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RESONANCE BANDS IN A TWO DEGREE OF FREEDOM HAMILTONIAN SYSTEM *

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In perturbations of integrable two degree of freedom Hamiltonian systems, the invariant (KAM) tori are typically separated by *zones of instability* or *resonance bands* inhabited by elliptic and hyperbolic periodic orbits and homoclinic orbits. We indicate how the Melnikov method or the method of averaging can asymptotically predict the widths of these bands in specific cases and we compare these predictions with numerical computations for a pair of linearly coupled simple pendula. We conclude that, even for low order resonances, the first order asymptotic results are generally useful only for very small coupling ($\varepsilon \lesssim 10^{-4}$).

1. Introduction

When an integrable two degree of freedom Hamiltonian system is subject to perturbation, the continuous families of invariant tori characteristic of the integrable limit typically break into Cantor sets of (KAM-) tori, which carry irrational flow, and 'resonance bands,' 'stochastic layers,' or 'zones of instability' (Birkhoff (1932) [1]). The former are the subject of the Kolmogorov–Arnold–Moser (KAM) theory [2], while the latter evolve from 'resonant' tori which carry orbits with rational frequency ratios (rotation numbers) and they contain elliptic and hyperbolic periodic orbits whose existence and number may be predicted in specific examples using perturbation methods, including that of Melnikov [3]. See Arnold [2] and Lichtenberg and Lieberman [4].

In Veerman and Holmes [5] we computed Melnikov functions for the linearly coupled pendula with Hamiltonian

$$H(q,p) = \frac{p_1^2}{2} + (1-\cos\omega q_1) + \frac{p_2^2}{2} + (1-\cos q_2) + \frac{\varepsilon}{2}(q_1-q_2)^2, \tag{1.1}$$

and proved that, in the case $\omega = 1$ and for each relatively prime pair (m,n) of *odd* integers and $\varepsilon \neq 0$ sufficiently small (depending on m and n), there exists exactly one elliptic and one hyperbolic periodic orbit with frequency ratio m/n. In that paper we remarked that the Melnikov functions can be used to

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predict the widths of each (m, n)-resonance band in the limit $\varepsilon \to 0$. In the present paper we perform the necessary computations and compare the results with numerical integrations for several (m, n), ε and ω values. We remark that the Melnikov method as used here is essentially a variant of the method of averaging (Hale [6]).

Resonance bandwidth results are important for several reasons, not least of which is their use in 'resonance overlap' calculations to predict the destruction of KAM tori (Walker and Ford [7], Chirikov [8], Lichtenberg and Lieberman [4]). Our results indicate that, while the predictions are asymptotically correct, nonetheless, for some low order resonances $(\frac{1}{3}, \frac{3}{5})$ they are significantly in error for coupling strengths as low as $\varepsilon = 10^{-4}$, *long before the bands in question overlap*.

In section 2 we review the relevant analytical methods: reduction and Melnikov's method, and then in section 3 we use the results of Veerman and Holmes [5], slightly generalized to $\omega \neq 1$, to compute resonance bandwidths to leading order. Since the expressions involve sums of infinite series, we estimate them by truncation after the leading term, obtaining a relative error of $\leq 0.1\%$. Finally, in section 4 we present numerical results and compare them with the asymptotic formulae.

For general background on the methods of this paper, see Melnikov [3], Greenspan and Holmes [9], Holmes and Marsden [10] and Guckenheimer and Holmes [11, chap. 4]. Related work has recently been done on the Volterra–Lotka equations of population dynamics by Blaine [12], who has also computed asymptotic formulae for resonance bandwidths. Lichtenberg and Lieberman [4, chap. 4] discuss the resonance overlap criterion of Chirikov and give examples of calculations for specific mappings. Walker and Ford [7] and Chirikov [8] also discuss some model problems in which bandwidths can be computed analytically in an integrable limit.

2. Reduction, resonance bands and Melnikov's method

The methods outlined in this section apply to two degree of freedom Hamiltonians of the form

$$H^{\varepsilon}(q, p) = F_1(q_1, p_1) + F_2(q_2, p_2) + \varepsilon H(\boldsymbol{q}, \boldsymbol{p}), 0 \leq \varepsilon \ll 1, \tag{2.1}$$

and, suitably generalized, to n degree of freedom systems (Holmes and Marsden [13, 14]). A canonical change of coordinates $(q_i, p_i) \to (I_i, \theta_i)$ permits us to rewrite (2.1) in action–angle coordinates (Goldstein [15]) in regions where the unperturbed ($\varepsilon = 0$) phase space is foliated by families of invariant tori:

$$H^{\varepsilon}(\boldsymbol{I}, \boldsymbol{\theta}) = F_1(I_1) + F_2(I_2) + \varepsilon H(\boldsymbol{I}, \boldsymbol{\theta}). \tag{2.2}$$

Thus, a typical unperturbed solution is characterized by actions I_1, I_2 and frequency ratio ω_1/ω_2, where $\omega_i = (\partial F_i/\partial I_i)(I_i)$.

The method of *reduction* (Whittaker [16], chap. 12, Birkhoff [17], chap. VI §3) allows us to restrict the flow of Hamilton's equations corresponding to (2.2) to a three-dimensional $(I_1, \theta_1, \theta_2)$ constant energy surface $H^{\varepsilon}(\boldsymbol{I}, \boldsymbol{\theta}) = h$ by inverting this equality to obtain

$$I_2 = L^0(I_1; h) + \varepsilon L^1(I_1, \theta_1, \theta_2; h) + \mathcal{O}(\varepsilon^2). \tag{2.3}$$

After a short computation one obtains the reduced system:

$$\left.\begin{aligned} I_1' &= \varepsilon \frac{\partial L^1}{\partial \theta_1}(I_1, \theta_1, \theta_2; h) \\ \theta_1' &= \frac{\partial L^0}{\partial I_1}(I_1; h) - \varepsilon \frac{\partial L^1}{\partial I_1}(I_1, \theta_1, \theta_2; h) \end{aligned}\right\} + \mathcal{O}(\varepsilon^2), \tag{2.4}$$

where $\mathrm{d}(\ )/\mathrm{d}\theta_2 \overset{\text{def}}{=} (\ )'$ and θ_2 has replaced t, the variable conjugate to $H^\varepsilon = h$. The functions L^0 and L^1 are expressed in terms of F_i, H as

$$L_0(I_1; h) = F_2^{-1}(h - F_1(I_1)); \quad L^1 = \frac{-H\left(I_1, \theta_1, L^0(I_1; h), \theta_2\right)}{\omega_2\left(L^0(I_1; h)\right)}, \tag{2.5}$$

(cf. Holmes and Marsden [10], Proposition 2.1) and thus we have

$$\frac{\partial L^0}{\partial I_1} = -\omega_1(I_1)/\omega_2\left(L^0(I_1; h)\right) \overset{\text{def}}{=} -\Omega_h(I_1). \tag{2.6}$$

We next make the transformation

$$\begin{aligned} I_1 &= I_1^{m,n} + \sqrt{\varepsilon} J, \\ \theta_1 &= \Omega_h(I_1^{m,n})\theta_2 - \psi = \frac{n\theta_2}{m} - \psi, \end{aligned} \tag{2.7}$$

where $I_1^{m,n}$ is a resonant action level on which $\Omega_h(I_1^{m,n}) = n/m$ for n, m relatively prime positive integers. We remark that the sign convention of (2.7) differs from that of Veerman and Holmes [5] and Guckenheimer and Holmes [11]. Using (2.7), (2.4) becomes

$$\left.\begin{aligned} J' &= \sqrt{\varepsilon}\, \frac{\partial L^1}{\partial \theta_1}\left(I_1^{m,n}, \frac{n\theta_2}{m} - \psi, \theta_2; h\right) \\ \psi' &= \sqrt{\varepsilon}\, \frac{\partial^2 L^0}{\partial I_1^2}(I_1^{m,n}; h) J \end{aligned}\right\} + \mathcal{O}(\varepsilon), \tag{2.8}$$

to which we may apply the averaging theorem [6, 11] to obtain

$$\left.\begin{aligned} J' &= \sqrt{\varepsilon}\, \frac{1}{2\pi m} \int_0^{2\pi m} \frac{\partial L^1}{\partial \theta_1}(\ldots)\, \mathrm{d}\theta_2 \\ \psi' &= \sqrt{\varepsilon}\, \frac{\partial^2 L^0}{\partial I_1^2}(\ldots) J \end{aligned}\right\} + \mathcal{O}(\varepsilon). \tag{2.9}$$

The averaging theorem guarantees that solutions of the truncated $\mathcal{O}(\sqrt{\varepsilon})$ system corresponding to (2.9) lying in the stable manifolds of hyperbolic fixed points remain $\mathcal{O}(\sqrt{\varepsilon})$ close for all times $t \geq 0$ to those of the full system, while solutions in unstable manifolds of such fixed points remain $\mathcal{O}(\sqrt{\varepsilon})$ close for all $t \leq 0$.

Hence, via (2.7), we can approximate solutions of the original problem which are asymptotic to hyperbolic periodic orbits.

Defining

$$M_h^{m,n}(\psi) = -\frac{1}{2\pi m}\int_0^{2\pi m}\frac{\partial L^1}{\partial\theta_1}\left(I_1^{m,n}, \frac{n\theta_2}{m}-\psi, \theta_2; h\right)\mathrm{d}\theta_2,$$

$$\bar{\omega}_h^{m,n} = \frac{\partial^2 L^0}{\partial I_1^2}(I_1^{m,n}; h), \tag{2.10}$$

we see that the $\mathcal{O}(\sqrt{\varepsilon})$ truncation of (2.9) is a single degree of freedom, autonomous (θ_2-independent) Hamiltonian system with energy

$$\mathscr{H}(J,\psi) = \sqrt{\varepsilon}\left\{\bar{\omega}_h^{m,n}\frac{J^2}{2} + V_h^{m,n}(\psi)\right\}, \tag{2.11}$$

where

$$\frac{\partial V_h^{m,n}}{\partial\psi}(\psi) = M_h^{m,n}(\psi). \tag{2.12}$$

Since L^1 is 2π-periodic in θ_1, $V_h^{m,n}$ is 2π-periodic in ψ and thus the phase portrait of (2.11) is similar to that of the simple pendulum. Elliptic and hyperbolic fixed points alternate on $J = 0$ ($I_1 = I_1^{m,n}$: eq. (2.7)) at zeros of $M_h^{m,n}(\psi)$, with the Hessian of $\mathscr{H}\sqrt{\varepsilon}\,\omega_h^{m,n}\partial M_h^{m,n}/\partial\psi > 0$ and < 0 respectively, and the separatrices of the latter have maximum separation

$$\Delta I(m,n;h) = 2\sqrt{\varepsilon}\left(\left|\frac{2}{\bar{\omega}_h^{m,n}}(V_{\max} - V_{\min})\right|\right)^{1/2}, \tag{2.13}$$

where $V_{\max}$ denotes the value of a maximum of $V_h^{n,n}(\psi)$ and $V_{\min}$ the value of a minimum (such values correspond to fixed points of (2.11)). Thus (2.13) provides an estimate, accurate to $\mathcal{O}(\varepsilon)$, of the (m,n) resonance bandwidth of (2.9) on the energy surface $H^\varepsilon = h$. Moreover, via the averaging theorem, (2.13) also provides an estimate to $\mathcal{O}(\varepsilon)$ of the maximum separation of orbits asymptotic to the hyperbolic periodic orbit of (2.4) as $\theta_2 \to \pm\infty$. This is the *resonance bandwidth*.

Finally we show how the expression $M_h^{m,n}(\psi)$ is related to the Melnikov functions of Holmes and Marsden [10] or Veerman and Holmes [5]. In the latter paper we defined the function

$$\begin{aligned}M(\theta_0; m,n,h) &= \int_0^{2\pi m}\{L^0, L^1\}(I_1(\theta_2), \theta_1(\theta_2), \theta_2+\theta_0; h)\,\mathrm{d}\Theta_2\\ &\equiv \frac{1}{\omega_2}\int_0^{mT_2}\{F_1, H\}\left(q_1(t), p_1(t), q_2\left(t+\frac{\theta_0}{\omega_2}\right), p_2\left(t+\frac{\theta_0}{\omega_2}\right)\right)\mathrm{d}t,\end{aligned} \tag{2.14}$$

where $\omega_2 = (\partial F_2/\partial I_2)(L^0(I_1^{m,n}; h)) = 2\pi/T_2$ is the unperturbed frequency of the second (F_2-) oscillator and $\{.,.\}$ denotes the Poisson bracket [2, 15]. Using the fact that $\{L^0, L^1\} = -(\partial L^0/\partial I_1)\times(\partial L^1/\partial\theta_1)$, since L^0 is θ_1-independent, we have from (2.10), (2.6) and (2.14)

$$M_h^{m,n}(\psi) = -\frac{1}{2\pi m}\left(-\frac{\omega_2}{\omega_1}\right)(-M(-\psi; m,n,h)) = -\frac{n}{2\pi m^2}M(-\psi; m,n,h). \tag{2.15}$$

We remark that an expression for the resonance bandwidth involving the averaged Poisson bracket (2.14) can be derived directly without appealing to reduction [18].

3. Linearly coupled pendula

Veerman and Holmes [5] computed the Melnikov function $M(t_0; m, n, h)$ of eq. (2.14) for the coupled pendula of eq. (1.1) in the case $\omega = 1$. A slight extension of those calculations shows that the unperturbed periods and actions of the (uncoupled) oscillators with energies h_1 and $h_2 = h - h_1$ are

$$T_1 = 4K\left(\frac{h_1}{2}\right)/\omega, \quad T_2 = 4K\left(\frac{h-h_1}{2}\right), \tag{3.1}$$

where $K(\cdot)$ and $E(\cdot)$ (to be used below) are the complete elliptic integrals of the first and second kinds, respectively. The resonance relationship $\omega_1/\omega_2 = T_2/T_1 = n/m$ leads to

$$nK\left(\frac{h_1}{2}\right) = \omega m K\left(\frac{h-h_1}{2}\right), \tag{3.2}$$

and, from (2.15) and eq. (3.26) of Veerman and Holmes [5] we have

$$M_h^{m,n}(\psi) = \frac{4T_1}{\pi m \omega} \sum_{\mu=0}^{\infty} \frac{\operatorname{sech}\left(\pi\left(\mu+\frac{1}{2}\right)m\tau_1\right)\operatorname{sech}\left(\pi\left(\mu+\frac{1}{2}\right)n\tau_2\right)}{(2\mu+1)} \sin\left((2\mu+1)m\psi\right), \tag{3.3}$$

where

$$\tau_i = K\left(\frac{1-h_i}{2}\right)\Big/K\left(\frac{h_i}{2}\right), \quad i = 1, 2, \tag{3.4}$$

and we have used the fact that $\psi = \omega_1 t_0 = 2\pi t_0/T_1$.

In Veerman and Holmes [5], we showed that M has simple zeros at $t_0 = kT_1/2m$, and hence $M_h^{m,n}$ has simple zeros at $\psi = k\pi/m$, $k = 0, 1, \ldots, 2m-1$ in each interval $0 \le \psi < 2\pi$. We now need to compute $V_{\max}$ and $V_{\min}$ over that interval, where $V_{\max}$ and $V_{\min}$ are the values of the maxima and minima of $V_h^{m,n} = \int M_h^{m,n}\,d\psi$. From (3.3) we have

$$V_{\max} - V_{\min} = \frac{8T_1}{\pi m^2 \omega} \sum_{\mu=0}^{\infty} \frac{\operatorname{sech}\left(\pi\left(\mu+\frac{1}{2}\right)m\tau_1\right)\operatorname{sech}\left(\pi\left(\mu+\frac{1}{2}\right)n\tau_2\right)}{(2\mu+1)^2}. \tag{3.5}$$

We also require

$$\bar{\omega}_h^{m,n} = \frac{\partial^2 L^0}{\partial I_1^2} = \frac{\partial}{\partial I_1}\left\{-\omega_1(I_1)/\omega_2\left(L^0(I_1; h)\right)\right\}\Big|_{I_1 = I_1^{m,n}}.$$

From Veerman and Holmes [5, eq. (3.4)], including $\omega \neq 1$, we have

$$\frac{\omega_1}{\omega_2} = \frac{T_2}{T_1} = \frac{\omega K((h-h_1)/2)}{K(h_1/2)}. \tag{3.6}$$

Using the derivative of K:

$$\frac{\mathrm{d}K}{\mathrm{d}\alpha}(\alpha)=\frac{1}{2\alpha(1-\alpha)}\left(E(\alpha)-(1-\alpha)K(\alpha)\right)$$

(cf. ref. 19), we compute from (3.1) and (3.6):

$$\bar{\omega}_h^{m,n}=-\frac{\mathrm{d}h_1}{\mathrm{d}I_1}\frac{\mathrm{d}}{\mathrm{d}h_1}\left(\frac{\omega_1}{\omega_2}(h_1)\right)=-\frac{\omega\pi}{2K(h_1/2)}\frac{\mathrm{d}}{\mathrm{d}h_1}\left(\frac{\omega_1}{\omega_2}(h_1)\right)$$

$$=\frac{-\pi\omega^2}{8K^3\left(\frac{h_1}{2}\right)}\left\{\frac{E\left(\frac{h}{2}-\frac{h_1}{2}\right)K\left(\frac{h_1}{2}\right)}{\left(\frac{h}{2}-\frac{h_1}{2}\right)\left(1-\left(\frac{h}{2}-\frac{h_1}{2}\right)\right)}+\frac{K\left(\frac{h}{2}-\frac{h_1}{2}\right)E\left(\frac{h_1}{2}\right)}{\frac{h_1}{2}\left(1-\frac{h_1}{2}\right)}-\frac{\frac{h}{2}K\left(\frac{h}{2}-\frac{h_1}{2}\right)K\left(\frac{h_1}{2}\right)}{\frac{h_1}{2}\left(\frac{h}{2}-\frac{h_1}{2}\right)}\right\}. \tag{3.7}$$

The general expression for bandwidth $\Delta I(m, n; h)$ can now be obtained from (3.5), (3.7) and (2.13). For our specific case we shall further simplify the expression by choosing $h=2$, so that we can use Legendre's identity

$$E(1-\alpha)K(\alpha)+E(\alpha)K(1-\alpha)-K(1-\alpha)K(\alpha)=\pi/2, \tag{3.8}$$

(Byrd and Friedman (1971)). Also, for $h=2$, we have

$$\tau_1=n/\omega m,\ \tau_2=\omega m/n, \tag{3.9}$$

from (3.2) and (3.4), so that (3.5), (3.7), (3.8–3.9) and (2.13) give

$$\Delta I(m,n,\omega;\ h=2)=\frac{64\sqrt{\varepsilon}K^2\left(\frac{h_1}{2}\right)}{\pi m\omega^2}\left\{\frac{1}{\pi}\left(\frac{h_1}{2}\right)\left(1-\frac{h_1}{2}\right)\right.$$
$$\left.\times\sum_{\mu=0}^{\infty}\frac{\operatorname{sech}\left(\pi\left(\mu+\frac{1}{2}\right)\frac{n}{\omega}\right)\operatorname{sech}\left(\pi\left(\mu+\frac{1}{2}\right)\omega m\right)}{(2\mu+1)^2}\right\}^{1/2}. \tag{3.10}$$

Finally we estimate the infinite sum of (3.10) by taking only the first term, the error being

$$R(m,n,\omega,2)=\sum_{\mu=1}^{\infty}\frac{\operatorname{sech}\left(\pi\left(\mu+\frac{1}{2}\right)\frac{n}{\omega}\right)\operatorname{sech}\left(\pi\left(\mu+\frac{1}{2}\right)\omega m\right)}{(2\mu+1)^2}, \tag{3.11}$$

which we can bound using $(2\mu+1)^2 \geq 9$ for $\mu \geq 1$ and $\operatorname{sech}(x) \leq 2\,\mathrm{e}^{-x}$, to obtain

$$R(m,n,\omega;2) \leq \tfrac{4}{9} \sum_{\mu=1}^{\infty} \mathrm{e}^{-\pi(\mu+\frac{1}{2})(n/\omega+\omega m)} = \tfrac{4}{9} \sum_{\nu=0}^{\infty} \mathrm{e}^{-\pi(n/\omega+\omega m)(\frac{3}{2}+\nu)}$$
$$= \tfrac{4}{9}\,\mathrm{e}^{-\frac{3}{2}\pi(n/\omega+\omega m)} \Big/ \left(1-\mathrm{e}^{-\pi(n/\omega+\omega m)}\right). \tag{3.12}$$

The relative error is therefore bounded above by

$$R_{\mathrm{e}} \leq \tfrac{4}{9}\,\mathrm{e}^{-\frac{3}{2}\pi(n/\omega+\omega m)} \left(\left(1-\mathrm{e}^{-\pi(n/\omega+\omega m)}\right)\left(\operatorname{sech}\left(\frac{\pi n}{2\omega}\right)\operatorname{sech}\left(\frac{\pi\omega m}{2}\right)\right)\right)^{-1},$$

or

$$R_{\mathrm{e}} \leq \tfrac{4}{9}\,\frac{\mathrm{e}^{-2\pi}(n/\omega+\omega m)}{\left(1-\mathrm{e}^{-\pi(n/\omega+\omega m)}\right)}, \tag{3.13}$$

using $\operatorname{sech}(x) \geq \mathrm{e}^{-x}$ for $x \geq 0$. Then using the inequality $n/\omega + \omega m \geq 2\sqrt{nm}$ we obtain

$$R_{\mathrm{e}} \leq \tfrac{4}{9}\,\mathrm{e}^{-2\pi(nm)^{1/2}}\left(\left(1-\mathrm{e}^{-2\pi(nm)^{1/2}}\right)\right)^{-1} \leq 0.1\%, \quad \text{for all } m, n \geq 1. \tag{3.14}$$

Then our final estimate for bandwidth for arbitrary (m,n) and $h=2$ is

$$\Delta I(m,n,\omega;2) = \frac{64\sqrt{\varepsilon}\,K^2\left(\frac{h_1}{2}\right)}{\pi m \omega^2}\left\{\frac{h_1}{2\pi}\left(1-\frac{h_1}{2}\right)\operatorname{sech}\left(\frac{\pi n}{2\omega}\right)\operatorname{sech}\left(\frac{\pi\omega m}{2}\right)\right\}^{1/2} + \mathcal{O}(\varepsilon), \tag{3.15}$$

with a maximum possible error of 0.05% in the $\mathcal{O}(\sqrt{\varepsilon})$ term (the square root effectively halves the relative error). Recall that $h_1 = h_1(m, n; h=2)$ is given by eq. (3.2).

4. Numerical computation

Using a fourth order Runge–Kutta scheme, we integrated Hamilton's equations corresponding to (1.1) in the original p_i, q_i coordinates and plotted Poincaré sections by the standard technique of recording values of p_1, q_1 each time $q_2 = 0$ with $p_2 \geq 0$. We checked conservation of total energy and, using a step size varying between 0.02 and 0.06 found that it remained within 0.05% of its starting value ($h=2$) over 10^4 to 5×10^5 steps. We estimate the maximum error in our bandwidth measurements to be of the order of 1%.

Since the bandwidths are measured in the (q_1, p_1)-Poincaré section, it is necessary to convert the width (3.15) in action space back to (q_1, p_1) space. As the Melnikov function indicates, a hyperbolic periodic point for each resonance band lies on the positive p_1 axis in the section, and since only odd resonances are excited to first order ($M_h^{m,n}(\psi) \equiv 0$ of m and/or n even), the maximum bandwidths occur on or near the negative p_1 axis (cf. fig. 2, below). Thus we compute Δp_1, using

$$\Delta p_1 = \left.\frac{\partial p_1}{\partial h_1}\right|_{q_1=0}\frac{\partial h_1}{\partial I_1}\cdot\Delta I = \frac{1}{\sqrt{2h_1}}\cdot\frac{\omega\pi}{2K(h_1/2)}\cdot\Delta I, \tag{4.1}$$

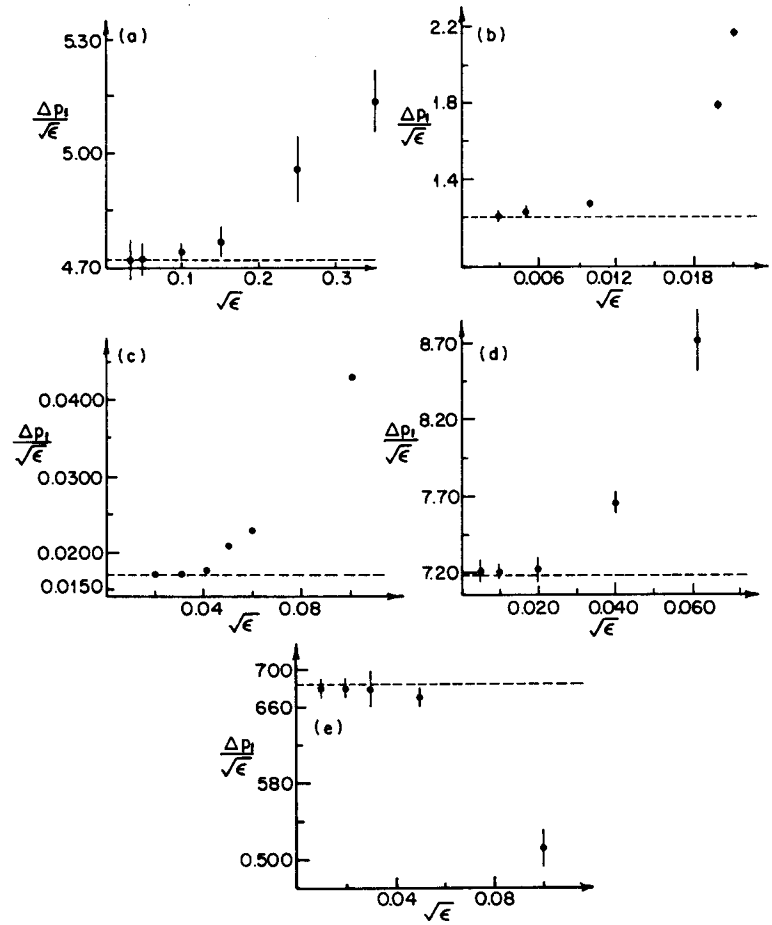

Fig. 1. Comparison of eq. (4.2) (– –) and numerical computation (I) of resonance bandwidths: a) $\omega = 1$, $m = n = 1$; b) $\omega = 1$, $m = 3$, $n = 1$; c) $\omega = 1$, $m = 5$, $n = 3$; d) $\omega = 0.5$, $m = n = 1$; e) $\omega = 0.225$, $m = 1$, $n = 3$.

so that we have

$$\Delta p_1(m, n, \omega; 2) = \frac{16\sqrt{\varepsilon} K\left(\frac{h_1}{2}\right)}{m\omega} \left\{ \frac{1}{\pi}\left(1 - \frac{h_1}{2}\right) \operatorname{sech}\left(\frac{\pi n}{2\omega}\right) \operatorname{sech}\left(\frac{\pi \omega m}{2}\right) \right\}^{1/2}, \tag{4.2}$$

to first order. In figs. 1(a–e), we plot $\Delta p_1(m, n, \omega; 2)/\sqrt{\varepsilon}$ for several low (m, n) and three values of ω and in fig. 2 we show a typical Poincaré section to illustrate the resonance bands. The widths were estimated by finding points (initial conditions) on $q_1 = 0$, $p_1 < 0$ whose orbits pass above and below the hyperbolic point on $q_1 = 0$, $p_1 > 0$ (see fig. 2). Computations were all performed in double precision FORTRAN on a VAX 11/750 using an AED 767 color graphics monitor in an interactive mode.

From this (rather limited) sample of computations, we conclude that, while the asymptotic formula (4.2) does provide a good estimate of resonance bandwidths as $\varepsilon \to 0$, it can be very poor for relatively small ε even for the low order resonances investigated here (see fig. 1b, for example, in which (4.2) is 50% in error for $\varepsilon = 4 \times 10^{-4}$). Note that the asymptotics depend on n and m, i.e.: for a *given* value of ε we cannot

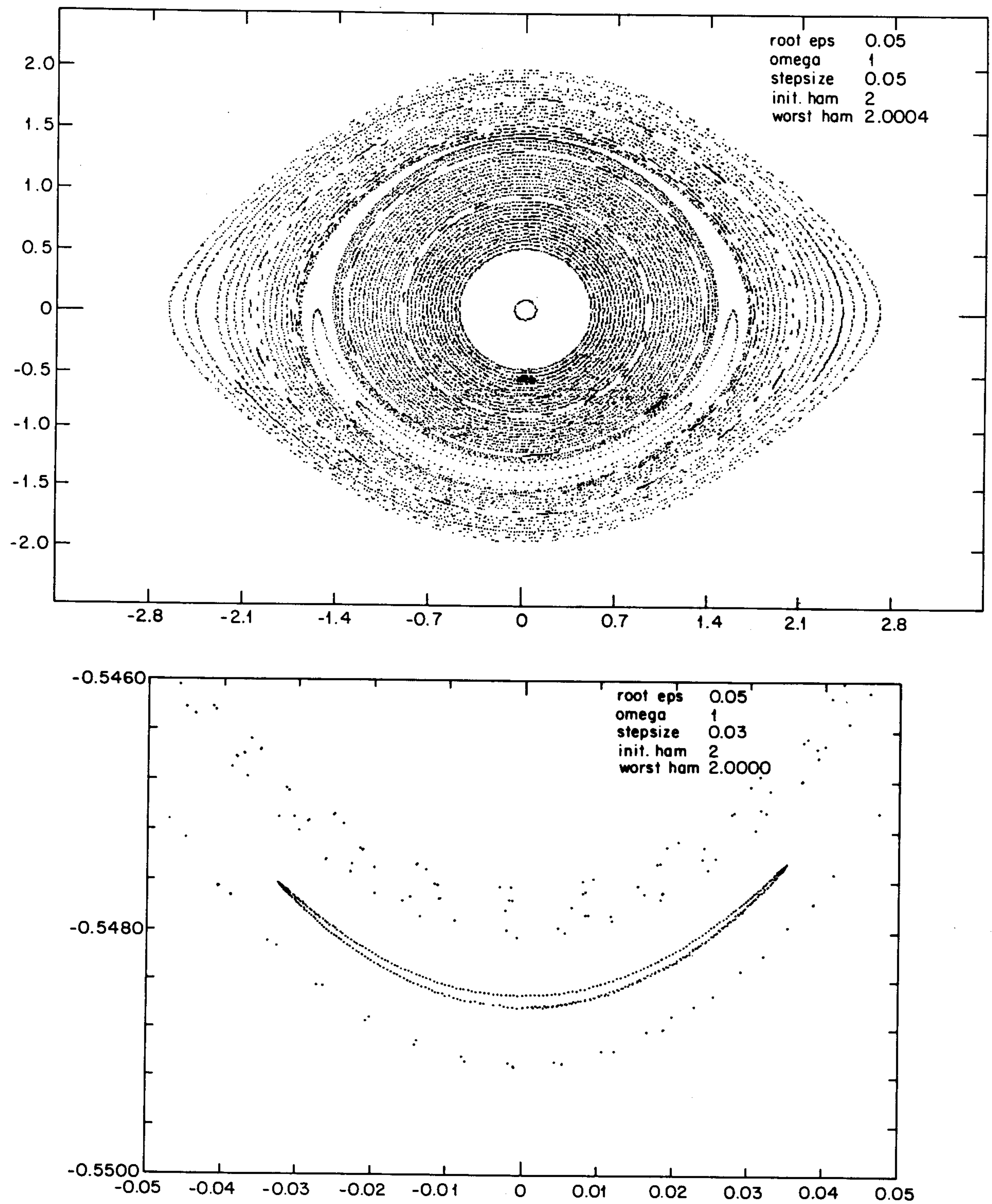

Fig. 2. Numerical computation of the (p_1, q_1) Poincaré section, showing (1/1) and (5/3) resonance bands. (5/3) picture (below) is an enlargement of the small boxed region on (1/1) picture (above).

calculate the bandwidths for all resonances. Perhaps more significant than the numerical values of ε, however, is the fact that such errors occur long before low order, neighboring, resonance bands appear to overlap. Thus, while in fig. 2 large sets of smooth invariant tori appear to exist between the resonant bands, the widths of the latter are already far from the $\mathcal{O}(\sqrt{\varepsilon})$ estimates of simple perturbation theory. This suggests that, in addition to taking higher order 'intermediate' resonances into account, as in Lichtenberg and Lieberman [4, §4.2], higher order ($\mathcal{O}(\varepsilon)$, etc.) terms should also be included in the estimates of individual bandwidths. Note in this connection that in figs. 1a, 1c and 1d, after the behavior $\Delta p \sim \sqrt{\varepsilon}$, there seems to be an indication of $\Delta p \sim \varepsilon$ behavior which is the next order of the perturbation theory.

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