TRIDIAGONAL MATRICES AND BOUNDARY CONDITIONS

J.J.P. VEERMAN* AND DAVID K. HAMMOND†

Abstract. We describe the spectra of certain tridiagonal matrices arising from differential equations commonly used for modeling flocking behavior. In particular we consider systems resulting from allowing an arbitrary boundary condition for the end of a one dimensional flock. We apply our results to demonstrate how asymptotic stability for consensus and flocking systems depends on the imposed boundary condition.

1. Introduction. The $n+1$ by $n+1$ tridiagonal matrix

$$A_{n+1} = \begin{pmatrix} b & 0 & 0 & & & \\ a & 0 & c & 0 & & \\ 0 & a & 0 & c & & \\ & & \ddots & \ddots & \ddots & \\ & & & a & 0 & c \\ & & & & a+e & d \end{pmatrix},$$

is of special interest in many (high-dimensional) problems with *local* interactions and internal translation symmetry but with no clear preferred rule for the boundary condition. We are interested in the spectrum and associated eigenvectors of this matrix. In particular in Section 4 we study how the spectrum depends on choices for the boundary conditions implied by d and e.

We will pay special attention to the following important subclass of these systems.

Definition 1.1. *If $b = a + c$ and $c = e + d$, the matrix A is called decentralized.*

One of the main applications of these matrices arises in the analysis of first and second order systems of ordinary differential equations in $\mathbb{R}^{n+1}$ such as

$$\dot{x} = -L(x-h), \quad \text{and} \tag{1.1}$$

$$\ddot{x} = -\alpha L(x-h) - \beta L\dot{x}. \tag{1.2}$$

Here L is the so-called directed graph Laplacian (e.g. [7]), given by $L = D - A$, where D is a diagonal matrix with i^{th} entry given by the i^{th} row sum of L. In the decentralized case $D = (a+c)I$, and L is given simply by

$$L = bI - A = (a+c)I - A \tag{1.3}$$

In (1.2) α and β are real numbers, and h is a constant vector with components h_k. Upon substitution of $z \equiv x - h$, the fixed points of Equations 1.1 and 1.2 are moved to the origin.

It is easy to prove that the systems in Equations 1.1 and 1.2 admit the solutions

$$x_k = x_0 + h_k \quad \text{and} \quad x_k = v_0 t + x_0 + h_k \tag{1.4}$$

(for the first order system and the second order system, respectively) for arbitrary reals x_0 and v_0 if and only if the system is decentralized.

The first order system given above is a simple model used to study 'consensus', while the second system models a simple instance of 'flocking' behavior. The latter is also used to study models for automated traffic on a single lane road. The interpretation d and e as specifying boundary conditions for these models can be understood as follows. Following the transformation $z \equiv x - h$, we may consider $z_k(t)$, for $0 \le k \le n$, as the transformed positions of the $n+1$ members of a "flock". Here $z_0(t)$ denotes the transformed position of the *leader*, the model 1.1 then specifies that $\dot{z}_0 = 0$, $\dot{z}_k = a(z_{k-1}-z_k)-c(z_k-z_{k+1})$ for $1 \le k < n$, and that $\dot{z}_n = (a+e)z_{n-1} + dz_n$. The terms $a(z_{k-1} - z_k)$ and $-c(z_k - z_{k+1})$ may be interpreted as control signals that are proportional to the displacement of z_k from its predecessor z_{k-1}, and from its successor z_{k+1}. The equation giving $\dot{z}_n$ is different as z_n has no successor, selection of the

*Maseeh Department of Mathematics & Statistics, Portland State University, Portland, OR 97201, USA
†Oregon Institute of Technology, Wilsonville, OR 97070

boundary condition consists of deciding what the behavior governing z_n should be. In the decentralized case, eliminating d shows that $\dot{z}_n = (a+e)(z_{n-1} - z_n)$, so that $\dot{z}_n$ is proportional to the difference from its predecessor, and e may be interpreted as the additional amount of the proportionality constant due to boundary effects. Interpretation of d and e for the second order system in 1.2 is similar. These problems are important examples of a more general class of problems where oscillators are coupled according to some large communication graph, and one wants to find out whether and how fast the system synchronizes. The asymptotic stability of both systems is discussed in Section 6.

One of the main motivations for this work came from earlier work [2] that led to the insight that in some important cases changes of boundary conditions did not give rise to appreciable changes in the dynamics of these systems (if the dimension was sufficiently high). This somewhat surprising discovery motivated the current investigation into how eigenvalues change as a function of the boundary condition. Indeed Corollaries 6.1 and 6.2 corroborate that at least the asymptotic stability of consensus systems and flock-formation systems is unchanged for a large range of boundary conditions.

The method presented here relies on the observation that the eigenvalue equation for A can be rewritten as a two-dimensional recursive system with appropriate boundary conditions. This procedure was first worked out in [3]. Here we give a considerably refined version of that argument, that allows us to draw more general conclusions. These conclusions are presented in Theorems 4.2, 4.3, and 4.4. The spectrum of tridiagonal matrices has also been considered by Yueh [10] who relies heavily on [4]. In that work however the parameter e is zero, and the emphasis is on analyzing certain isolated cases, while we attempt to give a comprehensive theory. The inclusion of the parameter e is necessary for our main application: decentralized systems. Related work has also been published by Willms [9], who considered tridiagonal matrices where the product of sub and super diagonal elements is constant, and Kouachi [5], who considered a similar condition where the product of sub and super diagonal elements alternates between two values. Both of these conditions exclude the case when $e \neq 0$ in the current work.

We assume a, b, c, d, e to be real. The cases where $a = 0$ or $c = 0$ are very degenerate. There are only 1 or 2 non-zero eigenvalues of A. We will not further discuss these cases. That leaves $a > 0$ and $c \neq 0$ as the general case to be studied. We will consider $a > 0$ and $c > 0$ and will assume this unless otherwise mentioned.

In Section 2 we derive a polynomial whose roots will eventually yield the eigenvalues. In the next Section we find the value of those roots. Then in Section 4 we use those results to characterize the spectrum of A. In Section 5 we apply this to the matrices associated with decentralized systems. In Section 6 we discuss the consequences for the asymptotic stability of decentralized systems of ordinary differential equations.

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2. Preliminary Calculations. We start by noting that A_{n+1} is block lower triangular. One block has dimension 1 and eigenvalue b. The other block has dimension n, in the following we will denote this by $n \times n$ matrix Q, given by

$$Q = \begin{pmatrix} 0 & a/\tau^2 & 0 & & \\ a & 0 & a/\tau^2 & & \\ & \ddots & \ddots & \ddots & \\ & & a & 0 & a/\tau^2 \\ & & & a+e & d \end{pmatrix},$$

where to facilitate calculations we have set $\tau^2 \equiv a/c$. The spectrum of A_{n+1} thus consists of union of the spectrum of Q and the the trivial eigenvalue b.

To find the spectrum of Q, we look for a number r and a n-vector v forming an eigenpair (r, v) as follows

$$\begin{array}{rrcl} & \dfrac{a}{\tau^2} v_2 & = & r v_1 \\ k \in \{2, \cdots, n-1\} & a v_{k-1} + \dfrac{a}{\tau^2} v_{k+1} & = & r v_k \\ & (a+e) v_{n-1} + d v_n & = & r v_n \end{array} \tag{2.1}$$

These equations may be considered as a recurrence relation, with appropriate boundary conditions. As we will show presently, this implies that the eigenvalues can be determined by the behavior of the roots of a particular order $2n+2$ polynomial.

LEMMA 2.1. *Let (r, v) an eigenpair for the matrix Q, and set*

$$P(y) = (ay^2 - d\tau y - e)y^{2n} + ey^2 + d\tau y - a. \tag{2.2}$$

If $P(y)$ has simple roots at $y = \pm 1$, then all of the eigenvalues and eigenvectors of Q are given by

$$r = \sqrt{ac}(y + y^{-1}) \quad \textit{and} \quad v_k = (\tau y)^k - \left(\frac{\tau}{y}\right)^k \tag{2.3}$$

where y is a root of $P(y)$, other than ± 1. If $P(y)$ has a repeated root (or roots) at $y = \epsilon$, for $\epsilon \in \{+1, -1\}$, then the eigenvalues and eigenvectors are given as above, with the addition of

$$r = \epsilon 2\sqrt{ac} \quad \textit{and} \quad v_k = k(\epsilon\tau)^{k-1}. \tag{2.4}$$

In addition, the set of roots of $P(y)$ is invariant under the transformation inv $: y \to y^{-1}$.

Proof. The equation $Qv = rv$ can be rewritten as

$$\begin{aligned} &\forall\, k \in \{1, \ldots, n\}\,, \begin{pmatrix} v_k \\ v_{k+1} \end{pmatrix} = C^k \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \\ &\text{and} \quad v_0 = 0 \quad \text{and} \quad \frac{a}{\tau^2} v_{n+1} - dv_n - ev_{n-1} = 0, \end{aligned} \tag{2.5}$$

where $C = \begin{pmatrix} 0 & 1 \\ -\tau^2 & r\tau^2/a \end{pmatrix}$.

Assume first that the eigenvalues of C are distinct, we will later show that this is equivalent to the condition that P has simple roots at $y = \pm 1$. Straightforward calculation gives the eigenvalues as $x_\pm = \frac{1}{2}\left(\frac{r\tau^2}{a} \pm \sqrt{\frac{r^2\tau^4}{a^2} - 4\tau^2}\right)$ with associated eigenvectors $\begin{pmatrix} 1 \\ x_+ \end{pmatrix}$ and $\begin{pmatrix} 1 \\ x_- \end{pmatrix}$. Expanding $\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = c_+ \begin{pmatrix} 1 \\ x_+ \end{pmatrix} + c_- \begin{pmatrix} 1 \\ x_- \end{pmatrix}$ and inserting into (2.5) gives $v_k = c_+ x_+^k + c_- x_-^k$. We may chose $c_+ = 1$ without loss of generality, the condition $v_0 = 0$ then implies $c_- = 1$, so that $v_k = x_+^k - x_-^k$. Now, set $y_\pm = \frac{1}{\tau} x_\pm$. The product $y_+ y_- = \frac{x_+ x_-}{\tau^2} = 1$, as $x_+ x_-$ is the product of the eigenvalues of C, equaling its determinant τ^2. This establishes $y_- = y_+^{-1}$. Now, denote y_+ by y. By above, $x_+ = \tau y$ and $x_- = \frac{\tau}{y}$, which establishes the second part of (2.3). The sum of the eigenvalues $x_+ + x_- = \text{trace}(C) = r\frac{\tau^2}{a}$, solving for r gives the first part of (2.3). Next, we cannot have $y = \pm 1$, as this would imply $y = 1/y$ which yields $x_+ = x_-$, whereas we have assumed the eigenvalues of C to be distinct. We must now show that y is a root of $P(y)$. Substituting the second part of (2.3) into the second boundary condition from (2.5) gives

$$\frac{a}{\tau^2}(\tau^{n+1}y^{n+1} - \tau^{n+1}y^{-(n+1)}) - d(\tau^n y^n - \tau^n y^{-n}) - e(\tau^{n-1}y^{n-1} - \tau^{n-1}y^{-(n-1)}) = 0, \tag{2.6}$$

multiplying by $\frac{y^{n+1}}{\tau^{n-1}}$ and reducing gives

$$(ay^2 - d\tau y - e)y^{2n} + ey^2 + d\tau y - a = 0 \tag{2.7}$$

which is equivalent to $P(y) = 0$.

We next consider when C has only a single eigenvalue. This occurs if $\frac{r^2\tau^4}{a^2} = 4\tau^2$, equivalent to $r = \epsilon\frac{2a}{\tau}$ for $\epsilon \in \{-1, +1\}$. In this case, $x_+ = x_- \equiv x = \epsilon\tau$. We compute $C\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix} + x\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, it follows that $C^k\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} kx^{k-1} \\ (k+1)x^k \end{pmatrix}$. Using $v_0 = 0$ and setting $v_1 = 1$ without loss of generality, this shows by (2.5) that $v_k = kx^{k-1}$. This establishes (2.4), if $r = \epsilon\frac{2a}{\tau}$ is an eigenvalue of Q. This will hold if the second boundary condition in 2.5 is satisfied, i.e. if $\frac{a}{\tau^2}(n+1)(\epsilon\tau)^n - dn(\epsilon\tau)^{n-1} - e(n-1)(\epsilon\tau)^{n-2} = 0$, which reduces to

$$n(a - d\tau\epsilon - e) + (a + e) = 0. \tag{2.8}$$

Finally, $P(y)$ will have a repeated root at $y = \epsilon$ if and only if $P'(\epsilon) = 0$. Straightforward calculation gives $P'(y) = (ay^2 - d\tau y - e)2ny^{2n-1} + (2ay - d\tau)y^{2n} + 2ey + d\tau$, so that $P'(\epsilon) = 2n\epsilon(a - d\tau\epsilon - e) + 2\epsilon(a + e)$.

Clearly $P'(\epsilon) = 0$ is equivalent to (2.8). Thus, if $P(y)$ has no repeated roots at $y = \pm 1$, C must have distinct eigenvalues for all r that are eigenvalues of Q, and so (2.3) holds for all such r. If $P(y)$ has repeated roots at either $y = 1$ or $y = -1$, or both, then the previous argument establishes that (2.4) holds for the one or two eigenvalues of Q given by the corresponding value of ϵ.

The last assertion follows as $y^{2n+2}P(y^{-1}) = -P(y)$, so $P(y) = 0$ iff $P(y^{-1}) = 0$. As $P(0) = -a \neq 0$, $y = 0$ is not a root. ∎

This lemma allow us in specific cases, namely when $P(y)$ factors, to obtain a simple explicit representation of the eigenvalues. Indeed in Yueh's paper [10] the emphasis is on these special cases. We give a number of examples that are commonly used in the literature. The remainder of the paper will then be devoted to obtain more general results. We note that all three examples are special cases of Theorem 4.2 part 2.

The first example is $c = a$ and $e = d = 0$. Here equation 2.7 factors as $y^{2n+2} - 1 = 0$. Thus after applying Lemma 2.1 we see that the eigenvalues for Q are given by $2a\cos(\frac{\pi k}{n+1})$ for $k \in \{1, \cdots n\}$. Our other two examples are of decentralized systems which are discussed in more detail in the Section 5. The first of these is $c = a$ and $e = 0$ and $d = a$. The polynomial equation now reduces to: $(y-1)(y^{2n+1}+1) = 0$. The roots at ± 1 must again be ignored and the eigenvalues of Q are $2a\cos(\frac{\pi(2k-1)}{2n+1})$ for $k \in \{1, \cdots n\}$. Finally we consider the case $c = a$ and $e = a$ and $d = 0$. The polynomial equation becomes: $(y^2 - 1)(y^{2n} + 1) = 0$. Eliminating ± 1 again, we get the eigenvalues $2a\cos(\frac{\pi(2k-1)}{2n})$ for $k \in \{1, \cdots n\}$.

3. The Roots of the Polynomial in Equation 2.2. We are interested in explicitly describing the roots of $P(y)$. As $P(y)$ always has the roots ± 1, and its roots are closed under inverses, we can often succinctly describe the root set by listing only n values, as below.

Definition 3.1. *The set of n numbers $\{y_1, y_2, ..., y_n\}$ is said to be root generating for $P(y)$ if all of the roots of $P(y)$ are given by y_i or y_i^{-1} for some i, or by ± 1.*

Proposition 3.2. *If $a + e = 0$ then we have the following root-generating set for $P(y)$:*

$$y_\ell = e^{\frac{\pi i \ell}{n}} \text{ for } \ell \in \{1, \cdots n-1\}\ , \quad y_n = \frac{1}{2a}\left(d\tau + sign(d\tau)\sqrt{d^2\tau^2 - 4a^2}\right)$$

Furthermore, if

1.	$2a < d\tau$	then	$y_n > 1$
2.	$-2a \leq d\tau \leq 2a$	then	$\lvert y_n\rvert = 1$
3.	$d\tau < -2a$	then	$y_n < -1$

Proof. Equation 2.7 factors to become:

$$(y^2 - \frac{d\tau}{a}y + 1)(y^{n-1} - y^{-(n+1)}) = 0$$

The roots at ± 1 can be discarded. The remaining roots are as stated in the Proposition. ∎

In fact, as we will see later, most of the roots of $P(y)$ lie on the unit circle. Looking for roots of the form $y = e^{i\phi}$ leads to the following :

Proposition 3.3. *$y = e^{i\phi}$ is a root of $P(y)$ iff ϕ is a solution of*

$$(e + a)\cos(n\phi)\sin(\phi) = (d\tau + (e - a)\cos(\phi))\sin(n\phi). \tag{3.1}$$

If $e^{i\phi} \neq \pm 1$, then $r = 2\sqrt{ac}\cos(\phi)$ is an eigenvalue of Q. If in addition $(e+a)\sin n\phi \neq 0$, then ϕ satisfies

$$\cot(n\phi)\sin(\phi) = \frac{d\tau}{e + a} + \frac{e - a}{e + a}\cos(\phi). \tag{3.2}$$

Proof. Multiplying both sides of $P(y) = 0$ by y^{-n-1} and rearranging gives the equivalent equation $a(y^{n+1} - y^{-(n+1)}) - d\tau(y^n - y^{-n}) + e(y^{-(n-1)} - y^{n-1}) = 0$. Substituting $y = e^{i\phi}$ and dividing by $2i$ gives $a\sin((n+1)\phi) - d\tau(\sin(n\phi)) + e\sin((n-1)\phi) = 0$, using the addition formula for sin and rearranging gives equation (3.1). If $(e + a)\sin(n\phi) \neq 0$, dividing by it gives equation (3.2). The statement about the eigenvalue of Q follows from Lemma 2.1. ∎

Definition 3.4. *The symbol ϕ_ℓ means a solution of Equation 3.2 in the interval $\left(\frac{(\ell-1)\pi}{n}, \frac{\ell\pi}{n}\right)$. The notation $y_\ell \approx L$ is reserved for: there is $\kappa > 1$ such that $y_\ell - L = O(\kappa^{-n})$ as n tends to ∞ (exponential convergence in n). We will furthermore denote the roots of $ay^2 - d\tau y - e$ as follows:*

$$y_\pm \equiv \frac{1}{2a}\left(d\tau \pm \sqrt{d^2\tau^2 + 4ae}\right) = \frac{1}{2\sqrt{ac}}\left(d \pm \sqrt{d^2 + 4ce}\right) \tag{3.3}$$

(We will choose the branch-cut for the root as the positive imaginary axis. So $\sqrt{x}$ will always have a non-negative real part.)

Proposition 3.5. *Let $-a < e \leq a$ and a, c, d, e fixed. Then, depending on the value of n, we have the following root generating set for $P(y)$:*

1.	$a - e + \frac{e+a}{n} < d\tau$	:	$y_\ell = e^{i\phi_\ell}$ for $\ell \in \{2, \cdots n\}$; $y_1 > 1, y_1 \in \mathbb{R}, y_1 \approx y_+$
2.	$-(a-e) - \frac{e+a}{n} \leq d\tau \leq a - e + \frac{e+a}{n}$	:	$y_\ell = e^{i\phi_\ell}$ for $\ell \in \{1, \cdots n\}$
3.	$d\tau < -(a-e) - \frac{e+a}{n}$	:	$y_\ell = e^{i\phi_\ell}$ for $\ell \in \{1, \cdots n-1\}$; $y_n < -1, y_n \in \mathbb{R}, y_n \approx y_-$

Proof. In each of the three cases we first look for roots of $P(y)$ of the form $y = e^{i\phi}$ for $\phi \in (0, \pi)$ using proposition 3.3, then find any remaining roots by other means. In Equation 3.1, $\sin n\phi = 0$ and $\sin\phi \neq 0$ does not give any solutions. We may thus investigate only the roots of Equation 3.2. See Figure 3.1. The left hand of that equation, $L(\phi) = \cot(n\phi)\sin(\phi)$, consists of n smooth decreasing branches on

$$\bigcup_{\ell=1}^{n} I_\ell \equiv \left[0, \frac{\pi}{n}\right) \cup \left(\frac{\pi}{n}, \frac{2\pi}{n}\right) \cdots \cup \left(\frac{(n-2)\pi}{n}, \frac{(n-1)\pi}{n}\right) \cup \left(\frac{(n-1)\pi}{n}, \pi\right]$$

whose ranges are $(-\infty, \frac{1}{n}]$ on I_1, $[-\frac{1}{n}, \infty)$ on I_n, and $(-\infty, \infty)$ in all other cases. The right hand side, $R(\phi) = \frac{d\tau}{e+a} + \frac{e-a}{e+a}\cos(\phi)$ is non-decreasing on $[0, \pi]$. Thus every interval I_ℓ has a root, except possibly the first and the last (see Figure 3.1).

We note that $L(0) = 1/n$, $L(\pi) = -1/n$, $R(0) = \frac{d\tau}{e+a} + \frac{e-a}{e+a}$, $R(\pi) = \frac{d\tau}{e+a} - \frac{e-a}{e+a}$. There will be a solution of Equation 3.2 in I_1 if $R(0) \leq L(0)$, which reduces to $d\tau \leq a - e + \frac{a+e}{n}$, similarly there will be a solution in I_n if $d\tau \geq -(a-e) - \frac{a+e}{n}$.

We now distinguish our three cases. In case 1, the condition for a root of Equation 3.2 in I_n is met, so we have the $n-1$ roots $y_\ell = e^{i\phi_\ell}$, for $2 \leq \ell \leq n$, of $P(y)$. By construction, none of these are equal to their reciprocals, which are also roots of $P(y)$. In addition ± 1 are roots, so we have identified $2n$ distinct roots, and must find two additional roots. These may be determined by observing that as $P(1) = 0$ and $\lim_{y\to\infty} P(y) = \infty$, if $P'(1) < 0$ there must be at least one real root of $P(y)$ in $(1, \infty)$. Calculating $P'(1) = 2n(a - e + \frac{a+e}{n} - d\tau)$ we see that $P'(1) < 0$ exactly under the condition of case 1. As the reciprocals of roots of $P(y)$ are also roots, and we have already identified the other $2n$ roots, there must be exactly one root in $(1, \infty)$, which we call y_1. We lastly observe that y_+ (from Definition 3.4) is real and greater than 1, by substituting $d\tau > a - e$ into Equation 3.3. It follows from Corollary 7.2 that $P(y)$ has a root exponentially close to y_+, this root must be y_1. We thus obtain n distinct roots of $P(y)$: $y_\ell = e^{i\phi_l}$ for $2 \leq \ell \leq n$ and $y_1 \approx y_+$. By construction none of these are equal to their reciprocals or to ± 1, so they must form a root generating set as $P(y)$ has at most $2n + 2$ distinct roots.

In case 2 the conditions are satisfied for Equation 3.2 to have solutions in I_1 and I_n. This yields n distinct roots of $P(y)$ of the form $y_\ell = e^{i\phi_\ell}$, which must form a root generating set.

In case 3 we have roots in all I_ℓ except in I_n. An almost identical argument to the one above now shows that there is a real root $y_n < -1$ which is exponentially close to y_-.
□

Proposition 3.6. *Let $e > a$ and a, c, d, e fixed. Then, depending on the value of n, have the*

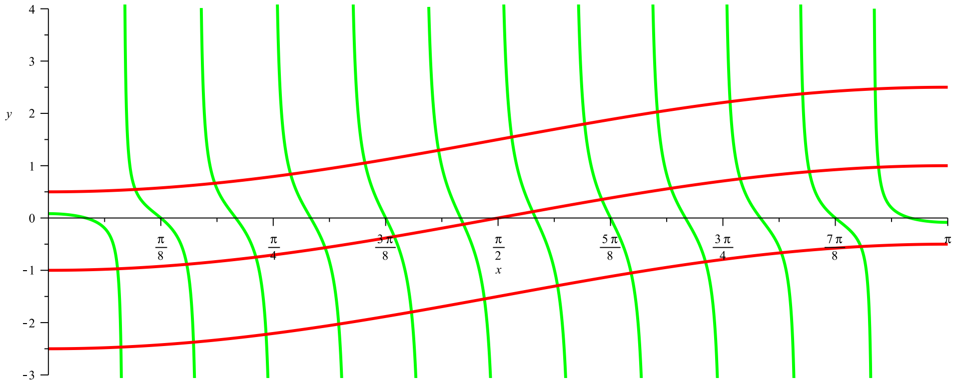

FIG. 3.1. The three cases of Proposition 3.5: The fast varying (green) plot is the left hand of Equation 3.2, and the slow varying (red) curves represent its right hand side in case 1, 2 , and 3 (from top to bottom).

following root generating set for $P(y)$:

1.	$-(a-e)-\frac{a+e}{n} \leq d\tau$ and $d\tau > a-e+\frac{a+e}{n}$	:	$y_\ell = e^{i\phi_\ell}$ for $\ell \in \{2, \cdots n\}$ and $y_1 \approx y_+ \geq \frac{e}{a}$
2.	$(a-e)+\frac{e+a}{n} < d\tau < -(a-e)-\frac{a+e}{n}$	:	$y_\ell = e^{i\phi_\ell}$ for $\ell \in \{2, \cdots n-1\}$ and $y_1 \approx y_+ \in [1, \frac{e}{a})$, $y_n \approx y_- \in (-\frac{e}{a}, -1]$
3.	$d\tau \leq (a-e)+\frac{a+e}{n}$ and $d\tau < -(a-e)-\frac{a+e}{n}$	:	$y_\ell = e^{i\phi_\ell}$ for $\ell \in \{1, \cdots n-1\}$ and $y_n \approx y_- \leq \frac{-e}{a}$
4.	$-(a-e)-\frac{a+e}{n} \leq d\tau \leq a-e+\frac{a+e}{n}$	:	$y_\ell = e^{i\phi_\ell}$ for $\ell \in \{1, \cdots n\}$

Proof. The proof is similar to that of the previous Proposition. It is again clear that in all these cases, the ranges of branches I_2 through I_{n-1} are all of $\mathbb{R}$ and so each of those branches must contain at least one solution.

Cases 1 and 3 can be resolved as in Proposition 3.5. The two conditions for case 1 exactly imply that Equation 3.2 has a solution in I_n, and that $P'(1) < 0$, implying the existence of a real root $y_1 \in (1, \infty)$. Together with the y_ℓ for $2 \leq \ell \leq n-2$ these produce a root generating set. Finally we observe that as

$$y_+ = \frac{d\tau + \sqrt{d^2\tau^2 + 4ae}}{2a} \geq \frac{e-a+e+a}{2a} \geq 1,$$

we may identify y_1 as the root of $P(y)$ close to y_+ implied by Corollary 7.2.

The two conditions for case 3 exactly imply that Equation 3.2 has a solution in I_1, and that $P'(-1) = 2n(-(a-e)-\frac{a+e}{n}-d\tau)$ is positive, which implies the existence of a real root $y_n \in (-\infty, -1)$, the remaining argument is similar to that for case 1.

In case 2, the conditions on the parameters imply both that $P'(1) < 0$ and that $P'(-1) > 0$, which implies the existence of real roots $y_1 > 1$ and $y_n < -1$ of $P(y)$. Together with y_ℓ for $2 \leq \ell \leq n-1$, these produce a root generating set. Finally, we verify the statements about the values of y_+ and y_-, which by definition are the roots of $f(y) \equiv ay^2 - d\tau y - e$. Simple calculation shows $f(1)$ and $f(-1)$ are negative, while $f(\frac{e}{a})$ and $f(-\frac{e}{a})$ are negative, implying $y_- \in (-\frac{e}{a}, -1)$ and $y_+ \in (1, \frac{e}{a})$.

Finally, in case 4 the conditions on the parameters imply that there are solutions to equation 3.2 on both I_1 and I_n, which as in the previous proposition produces a root generating set.
∎

We note that case 4 of the previous proposition cannot occur for large n (as $a-e < -(a-e)$). In particular, it is only possible for $n < \frac{e+a}{e-a}$.

In the next proposition, we must handle one of the parameter cases only for large n. The reasons for this are related to the possibility of multiple solutions of Equation 3.2 occurring within a single interval I_ℓ, as illustrated later in Figure 3.3.

PROPOSITION 3.7. *Let $e < -a$ and a, c, d, e fixed. Then, depending on the value of n we have the*

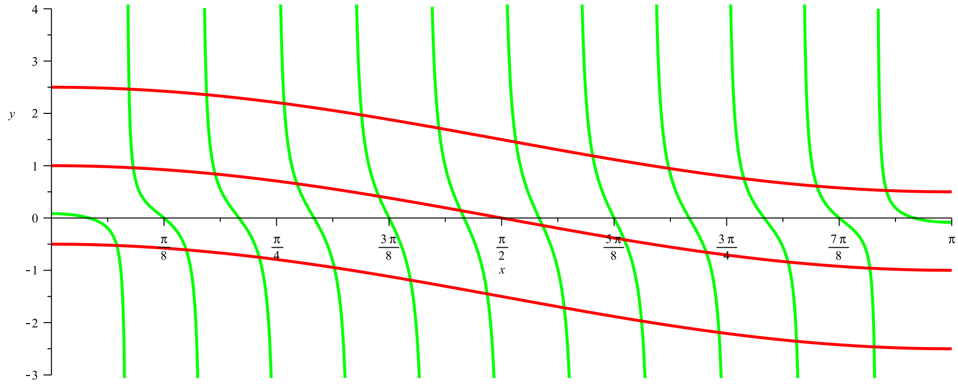

FIG. 3.2. The three cases of Proposition 3.6 (1, 2, 3 from top to bottom). See caption of Figure 3.1.

following root generating set for $P(y)$:

1. $d\tau < -(a-e) - \frac{a+e}{n}$ then $y_\ell = e^{i\phi_\ell}$ for $\ell \in \{2, \cdots n\}$ and $y_1 \approx y_- \leq \frac{e}{a}$
2. $(a-e) + \frac{a+e}{n} < d\tau$ then $y_\ell = e^{i\phi_\ell}$ for $\ell \in \{1, \cdots n-1\}$ and $y_n \approx y_+ \geq -\frac{e}{a}$

For n large enough, we have the following root generating set for $P(y)$:

3. $-(a-e) < d\tau < a-e$ then $y_\ell = e^{i\phi_\ell}$ for $\ell \in \{2, \cdots n-1\}$ and $|y_1|, |y_n| > 1$

Case 3 can be further subdivided as follows:

$$\begin{array}{lll}
3a. & -(a-e) < d\tau \leq -2\sqrt{|ae|} & : \; y_1 \approx \tilde{y}_1, y_n \approx \tilde{y}_n,\ \tilde{y}_1, \tilde{y}_n \in \left[\frac{e}{a}, -1\right] \\
3b. & -2\sqrt{|ae|} < d\tau < 2\sqrt{|ae|} & : \; y_1, y_n \text{ not real, } |y_1| \approx -\frac{e}{a} \approx |y_n| \\
3c. & 2\sqrt{|ae|} \leq d\tau < (a-e) & : \; y_1 \approx \tilde{y}_1, y_n \approx \tilde{y}_n,\ \tilde{y}_1, \tilde{y}_n \in \left[1, -\frac{e}{a}\right]
\end{array}$$

Proof. As before one easily sees that on each of the smooth branches I_2 through I_{n-1} the left hand of Equation 3.2 has range $\mathbb{R}$, yielding $n-2$ solutions with angles in $[0, \pi)$.

There will be a solution of Equation 3.2 in I_n if $R(\pi) \geq L(\pi)$, which (noting that $a+e < 0$) is equivalent to $d\tau \leq e - a - \frac{a+e}{n}$. This is ensured in case 1. The condition for case 1 also ensures that $P'(-1) > 0$, which as before implies the existence of a real root $y_1 \in (-\infty, -1)$ of $P(y)$. Together with the y_ℓ for $2 \leq \ell \leq n$, these form a root generating set. The assertion for case 2 is proved similarly.

For case 3, we are only guaranteed the existence of the $n-2$ solutions y_ℓ for $2 \leq \ell \leq n-1$, and must find two additional (reciprocal pairs of) roots of $P(y)$ that are not on the unit circle. Setting $f(y) \equiv ay^2 - d\tau y - e$ it is straightforward to verify the following table of values of f for the three cases mentioned above.

		$y = \frac{e}{a}$		$y=-1$	$y=0$	$y=1$		$y = \frac{-e}{a}$
1.	$d\tau \leq -(a-e)$	≤ 0		≤ 0	> 0	> 0		> 0
3a.	$-(a-e) < d\tau < -2\sqrt{\lvert ae\rvert}$	> 0	$f\left(\frac{d\tau}{2a}\right) \leq 0$	> 0	> 0	> 0		> 0
3b.	$-2\sqrt{\lvert ae\rvert} < d\tau \leq 2\sqrt{\lvert ae\rvert}$	> 0		> 0	> 0	> 0		> 0
3c.	$2\sqrt{\lvert ae\rvert} \leq d\tau < (a-e)$	> 0		> 0	> 0	> 0	$f\left(\frac{d\tau}{2a}\right) \leq 0$	> 0
2.	$d\tau \geq (a-e)$	> 0		> 0	> 0	≤ 0		≤ 0

Note that for $|y|$ large we have that f is always positive. Taking that into account we see from this table that in case 1, f has one real root less than or equal to $\frac{e}{a}$. Similarly in case 2, there is root greater than or equal to $\frac{-e}{a}$. In cases 3a and 3c there are two real valued solutions with absolute value greater than 1. Finally in case 3b the roots of f are complex conjugates with product $-e/a$, which by hypothesis is greater than 1. So also here f has two roots with absolute value greater than 1.

By Corollary 7.2 each of the larger than unity roots of f is approximated exponentially (in n) by a root of Equation 2.7. Concluding the proof for case 3, for sufficiently large n there will be two roots of $P(y)$ that are close to these roots of f, and thus lie outside of the unit circle. These complete the root generating set. ∎

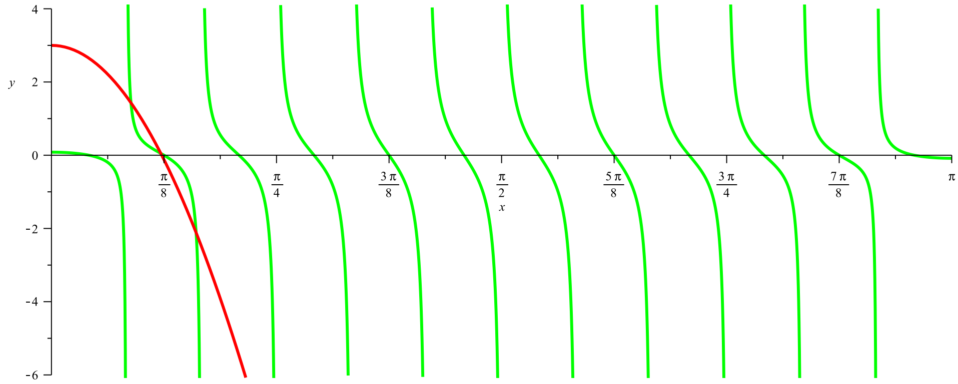

FIG. 3.3. Detail of case 2b of Proposition 3.7: $a = 1$, $e = -1.05$, and $d\tau = -1.9$. Here $n = 12$. The multiple roots in the second branch disappear for large n (while holding other parameters fixed).

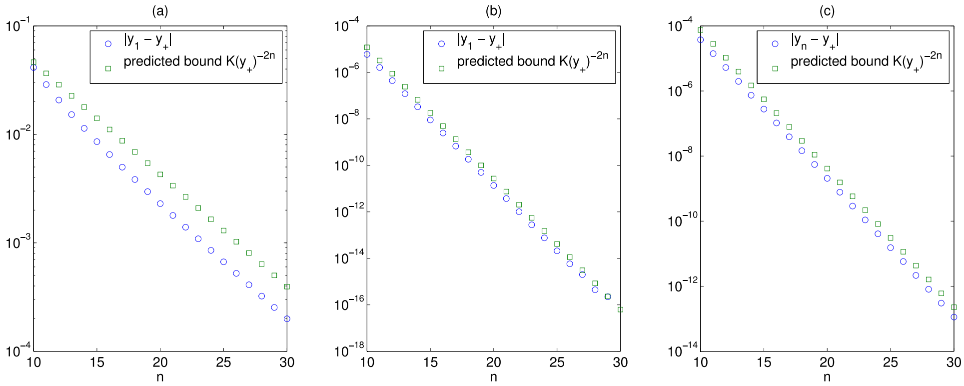

FIG. 3.4. *Exponential convergence for the approximated roots of $P(y)$. Parameters (a) $a = 7$, $c = 3$, $d = 4$, $e = 2$ (case 3.5-1) (b) $a = 4$, $c = 4$, $d = 4$, $e = 7$ (case 3.6-1) (c) $a = 6$, $c = 1$,$d = 6$, $e = -8$ (case 3.7-2)*

It perhaps worth pointing out that the above argument obtains $2n$ solutions of Equation 2.2. Together with the trivial roots ± 1 we therefore found all roots. It follows that even in this case 3 *provided n is large enough* Equation 3.2 has at most one solution in each interval I_ℓ. This however is *not true* for arbitrary n. See for instance Figure 3.3 where one can see 3 solutions in I_2.

Finally, in Figure 3.4 we explicitly illustrate the exponential convergence of the approximated roots of $P(y)$ for a few cases. These three figures were generated for parameter values corresponding to the cases in Propositions 3.5-1, 3.6-1 and 3.7-3 respectively, where there is a single approximate root (y_1 for (a),(b); y_n for (c)). For each value of n we computed y_1 (or y_n) numerically by a root bracketing algorithm. In figures (a), (b) we plot the exact value of $|y_1 - y_+|$ vs n, in figure (c) we plot the exact value of $|y_n - y_+|$. For comparison, we also plot values of the error bound $K(y_+)^{-2n}$, where $K = 2\frac{2|q(y_+)|}{|p'(y_+)|}$ as described in Corollary 7.2. Exponential convergence can be clearly seen, and the predicted error bounds show the observed convergence rate.

4. The Spectra. In this section we apply Equation 2.3 of Lemma 2.1 to the propositions of the previous section to obtain the spectrum of the $n+1$ by $n+1$ matrix A of Section 1. This gives us our main results. About the associated eigenvectors we remark here that those can be obtained using the same lemma. Note that wherever there is a double root in the polynomial equation 2.7 we obtain only one eigenvector. A generalized eigenvector (associated with a Jordan normal block of dimension 2 or higher) can be derived (see Yueh [10] for some examples). Since we are mainly interested in the spectrum we will not pursue this here.

DEFINITION 4.1. In this section we will denote, for $e \neq 0$,

$$r_\pm \equiv \frac{1}{2}\left[\left(1 - \frac{a}{e}\right) d \pm \left(1 + \frac{a}{e}\right)\sqrt{d^2 + 4ce}\right]. \tag{4.1}$$

When $e = 0$, by taking limits as $e \to 0$ we define[1]

$$r_- = d + \frac{ac}{d} \quad \text{if } d < 0$$
$$r_+ = d + \frac{ac}{d} \quad \text{if } d > 0.$$

In this section the symbol ψ_ℓ means a solution of Equation 3.2 in the interval $\left[\frac{(\ell-1)\pi}{n}, \frac{\ell\pi}{n}\right)$ (cf Definition 3.4).

THEOREM 4.2. *Let $-a \le e \le a$ and a, c, d, e fixed. Then for n large enough the $n+1$ eigenvalues $\{r_i\}_{i=0}^n$ of the matrix A are the following. First, $r_0 = b$. The other n eigenvalues are:*

1.	$(a-e)\sqrt{\frac{c}{a}} < d$	:	$r_\ell = 2\sqrt{ac}\cos\psi_\ell$, $\ell \in \{2, \cdots n\}$, $r_1 \approx r_+ > 2\sqrt{ac}$
2.	$-(a-e)\sqrt{\frac{c}{a}} \le d \le (a-e)\sqrt{\frac{c}{a}}$	:	$r_\ell = 2\sqrt{ac}\cos\psi_\ell$, $\ell \in \{1, \cdots n\}$
3.	$d < -(a-e)\sqrt{\frac{c}{a}}$	:	$r_\ell = 2\sqrt{ac}\cos\psi_\ell$, $\ell \in \{1, \cdots n-1\}$, $r_n \approx r_- < -2\sqrt{ac}$

When $a = -e$ then $\psi_\ell = (\ell-1)\pi/n$, otherwise $\psi_\ell \in ((\ell-1)\frac{\pi}{n}, \ell\frac{\pi}{n})$ (except possibly for $\ell = 1$ and $\ell = n$).

Proof. This follows from applying Proposition 2.1 to Propositions 3.2 and 3.5. We have substituted a/c for τ^2. r_+ and r_- are obtained by simplifying the expressions for $\sqrt{ac}(y_+ + y_+^{-1})$ and $\sqrt{ac}(y_- + y_-^{-1})$, respectively. ∎

The next two results follow in the same manner from Propositions 3.6 and 3.7. We omit the proofs since they are easy.

THEOREM 4.3. *Let $e > a$ and a, c, d, e fixed. Then for n large enough the $n+1$ eigenvalues $\{r_i\}_{i=0}^n$ of the matrix A are the following. First, $r_0 = b$. The other n eigenvalues are:*

1.	$-(a-e)\sqrt{\frac{c}{a}} \le d$	:	$r_\ell = 2\sqrt{ac}\cos\psi_\ell$, $\ell \in \{2, \cdots n\}$, $r_1 \approx r_+$
2.	$(a-e)\sqrt{\frac{c}{a}} < d < -(a-e)\sqrt{\frac{c}{a}}$	:	$r_\ell = 2\sqrt{ac}\cos\psi_\ell$, $\ell \in \{2, \cdots n-1\}$, $r_1 \approx r_+, r_n \approx r_-$
3.	$d \le (a-e)\sqrt{\frac{c}{a}}$	:	$r_\ell = 2\sqrt{ac}\cos\psi_\ell$, $\ell \in \{1, \cdots n-1\}$, $r_n \approx r_-$

Furthermore we also have that in these cases:

1. $r_+ \ge \sqrt{ac}\left(\frac{e}{a} + \frac{a}{e}\right)$
2. $r_- \in \left(-\sqrt{ac}\left(\frac{e}{a} + \frac{a}{e}\right), -2\sqrt{ac}\right]$ and $r_+ \in \left[2\sqrt{ac}, \sqrt{ac}\left(\frac{e}{a} + \frac{a}{e}\right)\right)$
3. $r_- \le -\sqrt{ac}\left(\frac{e}{a} + \frac{a}{e}\right)$

THEOREM 4.4. *Let $e < -a$ and a, c, d, e fixed. Then for n large enough the $n+1$ eigenvalues $\{r_i\}_{i=0}^n$ of the matrix A are the following. First, $r_0 = b$. The other n eigenvalues are:*

1.	$d \le -(a-e)\sqrt{\frac{c}{a}}$	:	$r_\ell = 2\sqrt{ac}\cos\psi_\ell$, $\ell \in \{2, \cdots n\}$, $r_1 \approx r_-$
2.	$-(a-e)\sqrt{\frac{c}{a}} < d < (a-e)\sqrt{\frac{c}{a}}$	:	$r_\ell = 2\sqrt{ac}\cos\psi_\ell$, $\ell \in \{2, \cdots n-1\}$, $r_1 \approx r_+, r_n \approx r_-$
3.	$(a-e)\sqrt{\frac{c}{a}} \le d$	:	$r_\ell = 2\sqrt{ac}\cos\psi_\ell$, $\ell \in \{1, \cdots n-1\}$, $r_n \approx r_+$

Furthermore we also have that in these cases:

1.	$d \le -(a-e)\sqrt{\frac{c}{a}}$	:	$r_- \le \sqrt{ac}\left(\frac{e}{a} + \frac{a}{e}\right)$
2*a*.	$-(a-e)\sqrt{\frac{c}{a}} < d \le -2\sqrt{\lvert ce\rvert}$	:	$r_-, r_+ \in \left(\sqrt{ac}\left(\frac{e}{a} + \frac{a}{e}\right), -2\sqrt{ac}\right]$
2*b*.	$-2\sqrt{\lvert ce\rvert} < d < 2\sqrt{\lvert ce\rvert}$	:	r_-, r_+ not real
2*c*.	$2\sqrt{\lvert ce\rvert} \le d < (a-e)\sqrt{\frac{c}{a}}$	:	$r_-, r_+ \in \left[2\sqrt{ac}, -\sqrt{ac}\left(\frac{e}{a} + \frac{a}{e}\right)\right)$
3.	$(a-e)\sqrt{\frac{c}{a}} \le d$	:	$r_+ \ge -\sqrt{ac}\left(\frac{e}{a} + \frac{a}{e}\right)$

[1]Note that r_+ (respectively r_-) is not defined if $e = 0$ and $d < 0$ (respectively $d > 0$); careful examination of the cases corresponding to these parameter values shows that the undefined symbols are not used.

As an illustration of these ideas we plot the solutions of Equation 2.7 and the spectrum of A in the case where $a = c = 1$, $e = -9/4$. We take $d \in \{2.95, 3.05, 3.3\}$ so that (since $\tau = 1$) we are in case 2b, 2c, and 3 respectively of Proposition 3.7 and Theorem 4.4. The results can be found in Figure 4.1. These results are numerical: the first three figures were obtained with the MAPLE "fsolve" routine, the last three were obtained from the former by applying Equation 2.3 to the roots to get the eigenvalues. We took $n = 100$. The eigenvalue b of the matrix A is not displayed.

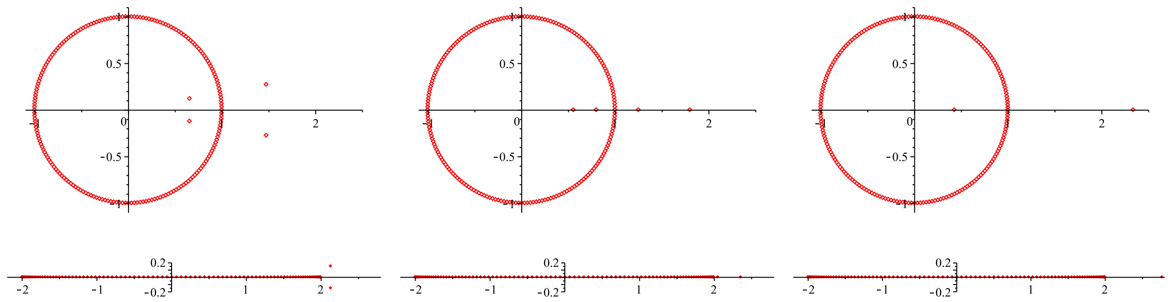

FIG. 4.1. Numerical illustration of the roots of $P(y)$ (top) and the eigenvalues (bottom), for three different parameter values (given in text)

5. The Decentralized Case. We look at the decentralized case defined in Definition 1.1.

LEMMA 5.1. *In the decentralized case the eigenvalues $r_\pm$ of Definition 4.1 become*

$$r_+ = a + c \quad \text{and} \quad r_- = -\left(\frac{ac}{e} + e\right), \qquad \text{if } c + e > 0$$
$$r_+ = -\left(\frac{ac}{e} + e\right) \quad \text{and} \quad r_- = a + c, \qquad \text{if } c + e < 0$$

Proof. Substituting $d = c - e$ into (4.1) gives $r_\pm = \frac{1}{2}\left[(1 - \frac{a}{c})(c - e) \pm (1 + \frac{a}{e})\sqrt{(c+e)^2}\right]$, using $\sqrt{(c+e)^2} = |c + e|$ and simplifying gives the result. ∎

THEOREM 5.2. *Let a, c, d, e fixed so that A is decentralized. Then for n large enough the $n+1$ eigenvalues $\{r_i\}_{i=0}^n$ of the matrix A are the following. First, $r_0 = a + c$. $n - 2$ eigenvalues are $r_\ell = 2\sqrt{ac}\cos\psi_\ell$ for $\ell \in \{2, \cdots n-1\}$ where the ψ_ℓ are solutions of Equation 3.2. The remaining two eigenvalues ($\ell = 1$ and $\ell = n$) either also satisfy that formula or else are exponentially close (in n) to the ones given in the table below. In the table we list the domain of e left of the colon, and the appropriate special eigenvalues (if any) right of the colon.*

$---$	$0 < c < a$	$c = a$	$a < c$
$e < -a$	$(-\infty, -a):\ -\left(\frac{ac}{e} + e\right)$	$(-\infty, -a):\ -a\left(\frac{a}{e} + \frac{e}{a}\right)$	$(-\infty, -\sqrt{ac}):\ \left\{a + c, -\left(\frac{ac}{e} + e\right)\right\}$ $[-\sqrt{ac}, -a):\ a + c$
$\lvert e\rvert \le a$	$[-a, -\sqrt{ac}):\ -(\frac{ac}{e} + e)$ $[-\sqrt{ac}, \sqrt{ac}]:\ \varnothing$ $(\sqrt{ac}, a]:\ -(\frac{ac}{e} + e)$	$[-a, a]:\ \varnothing$	$[-a, a]:\ a + c$
$a < e$	$(a, \infty):\ -(\frac{ac}{e} + e)$	$(a, \infty):\ \left\{-a\left(\frac{a}{e} + \frac{e}{a}\right), 2a\right\}$	$(a, \sqrt{ac}]:\ a + c$ $(\sqrt{ac}, \infty):\ \left\{-\left(\frac{ac}{e} + e\right), a + c\right\}$

Proof. We need to check in each case of the above table, which case of the appropriate Theorem in Section 4 applies. The result of that process is given in the table below. Each entry is a list of domains for e to the left of the colon together with the subcases of the relevant Theorem (to the right of the colon). Once we know which case applies, we list the appropriate special eigenvalues (if any) specified by those Theorems and Lemma 5.1 and that gives the table in the Theorem.

$---$	$0<c<a$	$c=a$	$a<c$
$e<-a$; Theorem 4.4	$(-\infty,-a):$ 4.4-3	$(-\infty,-a):$ 4.4-3	$(-\infty,-\sqrt{ac}):$ 4.4-2c $[-\sqrt{ac},-a):$ 4.4-3
$\|e\|\leq a$; Theorem 4.2	$[-a,-\sqrt{ac}):$ 4.2-1 $[-\sqrt{ac},\sqrt{ac}]:$ 4.2-2 $(\sqrt{ac},a]:$ 4.2-3	$[-a,a]:$ 4.2-2	$[-a,a]:$ 4.2-1
$a<e$; Theorem 4.3	$(a,\infty):$ 4.3-3	$(a,\infty):$ 4.3-2	$[a,\sqrt{ac}]:$ 4.3-1 $(\sqrt{ac},\infty):$ 4.3-2

The verification of the table in this proof is a tedious process. We will outline how to do that when $e<-a$. For other values of e the process is very similar.

Since A is decentralized we have (Definition 1.1) $d=c-e$. Recall that a and c are positive. So:

$$a>0, \quad c>0, \quad e<-a<0, \quad d=c-e$$

Expanding $0\leq(c+e)^2$ and subtracting $4ce$ gives $-4ce\leq c^2-2ce+c^2=(c-e)^2$. As e is negative, taking square roots implies $2\sqrt{c|e|}\leq c-e=d$. This implies that conditions for Theorem 4.4 cases 4.4-2c or 4.4-3 must hold. Case 4.4-3 applies if $(a-e)\sqrt{\frac{c}{a}}\leq c-e$, using $c=\sqrt{ac}\sqrt{\frac{c}{a}}$ and $a\sqrt{\frac{c}{a}}=\sqrt{ac}$, this condition is equivalent to

$$-\sqrt{ac}\left(\sqrt{\frac{c}{a}}-1\right)\leq e\left(\sqrt{\frac{c}{a}}-1\right). \tag{5.1}$$

If $c<a$ (corresponding to the first column of the above tables), $\sqrt{\frac{c}{a}}-1<0$ so (5.1) holds if $-\sqrt{ac}\geq e$, so case 4.4-3 holds for all $e\leq-\sqrt{ac}$. However, the appropriate set for e is actually smaller as we already have restricted e to $e<-a$, and $-a<-\sqrt{ac}$ (as $c<a$). So case 4.4-3 holds for $e\in(-\infty,-a)$.

If $c=a$ (corresponding to the second column of the above tables), (5.1) reduces to $0\leq 0$ which is true, implying that case 4.4-3 holds for all $e\in(-\infty,-a)$.

If $c>a$ (corresponding to the third column of the above tables), $\sqrt{\frac{c}{a}}-1>0$ so (5.1) holds if $-\sqrt{ac}\leq e$. This implies that case 4.4-3 holds for $e\in(-\sqrt{ac},-a)$, (which is not vacuous as $\sqrt{ac}<a$ for $a<c$), otherwise case 4.4-2c holds for $e\in(-\infty,-\sqrt{ac})$.

When translating from the table of conditions to the results for the "special eigenvalues", note that the case 4.4-3 produces the eigenvalue r_+, whose expression depends on the sign of $c+e$. It is straightforward to show that $c+e$ must be negative if $e<-a$ and $c\leq a$ (accounting for the first two columns of the above tables), and that $c+e$ is positive if $a<c$ but $e\in[-\sqrt{ac},-a)$ (accounting for the second case in the third column of the tables above). Note that the sign of $c+e$ is not determined for $e\in(-\infty,-\sqrt{ac})$, but this does not affect the eigenvalues in this case as case 4.4-2c produces both r_+ and r_- as special eigenvalues.

This finishes the classification of the decentralized spectra when $e<-a$. The strategy when $e\geq-a$ is the same. ◻

A special case of this, namely $\rho\in(0,1)$ and $a=1-\rho$, $c=\rho$, $e=\rho$ and $d=0$, was proved in [3].

6. Applications to Decentralized Systems of Differential Equations. In this section we will take up the asymptotic stability of the consensus (first order) and flocking (second order) given in Equations 1.1 and 1.2. In particular we prove that for both of these systems asymptotic stability does not depend on boundary conditions if these are 'reasonable' (in this case $|e|<a$).

The first order system has an eigenvalue 0, and the second order system has an eigenvalue 0 of multiplicity at least 2. These eigenvalues are associated with the solutions given in Equation 1.4. The systems are called asymptotically stable if those eigenvalues have multiplicities exactly 1 and 2, respectively, and if all other eigenvalues have negative real part.

Corollary 6.1. *Let a, c, d, e fixed, and so that A is decentralized. Then for n large enough the system in Equation 1.1 is asymptotically stable if $a+e>0$, and asymptotically unstable if $a+e<0$ and $c+e\neq 0$.*

Proof. The $n+1$ eigenvalues of $-L=A-(a+c)I$ associated with this systems are obtained by subtracting $a+c$ from those given by Theorem 5.2. One eigenvalue equals 0. Most other eigenvalues are given by $2\sqrt{ac}\cos\psi_\ell - (a+c) < 0$. There are at most 2 eigenvalues left and they are exponentially close to 0 or to $-\frac{ac}{e} - e - a - c = -\frac{(a+e)(c+e)}{e}$.

When $a+e<0$ and $c+e \neq 0$ then we must have $e < -a$, and so are among the cases in the top row of the table in Theorem 5.2. If $c+e<0$, then the (approximate) eigenvalue $-\frac{(a+e)(c+e)}{e}$ (which always appears in these cases) is greater than 0, implying asymptotic instability. If $c+e>0$, then the parameters satisfy $a < |e| < c$, so we are in the cases (top right of table in theorem 5.2) where $a+c$ is an approximate eigenvalue of A. Then as $a+e<0$, Corollary 7.4 implies that the actual eigenvalue of A that $a+c$ approximates is greater than $a+c$, implying that there is a positive eigenvalue for L.

When $a+e>0$ the approximate eigenvalue $-\frac{(a+e)(c+e)}{e}$ (if it occurs) is less than zero. This can be seen by noting that $-\frac{(a+e)(c+e)}{e} > 0$ only if $c+e$ and e have opposite signs, i.e. if $e<0$ and $|e|<c$. Looking at the table in Theorem 5.2, this is only possible for cases in the middle row. The only such case where $-\frac{(a+e)(c+e)}{e}$ is implied as an eigenvalue of L is when $e \in [-a, -\sqrt{ac}]$ and $c<a$, but we cannot have $e \in [-a, -\sqrt{ac}]$ as $|e| < c < \sqrt{ac}$. The only other potential cause of instability is the eigenvalue of L that is asymptotically equal to 0. However if $a+e>0$, Corollary 7.4 implies that then this eigenvalue is actually slightly less than 0. $\square$

A special case of the following result was first proved in [6]

Corollary 6.2. *Let a, c, d, e fixed, and so that A is decentralized. If both α and β are positive, then for n large enough the system of Equation 1.2 is asymptotically stable if $a+e>0$ and asymptotically unstable if $a+e<0$ and $c+e\neq 0$. If either α or β are negative, the system is asymptotically unstable.*

Proof. The eigenvalue equation for second order system can be written as follows:

$$\begin{pmatrix} \dot z \\ \ddot z \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\alpha L & -\beta L \end{pmatrix} \begin{pmatrix} z \\ \dot z \end{pmatrix} = \begin{pmatrix} \nu z \\ \nu \dot z \end{pmatrix}$$

The second equality yields 2 equations. The first of these is that $\dot z = \nu z$, the second is $\alpha(-Lz)+\beta(-L\dot z) = \nu \dot z$. Suppose λ is an eigenvalue of $-L$. Each eigenvalue λ gives rise to two eigenvalues $\nu_\pm$. To see this, let z be the corresponding eigenvector with $-Lz = \lambda z$, substituting it and $\dot z = \nu z$ into the second of the above equations gives $\alpha\lambda z + \beta\nu\lambda z = \nu^2 z$. Rearranging shows

$$(\nu^2 - \beta\lambda\nu - \alpha\lambda)z = 0 \quad \Rightarrow \quad \nu_\pm = \frac{1}{2}\left(\beta\lambda \pm \sqrt{\beta^2\lambda^2 + 4\alpha\lambda}\right)$$

Corollary 6.1 says that in all cases there are negative λ. Thus both α and β must be positive for the system to be asymptotically stable, otherwise at least one of $\nu_\pm$ will be positive. If α and β are both positive, by inspection we have stability precisely in those cases when λ is always negative (or zero with multiplicity 1), i.e. exactly where Corollary 6.1 insures stability for the first order system. $\square$

A few observations are in order here. Asymptotic stability is not the whole story. In fact as n becomes large, even for asymptotically stable systems the transients in Equations 1.1 and 1.2 may grow exponentially in n. This is due to the fact the eigenvectors are not normal and a dramatic example of this was given in [8]. Here we can see it expressed in the form of the eigenvectors given by Equation 2.3 of Lemma 2.1: if $\tau \neq 1$ the eigenvectors have an exponential behavior. When τ is small a long time will pass before a change in the velocity of the leader is felt at the back of the flock, and so coherence will be lost. On the other hand when τ is large a change will immediately amplify exponentially towards the back of the flock. These observations have been proved for $e = 1/2$ and $d = 0$ by [6]. But to address that problem in more generality, different concepts are needed. In [2], assuming some conjectures, we show that this phenomenon indeed appears to be independent of the boundary conditions (e and d). In that paper we also show that the behavior of the system 1.2 can be substantially improved if the position Laplacian and the velocity Laplacian are allowed to be different.

7. Appendix 1.

The results in this paper are based on the roots of the polynomial $P(z) = f(z)z^{2n} + g(z)$, where we introduce $f(z) = zy^2 - d\tau z - e$ and $g(z) = ez^2 + d\tau z - a$. The following results show that if $f(z)$ has an isolated root with magnitude larger than 1, then $P(z)$ must have a nearby root, where we consider possibly complex roots. Below, we denote the disc of radius ϵ by $D_\epsilon(\nu) \equiv \{z \in \mathbb{C} : |z-\nu| \leq \epsilon$.

Lemma 7.1. *Let $f(z)$ have an isolated root at $z=\nu$ with $|\nu|>1$, and suppose ϵ is such that $(|\lambda|-\epsilon)^{2n} > 2\frac{M}{A}$, where $M = \max_{z\in\partial D_\epsilon(\nu)} |g(z)|$ and $A = \min_{z\in\partial D_\epsilon(\nu)} |f(z)|$. In addition, assume ϵ is* *sufficiently small so that $|\nu| - \epsilon > 1$ and $f(z)$ has only one root in $D_\epsilon(\nu)$. Then $P(y)$ has exactly one root in $D_\epsilon(\nu)$.*

Proof. Set $Q(z) = z^{2n} f(z)$. Clearly, $Q(z)$ has exactly one root in $D_\epsilon(\nu)$. Then $P(z) = Q(z) + g(z)$ satisfies, for $z \in \partial D_\epsilon(\nu)$,

$$|Q(z) + g(z)| \geq |Q(z)| - |g(z)| \tag{7.1}$$

$$\geq (|\nu| - \epsilon)^{2n} A - M \tag{7.2}$$

$$> 2\frac{M}{A} A - M = M \tag{7.3}$$

This implies that $|-g(z)| < |P(z)|$ for $z \in \partial D_\epsilon(\nu)$. By Rouché's theorem (e.g. [1]), $P(z)$ and $P(z) - g(z) = Q(z)$ have the same number of zeros inside of $D_\epsilon(\nu)$, namely one. □

From this result we can demonstrate exponential convergence of the root of $P(z)$ to ν. In particular

Corollary 7.2. *Let $f(z)$ satisfy the conditions of Lemma 7.1, and let K be any constant satisfying $K > \frac{|g(\nu)|}{|f'(\nu)|}$. Then for n sufficiently large, $P(z)$ has a root z_1 satisfying $|z_1 - \nu| \leq 2K|\nu|^{-2n}$.*

Proof. Write $K = M/A$ where $M > |g(\nu)|$ and $A < |f'(\nu)|$. As $g(z)$ is continuous and $f(z)$ is differentiable with an isolated root at ν, for small enough ϵ (ensured by taking n large enough) we will have $|g(z)| \leq M$ and $|f(z)| > \epsilon A$ on $\partial D_\epsilon(\nu)$. Using these bounds on $f(z)$ and $g(z)$, the conclusion of the previous lemma holds, provided that $(|\nu| - \epsilon)^{2n} > 2\frac{M}{\epsilon A}$, which is equivalent to $\epsilon > 2K(|\nu| - \epsilon)^{-2n}$. This inequality will hold for $\epsilon = 2K|\nu|^{-2n}$. □

We remark that as n becomes large, we may take K approaching $\frac{|g(\nu)|}{|f'(\nu)|}$, and would expect the error $|z_1 - \nu|$ to be asymptotically close to $2\left|\frac{g(\nu)}{f'(\nu)}\right| |\nu|^{-2n}$. This is illustrated numerically in Figure 3.4.

In certain cases we will need to know whether z_1 is greater or less than ν. This is established by

Lemma 7.3. *Let $f(z) = az^2 - d\tau z - e$ and $g(z) = -z^2 f(z^{-1}) = ez^2 + d\tau z - a$. Suppose $f(z)$ has two real roots μ and ν such that $|\nu| > 1$ and $|\nu| > |\mu|$. Then for n large enough $P(y) = y^{2n} f(y) + g(y)$ has a real root z_1 with $z_1 \approx \nu$ and $\operatorname{sgn}(z_1 - \nu) = -\operatorname{sgn}\left((a+e)\nu^{1-2n}\right)$.*

Proof. We compute $P(\nu) = g(\nu) = -\nu^2 f(\nu^{-1})$ and $P'(\nu) = f'(\nu)\nu^{2n} + g'(\nu)$. As ν is an isolated root, $f'(\nu) \neq 0$, $|P'(z)|$ may be made arbitrarily large, in some neighborhood of ν, for n sufficiently large. Writing $P(\nu + a) = P(\nu) + \int_0^a P'(\nu + t)dt$, we see there must be a root of P in $(\nu, \nu + a)$, for sufficiently large n, if $P(\nu)$ and $aP'(\nu)$ have opposite signs. This implies

$$\operatorname{sgn}(z_1 - \nu) = \operatorname{sgn}\left(\frac{-P(\nu)}{P'(\nu)}\right) - \operatorname{sgn}\left(\frac{\nu^2 f(\nu^{-1})}{f'(\nu)\nu^{2n}}\right)$$

where in the denominator we have retained only the leading term. A short calculation that uses $\mu\nu = -\frac{e}{a}$, $a > 0$, and $\operatorname{sgn}(\nu - \mu) = \operatorname{sgn}(\nu)$, yields the second statement of the Corollary. □

Corollary 7.4. *Under the conditions of Lemma 7.3, the expressions $r_0 \equiv \sqrt{ac}(\nu + \nu^{-1})$ and $r_1 \equiv \sqrt{ac}(z_1 + z_1^{-1})$ satisfy*

$$\operatorname{sgn}(r_1 - r_0) = -\operatorname{sgn}(a + e)$$

Proof. This follows immediately as the function $f(x) = \sqrt{ac}(x + x^{-1})$ is increasing for $|x| > 1$, and because $|\nu| > 1$. □

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