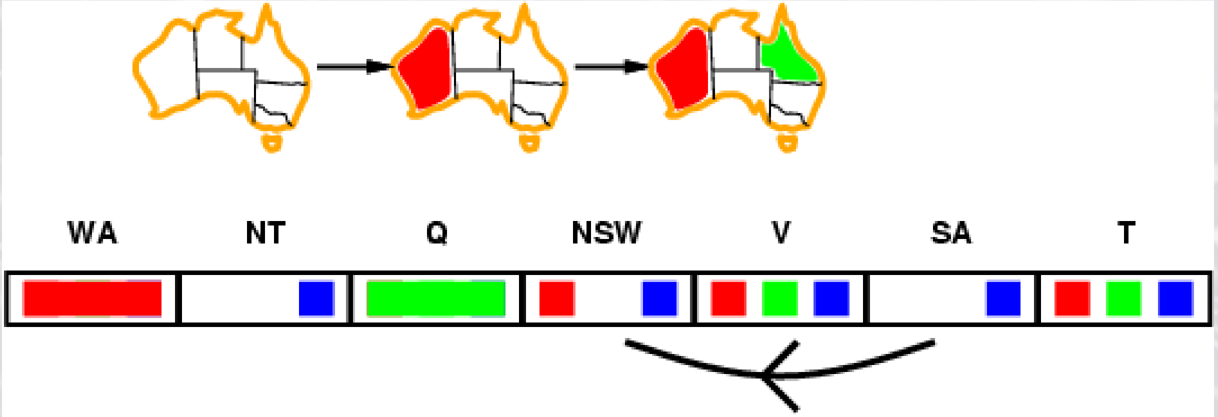
Node Consistency

- A single variable (in a CSP network) is **node-consistent** if all the values in the variable's domain satisfy the variable's unary constraints.
- E.g. Suppose South Australians dislike green; the variable SA starts with domain: {red, green, blue}, and we make it node-consistent by eliminating green, leaving SA with the reduced domain: {red, blue}.
- We say that a network is node-consistent if every variable in the network is node-consistent.

Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff

  for every value x of X there is some allowed y



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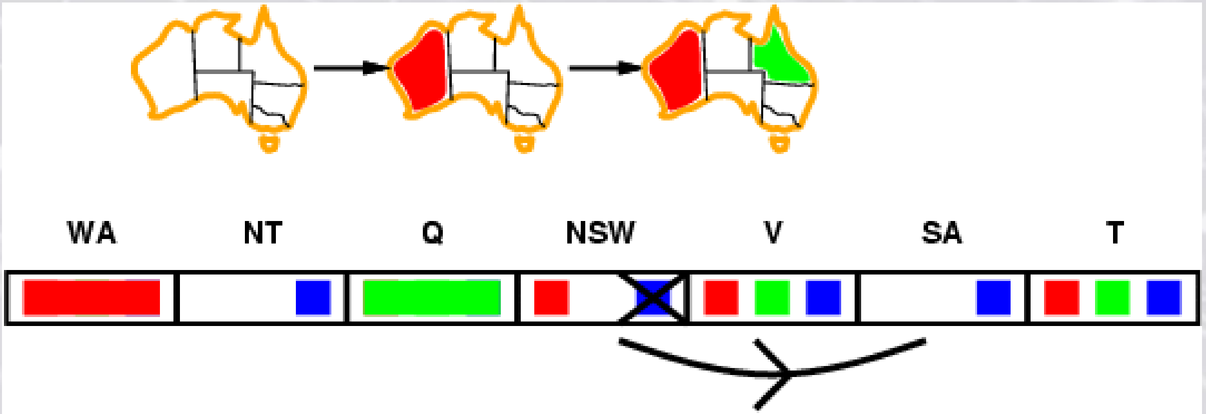

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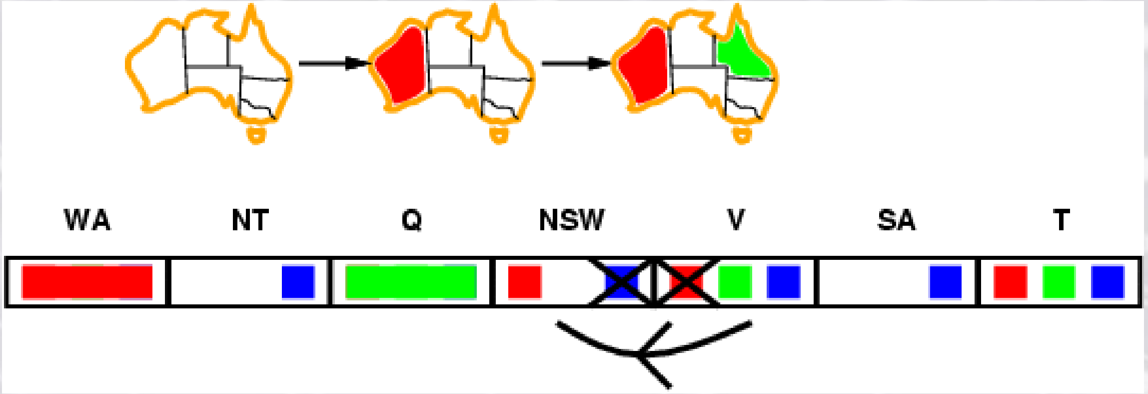

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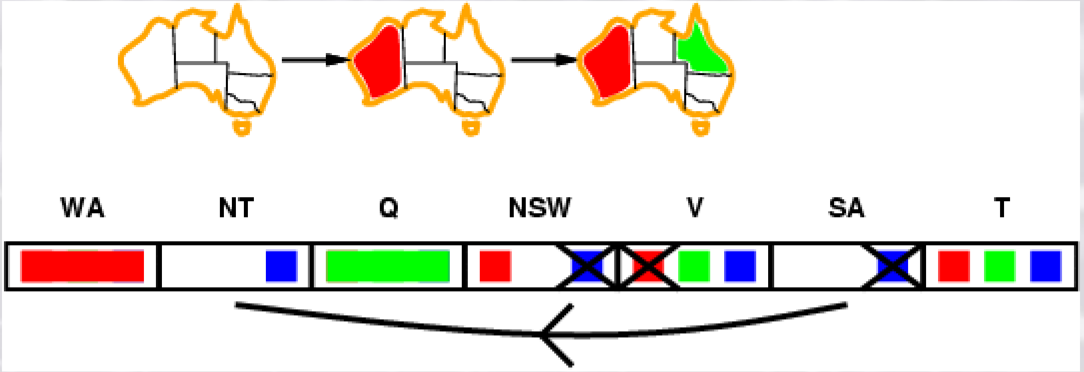


- If X loses a value, neighbors of X need to be rechecked
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

AC-3 Algorithm

- The most popular algorithm for arc consistency is called *AC-3*.

- The AC-3 algorithm maintains a queue of arcs to consider.

- Initially, the queue contains all the arcs in the CSP. AC-3 the pops off an arbitrary arc (X_i, X_j) from the queue and makes X_i arc-consistent with respect to X_j.

AC-3 Algorithm

- If this leaves D_i unchanged, the algorithm moves on to the next arc.
- If this revises D_i, then we add to the queue all arcs (X_k, X_i), where Xk is a neighbor of Xi.
- We need to do this because the change in D_i might enable further reductions in the domains of D_k.
- Continue this process…
- We end up with a CSP *equivalent* to the original CSP – but the arc-consistent CSP will in most cases be much faster to search.

Arc consistency algorithm AC-3

```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X_1, X_2, ..., X_n}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (X_i, X_j) ← REMOVE-FIRST(queue)
      if RM-INCONSISTENT-VALUES(X_i, X_j) then
         for each X_k in NEIGHBORS[X_i] do
            add (X_k, X_i) to queue
```

```
function RM-INCONSISTENT-VALUES( X_i, X_j) returns true iff remove a value
   removed ← false
   for each x in DOMAIN[X_i] do
      if no value y in DOMAIN[X_j] allows (x,y) to satisfy constraint(X_i, X_j)
         then delete x from DOMAIN[X_i];   removed ← true
   return removed
```

- Time complexity: O(#constraints |domain|^3)

Checking consistency of an arc is O(|domain|^2)

Path Consistency

- A two-variable set $\{X_i,X_j\}$ is **path-consistent** with respect to a third variable X_m, if, for every assignment $\{X_i=a, X_j=b\}$ consistent with the constraints on $\{X_i, X_j\}$, there is an assignment to Xm that satisfies the constrains on $\{X_i, X_m\}$ and $\{X_m, X_j\}$.

- This is called path-consistency, because one can think of it as looking at a path from X_i to X_j with X_m in the middle.

k-consistency

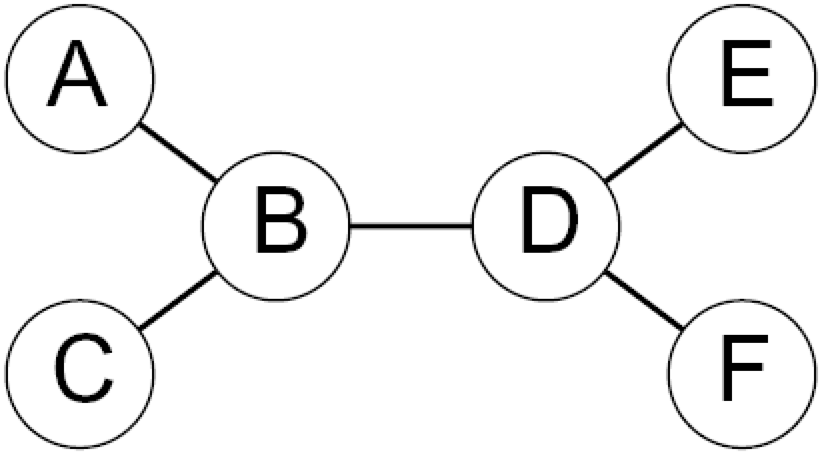
- A CSP is *k-consistent* if, for any set of k-1 variables, and for any consistent assignment to those variables, a consistent value can always be assigned to any kth variable
- 1-consistency is node consistency.
- 2-consistency is arc consistency.
- For binary constraint networks, 3-consistency is the same as *path consistency.*
- Getting k-consistency requires time and space exponential in k.
- *Strong k-consistency* means k'-consistency for all k' from 1 to k
  - Once strong k-consistency for k=#variables has been obtained, solution can be constructed trivially
- Tradeoff between propagation and branching.
- Practitioners usually use 2-consistency and less commonly 3-consistency.

Other techniques for CSPs

- Global constraints
  - E.g., Alldiff
  - E.g., Atmost(10,P1,P2,P3), i.e., sum of the 3 vars $\leq$ 10
  - Special propagation algorithms
    - Bounds propagation
      - E.g., number of people on two flight D1 = [0, 165] and D2 = [0, 385]
      - Constraint that the total number of people has to be at least 420
      - Propagating bounds constraints yields D1 = [35, 165] and D2 = [255, 385]
    - …
- Symmetry breaking (e.g. reduce search by imposing arbitrary ordering constraint).

Structured CSPs

Tree-structured CSPs



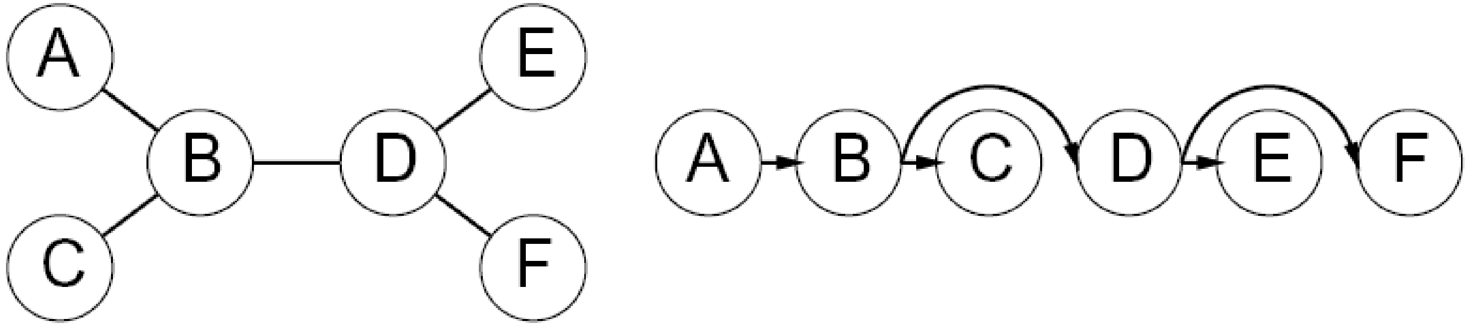
Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\, d^2)$ time

Compare to general CSPs, where worst-case time is $O(d^n)$

This property also applies to logical and probabilistic reasoning:
an important example of the relation between syntactic restrictions
and the complexity of reasoning.

Algorithm for tree-structured CSPs

1. Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering

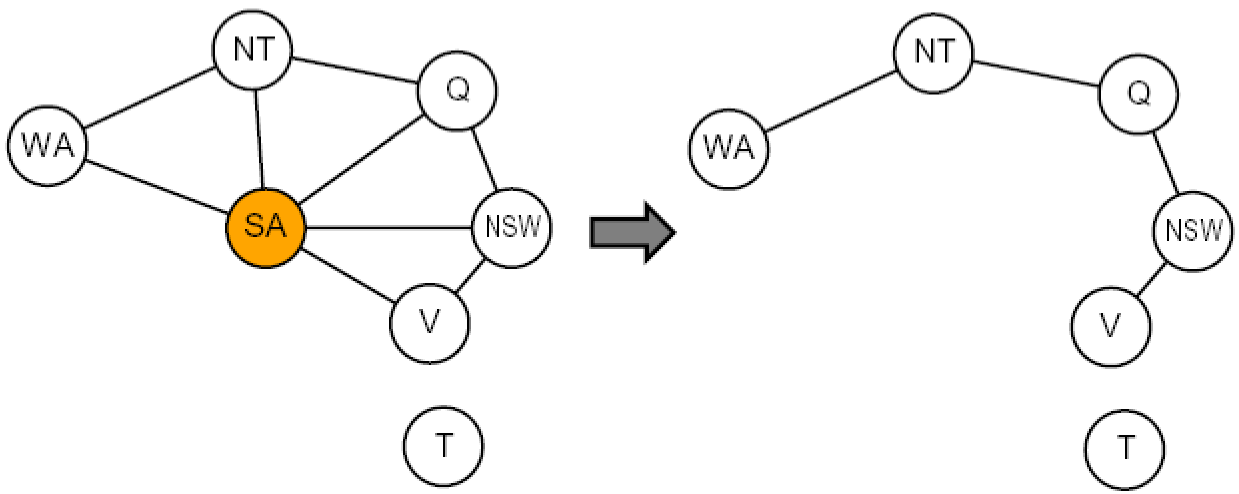


2. For j from n down to 2, apply $\textsc{RemoveInconsistent}(Parent(X_j), X_j)$

3. For j from 1 to n, assign X_j consistently with $Parent(X_j)$

Nearly tree-structured CSPs

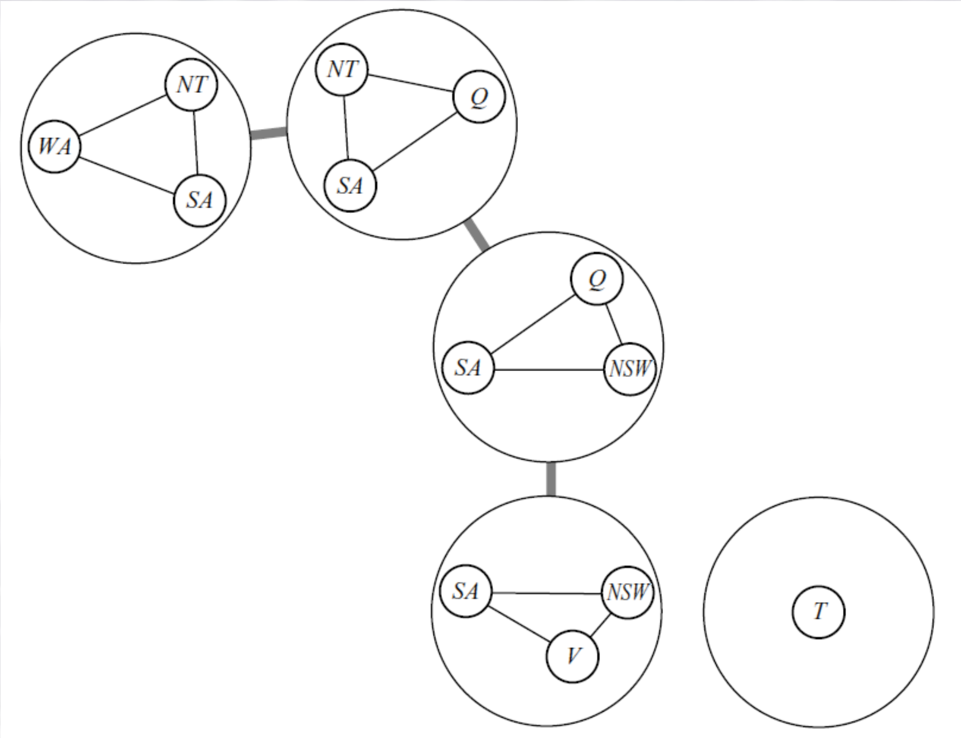
Conditioning: instantiate a variable, prune its neighbors' domains



Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree (Finding the minimum cutset is NP-complete.)

Cutset size c $\Rightarrow$ runtime $O(d^c \cdot (n-c)d^2)$, very fast for small c

Tree decomposition



- Every variable in original problem must appear in at least one subproblem
- If two variables are connected in the original problem, they must appear together (along with the constraint) in at least one subproblem
- If a variable occurs in two subproblems in the tree, it must appear in every subproblem on the path that connects the two

- Algorithm: solve for all solutions of each subproblem. Then, use the tree-structured algorithm, treating the subproblem solutions as variables for those subproblems.
- $O(nd^{w+1})$ where w is the *treewidth* (= one less than size of largest subproblem)
- Finding a tree decomposition of smallest treewidth is NP-complete, but good heuristic methods exists