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Signal Velocity in Oscillator Arrays

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Abstract. We investigate a system of coupled oscillators on the circle, which arises from a simple model for behavior of large numbers of autonomous vehicles where the acceleration of each vehicle depends on the relative positions and velocities between itself and a set of local neighbors. After describing necessary and sufficient conditions for asymptotic stability, we derive expressions for the phase velocity of propagation of disturbances in velocity through this system. We show that the high frequencies exhibit damping, which implies existence of well-defined *signal velocities* $c_+ > 0$ and $c_- < 0$ such that low frequency disturbances travel through the flock as $f_+(x - c_+t)$ in the direction of increasing agent numbers and $f_-(x - c_-t)$ in the other.

1 Introduction

This paper is part of a larger program to develop mathematical methods to quantitatively study models for flocking. The main motivation for the current work is to inform development of driverless cars to enable coherent motion at high speed, even under dense traffic conditions. We study models that assume that each car is programmed identically and that can observe relative velocities and positions of nearby cars. In this work we restrict our analysis to nearest neighbor interactions, however the methods we develop are also applicable to interactions involving more neighbors.

There are two main aspects in our analysis. The first is asymptotic stability, for a system with a fixed number of vehicles, which may be analyzed via the eigenvalues of the matrix associated with the first order differential equation. Section 3 is devoted to establishing necessary and sufficient conditions for a class of systems to be asymptotically stable. The second, more delicate aspect of the problem relates to controlling the growth of disturbances in the system as the number of vehicles becomes large. In this situation, even if all our systems are known to be asymptotically stable, transients may still grow exponentially in the number of cars. The spectrum of the linear operator does not help us to recognize this problem ([1]). A dramatic example of this can be found in [2] where eigenvalues have real part bounded from above by a negative number and yet transients grow exponentially in N. This kind of exponential growth underscores the need for different (non-spectral) methods to analyze these systems.

The main result of our paper represents one such alternative approach. We establish that for the parameter values of interest (e.g. asymptotically stable systems), solutions are well approximated by traveling wave signals (as described in [3]) with two distinct signal velocities, one positive (in direction of increasing agent number) and one negative.

Ever since the inception ([4], [5]) of the subject, systems with periodic boundary conditions have been popular ([6], [7], and [8]) because they tend to be easier to study. However the precise connection between these systems and more realistic systems with non-trivial boundary conditions has always been somewhat unclear. Our current program differs from earlier work in two crucial ways. The first is that we make precise what the impact of our analysis is for the (more realistic) systems on the line : namely in this paper we derive an expression for the velocity with which disturbances propagate in systems with periodic boundary, and in [9] we numerically verify that this holds on the line as well. The second is that we consider all possible nearest neighbor interactions: we do not impose symmetries. This turns out to be of the utmost importance: when we apply these ideas in [9] it turns out that the systems with the best performance are asymmetric. Asymmetric systems (though not the same as ours) have also been considered by [10] and with similar results. However their methods are perturbative, and spectral based. In [11] and [12] asymmetric interactions are also studied, and it was shown that in certain cases they may lead to exponential growth (in N) in the perturbation. In the later of these, the model is qualitatively different because absolute velocity feedback is assumed (their method is also perturbative and not global). Signal velocities were employed in earlier calculations namely [13] and [14]. In contrast to the model in our paper, these works considered *car-following* models where position and velocities of the neighbor *behind* the current car were not incorporated.

Our model is *strictly decentralized*, for two main reasons. First, in high speed, high/density traffic, small differences in measured absolute velocity may render that measurement useless, if not dangerous, for the feedback. Secondly, the desired velocity, even on the highway, may not be constant. For these reasons we limit ourselves to strictly *decentralized* models that only use information relative to the observers in the cars (see [15] and [2]). Many authors study models featuring a term proportional to velocity minus desired velocity (see e.g. [8], [10], [4], [6], [7], [5], and [12]).

2 Flocking Model

We consider a *decentralized* flock of N identical moving agents (e.g. cars), on a circle, where each agent's acceleration depends linearly on on the differences between its own relative position and velocity, and those its nearest neighbors. The system is designed to maintain a fixed spacing Δ between each vehicle. Letting x_k be the position of the k^{th} agent, we have

$$\begin{aligned} \ddot{x}_k =& g_x\left(\rho_{x,1}(x_{k+1}-x_k-\Delta)+\rho_{x,-1}(x_{k-1}-x_k+\Delta)\right)+ \\ & g_v\left(\rho_{v,1}(\dot{x}_{k+1}-\dot{x}_k)+\rho_{v,-1}(\dot{x}_{k-1}-\dot{x}_k)\right). \end{aligned} \tag{1}$$

For example, the term above involving $g_x\rho_{x,1}$ tells the kth car to accelerate if distance between itself and the car in front is larger than Δ, so it needs to "catch up". Similarly the terms involving $\rho_{x,-1}$, $\rho_{v,1}$ and $\rho_{v,-1}$ contribute acceleration if the spacing of the car behind is not Δ, or if there is a difference between the kth cars velocity and that of the neighbors'. We enforce the circular boundary by setting $x_k = x_{k+N}$.

We simplify the above system with the change of variables $z_k = x_k - k\Delta$ (see [15] for more details), here z_k represents the deviation from the equilibrium position within the flock. Let $\mathcal{N} = \{-1, 0, 1\}$. Expanding the above system gives the system we call S_N^* :

$$\ddot{z}_k = g_x \sum_{j\in\mathcal{N}} \rho_{x,j} z_{k+j} + g_v \sum_{j\in\mathcal{N}} \rho_{v,j} \dot{z}_{k+j} \tag{2}$$

Comparing Equations (1) and (2) shows that $\rho_{x,0} = -(\rho_{x,1} + \rho_{x,-1})$ and $\rho_{v,0} = -(\rho_{v,1} + \rho_{v,-1})$.

By introducing matrices L_x and L_v whose rows consist of appropriately circularly shifted copies of the vectors $(\rho_{x,-1}, \rho_{x,0}, \rho_{x,1}, 0, ..., 0)$ or $(\rho_{v,-1}, \rho_{v,0}, \rho_{v,1}, 0, ..., 0)$ respectively, we may write this system in vector form as $\ddot{z} = g_x L_x z + g_v L_v \dot{z}$. The $N \times N$ matrices L_x and L_v so defined so above are circulant matrices with row sums equal to zero, will refer to them as Laplacian matrices. These allow us to write the equations of S_N^* as a first order system:

$$\frac{d}{dt}\begin{pmatrix} z \\ \dot{z} \end{pmatrix} = M_N \begin{pmatrix} z \\ \dot{z} \end{pmatrix} \equiv \begin{pmatrix} 0 & I \\ g_x L_x & g_v L_v \end{pmatrix} \begin{pmatrix} z \\ \dot{z} \end{pmatrix} \tag{3}$$

This system has a 2-dimensional family of coherent solutions, where $z_i(t) = v_0 t + x_0$ for $i = 1...N$, where v_0 and x_0 are arbitrary elements of $\mathbb{R}$. These correspond to the generalized eigenspace of M_N for the eigenvalue 0. It is easy to see that all solutions converge to one of these coherent solutions if and only if all other eigenvalues of M_N have negative real part. With a slight abuse of notation we will call this case asymptotically stable (see [17] for precise definitions):

Definition 1 *S_N^* is asymptotically stable if M_N from (3) has a single eigenvalue of 0 with algebraic multiplicity 2, and other eigenvalues with strictly negative real parts.*

An important property of all circulant matrices is that their eigenvectors are exactly given by the discrete Fourier transform (see [16]). This will play a fundamental role in our analysis. Throughout the rest of the paper we will extensively use the notation that $\phi = \frac{2\pi m}{N}$ (so ϕ implicitly depends on m). We define the m^{th} discrete Fourier vector $w_m \equiv \dfrac{1}{\sqrt{N}} \left(1, e^{i\phi}, e^{i2\phi}, \cdots e^{i(N-1)\phi}\right)^T$, for each m this is an eigenvector of both $g_x L_x$ and $g_v L_v$. The eigenvalues can be found by a straightforward calculation showing that $g_x L_x w_m = \lambda_{x,m} w_m$ and $g_v L_v w_m = \lambda_{v,m} w_m$, where

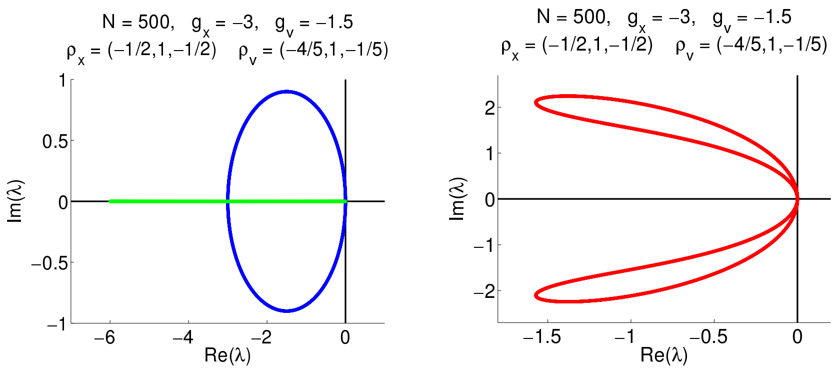
$$\lambda_x(\phi) \equiv g_x \sum_{j\in\mathcal{N}} \rho_{x,j}\, e^{ij\phi} \quad \text{and} \quad \lambda_v(\phi) \equiv g_v \sum_{j\in\mathcal{N}} \rho_{v,j}\, e^{ij\phi}. \tag{4}$$

Much of our later analysis relies on examining the eigenvalues of M_N for small ϕ (i.e. $m << N$). To that end we define the moments of $g_x \rho_x$ and $g_v \rho_v$ by

$$I_{x,\ell} \equiv g_x \sum_{j\in\mathcal{N}} \rho_{x,j}\, j^\ell \quad \text{and} \quad I_{v,\ell} \equiv g_v \sum_{j\in\mathcal{N}} \rho_{v,j}\, j^\ell. \tag{5}$$

and observe that $\lambda_{x,m}$ and $\lambda_{v,m}$ have the Taylor expansion

$$\lambda_x(\phi) = i\phi I_{x,1} - \frac{\phi^2}{2} I_{x,2} - ... \quad \text{and} \quad \lambda_v(\phi) = i\phi I_{v,1} - \frac{\phi^2}{2} I_{v,2} - ... \tag{6}$$



(a) (b)

Fig. 1. Representative figures for the calculation of the eigenvalues for 500 agents, for parameters giving an asymptitically stable system. The values of the parameters are given in the figures. (a) Green line: $\lambda_{x,m}$, Blue ellipse: $\lambda_{v,m}$. (b) The eigenvalues $\nu_{m,\pm}$ of M_N of Proposition 1.

3 Asymptotic Stability

If the system S_N^* is not asymptotically stable, then for fixed N as $t \to \infty$ the behaviour will be dominated by exponential growth. Parameters yielding an unstable system are clearly inappropriate for control of an actual vehicle flock. As the ultimate goal of our work is to understand transient behaviour for control of flocks, we wish to restrict our analysis to asymptotically stable systems. In this section we examine the eigenvalues of M_N, in order to determine conditions to ensure asymptotic stability.

Proposition 1 *The eigenvalues $\nu_{m\pm}$ ($m \in \{0, \cdots N-1\}$) of M_N are given by the solutions of $\nu^2 - \lambda_{v,m}\nu - \lambda_{x,m} = 0$, i.e. $\nu_{m\pm} = \frac{\lambda_{v,m}}{2} \pm \sqrt{\frac{\lambda_{v,m}^2}{4} + \lambda_{x,m}}$ with associated eigenvectors given by $\begin{pmatrix} w_m \\ \nu_{m\pm} w_m \end{pmatrix}$.*

Proof: Let ν be an eigenvalue of M_N, with eigenvector written as $\begin{pmatrix} q \\ u \end{pmatrix}$. Then $\begin{pmatrix} 0 & I \\ g_x L_x & g_v L_v \end{pmatrix} \begin{pmatrix} q \\ u \end{pmatrix} = \nu \begin{pmatrix} q \\ u \end{pmatrix}$, which implies first that $u = \nu q$ and then that $(g_x L_x + g_v L_v \nu) q = \nu^2 q$. Using the eigenvalues and eigenvectors of $g_x L_x$ and $g_v L_v$ from (4) shows that $\begin{pmatrix} w_m \\ \nu_{m\pm} w_m \end{pmatrix}$ is an eigenvector of M with eigenvalues $\nu_{m\pm}$ if the latter are the solutions of $\nu^2 = \lambda_{x,m} + \nu \lambda_{v,m}$. ■

It is instructive to look at the graphs of the functions λ_x and λ_v, which define closed curves in the complex plane. We also define the curves $\gamma_\pm$ to be the set of all complex numbers ν satisfying $\nu^2 - \lambda_v(\phi)\nu - \lambda_x(\phi) = 0$ for some ϕ. As N grows, the (discrete) set of eigenvalues $\lambda_{x,m}$ and $\lambda_{v,m}$ fills out the curves λ_v and λ_x, and similarly the eigenvalues $\nu_{m,\pm}$ of M_N fill out the curves $\gamma_\pm$. These are illustrated in Figure 1 for parameters giving a stable system. It can be seen that the graph of $\gamma_\pm$ has four separate branches near the origin. Our first necessary condition for asymptotic stability is based on ensuring that none of these branches have initial direction crossing into the positive real half-plane.

Proposition 2 *If $I_{x,1} \neq 0$, then for large enough N, S_N^* is not asymptotically stable.*

Proof: Proposition 1 and Equation 6 imply that for large N and small m the eigenvalues of M satisfy:

$$\nu_\pm(\phi) \approx \frac{i\phi I_{v,1}}{2} \pm \sqrt{\frac{-\phi^2 I_{v,1}^2}{4} + i\phi I_{x,1}}$$

If $I_{x,1}$ is not zero, then for small enough ϕ, the second term under the square root dominates. This gives rise to

$$\nu_\pm(\phi) = \pm\sqrt{i\phi I_{x,1}} + \mathcal{O}(|\phi|)$$

where ϕ can be positive or negative. Therefore this has four branches near the origin, two of which have positive real part. ∎

Proposition 3 *Suppose S_N^* satisfies $\rho_{x,1} = \rho_{x,-1}$. Then it is asymptotically stable for all N if and only if $\mathrm{Re}(\lambda_x(\phi)) < 0$ and $\mathrm{Re}(\lambda_v(\phi)) < 0$ for all $\phi \neq 0$. Instability occurs for large enough N if either opposite inequality holds for some $\phi \neq 0$.*

Proof: By the previous Proposition we must have $I_{x,1} = g_x(\rho_{x,1} - \rho_{x,-1}) = 0$. If $g_x = 0$, Proposition 1 yields many eigenvalues equal to zero. So a necessary condition for stability is $\rho_{x,1} = \rho_{x,-1}$. This implies $\mathrm{Im}(\lambda_x(\phi)) = 0$.

By the Routh-Hurwitz criterion applied to the equation $\nu^2 - \lambda_v(\phi)\nu - \lambda_x(\phi) = 0$ with complex coefficients (see [18]), we see that all nonzero $\nu_\pm(\phi)$ have negative real parts if and only if for all $\phi \in \{1, \cdots N-1\}\frac{2\pi}{N}$ we have :

$$\begin{aligned}
\mathrm{Re}(\lambda_v) &< 0 \\
2\,\mathrm{Re}(\lambda_x) &< |\lambda_v|^2 \\
\mathrm{Re}(\lambda_x)\,\mathrm{Re}(\lambda_v) + \mathrm{Im}(\lambda_x)\,\mathrm{Im}(\lambda_{v,m}) &> 0 \\
\mathrm{Re}(\lambda_x)[\mathrm{Re}(\lambda_v)]^2 + \mathrm{Re}(\lambda_v)\,\mathrm{Im}(\lambda_x)\,\mathrm{Im}(\lambda_v) + [\mathrm{Im}(\lambda_x)]^2 &< 0
\end{aligned}$$

The condition on $\rho_{x,1}$ and $\rho_{x,-1}$ implies $\mathrm{Im}(\lambda_x(\phi)) = 0$. Thus these equations reduce to: for all $\phi \neq 0$, $\mathrm{Re}(\lambda_x(\phi)) < 0$ and $\mathrm{Re}(\lambda_v(\phi)) < 0$. (The case of $\phi = 0$ is excluded because the zero eigenvalue with multiplicity 2 is excluded from our definition of asymptotic stability).

On the other hand, if either $\mathrm{Re}(\lambda_x(\phi)) > 0$ or $\mathrm{Re}(\lambda_v(\phi)) > 0$ for some $\phi \neq 0$, then for N large enough there must be some N and m so that $\mathrm{Re}(\lambda_{x,m}) > 0$ or $\mathrm{Re}(\lambda_{v,m}) > 0$, in which case S_N^* is asymptotically unstable. ∎

We may now state and prove the main theorem of this section.

Theorem 1 *The system $\mathcal{S}_N^*$ is asymptotically stable for all N if and only if $\rho_{x,1} = \rho_{x,-1}$, $g_x\rho_{x,0} < 0$, and $g_v\rho_{v,0} < 0$.*

Proof: If S_N^* is asymptotically stable then $I_{x,1} = g_x(\rho_{x,1} - \rho_{x,-1}) = 0$ by Proposition 2. If $g_x = 0$ then Proposition 1 implies that M has many eigenvalues equal to zero. Thus $\rho_{x,1} = \rho_{x,-1}$. The remaining two condition are implied by Proposition 3.

To prove the other direction, let $\rho_{x,-1} = \rho_{x,1}$, $g_x\rho_{x,0} < 0$ and $g_v\rho_{v,0} < 0$. The same calculation as above shows $\mathrm{Re}(\lambda_x(\phi)) < 0$ and $\mathrm{Re}(\lambda_v(\phi)) < 0$ for $\phi \neq 0$, then Proposition 3 implies S_N^* is asymptotically stable. ∎

According to Theorem 1, there are two ways that the parameters may lead to S_N^* being asymptotically unstable. The first is to set $\rho_{x,1} \neq \rho_{x,-1}$. This causes the curve on which the eigenvalues lie to have four branches near the origin, two of which lie in the positive real half plane. This is discussed in the proof of Proposition 2 and illustrated in Figure 2 (a) and (b). The second way is for one (or both) of g_x or g_v to

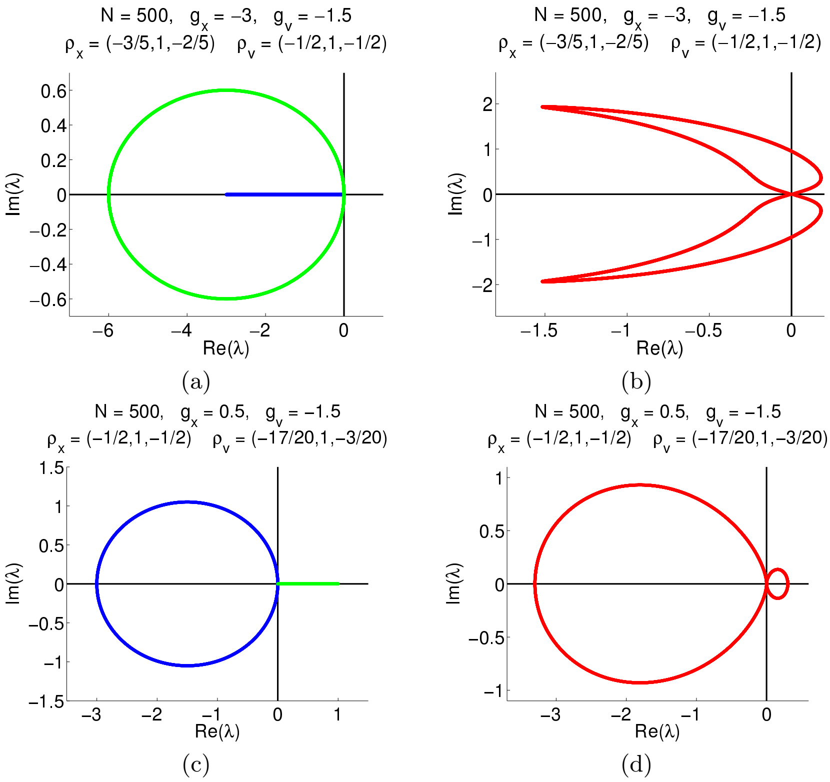


Fig. 2. Representative figures for the calculation of the eigenvalues for 500 agents, for parameters giving asymptotically unstable system (a) Green: $\lambda_{x,m}$, Blue: $\lambda_{v,m}$. (b) $\nu_{m,\pm}$ (c), (d) : similar, for a second set of parameters giving an unstable system.

be positive. This is illustrated in Figure 2 (c) and (d). Note that positive values for g_x or g_v would correspond to terms in Equation (2) contributing acceleration in the opposite direction from that needed to return the agent to equilibrium relative to its neighbors.

4 Signal Velocities

This section concerns the signal velocity for asymptotically stable systems, i.e. the velocity with which disturbances (such as a short pulse) propagate through the flock. From now on we will restrict our attention to (stable) systems satisfying the conditions the conclusions of Theorem 1.

We first examine the phase velocities of solutions of S_N^* arising from eigenvectors of M_N. According to proposition 1, there are two such eigen solutions associated with the wavenumber m, namely

$$z_k(t) = e^{(\nu_{m\pm})t} e^{ik\phi} = e^{\mathrm{Re}(\nu_{m\pm})\,t} e^{i(\mathrm{Im}(\nu_{m\pm})t+k\phi)} = e^{\mathrm{Re}(\nu_{m\pm})\,t} e^{i(\mathrm{Im}(\nu_{m\pm})t-(-\phi)k)} \tag{7}$$

These each have the form of an exponentially damping (in time) term times a time-varying sinusoid of the form $e^{i\omega t-bk}$, where we identify $\omega = Im(\nu_{m\pm})$ and $b = -\phi$. In general, the phase velocity of the time-varying sinusoid $f(x,t) = e^{i(\omega t-bx)}$ on the real line is defined as the rate of change of the locus of points of constant phase: $\omega t - bx = c$, which gives the phase velocity ω/b. In analogy with this, we define the phase velocity in units of number of agents per time for the solution $z_k(t) = Ae^{i(\omega t - bk)}e^{-at}$ to be ω/b. This can yield the phase velocities of the eigensolutions in (7), in particular :

Lemma 1 *For S_N^* as in Theorem 1, phase velocities are given by (for $1 \leq m \leq \frac{N}{2}$) :*

$$c_{m+} = \frac{-\operatorname{Im}(\nu_{m-})}{\phi} > 0 \quad \text{and} \quad c_{m-} = \frac{-\operatorname{Im}(\nu_{m+})}{\phi} < 0 \tag{8}$$

Proof: Identifying $\omega = Im(\nu_{m\pm})$ and $b = -\phi$ from (7) in the above definition of phase velocities gives the expressions for c_{m+} and c_{m-}. These expressions will have opposite signs if $Im(\nu_{m-})$ and $Im(\nu_{m+})$ have opposite signs, this is true and is shown below in Lemma 2. We are now free to redefine the subscripts "+" and "−" so that ν_{m+} has positive imaginary part, and ν_{m-} has negative imaginary part, which gives the desired result. ■

Lemma 2 *The values* $\operatorname{Im}(\nu_{m-})$ *and* $\operatorname{Im}(\nu_{m+})$ *have opposite signs if $m \neq 0$.*

Proof: $\nu_{m\pm}$ are the eigenvalues of the system that we assume to be stable. Set $\nu_{m+} = \alpha_1 + i\beta_1$ and $\nu_{m-} = \alpha_2 + i\beta_2$ with $\alpha_i < 0$. By Proposition 1 $\nu_{m\pm}$ are the roots of $\nu^2 - \nu\lambda_{v,m} - \lambda_{x,m}$, and

$$(\nu - \mu_1)(\nu - \mu_2) = \nu^2 - (\alpha_1 + \alpha_2 + i(\beta_1 + \beta_2))\nu + \alpha_1\alpha_2 - \beta_1\beta_2 + i(\alpha_1\beta_2 + \alpha_2\beta_1),$$

So we can identify $\lambda_{x,m} = -(\alpha_1\alpha_2 - \beta_1\beta_2) + i(\alpha_1\beta_2 + \alpha_2\beta_1)$. By Proposition 2, we have $I_{x,1} = 0$ which implies $\operatorname{Im}(\lambda_{x,m}) = 0$. Therefore $\alpha_1\beta_2 + \alpha_2\beta_1 = 0$ and so the β_i must have opposite signs. ■

Any solution to the system S_N^* consists of sums of eigensolutions as in (7). Crucially, if the phase velocities c_{m+} and c_{m-} were in fact constants not depending on m, then by grouping any solution into two sums, one over eigensolutions corresponding to ν_{m+} (for varying m) and another over eigensolutions corresponding to ν_{m-}, one sees that all solutions of S_N^* would behave as a sum of two travelling waves in opposite directions, with the specified velocities c_{m+} and c_{m-}. The main result of this paper shows that this statement still approximately holds, even though the $c_{m\pm}$ do depend on m. Intuitively, this till holds because eigensolutions with larger m, where $c_{m\pm}$ deviates from $c_{0\pm}$, will experience greater exponential (in time) damping, and may thus be ignored. The remainder of our paper is devoted to careful analysis of these two effects, beginning with expansions of $\nu_{m\pm}$ and $c_{m\pm}$ valid for $m << N$.

We first define

$$a \equiv \frac{I_{v,1}^2}{4} + \frac{I_{x,2}}{2} = \frac{(\rho_{v,0} + 2\rho_{v,1})^2 g_v^2}{4} + \frac{-g_x \rho_{x,0}}{2},$$

for stable systems we have $a > 0$. Without loss of generality we re-scale g_x and g_v so that the values of $\rho_{x,0}$ and $\rho_{v,0}$ are 1 from now on.

Proposition 4 *Let the parameters of S_N^* satisfy the hypothesis of Theorem 1. Then the eigenvalues $\nu_{m\varepsilon}$ of M_N can be expanded as (with $\varepsilon = \pm 1$ and $\phi \equiv \frac{2\pi m}{N}$):*

$$\nu_{m\varepsilon} = i\phi\left(\frac{I_{v,1}}{2} + \varepsilon a^{1/2}\right) + \phi^2\left(-\frac{I_{v,2}}{4} - \varepsilon\frac{\left(\frac{I_{v,1}I_{v,2}}{4} + \frac{I_{x,3}}{6}\right)}{2a^{1/2}}\right) +$$

$$i\phi^3\left(-\frac{I_{v,3}}{12} - \varepsilon\frac{\left(\frac{I_{v,1}I_{v,3}}{12} + \frac{I_{x,4}}{24} + \frac{I_{v,2}^2}{16}\right)}{2a^{1/2}} + \varepsilon\frac{\left(\frac{I_{v,1}I_{v,2}}{4} + \frac{I_{x,3}}{6}\right)^2}{8a^{3/2}}\right) +$$

$$\phi^4\left(\frac{I_{v,4}}{48} + \varepsilon\frac{\left(\frac{3I_{v,2}I_{v,3}}{48} + \frac{I_{x,5}}{125}\right)}{2a^{1/2}} - \varepsilon\frac{\left(\frac{I_{v,1}I_{v,2}}{4} + \frac{I_{x,3}}{6}\right)\left(\frac{I_{v,1}I_{v,3}}{12} + \frac{I_{x,4}}{24} + \frac{I_{v,2}^2}{16}\right)}{4a^{3/2}} + \varepsilon\frac{\left(\frac{I_{v,1}I_{v,2}}{4} + \frac{I_{x,3}}{6}\right)^3}{16a^{5/2}}\right)$$

plus higher orders.

Proof: Expand $\nu_{m\pm}$ given in Proposition 1 in powers of ϕ using

$$a \neq 0 \Rightarrow \sqrt{z-a} = \pm i\sqrt{a}\left(1 - \frac{z}{2a} - \frac{z^2}{8a^2} - \frac{z^3}{16a^3}\cdots\right)$$

After a substantial but straightforward calculation the result is obtained. ■

Lemma 3 *The phase velocities $c_{m\varepsilon}$ of Lemma 1 can be expanded as ($\varepsilon \in \{-1,1\}$):*

$$c_{m\varepsilon} = -\frac{g_v(1+2\rho_{v,1})}{2} + \varepsilon\sqrt{\frac{g_v^2(1+2\rho_{v,1})^2}{4} - \frac{g_x}{2}} + \phi^2\left(\frac{g_v(1+2\rho_{v,1})}{12}\right.$$
$$\left. -\varepsilon\frac{2g_v^2(1+2\rho_{v,1})^2 - g_x + \frac{3}{2}g_v^2}{24[g_v^2(1+2\rho_{v,1})^2 - 2g_x]^{1/2}} + \varepsilon\frac{g_v^2(1+2\rho_{v,1})}{16[g_v^2(1+2\rho_{v,1})^2 - 2g_x]^{3/2}}\right) + \mathcal{O}(\phi^4)$$

The real parts of the associated eigenvalues can be expanded as:

$$\mathrm{Re}(\nu_{m\varepsilon}) = \phi^2\left(\frac{g_v}{4} + \varepsilon\frac{g_v^2(1+2\rho_{v,1})}{4[g_v^2(1+2\rho_{v,1})^2 - 2g_x]^{1/2}}\right) + \mathcal{O}(\phi^4)$$

Proof: With the reduction $\rho_{x,0} = \rho_{v,0} = 1$, Theorem 1 implies $\rho_{x,1} = \rho_{x,-1} = -\frac{1}{2}$, and the decentralized condition implies $\rho_{v,-1} = -(1+\rho_{v,1})$. We can then compute all of the moments $I_{x,j} = (-1)^j(-\frac{1}{2}) + 1^j(-\frac{1}{2})$ and $I_{v,j} = (-1)^j(-(1+\rho_{v,1})) + 1^j\rho_{v,1}$. It follows that $I_{x,j} = 0$ if j is even and $I_{x,j} = 1$ if j is odd, and that $I_{v,j} = -1$ if j is even and $I_{v,j} = 1 + 2\rho_{v,j}$ if j is odd.

Substituting the expansion from Proposition 4 into the expressions for the phase velocity $c_{m\pm} = -\frac{\mathrm{Im}(\nu_{m\mp})}{\phi}$ from Lemma 1, and using the above expressions for the moments $I_{x,j}$ and $I_{v,j}$ gives the desired expansion. ■

It (conveniently) turns out that very often the greatest phase velocities are associated with the lowest wave numbers. A typical case is seen in Figure 3 (a). One can show that in those asymptotically stable cases where $\rho_{v,1}$ is close to -1/2, we have that $c_{m\pm}$ has a local maximum at $m = 0$. In fact Lemma 3 implies that for $\rho_{v,1} = -1/2$, we have $c_{m\varepsilon} = \varepsilon\sqrt{\frac{-g_x}{2}} + \varepsilon\phi^2\left(\frac{g_x - \frac{3}{2}g_v^2}{24\sqrt{-2g_x}}\right) + \cdots$, which has a local maximum at $m = 0$.

Using Proposition 1 and noting that $(w_m)_k \propto e^{i\phi k}$, where $\phi \equiv \frac{2\pi m}{N}$, one can see that for any set of initial conditions $z_k(0)$ and $\dot{z}_k(0)$, there are unique constants a_m and b_m so that the solution of the system S_N^* has the form

$$z_k(t) = \sum_{m=-N/2}^{N/2} a_m e^{i\phi k}\, e^{\nu_{m-}t} + \sum_{m=-N/2}^{N/2} b_m e^{i\phi k}\, e^{\nu_{m+}t} \tag{9}$$

The a_m and b_m are related to the inverse of the discrete Fourier transform of $z_k(0)$ and $\dot{z}_k(0)$. For example, if the b_m are zero, then (see [19], section 4.4.1.3)

$$a_m = N^{-1}\sum_{k=-N/2}^{N/2} z_k(0)e^{-i\phi k} \tag{10}$$

Our main result is that the first sum in Equation 9 may be approximated by a traveling wave with a single *signal velocity* equal to c_+ (see [3]). Likewise, the second sum represents a signal traveling in the direction of decreasing agent number with signal velocity c_-. These two signal velocities are defined as (with $c_{m\pm}$ from Lemma 3):

$$c_\pm = c_{0\pm} = -\frac{g_v(1+2\rho_{v,1})}{2} \pm \sqrt{\frac{g_v^2(1+2\rho_{v,1})^2}{4} - \frac{g_x}{2}} \tag{11}$$

We first need a few simple inequalities to facilitate the proof of the next proposition. We list them here without proof.

Lemma 4 *i:* $\forall\, t,\, s \in \mathbb{R}$ *we have* $|e^{is} - e^{it}| < |s-t|$.
ii: $\forall\, t,\, s \le 0$ *we have* $|e^{s} - e^{t}| < |s-t|$.
Below N is large, $q > 0$ *is a constant. For C large (independent of N), we have:*
iii: $\sum_{|m|<N^\alpha, m\neq 0} |m|^{2-q} < CN^{(3-q)\alpha}$.
iv: $\sum_{|m|>N^\alpha} |m|^{-1-q} < CN^{-\alpha q}$.
v: $\sum_{|m|<N^\alpha, m\neq 0} |m|^{1-q} < CN^{(2-q)\alpha}$.

Proposition 5 *Suppose* S_N^* *satisfies Theorem 1. Let* $\alpha \in (0,1)$. *Suppose the system* S_N^* *has initial condition as in Equation 9 for* $t = 0$, *but with* $b_m = 0$. *Suppose there are* $q, M > 0$ *such that* $|a_m| < N^{-1}M|m|^{-1-q}$. *Then for large N there are a function* f_+ *and constants* K_i *such that for large N*

$$|z_k(t) - f_+(k - c_+t)| < K_1 N^{(2-q)\alpha-3} t + K_2 N^{-\alpha q - 1}$$

where c_+ *is the signal velocity defined in Equation 11.*

Proof: Equation 9 describes the signal after time t. We use the hypotheses of the Proposition, Lemma's 1 and 3, and Equation 11 to write (recall that $\phi = \frac{2\pi m}{N}$):

$$z_k(t) = \left(\sum_{|m|<N^\alpha} + \sum_{|m|\in[N^\alpha, N/2)} \right) a_m\, e^{\mathrm{Re}(\nu_{m-})t}\, e^{i\phi(k-c_+t)}\, e^{i\phi(c_+t - c_{m+}t)}. \tag{12}$$

For simplicity we take N odd so that the sum $\sum_{m=0}^{N-1}$ can be written as $\sum_{|m|<N/2}$ and which in turn can be split up as indicated. We will abbreviate these respective sums from now on as $\sum_<$ and $\sum_>$. We indicate the total sum $\sum_< + \sum_>$ by $\sum$.

Define:

$$\begin{aligned} g(t, k - c_+t) &= \textstyle\sum a_m\, e^{\mathrm{Re}(\nu_{m-})t}\, e^{i\phi(k-c_+t)} \\ f_+(k - c_+t) &= \textstyle\sum a_m\, e^{i\phi(k-c_+t)} \\ A &= |z_k(t) - g(t, k-c_+t)| \\ B &= |g(t, k-c_+t) - f_+(k-c_+t)| \end{aligned}$$

We wish to estimate $|z_k(t) - f_+(k-c_+t)| \le A + B$. For A and B we obtain:

$$\begin{aligned} A &= \left| \left(\textstyle\sum_< + \sum_>\right) a_m\, e^{\mathrm{Re}(\nu_{m-})t}\, e^{i\phi(k-c_+t)} \left(e^{i\phi(c_+t-c_{m+}t)} - 1\right)\right| \\ B &= \left| \left(\textstyle\sum_< + \sum_>\right) a_m\, e^{i\phi(k-c_+t)} \left(e^{\mathrm{Re}(\nu_{m-})t} - 1\right)\right| \end{aligned}$$

Note that for both A and B the terms for $m = 0$ above are zero as $c_{0+} = c_+$ and $\mathrm{Re}(\nu_{0-}) = 0$. For the estimate of A, we start with the first sum ($\sum_<$). By Lemma 4 i and then Lemma 3 we can say $\left|e^{i\phi(c_+t-c_{m+}t)} - 1\right| \le |t\phi(c_+ - c_{m+})| \le tC_1|m|^3/N^3$. Because we are assuming the system is stable we have $\left|e^{\mathrm{Re}(\nu_{m-})t}\right| < 1$. For the second sum ($\sum_>$) we bound $\left|e^{i\phi(c_+t-c_{m+}t)}-1\right|$ by a constant, and again $\left|e^{\mathrm{Re}(\nu_{m-})t}\right|<1$. When we use the assumption on the a_m, this gives:

$$A < C_1 N^{-4} \sum_{<} |m|^{2-q}\, t + C_2 N^{-1} \sum_{>} |m|^{-1-q},$$

where here as well as below, the C_i are constants independent of N. Applying Lemma 4 iii and iv to this gives:

$$A < C_3 N^{(3-q)\alpha-4} t + C_4 N^{-\alpha q-1}$$

The estimate on B proceeds in much the same way. Start with $\sum_<$ and use Lemma 4 ii and then Lemma 3 on $\left|e^{\mathrm{Re}(\nu_{m-})t}-1\right|$. For $\sum_>$ we use that $\left|e^{\mathrm{Re}(\nu_{m-})t}-1\right|$ is less than a constant. Again using the assumption on the a_m gives:

$$B < C_5 N^{-3} \sum_{<} |m|^{1-q}\, t + C_6 N^{-1} \sum_{>} |m|^{-1-q}$$

Applying Lemma 4 iv and v to this gives:

$$B < C_7 N^{(2-q)\alpha-3} t + C_8 N^{-\alpha q-1}$$

If we sum A and B while retaining only the leading terms and adapting the constants if necessary, we get the formula given by the Proposition. ■

The same calculation can be done in the case the b_m are not zero. This establishes the main result of the paper, which shows that in general $z_k(t)$ is well approximated by two traveling waves in opposite directions.

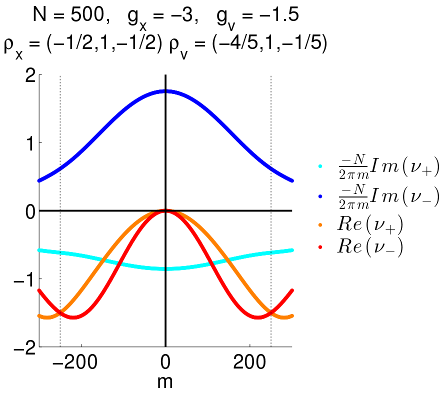
Theorem 2 *Suppose the systems S_N^* are asymptotically stable (Theorem 1). Let $\alpha \in (0,1)$ and $q, M > 0$ such that $|a_m|$ and $|b_m|$ are less than $N^{-1}M|m|^{-1-q}$. Then for large N, there are functions f_+ and f_- and constants K_i such that*

$$|z_k(t) - f_+(k-c_+t) - f_-(k-c_-t)| < K_3 N^{(2-q)\alpha-3} t + K_4 N^{-\alpha q-1}$$

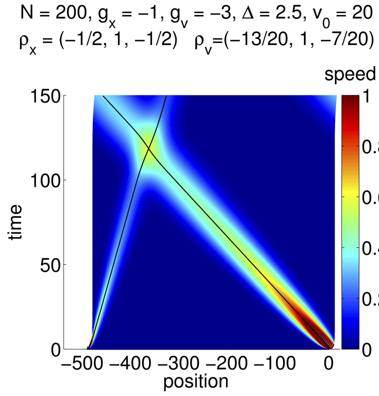
where $c_\pm$ are the signal velocities defined in Equation 11.

Finally, we test our prediction of the signal velocity in a numerical experiment. Our theory described the error due to approximating the disturbance signal subject to a constraint on the decay of the Fourier coefficients of the initial disturbance. If we take α and $q>0$ such that $(2-q)\alpha<1$, the Theorem predicts that for times of order $o(N^2)$ the disturbance propagates as $f_+(k-c_+t)+f_-(k-c_-t)$ with an error that asymptotically goes to zero for large N, and where $c_\pm$ can be explicitly calculated from the parameters. Note that the travel time for a signal to go once around the ring is proportional to N, well within the allowed time frame.

Numerically, we assign agent number 0 at time $t=0$ a different initial velocity from the others. We note that even though this type of impulse disturbance does not have the Fourier coefficient decay required by our theory, we nonetheless observe two distinct signal velocities as predicted. The result can be seen in Figure 3(b). That signal propagates forward through the flock as well as backwards. In the figure we color coded according to the speed of the agents, who are stationary until the signal reaches them. In black we mark when the signal is predicted to arrive, according to the theoretically predicted signal velocities. One can see the excellent agreement.



(a)



(b)

Fig. 3. (a) *Light blue:* $\frac{-\operatorname{Im}(\nu_{m+})}{\phi}$, *Blue:* $\frac{-\operatorname{Im}(\nu_{m-})}{\phi}$, *Orange:* $\operatorname{Re}(\nu_{m+})$, *Red:* $\operatorname{Re}(\nu_{m-})$. (b) *The orbits of 200 cars (Δ is the desired distance between cars). At time 0 agent 0 receives a different initial condition. They are color coded according to the velocity of the agent. Black curves indicate the theoretical position of the wavefront calculated via the signal velocity. Note that these velocities depend on the direction, and that the signal velocity is measured in number of cars per time unit. Due to the different velocities of the cars, these curves are not straight lines.*

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