Soliton stability in a Z(2) field theory

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Abstract

We investigate the stability of the coupled soliton solutions of a two-component Z(2) vector field model, in contraposition to similar solutions of a Z(2)×Z(2) model recently introduced. We demonstrate that the coupled soliton solutions of the Z(2) model are classically unstable.

Z(2) field theoretical models play a very important role in condensed matter physics. They have been used to describe a wide range of physical systems exhibiting phase transitions involving break of Z(2) symmetry. Examples [1] of such systems are uniaxial antiferromagnets like Rb_2NiF_4 or K_2MnF_4, systems presenting order-disorder transitions on bipartite lattices like in β-brass, or liquid-gas transitions, etc. On the other hand, topological defects may be generated in phase transitions involving broken symmetry. They are low-energy, spatially localized, solutions of the field equations, which are topologically stable. They are of fundamental importance for a variety of physical phenomena in the systems where they appear. As an example of their importance, we mention the quasi-one-dimensional organic system trans-polyacetylene. The relevant topological defect here, the soliton, is responsible for a tremendous increase in the conductivity to almost metallic levels of this insulator when charged solitons are introduced by doping [2].

In this work we are interested in double soliton solutions for coupled scalar fields in two-dimensional spacetime. Such solutions have been recently investigated in a class of systems defined by a very specific potential [3, 4, 5]. These works have shown that there are solutions of the second order equations of motion that are also solutions of some first order differential equations. Also, the important issue of the stability of the soliton solutions has been addressed [4]: it was found that the soliton solutions of those systems, if they exist, are intrinsically stable when they also satisfy the first order equations. This is also important for condensed matter systems. For instance, there is evidence that solitons in coupled scalar field theories may be important to describe ferroelectric crystals [5] and hydrogen-bonded chains [6]. And it is known that stable solutions play relevant role at the quantum level.

Other issues concerning stability of the soliton solutions for coupled scalar fields have recently been considered in [7], for the class of systems introduced in [3, 4, 5]. As we know, however, in the past a Z(2) coupled scalar field model was shown to present very similar coupled soliton solutions [8]. Furthermore, the motivations presented in that work are closely related to the basic motivations introduced in the more recent works [3, 4, 5]. For this reason, because that Z(2) model and its soliton solutions are closely related to the models introduced in [3, 4, 5], it seems important to investigate the classical or linear stability of the soliton solutions found in [8] in order to identify possible distinctions in these two approaches. This is our main motivation, and here we present a detailed analysis of the stability of the coupled soliton solitons found in [8] showing that these solutions are always unstable.

We start with the Lagrangian density [8]

$$\mathcal{L} = \frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi + \frac{1}{2}\partial_\alpha\chi\partial^\alpha\chi - U(\phi,\chi), \tag{1}$$

where the potential is

$$U(\phi,\chi) = \lambda\phi^4 + \lambda\chi^4 + 2\lambda\phi^2\chi^2 - (\mu+\nu)\phi^2 - (\mu-\nu)\chi^2 - \gamma. \tag{2}$$

The gradient of the potential with respect to the fields is given by

$$\nabla_{\phi,\chi}U = \begin{pmatrix} -2(\mu+\nu)\phi + 4\lambda\phi\chi^2 + 4\lambda\phi^3 \\ -2(\mu-\nu)\chi + 4\lambda\phi^2\chi + 4\lambda\chi^3 \end{pmatrix} \quad . \tag{3}$$

The potential has stationary points at $(\phi,\chi) = (\pm\sqrt{(\mu+\nu)/2\lambda}, 0)$. These points are (non-degenerate) minima when the Hessian of the potential is definite positive. The Hessian is given by:

$$\mathrm{Hess}\,U = \begin{pmatrix} 12\lambda\phi^2 + 4\lambda\chi^2 - 2(\mu+\nu) & 8\lambda\phi\chi \\ 8\lambda\phi\chi & 12\lambda\chi^2 + 4\lambda\phi^2 - 2(\mu-\nu) \end{pmatrix} \quad . \tag{4}$$

Substituting the value of the stationary points, we see that

$$\mathrm{Hess}\,U = \begin{pmatrix} 4(\mu+\nu) & 0 \\ 0 & 4\nu \end{pmatrix} \quad . \tag{5}$$

Thus, in order for the stationary points above to be minima of the potential, we require

$$\begin{array}{c} \nu > 0 \\ \mu > -\nu. \end{array} \tag{6}$$

The equations of motion corresponding to the Lagrangian are obtained as the Euler-Lagrange equations of the functional $\int \mathcal{L}$. We are looking for static soliton solutions, and may thus neglect time. The functional becomes:

$$E_c[\phi,\chi] = \int dx \{ \frac{1}{2}(\frac{\partial\phi}{\partial x})^2 + \frac{1}{2}(\frac{\partial\chi}{\partial x})^2 + U(\phi,\chi)\} \ . \tag{7}$$

The Euler-Lagrange equations are (we write $\triangle_x$ for $\frac{d^2}{dx^2}$)

$$-\triangle_x \begin{pmatrix} \phi \\ \chi \end{pmatrix} - \nabla_{\phi,\chi} U = 0 \ . \tag{8}$$

The soliton solutions connect the two minima of the potential at $(\phi,\chi) = (\pm\sqrt{(\mu+\nu)/2\lambda}, 0)$. There are two sets of static soliton solutions:

$$\bar{\phi} = \pm\sqrt{\frac{\mu+\nu}{2\lambda}} \tanh \sqrt{\mu+\nu}\, x \tag{9}$$

$$\bar{\chi} = 0, \tag{10}$$

which together with equation (6) requires that

$$\begin{array}{c} \nu > 0 \\ \mu+\nu > 0 \\ \lambda > 0. \end{array} \tag{11}$$

The second set of solutions is given by

$$\bar{\phi} = \pm\sqrt{\frac{\mu+\nu}{2\lambda}} \tanh \sqrt{4\nu}\, x \tag{12}$$

$$\bar{\chi} = \pm\sqrt{\frac{\mu-3\nu}{2\lambda}} \text{sech} \sqrt{4\nu}\, x \ , \tag{13}$$

implying in this case (with equation (6))

$$\begin{array}{c} \nu > 0 \\ \lambda > 0 \\ \mu-3\nu > 0 \ . \end{array} \tag{14}$$

The first pair of solutions can be investigated easily: The calculations follow the same steps already introduced in [3] for the related pair of solutions. Therefore, here we will focus attention on the stability analysis of the coupled solitons of the second solution set.

Classical stability may be discussed in the following way: If we are to have stable solitons, the second variation of $E_c[\phi, \chi]$ evaluated at the solution should be a positive differential operator. We obtain thus

$$\mathrm{Hess}(E_c[\phi,\chi]) = \begin{pmatrix} -\triangle_x & 0 \\ 0 & -\triangle_x \end{pmatrix} + \mathrm{Hess}\, U \ , \tag{15}$$

as can be most easily seen from equation (8) and noting that $\triangle$ is linear. We will call this operator $\hat{S}$. Its lowest eigenvalue will be denoted by $E_0(\mu, \nu, \lambda)$. We will show that

$$E_0(\mu, \nu, \lambda) < 0 \quad . \tag{16}$$

This way we establish that the soliton solutions are always unstable.

For the second solution pair we get the Hessian

$$\mathrm{Hess}(E_c[\phi,\chi]) = \begin{pmatrix} 12\lambda\bar{\phi}^2 + 4\lambda\bar{\chi}^2 - 2(\mu+\nu) & 8\lambda\bar{\phi}\bar{\chi} \\ 8\lambda\bar{\phi}\bar{\chi} & 12\lambda\bar{\chi}^2 + 4\lambda\bar{\phi}^2 - 2(\mu-\nu) \end{pmatrix} \tag{17}$$

In order to decouple the corresponding eigenvalue equations we need to diagonalize the above matrix. After some algebra we find for its eigenvalues

$$\begin{aligned} V_\pm &= 2\mu - 12\nu + 16\nu f \pm 2\sqrt{16\nu^2 f^2 + 4\nu(\mu - 5\nu) f + (\mu - 2\nu)^2} \\ &= 2\nu\{\delta - 6 + 8f \pm \sqrt{16 f^2 + 4(\delta - 5) f + (\delta - 2)^2}\} \ , \end{aligned} \tag{18}$$

where $f = f(x)$ stands for $\tanh^2(\sqrt{4\nu}x)$ and $\delta = \mu/\nu$. Notice that $f(x)$ varies in $[0, 1)$, and $\nu > 0$ and $\delta > 3$ are parameters (see equation (14)). The operator $\hat{S}$ is now

$$\hat{S} = \begin{pmatrix} -\triangle_x + V_+ & 0 \\ 0 & -\triangle_x + V_- \end{pmatrix} . \tag{19}$$

Notice that $V_\pm$ now only depends on δ and ν. To eliminate the dependence on ν, write

$$V_\pm(x) = 4\nu U_\pm(\sqrt{4\nu}x) \quad ,$$

and in the operator subsitute $x = y/\sqrt{4\nu}$. It is easy to see that $\hat{S}$ now becomes

$$\hat{S} = 4\nu \begin{pmatrix} -\triangle_y + U_+(y) & 0 \\ 0 & -\triangle_y + U_-(y) \end{pmatrix} . \tag{20}$$

Now restrict attention to U_- and drop the subscript. We write

$$U_\delta = \frac{1}{2}\delta - 3 + 4f - \frac{1}{2}\sqrt{16f^2 + 4(\delta - 5)f + (\delta - 2)^2} \tag{21}$$

where $f = \tanh^2(y)$ and $\delta > 3$.

We wish to derive an upper estimate for the lowest eigenvalue of the equation

$$(-\triangle_y + U_\delta(y))\psi(y) = \varepsilon(\delta)\psi(y) \ . \tag{22}$$

Notice that $U_\delta(-\infty) = U_\delta(\infty) = 1$ and that U_δ is well-shaped. It is known that in the one-dimensional case there is *always* at least a bound eigenstate [9]. That is, there is an eigenfunction ψ with associated eigenvalue less than 1, and with the property that $\int \psi^* \psi dy = 1$. Our estimate relies on the following observation. Let U_1 and U_2 be two potentials as above, but with the property that for all y: $U_1(y) \leq U_2(y)$. The associated eigenvalues, λ_1 and λ_2 then satisfy the same relation, $\lambda_1 \leq \lambda_2$.

For $\delta > 3$, the potential $U_\delta(y)$ is a (weakly) decreasing function of δ:

$$\frac{\partial U_\delta}{\partial \delta} \leq 0 \quad .$$

It then follows that if we denote by λ_δ the lowest eigenvalue associated with U_δ

$$\lambda_\delta \leq \lambda_3 \ , \tag{23}$$

where λ_3 stands for the case $\delta = 3$. It is an easy calculation to show that

$$U_3(y) = 1 - \frac{2}{\cosh^2(y)}. \tag{24}$$

The corresponding eigenvalue equation

$$(-\triangle_y + U_3(y))\psi(y) = \varepsilon(3)\psi(y) \tag{25}$$

is easily solvable – see [3] and [10]. The calculations lead to only one bound state at $\varepsilon_0(3) = 0$ and a continuous spectrum $\varepsilon_c(3) > 1$. Therefore $\lambda_3 = 0$ and by (23)

$$\lambda_\delta \leq 0 \ . \tag{26}$$

In fact, we can show that we have here a strict inequality. Notice that $H_\delta = -\Delta_y + U_\delta$ implies that $H_\delta = H_3 + (U_\delta - U_3)$. Now take ψ_0 to be the groundstate eigenfunction of H_3. It follows that

$$\begin{aligned}\int_{-\infty}^{+\infty} \psi_0^*(y) H_\delta \psi_0(y) dy &= \int_{-\infty}^{+\infty} \psi_0^*(y) H_3 \psi_0(y) dy + \\ &\int_{-\infty}^{+\infty} \psi_0^*(y)[U_\delta(y) - U_3(y)]\psi_0(y) dy \ .\end{aligned}$$

Since $(U_\delta(y) - U_3(y)) \leq 0$ for $y \in (-\infty, +\infty)$, we have that $\int_{-\infty}^{+\infty} \psi_0^*(y)[U_\delta(y) - U_3(y)]\psi_0(y)dy < 0$. Also, since ψ_0 is not the groundstate of H_δ for $\delta > 3$, then $\int_{-\infty}^{+\infty} \psi_0^*(y) H_\delta \psi_0(y) dy > \lambda_\delta$. With $\int_{-\infty}^{+\infty} \psi_0^*(y) H_3 \psi_0(y) dy = 0$ it follows that

$$\lambda_\delta < 0 \ . \tag{27}$$

Alternatively, we see that the limit $\delta \to 3$ transforms the second pair of solutions (12) and (13) back to the first pair (9) and (10), for which we have $\lambda_3 = 0$. Thus, unicity of the ground state allows writing $\lambda_\delta \leq 0$ for $\delta \geq 3$, or better $\lambda_\delta < 0$ for $\delta > 3$, which is the region in parameter space where the second pair of solutions appears. This concludes our demonstration that $E_0(\mu, \nu, \lambda) < 0$ for all parameter values that respect equation (14), and thus that the soliton solutions discussed here are always unstable.

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