THE TOPOLOGY OF SURFACE MEDIATRICES

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Abstract. Given a pair of distinct points p and q in a metric space with distance d, the mediatrix is the set of points x such that $d(x,p) = d(x,q)$. In this paper, we examine the topological structure of mediatrices in connected, compact, closed 2-manifolds whose distance function is inherited from a Riemannian metric. We determine that such mediatrices are, up to homeomorphism, finite, closed simplicial 1-complexes with an even number of incipient edges emanating from each vertex. Using this and results from [7], we give the classification up to homeomorphism of mediatrices on genus 1 tori (and on projective planes) and outline a method which may possibly be used to classify mediatrices on higher-genus surfaces.

Let M be a compact, connected, n-dimensional Riemannian manifold. For any $p, q \in M$, let the distance $d(p,q)$ from p to q be defined as usual to be the infimum of the lengths of all piecewise differentiable curves in M from p to q. For any $p, q \in M$, the *mediatrix* L_{pq} is the set of all points which are equidistant from p and q:

$$L_{pq} = \{x \in M \mid d(x,p) = d(x,q)\}.$$

In [7], some topological restrictions placed on L_{pq} by the topology of M were found. In this paper, we focus on the particular case in which M is a 2-manifold to determine what can said about L in that case. In particular, consider two mediatrices L_{pq} and L'_{pq} in a given manifold M equivalent if there is homeomorphisms $\phi : L_{pq} \to L'_{pq}$. (Note that the homeomorphism ϕ is not required to preserve the surface M.) We investigate the question of which classes of mediatrices can occur on a surface if the metric $d(.,.)$ and the points p and q are allowed to vary.

In Section 1, we examine the local structure of L and show that L is a finite closed simplicial 1-complex. Next, in Section 2, we use this to classify mediatrices on genus 1 tori up to homeomorphism, and in Section 3, we discuss the classification of mediatrices on surfaces of higher genus. We conclude with an outline of some open questions relating to the classification of mediatrices.

1. Surface mediatrices as simplicial 1-complexes

In this section, we let M denote a compact, connected, 2-dimensional Riemannian manifold with associated distance function $d : M \times M \to \mathbb{R}$, as defined above. We refer to a mediatrix on such a manifold M as a *surface mediatrix*, and we continue to denote the mediatrix associated with distinct points $p, q \in M$ by L_{pq}.

The main result that we establish in this section is the following:

Theorem 1.1. *Any surface mediatrix is homeomorphic to a closed finite simplicial 1-complex.*

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The proof of this theorem will take up most of this section. We begin by examining the local structure of a surface mediatrix $L_{pq} \subset M$.

At any point $x \in L_{pq}$ outside the cut loci of both p and q, the distance function $d_{pq}(y) = d(y,p) - d(y,q)$ is differentiable and nonsingular (see [6], for example). Since L_{pq} is the zero set of d_{pq}, then by the Implicit Function Theorem, x has a neighborhood in L_{pq} which is diffeomorphic to an open interval in $\mathbb{R}$. However, at points in either cut locus, the function d_{pq} may fail to be differentiable. At such points, the Implicit Function Theorem does not apply, so we will need another technique to analyze the local structure of L_{pq}. We use a technique somewhat similar to that used by Myers in [5] in examining the structure of the cut locus C_p for a point p on a surface. However, while he looks at geodesics near p (and not near a point on the cut locus) for his purposes, we instead look at geodesics near a point on a mediatrix (and not near the points p or q defining the mediatrices).

For this, we consider the tangent space T_xM at an arbitrary point $x \in L_{pq}$, or, more specifically, the unit circle S_xM within that tangent space:

$$S_xM = \{\mathbf{v} \in T_xM \mid |\mathbf{v}| = 1\}.$$

Thinking of S_xM as the set of "directions" for geodesics at x, we single out those directions which give rise to minimal geodesics to p and to q: let

$$\Theta_p = \{\mathbf{v} \in S_xM \mid t \mapsto \exp_x(t\mathbf{v}) \text{ is a minimal geodesic to } p\}$$
$$\Theta_q = \{\mathbf{v} \in S_xM \mid t \mapsto \exp_x(t\mathbf{v}) \text{ is a minimal geodesic to } q\}.$$

We will soon examine how L_{pq} is situated with respect to these minimal geodesic directions, but first we have some preliminary lemmas concerning Θ_p and Θ_q.

Lemma 1.2. *The sets Θ_p and Θ_q are disjoint compact subsets of S_xM.*

Proof. The sets Θ_p and Θ_q are disjoint because if $\mathbf{v} \in \Theta_p \cap \Theta_q$, then $t \mapsto \exp_x(t\mathbf{v})$ is a minimal geodesic going both to p and to q. However, since $x \in L_{pq}$ is equidistant from p and q, this would imply that $p = q$, contrary to our assumption that p and q are distinct.

To show that Θ_p is compact, we need to show that it is closed. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots\}$ be a sequence of vectors in Θ_p converging to a vector $\mathbf{v} \in T_xM$. Then $\mathbf{v} \in S_xM$ by the continuity of the norm. Now by geodesic completeness (M is compact), the image of $\mathbb{R}$ under the map $t \mapsto \exp_x(t\mathbf{v})$ is a geodesic. The length of all the geodesic segments $t \mapsto \exp_x(t\mathbf{v}_i)$ from x to p is the same, namely $d(x,p) = r$. The continuity of the map $\phi : (t, \mathbf{v}) \mapsto \exp_x(t\mathbf{v})$ implies that $\exp_x(r\mathbf{v})$ equals p. This means that $\mathbf{v} \in \Theta_p$, so $\Theta_p \subset S_xM$ is closed and hence compact. The set Θ_q is compact as well by the same argument. $\square$

In the above proof we have really used sequential compactness. We are allowed to do this since for a metrizable space it is equivalent to compactness.

We can now isolate the regions in which L_{pq} lies, at least locally. First of all, L_{pq} will not intersect geodesics from x to p or from $xinL_{pq}$ to q, except at x, by the following lemma, proved in [7]:

Lemma 1.3. *Let $p, q \in M$ with $p \neq q$. Suppose x and y are (not necessarily distinct) points in a mediatrix L_{pq}. Let γ be a minimizing path connecting the* *points x and p or q and η a minimizing path connecting y to either p or q. Then* $\overset{\circ}{\gamma} \cap \overset{\circ}{\eta} = \emptyset$.

As usual $\overset{\circ}{\gamma}$ denotes the interior of γ.

Since L_{pq} does not intersect these minimal geodesics, we can use the sets Θ_p and Θ_q to divide up the tangent space into regions corresponding (via $\exp_x : T_xM \to M$) to those where L_{pq} might lie locally. More specifically, we have proved the following lemma.

Lemma 1.4. *Let $\rho > 0$ be chosen smaller than the radius of injectivity of the exponential map at $x \in L_{pq}$, so that $\exp_x$ maps the disk of radius ρ centered at the origin in T_xM diffeomorphically onto its image in M. Also, choose ρ small enough that $\rho < d(x,p) = d(x,q)$. Then*

$$L_{pq} \cap \{\exp_x(t\Theta_p) \mid 0 \le t < \rho\} = \{x\}$$
$$\text{and } L_{pq} \cap \{\exp_x(t\Theta_q) \mid 0 \le t < \rho\} = \{x\}.$$

Near $x \in L_{pq}$ then, the mediatrix lies, roughly speaking, only in directions "between" a direction in Θ_p and a direction in Θ_q. In order to establish a precise notion of "betweenness" of directions, we choose an orthonormal basis of T_xM. With respect to this basis, we can write any vector $\mathbf{v} \in T_xM$ as

$$\mathbf{v} = (r\cos\theta, r\sin\theta) \in T_xM,$$

where as usual, $\theta \in \mathbb{R}$ and $r \ge 0$. This leads us to the following definition.

Definition 1.5. *A* wedge of radius ρ *is a set $W_\rho \subset T_xM$ with the following two properties:*

(1) *It can be written as*

$$W_\rho = \{(r\cos\theta, r\sin\theta) \in T_xM \mid 0 \le r \le \rho \textit{ and } \theta_p \le \theta \le \theta_q\}$$
$$\text{or } W_\rho = \{(r\cos\theta, r\sin\theta) \in T_xM \mid 0 \le r \le \rho \textit{ and } \theta_q \le \theta \le \theta_p\},$$

for some $\theta_p \in \Theta_p$ and some $\theta_q \in \Theta_q$.

(2) *The intersections* $\mathrm{int}(W_\rho) \cap \Theta_p$ *and* $\mathrm{int}(W_\rho) \cap \Theta_q$ *are both empty.*

In other words, W_ρ is a sector of the disk of radius ρ centered at the origin in T_xM which lies between two minimal geodesic directions, one to p and the other to q, and which contains no minimal geodesic directions in its interior. The wedges are the shaded regions in the picture below.

A first thing to notice about these wedges is that we cannot have infinitely many of them.

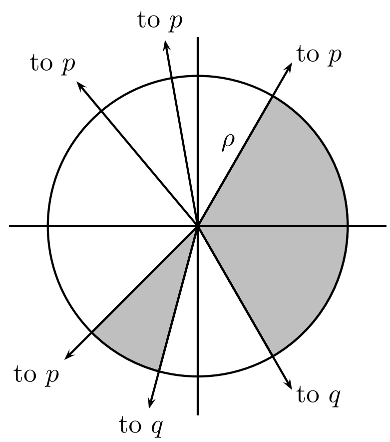


Figure 1.6: The definition of wedges

Lemma 1.7. *The are only finitely many wedges of any given radius.*

Proof. Assume we have an infinite sequence of distinct wedges of radius ρ. Then one of the limiting angles determining each wedge must give rise to a minimal geodesic to p, and the other to a minimal geodesic to q. From this, we can construct, as follows, two sequences of vectors

$$\mathbf{v}_{p,i} = (\rho\cos\theta_{p,i}, \rho\sin\theta_{p,i})$$
$$\mathbf{v}_{q,i} = (\rho\cos\theta_{q,i}, \rho\sin\theta_{q,i})$$

on the circle of radius ρ in T_xM, the first giving rise to minimal geodesics $t \mapsto \exp(t\mathbf{v}_p)$ to p, and the second giving rise to minimal geodesics $t \mapsto \exp(t\mathbf{v}_q)$ to q.

Starting, say, at $(\rho, 0)$, we proceed counterclockwise along the circle of radius ρ in T_xM until we hit a new wedge. We take $\theta_{p,1}$ (which determines $\mathbf{v}_{p,1}$) to be the boundary angle of that wedge giving rise to a minimal geodesic to p. Then we continue counterclockwise along the circle of radius ρ until we come to another wedge. From this second wedge, we take $\theta_{q,1}$ (which determines $\mathbf{v}_{q,1}$) to be the boundary angle of that wedge giving rise to a minimal geodesic to q. We then continue counterclockwise along the circle of radius ρ to obtain $\theta_{p,2}$, then $\theta_{q,2}$, and so forth. Since we are assuming there are infinitely many wedges, this gives us two infinite sequences of vectors, $\{\mathbf{v}_{p,i}\}$ and $\{\mathbf{v}_{q,i}\}$.

By the compactness of the circle of radius ρ, there must be subsequences $\{\mathbf{v}_{p,i_j}\}$ and $\{\mathbf{v}_{q,i_k}\}$ which converge, but by the very construction of the two sequences (with any term in one being "between" the two adjacent terms in the other), these two subsequences must converge to a common limit $\mathbf{v}$. This is a contradiction, since that common limit must give rise to a minimal geodesic $t \mapsto \exp(t\mathbf{v})$ to both p and q. $\square$

Next we observe that we can actually add to Lemma 1.4: near the point x, not only does the mediatrix L_{pq} not intersect the exponential of the "sides" of the wedge, as in the lemma, but L_{pq} lies within the wedges, or more correctly, within the image of the union of the wedges under the exponential map. In the following D_ρ denotes the open disc $D_\rho \subset T_xM$ of radius ρ.

Lemma 1.8. *For suitably small $\rho > 0$, the set $L_{pq} \cap \exp_x(D_\rho)$ is contained in*

$$\bigcup_{i=1}^{k} \exp_x(W_{\rho,i}) \quad ,$$

where $W_{\rho,1}, \ldots, W_{\rho,k}$ are the (finitely many, by Lemma 1.7) wedges of radius ρ.

Proof. In order to obtain a contradiction, let us assume that for all $\rho > 0$, the set $L_{pq} \cap \exp_x(D_\rho - \bigcup_{i=1}^{k} W_{\rho,i})$ contains some point other than x. Since $L_{pq} \cap (t\Theta_p \cup t\Theta_q) = \emptyset$ for all t with $0 < t < \rho$. This assumption implies that

$$L_{pq} \cap \exp_x \left(D_\rho - \left(\cup_{i=1}^{k} W_{\rho,i} \cup_t t\Theta_p \cup_t t\Theta_q\right)\right)$$

contains some point other than x. Now since there are only finitely many wedges, this set is contained in of the union of finitely many sectors of the disc D_ρ which are of the form

$$\begin{aligned} S_p &= \{(r\cos\theta, r\sin\theta) \in T_xM \mid 0 \le r < \rho \text{ and } \theta_{p,1} < \theta < \theta_{p,2}\} \\ \text{or } S_q &= \{(r\cos\theta, r\sin\theta) \in T_xM \mid 0 \le r < \rho \text{ and } \theta_{q,1} < \theta < \theta_{q,2}\}, \end{aligned}$$

where $\theta_{p,1}, \theta_{p,2} \in \Theta_p$ and $\theta_{q,1}, \theta_{q,2} \in \Theta_q$ are limiting angles for some wedges (of radius ρ). From the definition of the wedges it follows that we can impose the additional restrictions that $S_p \cap \Theta_q = \emptyset$ and $S_q \cap \Theta_p = \emptyset$. For example, see the unshaded regions inside D_ρ in Figure 1.6 above.

Since there are only finitely many such sectors, the assumption that for all $\rho > 0$, the set $L_{pq} \cap \exp_x(D_\rho - \bigcup_{i=1}^{k} W_{\rho,i})$ contains some point other than x allows us to find a sequence $\{x_i\}$ of points in $L_{pq} - \{x\}$ converging to x and contained entirely within the exponential of a single such sector. We denote this sector by S_p, and without loss of generality assume it is of the first form above, so

$$S_p = \{(r\cos\theta, r\sin\theta) \in T_xM \mid 0 \le r < \rho \text{ and } \theta_{p,1} < \theta < \theta_{p,2}\}$$

and $\{x_i\} \subset \exp_x(S_p)$.

From each point x_i in this sequence, there is at least one minimal geodesic γ_i to q, which can be parametrized as $\gamma_i(t) = \exp_{x_i}(t\mathbf{v}_i)$ for some unit vector $\mathbf{v}_i \in T_{x_i}M$. Now we choose $\rho > 0$ small enough that TM is trivial when restricted to the neighborhood $\exp_x(\overline{D}_\rho)$ of x, where $\overline{D}_\rho \subset T_xM$ is the closed disc of radius ρ centered at the origin in T_xM. Then TM restricted to $\exp_x(\overline{D}_\rho)$ is diffeomorphic to $\overline{D} \times \mathbb{R}^2$, where $\overline{D}$ is a closed 2-disc, and the sequence $(x_i, \mathbf{v}_i)$ can be thought of as lying in $\overline{D} \times S^1$. By the compactness of $\overline{D} \times S^1$, there is a subsequence $(x_{i_j}, \mathbf{v}_{i_j})$ converging to $(x, \mathbf{v}) \in T_xM$ for some $\mathbf{v} \in T_xM$.

Since $x_{i_j} \to x$, then $d(x_{i_j}, q) \to d(x, q)$, meaning that the geodesic $\gamma(t) = \exp_x(t\mathbf{v})$ to which the geodesics $\gamma_{i_j}(t)$ converge is a minimal geodesic from x to q. This implies that $\mathbf{v} \in \Theta_q$, from which it follows that $\mathbf{v} \notin \overline{S_p}$. Consequently,

$$d(\exp_x \mathbf{v}, \exp_x \overline{S_p}) := \inf_{\mathbf{s} \in \overline{S}} d(\exp_x \mathbf{v}, \exp_x \mathbf{s}) > 0.$$

Because $x_{i_j} \to x$ and $\mathbf{v}_{i_j} \to \mathbf{v}$, then by the continuity of the exponential map, we have $\gamma_{i_j}(t) \to \gamma(t)$ for all $t \in [0, \rho]$, uniformly since $[0, \rho]$ is compact. Given any $\varepsilon > 0$ then, $d(\gamma_{i_j}(t), \gamma(t)) < \varepsilon$ for all $t \in [0, \rho]$ for suitably large j. In particular, this holds for $\varepsilon = d(\exp_x \mathbf{v}, \exp_x \overline{S_p})$. For suitably large j for this ε, however, $\gamma_{i_j}(0) = x_{i_j} \in S_p$ and $\gamma_{i_j}(\rho) \notin S_p$, and it is apparent from the picture below that this implies that the minimal geodesic γ_{i_j} to q must intersect one of the "sides" of the sector S_p at some point:

$$\gamma_{i_j}(t) = \exp_x(r\cos\theta_{p_1}, r\sin\theta_{p_1})$$
$$\text{or } \gamma_{i_j}(t) = \exp_x(r\cos\theta_{p_2}, r\sin\theta_{p_2})$$

for some $r, t \in (0,\rho)$. In Figure 1.9, the dotted lines represent the ε-neighborhood of $\{\gamma(t)\,|\,t \in [0,\rho]\}$ which must contain the curve $\{\gamma_{i_j}(t)\,|\,t \in [0,\rho]\}$.

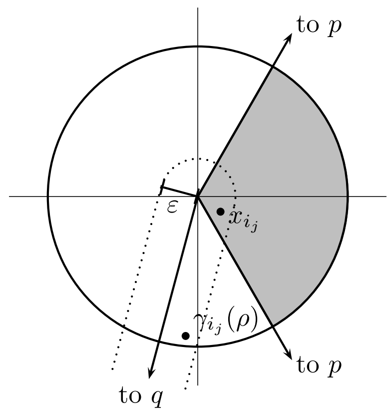


Figure 1.9: An ε-neighborhood of a radial geodesic.

This is a contradiction by Lemma 1.3, and so the lemma is proved. □

We now determine what L_{pq} looks like within each wedge. For any wedge $W_{\rho,m}$, we call the set $\exp_x^{-1}(L_{pq}) \cap W_{\rho,m} - \{0\}$ the *m-th spoke* of radius ρ at x, and we denote it by L_m. Also, let us assume the number of wedges at x is k.

In the proof that follows there is a technical difficulty arising from the fact that the lift of geodesics not based at x are no longer necessarily straight lines. However, the segments contained in D_ρ differ from straight lines only by small amounts. The next few remarks make this precise.

Let D_ρ be a disk of radius ρ in T_xM. We will use geodesic coordinates at x in M. Let $\{v_1, v_2\}$ an orthonormal basis for T_xM and let $||\cdot||$ denote the (Riemannian) norm in T_xM. Note that if we lift the ρ-neighborhood of x in M by $\exp_x^{-1}$ we obtain the ρ-neighborhood D_ρ of the origin in T_xM. We can now choose a local parametrization ϕ of M in a neighborhood of x as follows:

$$\phi : D_\rho \cap \mathbb{R}^2 \simeq D_\rho \cap T_xM \;\to M$$
$$\phi(x_1, x_2) = \exp_x(x_1v_1 + x_2v_2) \quad .$$

Next suppose that $\gamma(t)$ is a geodesic (parametrized by arc-length) such that $\gamma(0)$ is in a neighborhood of x.

Proposition 1.10. *There is a $C > 0$ such that if $\rho > 0$ is small enough, then for any point x in M and any geodesic $\gamma(t) = \phi(\vec{x}(t))$ restricted to a ρ-neighborhood of x and with $\gamma(0) = \phi(\vec{x}(0))$ and $\dot\gamma(0) = d\phi_{\vec{x}(t)}(\dot{\vec{x}}(0))$, we have that in D_ρ*

$$||\vec{x}(t) - (\dot{\vec{x}}(0)t + \vec{x}(0))|| < Ct^2\rho$$

(where along the geodesic segment $|t| < 2\rho$ by hypothesis).

This result appears well-known, although we haven't been able to find this exact statement in the literature. It is slightly different from a statement that can be found in [3] (we will also use their version). For completeness we give a proof (different from the one in [3]) in the appendix. Also this is not quite the same as the statement found in [2] section 5.2, although the constant C is related to the sectional curvature.

We will also need to characterize Lipschitz functions in what follows. Let $f : \mathbb{R} \to \mathbb{R}$ and define the following subsets of $\mathbb{R}^2$

$$G_{x>x_0} = \{(x, f(x)) \,|\, x > x_0\} \ , \ G_{x<x_0} = \{(x, f(x)) \,|\, x < x_0\} \quad .$$

The half-cone C_{x_0,θ_0} is defined as

$$C_{x_0,\theta_0} = \{(x_0, f(x_0)) + (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \,|\, r \geq 0, -\theta_0 \leq \theta \leq \theta_0\} \quad .$$

If for all x_0 we have that $G_{x>x_0} \subset C_{x_0,\theta_0}$ with $\theta_0 \in (0, \pi/2)$, then of course f is Lipschitz. This is the forward cone criterion.

Similarly we can define a backward cone criterion for f to be Lipschitz. Suppose that this time $\theta_0 \in (\pi/2, \pi)$ and we have that for all x_0: $G_{x<x_0} \subset \overline{\mathbb{R}^2 - C_{x_0,\theta_0}}$ (the closure of the complement), then again f is Lipschitz.

Proposition 1.11. *For any m with $1 \leq m \leq k$, the spoke L_m is, for all suitably small $\rho > 0$, diffeomorphic to the graph $\{(t, f(t)) \mid t \in (0,1)\}$ of some Lipschitz function $f : (0,1) \to \mathbb{R}$.*

Proof. In part A we make the argument for the case in which the angular width (or "aperture") of $W_{\rho,m}$ is less than π; the case for wider wedges is dealt with in part B of this proof.

A): By choosing an appropriate basis for T_xM then, we can arrange for the sides of the wedge $W_{\rho,m}$ to be at angles $-\theta_0$ and θ_0 for some angle θ_0 with $0 < \theta_0 < \pi/2$, as in Figure 1.12. Without loss of generality, we will assume that the geodesic in the direction of $-\theta_0$ goes to q, and the geodesic in the direction of θ_0 goes to p.

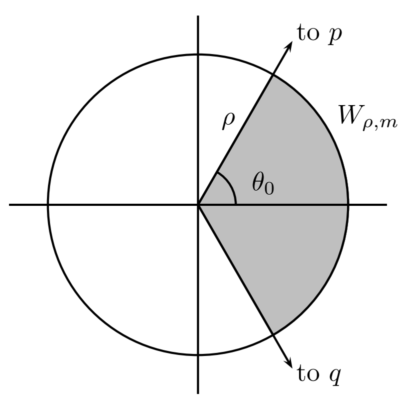


Figure 1.12: The angle θ_0.

We now examine the intersection of the m-th spoke L_m with vertical lines ℓ_ε given in the tangent space T_xM by setting the horizontal component equal to ε, for small positive values of $\varepsilon < \rho$, as pictured by the dashed line in Figure 1.13.

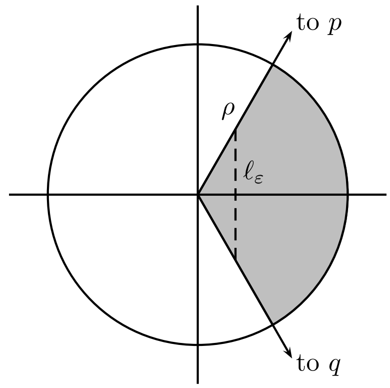


Figure 1.13: The definition of ℓ_ε.

We claim that, for suitably small positive values of ε, $L_m \cap \ell_\varepsilon$ contains exactly one point.

We know that it must contain at least one point since, by Lemma 1.3, the geodesic $t \mapsto \exp_x((t\cos\theta_0, t\sin\theta_0))$ is in L_p for $0 < t < \rho$, and similarly the geodesic $t \mapsto \exp_x((t\cos -\theta_0, t\sin -\theta_0))$ is in L_q for $0 < t < \rho$. By Lemma 2.6 of [7], the set L_{pq} separates M into L_p and L_q, so it follows that $\exp_x(\ell_\varepsilon)$ must pass through L_{pq} somewhere in the exponential of the wedge.

To show that $L_m \cap \ell_\varepsilon$ contains no more than one point, let us assume that there is a sequence $\{\varepsilon_i\}_{i=1}^\infty$ with $\lim_{i\to\infty} \varepsilon_i = 0$ such that each ℓ_{ε_i} contains two distinct points in L_m, in order to obtain a contradiction. Then, as in the proof of Lemma 1.8, we obtain two sequences $(a_i, \mathbf{v}_i)$ and $(b_i, \mathbf{w}_i)$ of elements in TM (continuing to use the same local trivialization), with the properties that (see Figure 1.14):

- There is a pair of points a_i, b_i is contained in $L_m \cap \ell_{\varepsilon_i}$ with the vertical component of a_i being greater than that of b_i.
- The map $t \mapsto \exp_{a_i}(t\mathbf{v}_i)$ is a minimal geodesic from a_i to q, and the map $t \mapsto \exp_{b_i}(t\mathbf{w}_i)$ is a minimal geodesic from b_i to p.
- $\lim_{i\to\infty} \mathbf{v}_i = (\cos -\theta_0, \sin -\theta_0)$ and $\lim_{i\to\infty} \mathbf{w}_i = (\cos\theta_0, \sin\theta_0)$.

(Note that the last property follows because — as in Lemma 1.8 — by geodesic completeness, for all i, $\mathbf{v}_i' = \exp_x^{-1}(\exp_{a_i}(t\mathbf{v}_i)) \in W_{\rho,m}$ and $\lim_{i\to\infty} \mathbf{v}_i \in \Theta_p$.) Now if $t \mapsto \exp_{a_i}(t\mathbf{v}_i)$ and $t \mapsto \exp_{b_i}(t\mathbf{w}_i)$ intersect, then by Lemma 1.3 we have a contradiction.

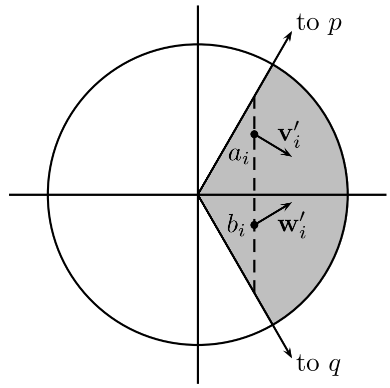


Figure 1.14: The vectors $\exp_x^{-1}(\exp_{a_i}(t\mathbf{v}_i))$ and $\exp_x^{-1}(\exp_{a_i}(t\mathbf{w}_i))$ in $W_{\rho,m}$.

Let us first examine the case in which M is flat (see Figure 1.15). In this case, locally we have that $M = T_xM$ and all geodesics are simply straight lines. Because of the third property above, we can choose a positive integer N, such that for all $i > N$, the slope of the line emanating in the direction of $\mathbf{v}_i$ is negative and close to $\mathbf{v}$, and the slope of the line emanating in the direction of $\mathbf{w}_i$ is positive and close to $\mathbf{w}$. For such an i, the two minimal geodesics $t \mapsto \exp_{a_i}(t\mathbf{v}_i)$ and $t \mapsto \exp_{a_i}(t\mathbf{w}_i)$ will intersect at some positive time t. Increasing i further will cause the limit of the distance between a_i and b_i to tend to zero, which means that the point of intersection of these two geodesics can be made to be within the wedge $W_{\rho,m}$. This gives a contradiction with Lemma 1.3.

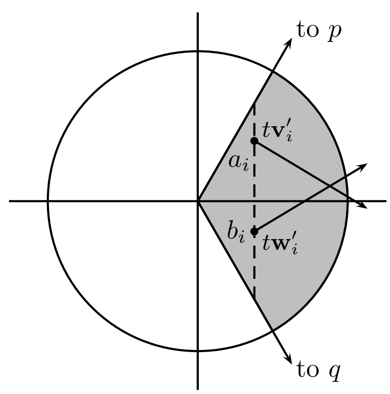


Figure 1.15: Minimal geodesics connecting a_i with q, and b_i with p.

The argument for the case when M is not flat is the same, but a little more care has to be taken since the geodesics not based at x are no longer necessarily straight lines. However, by virtue of Proposition 1.10, for ρ small enough their lifts under $\exp_x^{-1}$ can be made arbitrarily close to straight segments in D_ρ. Thus their lifts in the tangent space intersect. Back in the manifold the geodesics must then also intersect, again contradicting Lemma 1.3. The situation is depicted in Figure 1.16.

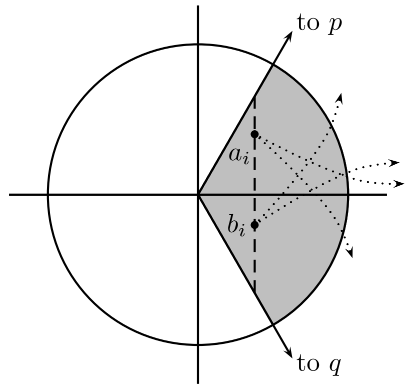


Figure 1.16: The effect of curvature on geodesics that miss the origin.

Therefore L_m intersects each ℓ_ε exactly once, for all suitably small values of ε, so L_m is diffeomorphic to the graph of some function. Furthermore, this graph does not stray outside the sides of the wedge, so the function is Lipschitz at the origin, the Lipschitz constant coming from the angle forming the wedge $W_{\rho,m}$.

To see that L_m satisfies the forward cone criterion one must apply the same argument at other points of L_m. However, suppose that $a \in L_m$ and η and γ are the lifts to D_ρ of the shortest paths from a to p and q, respectively. Then using the Proposition 1.10 again, η and γ are very nearly straight segments and their slope for reasons of continuity and completeness, is close to that of the sides of $W_{\rho,m}$. Thus the angle of a given wedge will vary continously along any path proceeding for a suitably short positive distance from x into that wedge. This means that the Lipschitz constant varies continuously in a neighborhood of x within the wedge. Consequently, on suitably small closed ball, there will be a maximum Lipschitz constant K over all points in the ball. Therefore, in a suitably small closed ball around the origin in T_xM, L_m is diffeomorphic to the graph of a Lipschitz function (with Lipschitz constant K).

B): The second part of this proof concerns the situation where $W_{\rho,m}$ has aperture π or bigger ($\theta_0 \in [\pi/2, \pi)$). Here there are two possibilities. The first is that the aperture is greater than π or $\theta_0 > \pi/2$. In this case we reason very much like before except we use the closure of the complement of C_{x_0,θ_0}. This backward cone has aperture less than π and the result now follows by using the backward cone criterion.

The last case is the awkward one where $\theta_0 = \pi/2$. For the following argument we refer to Figure 1.17. Consider again $D_\rho \subset T_xM$ and $x \in L_{pq}$ as defined before, but suppose that $\theta_0 = \pi/2$. In $W_{\rho,m}$ find another point $a \in L_{pq}$. Clearly the shortest geodesics from a to p respectively q restricted to D_ρ must be very close to straight segments parallel to those emanating from the origin. Suppose that $b \in L_m$ as pictured in Figure 1.17. Recall that a constant inner product (given by the Riemannian metric at x) measures distance in $D_\rho \subset T_xM$. By the already mentioned result given in [3], lengths of curves in the the tangent space differ very little from the length of their projections onto the manifold. In fact the difference is given by the same expression as in our Proposition 1.10: cubic in ρ. By projecting b orthogonally to b' on the vertical axis, we see from the tangent space picture in

Figure 1.17 that there is a positive constant c for which

$$d(b,q) - d(b,p) \approx d(b',q) - d(b',p) < -c\rho \quad .$$

So therefore $b \notin L_m$, so that in fact $L_m \subset C_{0,\theta_0}$ where $\theta_0 < \pi/2$. With some more work one sees that this argument applies at points of L_m in a small ball around x. $\square$

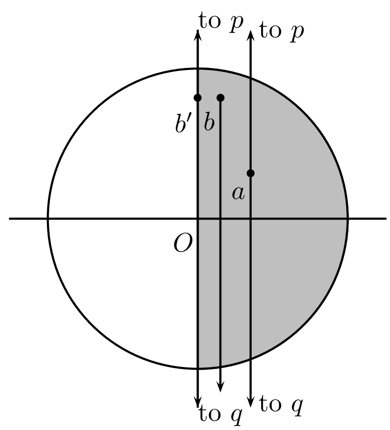


Figure 1.17: The construction of a cone in the "differentiable" case.

Corollary 1.18. *The number of spokes emanating from any given vertex is finite and even.*

Remark 1.19. Note that two of these spokes may actually be different parts of the same edge globally in the simplicial complex, but we count such spokes as being distinct.

Proof of Corollary 1.18. In the proof of Proposition 1.11, we showed that there is exactly one spoke for each wedge at a given vertex. By Lemma 1.7, there are only finitely many wedges about any given vertex, and a short inductive argument based on the definition of a wedge implies that there can be only an even number of wedges. $\square$

Note that this implies that there are no vertices with exactly one edge emanating from them, so the simplicial 1-complex L_{pq} is closed.

Also, we have the following result.

Corollary 1.20. *There are only finitely many vertices in L_{pq} with more than two spokes emanating from them.*

Proof. If there were infinitely many such vertices, they would have an accumulation point x by the compactness of M. But then near x, the mediatrix L_{pq} would not be locally the graph of a Lipschitz function, contradicting Proposition 1.11. $\square$

Since we are only discussing L_{pq} up to homeomorphism ($\phi : L_{pq} \to L_{pq}$), we do not need to consider as vertices those points with only two spokes, unless those two edges happen to be the same globally. Disregarding such points as vertices then, we can finally prove our main result.

Theorem 1.21. *Surface mediatrices are homeomorphic to finite, closed, simplicial 1-complexes.*

Proof. We have that the number of vertices is finite. Since at each vertex the number of spokes is finite, and each point has at least two spokes associated with it, we have associated a finite closed simplicial 1-complex with a mediatrix L. We now use Proposition 1.11 to map each edge continuously to a standard interval in the real line. Consider a vertex x in L (with more than two spokes emanating from it) and choose a spoke S based at x. Around any point a of L which is not a vertex there is an open neighborhood of a in L which is homeomorphic to an open interval (by Proposition 1.11). Since L is closed we can keep doing this until we hit another vertex. Thus an edge (including endpoints) is homeomorphic to a closed interval. □

Remark 1.22. It is important to note that we have not quite proved the Lipschitz character of the complex, since two spokes emanating from the same vertex could still form a cusp. In the tangent space this could look like the graph of $\sqrt{|x|}$, for example. The fact that this cannot happen is relatively easy to prove, but not immediately relevant for the current discourse.

2. Torus mediatrices

We can now use the results from Section 1, combined with results from [7] to turn our attention to the classification of mediatrices on a torus up to homeomorphism. We first note as an aside, however, that combining the results from Section 1 with those from [7], we can now readily classify mediatrices on spheres up to homeomorphism.

Proposition 2.1. *Let L be a mediatrix on a sphere S^2 whose distance function is inherited from a Riemannian metric. Then L is homeomorphic to S^1.*

Proof. By Theorem 1.21, L is a finite closed 1-simplex, and by Corollary 4.4 of [7], we have $H_1(L;\mathbb{Z}_2) = \mathbb{Z}_2$. Together these imply that L is homeomorphic to S^1. □

On S^2 then, there is only one type of mediatrix up to homeomorphism. Since mediatrices on Riemannian manifolds are minimally separating, this classification on S^2 follows immediately from the Jordan Curve separation theorem, which should not surprise us since a key ingredient used in our proof (see [7]) was Lefschetz Duality, a very general version of the Jordan separation theorem.

We now address the case of mediatrices on genus 1 tori, where the situation is more complicated. By Theorem 1.21, we have that a mediatrix L in a torus T is homeomorphic to a finite closed simplicial 1-complex.

Let us denote the dimension of $H_k(L;\mathbb{Z}_2)$ by b_k (the k-th Betti number of L), and let us also denote by v the number of vertices in L and by e the number of edges in L. Instead of the word "spoke", we will use the more standard "incipient edge" in this section. Also we will will call a point a non-trivial vertex if and only if it has more than 2 incipient edges associated with it. The number of non-trivial vertices will be denoted by v^*, and the number of edges attached to non-trivial vertices by e^*. (Observe that trivial vertices have a neighborhood in L which is homeomorphic to an open interval in $\mathbb{R}$.)

Lemma 2.2. *Let L be a mediatrix on a genus 1 torus T whose distance function is inherited from a Riemannian metric. Then L has the following properties:*
i): $1 \leq b_1 \leq 3$, $1 \leq b_0$.
ii): The number of incipient edges at a non-trivial vertex is even and at least 4.
iii): $1 - b_0 \leq e - v \leq 3 - b_0 \leq 2$.
iv): $e^* \geq 2v^*$.
v): $v^* \leq 3 - b_0 \leq 2$.
vi): $e^* \leq 4$.

Proof. i) and ii) are implied by Theorem 1.21 and Corollary 1.18, respectively. iii) follows from item i) and the fact that $e-v$ (the graph-theoretical Euler characteristic of L) equals $b_1 - b_0$ (the homological Euler characteristic). iv) follows from ii) and the observation that each incipient edge counts as half an edge. v) follows from a calculation: The only complexes without non-trivial vertices are disjoint unions of circles, for which $e - v = 0$. So using that we have:

$$v^* = 2v^* - v^* \leq e^* - v^* = e - v = b_1 - b_0 \leq 3 - 1 = 2.$$

Finally, vi) follows by noting that $e^* - v^* = e - v$ and combining iii) and v). $\square$

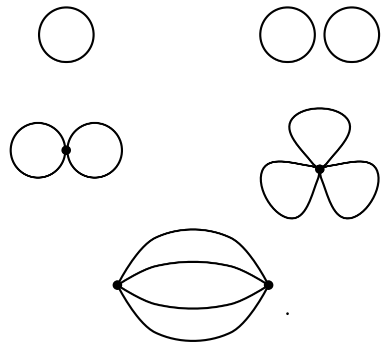
In the following we denote a torus with an arbitrary (smooth) Riemannian metric by T. Suppose that α and β are elements of $H_1(T;\mathbb{Z}_2)$. The $\mathbb{Z}_2$-intersection number $\alpha \bullet \beta$ by definition equals the number — modulo 2 — of transversal intersections of loops representing α and β. If the intersections are not transversal, one needs to perturb them (within their respective classes) until intersections are transversal. The intersection numbers are topological invariants (see [4]). The following is needed in the proof of the main theorem of this section.

Proposition 2.3. *Let L be a separating set in a genus 1 torus T whose distance function is inherited from a Riemannian metric. If L contains three closed loops and there is a proper subset L' of L containing two non-intersecting closed loops, then L is not minimal.*

Proof. By assumption we have that $H_1(L;\mathbb{Z}_2) \supseteq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, a basis for which is given by the homology classes of each of the three loops. Also, we have $H_1(T;\mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $i : L \to T$ be the inclusion map, which induces a map $i_* : H_1(L;\mathbb{Z}_2) \to H_1(T;\mathbb{Z}_2)$. By Theorem 4.2 of [7], if L' is any proper subset of L, then the induced map $(i|_{L'})_* : H_1(L';\mathbb{Z}_2) \to H_1(T;\mathbb{Z}_2)$ of i restricted to L' must be injective. This implies that none of the three elements $i_*(1,0,0)$, $i_*(0,1,0)$, and $i_*(0,0,1)$ can equal zero, and furthermore that no two of these elements can be the same.

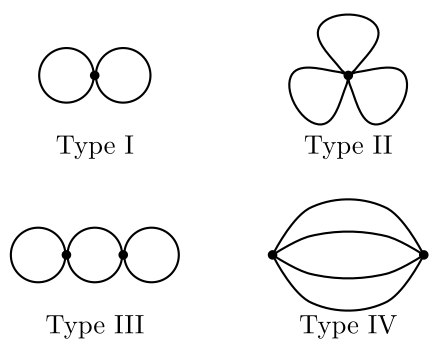
Note that $H_1(T;\mathbb{Z}_2)$ has exactly 3 non-zero, distinct elements. Represent $(1,0) \in H_1(T;\mathbb{Z}_2)$ by the longitude in the standard picture and $(1,0) \in H_1(T;\mathbb{Z}_2)$ by the latitude. One sees that the $\mathbb{Z}_2$-intersection number of any two distinct nonzero elements in $H_1(T;\mathbb{Z}_2)$ equals 1. However, according to the hypothesis of the proposition, two closed loops, say the ones representing $(1,0,0) \in H_1(L;\mathbb{Z}_2)$ and $(0,0,1) \in H_1(L;\mathbb{Z}_2)$ do not intersect. So $i_*(1,0,0)$ and $i_*(0,0,1)$ must have intersection number 0, since i is an inclusion map. Consequently, $i_*(1,0,0)$ and $i_*(0,0,1)$ cannot be distinct nonzero elements of $H_1(T;\mathbb{Z}_2)$ as required. $\square$

Theorem 2.4. *Let L be a mediatrix in a genus 1 torus T whose distance function is inherited from a Riemannian metric. Then L is homeomorphic to one of the following five spaces:*

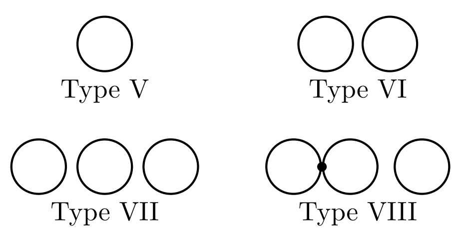


Proof. In part A) of the proof we assemble a list of candidate-mediatrices that must contain all topological types. In part B) we construct an example of each of these types, there showing that the list is complete.

A): Let us assume for the moment that L has no components which are circles, so that we may dispense with all trivial vertices. Since b_0 counts the number of components of L, item v) of Lemma 2.2 tells us that in this case L has only one component and either one or two non-trivial vertices. Items iv) and vi) imply that if L has two vertices, each of them must have 4 incipient edges. There are now only a small number of possibilities left, and these are easily reduced to the following topological types.



To these 4 possibilities, we can add those obtained by allowing for disjoint circles as well (being careful not to exceed the bound that $b_1 \leq 3$). This gives us another 4 possibilities for the topological type of L:



Types III, VII, and VIII, although they pass all the tests of Lemma 2.2, cannot occur as mediatrices on a torus, since they violate Proposition 2.3.

B): The remaining five possibilities (spaces homeomorphic to the Type I, II, IV, V, and VI spaces) are constructed below on various tori as L_{pq} for the indicated pairs of points p,and q. Except for the Type II space, all these constructions are self-explanatory.

In the construction of the Type II space, we start with an equilateral triangle T inscribed in a circle (so that they have the same center, which we call x). Add an isometric triangle T' to obtain a parallelogram P (the grey area) and identify sides as indicated to obtain a torus. Now reflect P in the center x, so that we obtain the unshaded torus P'. The point q is the reflection of p in x. We leave it to the reader to conclude that the symmetries imply that the set L_{pq} contains x and that there are 6 incipient edges associated with x.

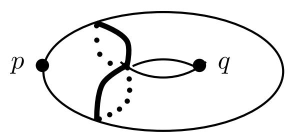


Figure 2.4a: A Type I space

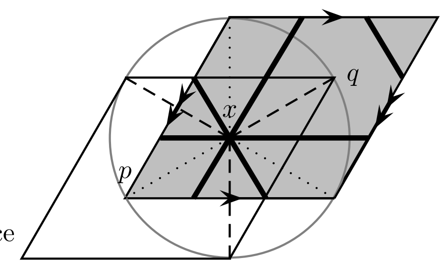


Figure 2.4b: A Type II space

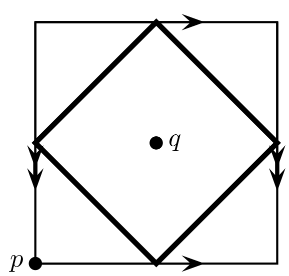


Figure 2.4c: A Type IV space

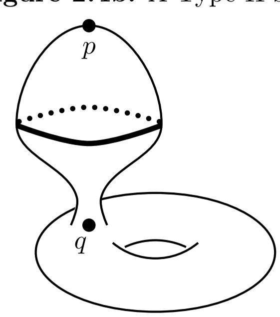


Figure 2.4d: A Type V space

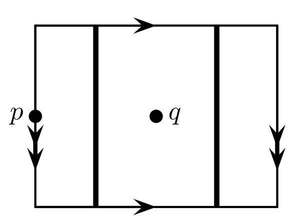


Figure 2.4e: A Type VI space

This completes the classification of mediatrices on tori up to homeomorphism. □

The techniques employed in Lemma 2.2 can also be used to generate a list of possible topological types of mediatrices in a real projective plane $\mathbb{R}P^2$. It turns out that there are two possibilities, namely Types V and I from the above list.

Proposition 2.5. *Let L be a mediatrix in a real projective plane $\mathbb{R}P^2$ whose distance function is inherited from a Riemannian metric. Then L is homeomorphic to one of the following two spaces:*

3. Higher genus surfaces

Having classified mediatrices on 2-spheres, real projective planes, and tori, we turn now to connected sums of tori, namely the orientable higher genus surfaces. It is possible that the direct method of obtaining an upper bound on the number of vertices, enumerating the possibilities, and then either constructing and eliminating the possibilities will produce a classification of mediatrices on a surface of higher genus. However, it is difficult to see how through this method a pattern might emerge which would, for example, provide a formula for the number of distinct mediatrices up to homeomorphism possible on a surface of genus g.

An inductive approach seems more promising for such a classification. For example, any type of mediatrix which can occur on a surface of genus less than g can also occur on a surface of genus g. This can be seen using a connected sum construction. If, then, we wanted to exhibit, say, a Type I torus mediatrix on a genus 2 torus instead, we could simply take the connected sum as indicated below.

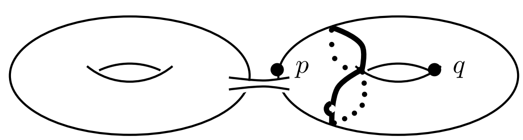


Figure 3.1: Survival of mediatrices in higher genus.

As shown in the picture, this may cause a slight perturbation in some part of the mediatrix, but as there are only finitely many vertices (with more than two edges emanating from them), the connected sum can be arranged in such a way as not to affect the mediatrix in a neighborhood of the vertices. Thus the homeomorphism type of the mediatrix is preserved.

The same procedure then can be seen in general to produce a mediatrix on a surface of genus g with the homeomorphism type of any mediatrix on a surface of genus less than g. This means that we do not "lose" mediatrices when the genus is increased, so we can focus our attention on those new possibilities which could not occur before.

For example, let T_g be a genus g torus constructed in the usual fashion by identifying sides of a $4g$-gon. Then $H_1(T_g;\mathbb{Z}_2) = \mathbb{Z}^{2g}$, and the sides of the $4g$-gon form a basis $\{a_1,\ldots,a_g,b_1,\ldots,b_g\}$ for $H_1(T_g;\mathbb{Z}_2)$. If L is a mediatrix on T_g and $i_* : H_1(L;\mathbb{Z}_2) \to H_1(T_g;\mathbb{Z}_2)$ is the map induced by the inclusion, then we do not find any new topological possibilities for mediatrices by considering cases in which the image of i_* is contained in a subset of $H_1(T_g;\mathbb{Z}_2)$ generated by $\{a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_g,b_1,\ldots,b_{j-1},b_{j+1},\ldots,b_g\}$ for some j. In any such case, L "misses" a handle on T_g and so lies on a subset of lower genus, and we have already accounted for all matrices which occur on surface of lower genus.

In addition, it is possible to produce a mediatrix on a surface of genus g whose topological type is that of the disjoint union of any mediatrix on a surface of genus less than g with a circle. We merely glue on a handle with one end in L_p and the other in L_q, as indicated in the picture below.

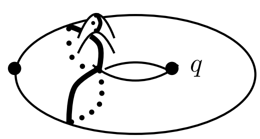


Figure 3.2: Construction of more complicated mediatrices in higher genus.

Intersection number arguments imply more topological restrictions on mediatrices in higher genus surfaces, however. For example, they show that on a surface of genus g, any mediatrix consisting only of disjoint circles can contain at most $g+1$ such circles.

Continuing to find and classify such techniques for arriving at a mediatrix on a surface of genus g from one on a surface of genus less than g may lead to an inductive classification of mediatrices on connected, compact, closed, oriented 2-manifolds, as well as to an exact count of the number of possible types up to homeomorphism for each genus.

Conjecture 3.3. *The set of homeomorphism classes of mediatrices possible on any given connected, compact, closed,* 2*-manifold* M *(allowing the metric to vary), is equal to the set of homeomorphism classes of minimally separating, finite, closed simplicial 1-complexes contained in* M *whose vertices all have an even number of incipient edges.*

We should note that the techniques we have used here carry over directly to non-orientable surfaces. The classification will, of course, be somewhat different since the homology is different (along with the $\mathbb{Z}_2$-intersection pairings), but the general theorems still apply.

4. Conclusion

We have shown that mediatrices on connected, compact, closed 2-manifolds are finite closed simplicial 1-complexes whose vertices all have an even number of incipient edges emanating from them. Combining this with the result from [7] that mediatrices are minimally separating and with the techniques used in that paper, we have a complete classification of mediatrices in 2-spheres, projective planes, and tori up to homeomorphism. In addition we have outlined an inductive approach which may, if developed further, yield a classification of mediatrices on surfaces of any genus g.

There are various natural unanswered questions associated with mediatrices in compact connected Riemann surfaces. We list some important ones here. The method used in Proposition 1.11 to prove that mediatrices in a surface are locally homeomorphic to $\mathbb{R}$, appears to indicate that the homeomorphism has a Lipschitz quality. According to Remark 1.22 one would only have to prove that the mediatrix admits no cusps. A little harder but also reasonable, is the suspicion that the spokes are "radially differentiable". To see what this means, consider a spoke L_m in a wedge $W_{\rho,m}$ based at the point $x \in L_{pq}$, where L_{pq} is a mediatrix in M. From the material in Section 1, it is clear that for small enough ρ, we can parametrize the m-th spoke in polar coordinates by giving the angle as function $\theta_m : (0, \rho) \to (-\pi, \pi)$ for an appropriate choice of the horizontal axis. Thus the locus of L_m in $W_{\rho,m}$ can be given in polar as well as Cartesian coordinates as:

$$L_m = \{(r, \theta_m(r)) \,|r \in (0, \rho)\} = \{(x, y_m(x)) \,|x \in (0, \rho \sin \theta_0)\} \quad .$$

We call L_{pq} radially differentiable at x if for every spoke associated with x we have

$$\lim_{r \to 0} \theta_m(r) = \theta_m$$

exists. This definition is reasonable since by fixing an m and choosing the horizontal axis in T_xM in such a way that $\theta_m = 0$, we see that:

$$\lim_{x \to 0} \frac{y_m(x)}{x} = \lim_{r \to 0} \frac{r \sin \theta_m(r)}{r \cos \theta_m(r)} = 0 \quad .$$

Work is currently in progress to prove that mediatrices in compact 2-dimensional Riemannian manifold are both Lipschitz and radially differentiable.

Finally we wish to remark that the methods employed in Section 1 are *emphatically* two-dimensional and do not seem to generalize. It is, at this point not even clear if a mediatrix in a three-dimensional manifold is locally homeomorphic to $\mathbb{R}^2$ except at a set of points of dimension at most 1. However, for real analytic manifolds of dimension n it should be possible to prove that mediatrices are triangulable. (Buchner [1] has proved that the cut locus of a real analytic Riemannian n-dimensional manifold is triangulable. His proof uses Hironaka's theory of subanalytic sets.)

5. Appendix

The purpose of this appendix is to prove Proposition 1.10 (and we follow the notation of that remark). For the purpose of this appendix alone we denote the base-point by p and the local coordinates by $\vec{x}$. To start, the metric is defined by (following [2])

$$g_{ij}(\vec{x}) =< d\phi_{\vec{x}} v_i, d\phi_{\vec{x}} v_j > \quad .$$

Since exp is tangent to the identity, the choice of the coordinate system implies that $g(p)$ equals the identity. Since the derivatives of g are uniformly bounded (X is compact), we see that for r small enough $d\phi$ is injective. By compactness of X, $r > 0$ can be chosen independent of the base-point p. Since g is positive definite in the v_i, there exists $K > 0$ such that for any geodesic $\gamma(t) = \phi(\vec{x}(t))$:

$$< \dot{\gamma}(t), \dot{\gamma}(t) >= \Sigma g_{ij} \dot{x}_i \dot{x}_j = 1 \;\Rightarrow\; \forall\, i : \;\; |\dot{x}_i(t)| < \sqrt{K} \;\Rightarrow\; \forall\, i, j : \;\; |\dot{x}_i(t) \dot{x}_j(t)| < K.$$

In local coordinates the geodesic satisfies (where Γ_{ij}^k are the Christoffel symbols):

$$\forall\, k: \quad \ddot{x}_k + \Sigma_{ij}\, \Gamma_{ij}^k \dot{x}_i \dot{x}_j = 0 \quad .$$

The coordinates we defined form a geodesic frame. Therefore $\Gamma_{ij}^k(0) = 0$. We assume the metric is twice continuously differentiable. Therefore by compactness of X there is a constant (independent of the base-point) such that $\forall q \in X$ with $d(q,p) < r$, we have $|\Gamma(q)| < Qr$. For each component this gives:

$$|\ddot{x}_k| < d^2 KQr \quad ,$$

where d is the dimension of the manifold (2 in the body of this work). Upon integration this gives

$$|\dot{x}_k(t) - \dot{x}_k(0)| < d^2 KQtr \quad .$$

Integrating once more gives

$$|x_k(t) - (\dot{x}_k(0)t + x_k(0))| < \frac{1}{2} d^2 KQt^2 r \quad .$$

With the Euclidean $\mathbb{R}^d$ norm $||.||$ this gives:

$$||\vec{x}(t) - (\dot{\vec{x}}(0)t + \vec{x}(0))|| < \frac{1}{2} d^2 \sqrt{d} KQt^2 r \quad .$$

Noting that $t < 2r$ and setting the constant C of the remark to $C = \frac{1}{2} d^2 \sqrt{d} KL$ finishes the proof. $\square$

Acknowledgement We thank Sergei Anisov for alerting us to an error and Detlef Gromoll for useful conversations.

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