MINIMALLY SEPARATING SETS, MEDIATRICES, AND BRILLOUIN SPACES

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Abstract. Brillouin zones and their boundaries were studied in [16] because they play an important role in focal decomposition as first defined by Peixoto in [9] and in physics ([1] and [3]). In so-called Brillouin spaces, the boundaries of the Brillouin zones have certain regularity properties which imply that they consist of pieces of mediatrices (or equidistant sets).

The purpose of this note is two-fold. First, we give some simple conditions on a metric space which are sufficient for it to be a Brillouin space. These conditions show, for example, that all compact, connected Riemannian manifolds with their usual distance functions are Brillouin spaces. Second, we exhibit a restriction on the $\mathbb{Z}_2$-homology of mediatrices in such manifolds in terms of the $\mathbb{Z}_2$-homology of the manifolds themselves, based on the fact that they are Brillouin spaces. (This will used to obtain a classification up to homeomorphim of surface mediatrices in [15]).

This note begins with some preliminaries in Section 1, where we define the relevant concepts and provide some background and motivation for the questions being considered here. In Section 2, we list a simple set of conditions on a metric space and prove that these conditions suffice for it to be a Brillouin space. In Section 3, we give some examples and mention a result for use in a later paper. We then investigate in Section 4 the homological restrictions placed on a mediatrix by the topology of the surrounding Brillouin space. The paper then closes with some concluding remarks in Section 5 and some acknowledgements.

1. Preliminaries

Let (X, d) be a path-connected metric space. The central object of study in this paper is the *mediatrix*, namely:

Definition 1.1. *For any $p, q \in X$, the* mediatrix L_{pq} *is defined to be the set of all points which are equidistant from p and q:*

$$L_{pq} := \{x \in X \mid d(x, p) = d(x, q)\}.$$

Thus mediatrices are level sets of $f(x) = d(x, p) - d(x, q)$. We note in passing that there is an interesting relation between the mediatrix and the cut loci of p and q if (X, d) is an n-dimensional smooth complete Riemannian manifold. The function $f(x)$ is differentiable with non-zero derivative if x is outside the cut locus of p or q. (The cut locus of p is "the locus of the points for which the geodesics starting from p stop being globally minimizing", according to [4]). The Implicit Function Theorem thus guarantees that mediatrices are locally $(n-1)$-dimensional manifolds except perhaps at a point which is also in the cut locus of p or the cut locus of q.

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Upon examining mediatrices in some common examples of path-connected metric spaces (such as Euclidean space $\mathbb{R}^n$, the n-sphere S^n, hyperbolic n-space, and their quotients, all with their usual metrics), one notices that mediatrices in these spaces appear all to be *minimally separating*, in the following sense (a similar definition can be found in [18], page 43).

Definition 1.2. *A subset L of a connected topological space X is* separating *if $X - L$ consists of more than one connected component. If L is separating but no proper subset of L is separating, then L is* minimally separating.

A simple counterexample, however, shows that this property of mediatrices does not hold for all spaces.

Example 1.3. Consider a figure in the shape of a "Y", with the usual notion of distance inherited from $\mathbb{R}^2$. If p is a point on the top left branch and q is a point on the top right branch the same distance from the center vertex of the figure, then the mediatrix L_{pq} consists of the entire lower branch of the figure, which is separating but certainly not minimally separating. Notice also that the center point of the figure *is* a minimally separating set, and its complement consists of three components.

In [16], *Brillouin spaces* were defined. These are spaces in which mediatrices have certain desirable properties, the most important of which is that they are minimally separating. Before recalling the exact definition of such a space, we give definitions of its other characteristic properties. The first such property is that it is *proper*.

Definition 1.4. *A metric space (X, d) is* proper *if the function $d_p = d(p, \cdot\,)$ is proper (i.e., inverse images of compact sets are compact) for all $p \in X$.*

The other defining property of Brillouin spaces is somewhat more technical. We use the following notation for open balls (or open discs) and spheres: for any $p \in X$ and any positive real number r,

$$D(x; r) := \{x \in X \mid d(x, p) < x\}$$
$$\partial D(x; r) := \{x \in X \mid d(x, p) = x\}.$$

With this notation, we can now define the other characteristic property of Brillouin spaces.

Definition 1.5. *A metric space (X, d) is called* metrically consistent *if, given any $x \in X$ and any positive real number R, the following property holds for all sufficiently small positive real r: for each $a \in \partial D(x; R)$, there is a $z \in \partial D(x; r)$ satisfying $D(z; d(z, a)) \subseteq D(x; R)$ and $\partial D(z; d(z, a)) \cap \partial D(x; R) = \{a\}$.*

The following picture illustrates the condition in this definition.

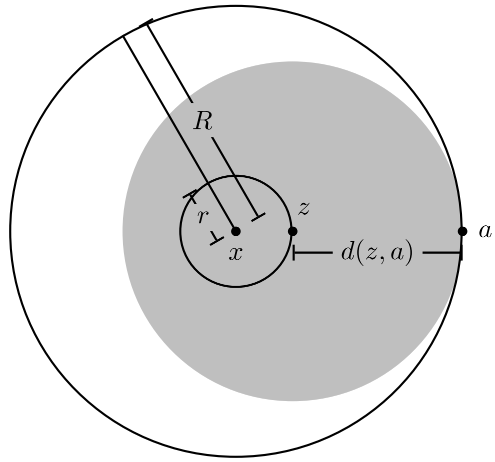

Figure 1.6. Defining metrical consistency

We have now listed the defining properties of Brillouin spaces, whose precise definition we now recall.

Definition 1.7. *A path-connected, proper, metric space* (X, d) *is called a* Brillouin space *if*
i) it is metrically consistent and
ii) for all $p, q \in X$*, the mediatrix* L_{pq} *is minimally separating.*

In the next section, we will prove that a large class of metric spaces, including all compact, connected Riemannian manifolds, are Brillouin spaces. In [16], this was established only for those manifold whose curvature is constant. While the property of having minimally separating mediatrices is interesting in its own right, we should mention here another property which was shown to hold for Brillouin spaces in [16], and which makes the characterization of a class of Brillouin spaces in the next section all the more useful.

The notion of a *Brillouin zone* was introduced by Brillouin in the 1930's and plays an important role in solid state physics (see [3] or [1]). Brillouin zones also play an important role in the study of focal decomposition as defined by Peixoto [9]. Let (X, d) be a path-connected metric space, and let $S \subset X$ be a discrete indexed set $\{x_i\}_{i\in\mathbb{N}}$. In geometry, the set $b_1(x_n)$ is the Dirichlet domain associated with x_n, and in computational geometry, this set is often referred to as a Voronoi cell (see [10]). Loosely speaking, the set $b_m(x_n)$ is the set of points x such that every continuous path from x_n to x crosses at least $m-1$ mediatrices $L_{x_n x_i}$, and such that there exists a path from x_n to x which crosses exactly $m-1$ of these mediatrices. Much of the physical and mathematical interest of Brillouin zones is derived from the fact that in a precise sense (see [16]), these Brillouin Zones tile the underlying space in a very regular fashion. This was proved in the case of $\mathbb{R}^2$ with a lattice by Bieberbach (see [2]). In [16] this was proved to hold for all Brillouin spaces (equipped with a discrete set).

2. Characterization of a class of Brillouin spaces

In this section, we exhibit a large class of metric spaces which have the Brillouin property, a class which includes all compact, connected Riemannian manifolds. In order to describe this class, we first recall some basic facts about paths in metric spaces.

On any metric space (X, d), there is a notion of the *length* ℓ of a continuous path $\gamma : [0, 1] \to X$, given by

$$\ell(\gamma) = \sup \left\{ \sum_{i=0}^{k-1} d\big(\gamma(t_i), \gamma(t_{i+1})\big) \;\middle|\; 0 = t_0 \le t_1 \le t_2 \le \cdots \le t_n = 1 \right\},$$

where the supremum is taken over all possible partitions of the interval $[0, 1]$. In this paper, we will assume all paths to be continuous unless otherwise noted.

As usual then, for any two points $p, q \in X$, we say that a path $\gamma : [0, 1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$ is a *shortest path* from p to q if $\ell(\gamma) \le \ell(\alpha)$ for any path $\alpha : [0, 1] \to X$ with $\alpha(0) = p$ and $\alpha(1) = q$.

It is easy to see that parametrizations of shortest paths must be one-to-one, so that if $\gamma : [0, 1] \to X$ is a shortest path, we can speak unambiguously of the point $\gamma^{-1}(x)$ for any $x \in \gamma\big([0, 1]\big)$. Given any two points $x, y \in \gamma\big([0, 1]\big)$ then, we denote by $\gamma_{x,y} : [0, 1] \to X$ the path γ restricted to the subpath from x to y:

$$\gamma_{x,y}(t) = \gamma\big((1 - t)\gamma^{-1}(x) + t\gamma^{-1}(y)\big).$$

Also, if p, q, $x \in X$, then given a path $\alpha : [0, 1] \to X$ from p to x and a path $\beta : [0, 1] \to X$ from x to q, we define $\alpha * \beta : [0, 1] \to X$ to be their concatenation defined in the usual way. It is immediate that $\ell(\alpha * \beta) = \ell(\alpha) + \ell(\beta)$.

We can now describe a class of metric spaces which are Brillouin.

Theorem 2.1. *Let (X, d) be a proper, path-connected metric space satisfying the following conditions:*

(1) *Between any two points $p, q \in X$, there is always a shortest path whose length equals $d(p, q)$.*
(2) *Let $p, x, y \in X$ with $x \ne y$. If two shortest paths, one from p to x and the other from p to y, have a common segment, then one of the paths is a subset of the other.*

Then (X, d) is Brillouin.

Remark 2.2. Gromov gives in [6] a generalized version of the Hopf-Rinow Theorem which he shows that Condition (1) in Theorem 2.1 is satisfied by any path-connected, complete, locally compact, path metric space. (X is called a path metric space if for all a and b in X, $d(a, b)$ equals the infimum of the lengths of the paths that join these points.)

Before we turn to the proof of this theorem, we need an onslaught of lemmas. The proofs of the first three of these are rather straightforward combinations of the two hypotheses in the theorem and the triangle inequality. The fourth relies on continuity of the distance function.

Lemma 2.3. *Let (X, d) be a proper, path-connected metric space satisfying the hypotheses of Theorem 2.1. Then any path in X which is a subset of a shortest path in X is itself a shortest path in X.*

Proof. Let $\gamma : [0,1] \to X$ be a shortest path from $p \in X$ to $q \in X$. Any path which is a subset of γ can be written as $\gamma_{x,y}$ for some $x, y \in \gamma\big([0,1]\big)$. Now let $\alpha : [0,1] \to X$ be any path from x to y. Then $(\gamma_{p,x} * \alpha) * \gamma_{y,q}$ is a path from p to q and therefore must have length greater than or equal to the length of γ, so:

$$\ell(\gamma_{p,x}) + \ell(\alpha) + \ell(\gamma_{y,q}) \geq \ell(\gamma_{p,x}) + \ell(\gamma_{x,y}) + \ell(\gamma_{y,q}) \quad ,$$

which gives us that $\gamma_{x,y}$ is a shortest path. $\square$

Corollary 2.4. *Let (X, d) be a proper, path-connected metric space satisfying the hypotheses of Theorem 2.1. If $\gamma : [0,1] \to X$ is a shortest path from $p \in X$ to $q \in X$ and if $x \in \gamma([0,1])$, then $d(p,q) = d(p,x) + d(x,q)$.*

Proof. We have that $\gamma = \gamma_{p,x} * \gamma_{x,q}$, so

$$\ell(\gamma) = \ell(\gamma_{p,x}) + \ell(\gamma_{x,q}).$$

Now γ is a shortest path, so $\ell(\gamma) = d(p,q)$. By Lemma 2.3, the paths $\gamma_{p,x}$ and $\gamma_{x,q}$ are also shortest paths, so their lengths are $d(p,x)$ and $d(x,q)$ respectively, from which the corollary follows. $\square$

For any $p, q \in X$, we define the following two subsets of X:

$$L_p = \{x \in X \mid d(p,x) - d(q,x) < 0\}$$
$$L_q = \{x \in X \mid d(q,x) - d(p,x) < 0\}.$$

In other words, L_p is the set of points in X which are closer to p and L_q is the set of those which are closer to q. Note that L_p (L_q) is not empty since it contains p (q). For a path $\gamma : [0,1] \to X$ we denote the image of $\gamma : (0,1) \to X$ by $\overset{\circ}{\gamma}$.

Lemma 2.5. *Let (X, d) be a metric space which satisfies the hypotheses of Theorem 2.1, and let $p, q \in X$ with $p \neq q$. If $x \in X$ satisfies $d(x,p) \leq d(x,q)$, then for any shortest path $\gamma : [0,1] \to X$ from x to p, we have $\overset{\circ}{\gamma} \subset L_p$.*

Proof. Let $\gamma : [0,1] \to X$ be a shortest path from x to p. Then $\gamma(1) = p \in L_p$. Now let $y \in \overset{\circ}{\gamma}$, and let $\alpha : [0,1] \to X$ be a shortest path from y to q. Note that the segment $\gamma_{y,p} \subset \gamma$ is minimizing due to Lemma 2.3. We need to show that $y \in L_p$ or $d(y,q) > d(y,p)$.

There are two possibilities. Either α and $\gamma_{y,p}$ share a segment or not. In the first case, since both are minimizing segments, one must *strictly* (since $p \neq q$) contain the other. Now, because $x \in \overline{L_p}$, α must contain $\gamma_{y,p}$ (forming a geodesic segment *xypq*). Since α is minimizing, by Corollary 2.4,

$$d(y,q) = d(y,p) + d(p,q) > d(y,p).$$

In the other case, α and $\gamma_{y,p}$ do not share a segment. But $\gamma_{x,y} * \alpha$ still contains a segment in common with the shortest path γ from x to p, and thus it cannot itself be a shortest path from x to q (by the second hypothesis of Theorem 2.1). This means that

$$\ell(\gamma_{x,y}) + \ell(\alpha) = \ell(\gamma_{x,y} * \alpha) > d(x,q) \geq d(x,p).$$

Since α and γ are shortest paths, Corollary 2.4 implies that

$$d(x,y) + d(y,q) = \ell(\gamma_{x,y} * \alpha) \text{ and } d(x,p) = d(x,y) + d(y,p).$$

Combining this with the previous equation gives the required result. □

Since the distance function d is continuous, L_p and L_q are both open sets, so L_{pq}, which equals $X - (L_p \cup L_q)$, is closed. Also, the sets L_p and L_q are both path-connected (hence connected): any points $x, y \in L_p$, for example, can be connected by a path which is the concatenation of a shortest path from x to p with a shortest path from p to y. By Lemma 2.5, such a path is contained entirely in L_p.

We summarize this argument in the following lemma.

Lemma 2.6. *Let (X, d) be a metric space satisfying the hypotheses of Theorem 2.1, and let $p, q \in X$. The mediatrix L_{pq} is a closed subset of X which separates X into two connected components, namely L_p and L_q, each of which is path-connected.*

We are now in a position to prove the main theorem (Theorem 2.1). For convenience, we split the statement into two propositions.

Proposition 2.7. *Let (X, d) be a metric space satisfying the hypotheses of Theorem 2.1, and let $p, q \in X$. The mediatrix L_{pq} is* minimally *separating.*

Proof. Take any $x \in L_{pq}$ and let L_0 denote the set $L_{pq} - \{x\}$ obtained by removing x from L_{pq}. We claim that the set $X - L_0$ is path-connected (and hence connected). It suffices to show that p and q can be joined by a path in $X - L_0$. By Lemma 2.5, a shortest path from p to x does not intersect $X - L_0$ and a shortest path from x to q also does not intersect $X - L_0$. The concatenation of these two paths is a path in $X - L_0$ joining p to q, which shows that L_{pq} is minimally separating. □

Proposition 2.8. *Let (X, d) be a metric space satisfying the hypotheses of Theorem 2.1. Then X is metrically consistent.*

Proof. Let $x \in X$ and $R > r > 0$. For any $a \in \partial D(x; r)$, let $\gamma : [0, 1] \to X$ be a shortest path from x to a. Since the distance function d is continuous and $R = d(x, a) > r > d(x, x) = 0$ the Intermediate Value Theorem implies that there must be some point $z \in \gamma([0, 1])$ with $d(x, z) = r$, i.e., with $z \in \partial D(x; r)$. We use this point z to show that (X, d) is metrically consistent. For the remainder of this proof, compare with Figure 1.6.

By Corollary 2.4, we have that $d(z, a) = R - r$. For any point $p \in D(z; d(z, a)) = D(z; R - r)$ then,

$$\begin{aligned} d(x, p) &\leq d(x, z) + d(z, p) \\ &= r + (R - r) \\ &= R, \end{aligned}$$

so $p \in D(x; R)$, meaning that $D(z; d(z, a)) \subseteq D(x; R)$ as required.

By design, $a \in \partial D(z; d(z, a)) \cap \partial D(x; R)$. To show that a is the only point in this intersection, assume that $a \neq q \in \partial D(z; d(z, a)) \cap \partial D(x; R)$. Let $\alpha : [0, 1] \to X$ be a shortest path from z to q. Then $\ell(\gamma_{x,z}) = d(x, z) = r$ and $\ell(\alpha) = d(z, q) = d(z, a) = R - r$, so $\ell(\gamma_{x,z} * \alpha) = R$. Therefore $\gamma_{x,z} * \alpha$ is a shortest path from x to q which shares a common subpath (namely $\gamma_{x,z}$) with the shortest path γ from x to a. By the second hypothesis of Theorem 2.1, either $\gamma\big([0, 1]\big)$ is a subset of $\gamma_{x,z} * \alpha\ \big([0, 1]\big)$ or vice versa.

This means that either $\gamma(1) = a \in \gamma_{x,z} * \alpha \ \big([0,1]\big)$ or else $\gamma_{x,z} * \alpha\ (1) = q \in \gamma\big([0,1]\big)$. Unless $q = a$, then by Corollary 2.4, the first case implies that $d(x,q) > R$ and the second that $d(x,q) < R$, both of which are contradictions. Therefore $q = a$, meaning that $\partial D(z; d(z,a)) \cap \partial D(x; R) = \{a\}$, which shows that (X, d) is metrically consistent. □

Theorem 2.1 follows directly from the last two propositions. As an immediate corollary to Theorem 2.1, we have that compact, connected Riemannian manifolds are Brillouin spaces. (Note that in [16] this was established only for Riemannian manifolds of constant curvature.)

Corollary 2.9. *Let M be a compact, connected Riemannian manifold with the usual distance function d associated with the Riemannian metric. Then M is a Brillouin space.*

Proof. Since M is a manifold, connectedness implies path-connectedness. Also, for any $p \in M$, the function $d_p = d(p, \cdot\,)$ is continuous (see [11], page 26, for example). Since any continuous function on a compact manifold is proper, the function d_p is proper for any $p \in M$, meaning that M is itself proper.

Property (1) of Theorem 2.1 follows from the Hopf-Rinow Theorem (see, for example, [11], page 84), and Property (2) is a standard property of geodesics which follows from the fundamental existence and uniqueness theorem of solutions to differential equations. □

For future reference (we will need it in [15]) we prove a slightly stronger version of Lemma 2.5.

Lemma 2.10. *Let (X, d) be a path-connected metric space, and let p and q be points in X. Suppose x and y are (not necessarily distinct) points in the mediatrix L_{pq}. Let γ be a minimizing path connecting the points x and p or q and η a minimizing path connecting y to either p or q. Then $\mathring{\gamma} \cap L = \mathring{\eta} \cap L = \emptyset$ and $\mathring{\gamma} \cap \mathring{\eta} = \emptyset$, or else $\gamma = \eta$.*

Proof: The first equality was proved in Lemma 2.5. For the second, first assume that γ lands in p and η lands in q (or vice versa). Lemma 2.5 implies that $\mathring{\gamma}$ and $\mathring{\eta}$ lie in distinct components of $X - L_{pq}$. Hence their intersection is empty.

Now assume that γ and η land in the same point, say p. Assume that the lemma is false. So there is a point $z \in \mathring{\gamma} \cap \mathring{\eta}$. By Lemma 2.3, the sub-path along γ from x to z, and the one from z to p are shortest paths with respective lengths $d(x, z)$ and $d(z, p)$. The path along η from z to p is also a shortest path of length $d(z, p)$. Thus from x to p there are two distinct shortest paths having a common segment, contradicting Lemma 2.3. □

3. Miscellaneous examples

In this section we exhibit some more examples and collect some results for later use.

Even though mediatrices are by their definition closed sets, it is easy to see that any minimally separating set K in a metric space (X, d) is closed (see also [18]). To see this, note that separating implies that there are disjoint nonempty open sets

V_1 and V_2 such that

$$V_1 \cup V_2 \cup K = X \quad ,$$

and since $X - (V_1 \cup V_2)$ is already separating, minimality implies that

$$K = X - (V_1 \cup V_2) \quad ,$$

so that K is closed in X.

Example 3.1. We have already seen in Example 1.3 that the complement of a minimally separating set does not necessarily have two components (although in the case of mediatrices in Brillouin spaces this is the case). In fact, the classical "Lakes of Wada" construction as described in [7] gives rise to a minimally separating set whose complement has three connected components. To see that that the resulting set K is *minimally* separating, note that if there is an $x \in K$ such that $K' = K - \{x\}$ also separates, there must be two nonempty open sets V_1 and V_2 such that $V_1 \cup V_2 \cup K' = X$. Without loss of generality we may assume that $x \in V_1$. But x is also contained in the boundary of each of the three components of $X - K$. Since each of these components is a connected open set, it turns out that V_1 and V_2 must separate at least one of these, which is a contradiction.

Example 3.2. Equip $\mathbb{R}^2$ with the distance function

$$d(x,y) = \left(|y_1 - x_1|^k + |y_2 - x_2|^k\right)^{\frac{1}{k}} \quad ,$$

where k is a real number greater than 1. (For $k < 1$ this expression does not satisfy the triangle inequality.) Note that for $k \neq 2$, this space is not a Riemannian manifold. In Section 5 of [16] it was stated that the mediatrices in this metric space are minimally separating. With a little extra work, one can show that the space is metrically consistent. However, we can also use Theorem 2.1 to show this. To do this, it is clearly sufficient to show that the unique shortest path between two points in the plane is a line segment. (Notice that this implies that in this case mediatrices are *not* orthogonal to shortest paths.) For $k = 1$ — as explained in [16], Example 2.5 — there many more shortest paths. Here follows an outline of that argument.

We restrict to differentiable paths $\gamma(x) = (x, f(x))$ connecting $a = (a_1, f(a_1))$ to $b = (b_1, f(b_1))$, where $a_1 < b_1$, such that $f : [a_1, b_1] \to \mathbb{R}$ has positive derivative. (The more general case is left to the reader.) Now let $a_1 = x_1 < x_2 < \cdots < x_n = b_1$. From the triangle inequality and the earlier definition of of length one derives that

$$\ell(\gamma) \geq \sum_{i=1}^{n} d\big(\gamma(x_i), \gamma(x_{i+1})\big) \geq d(a,b) \quad .$$

In fact, with a little thought, one sees that (using the same definition of length)

$$\ell(\gamma) = \lim_{\max |x_{i+1} - x_i| \to 0} \sum_{i=1}^{n} d\big(\gamma(x_i), \gamma(x_{i+1})\big) = \int_{a_1}^{b_1} \left(1 + f'(x)^k\right)^{\frac{1}{k}} \, dx \quad .$$

For $k > 1$, the Euler-Lagrange equation implies that $f''(x) = 0$. (For $k = 1$, the first variation equals 0.)

4. Mediatrices in Riemannian manifolds

In this section, we give a partial description of the $\mathbb{Z}_2$-homology of mediatrices in compact, connected, Riemannian manifolds whose distance function is inherited from the Riemannian structure. In an accompanying paper we show that surface mediatrices are finite, closed, simplicial 1-complexes, and this together with Theorem 4.2 leads to a classification up to homeomorphism of mediatrices in some surfaces such as a torus (see [15]).

By Corollary 2.9, any compact, connected, Riemannian manifold is a Brillouin space, meaning that all mediatrices in such a manifold are minimally separating. Our general approach is somewhat reminiscent of methods used by Whitehead [17] and Myers [8] to prove that the cut locus of a 2-dimensional manifold has the same 1-dimensional Betti number modulo 2 as the underlying manifold.

Lemma 4.1. *Let M be a compact, connected, n-dimensional Riemannian manifold whose distance function is that associated with the Riemannian metric. Then for any $p, q \in M$, the mediatrix L_{pq} contains no open sets.*

Proof. If L_{pq} were to contain an open n-ball around some point $x \in L_{pq}$. If $\gamma : [0,1] \to M$ is a shortest path from x to p, then $\gamma\big((0,1]\big) \subset L_p$ by Lemma 2.5. The continuity of γ implies that the inverse image under γ of any open n-ball around x is an open subset of $[0,1]$ containing 0. In particular, this means then that any open n-ball around x contains points in $\gamma\big((0,1]\big) \subset L_p$ and hence outside of L_{pq}, so L_{pq} cannot contain any n-cell. □

It immediately follows from this theorem that L_{pq} contains no n-cells and that $H_n(L_{pq};\mathbb{Z}_2) = 0$. Since L_{pq} is a subset of an n-dimensional manifold, this means that $H_k(L_{pq};\mathbb{Z}_2) = 0$ for all $k \geq n$.

Theorem 4.2. *Let M be a compact, connected, n-dimensional Riemannian manifold, whose distance function is that associated with the Riemannian metric. Then for any $p, q \in M$,*

$$1 \leq \dim H_{n-1}(L_{pq};\mathbb{Z}_2) \leq \dim H_{n-1}(M;\mathbb{Z}_2) + 1.$$

Also, if L' is any proper subset of L_{pq} and $i : L' \to M$ is its inclusion map, then the induced map $i_{n-1} : H_{n-1}(L';\mathbb{Z}_2) \to H_{n-1}(M;\mathbb{Z}_2)$ is injective.

Remark 4.3. Note that by Poincaré Duality $H_{n-1}(M;\mathbb{Z}_2) = H^1(M;\mathbb{Z}_2)$ which is isomorphic to $H_1(M;\mathbb{Z}_2)$ (since $\mathbb{Z}_2$ is a field). Thus the bound $\dim H_{n-1}(M;\mathbb{Z}_2)+1$ above could be replaced by $\dim H_1(M;\mathbb{Z}_2) + 1$, although the same is not true for $\dim H_{n-1}(L_{pq};\mathbb{Z}_2)$, since L_{pq} is in general not a manifold.

Proof of Theorem 4.2. The relative homology sequence of the pair (M, L_{pq}) is:

$$\cdots \longrightarrow H_n(L_{pq};\mathbb{Z}_2) \xrightarrow{i_n} H_n(M;\mathbb{Z}_2) \xrightarrow{p_n} H_n(M, L_{pq};\mathbb{Z}_2)$$
$$\xrightarrow{\partial_n} H_{n-1}(L_{pq};\mathbb{Z}_2) \xrightarrow{i_{n-1}} H_{n-1}(M;\mathbb{Z}_2) \xrightarrow{p_{n-1}} \cdots .$$

Now by Lemma 4.1, $H_n(L_{pq};\mathbb{Z}_2) = 0$. Since M is a connected manifold, we have that $H_n(M;\mathbb{Z}_2) = \mathbb{Z}_2$. Furthermore, by Lemma 2.6, L_{pq} is closed and therefore compact, so Lefschetz Duality (see, for example, [12]) implies that $H_n(M, L_{pq};\mathbb{Z}_2) = H^0(M - L_{pq};\mathbb{Z}_2)$, which is $\mathbb{Z}_2 \times \mathbb{Z}_2$ since Lemma 2.6 also shows that $M - L_{pq}$ has two components.

We have then the sequence

$$\cdots \longrightarrow 0 \xrightarrow{i_n} \mathbb{Z}_2 \xrightarrow{p_n} \mathbb{Z}_2 \times \mathbb{Z}_2$$
$$\xrightarrow{\partial_n} H_{n-1}(L_{pq};\mathbb{Z}_2) \xrightarrow{i_{n-1}} H_{n-1}(M;\mathbb{Z}_2) \xrightarrow{p_{n-1}} \cdots .$$

By the exactness of the sequence, $\ker(i_{n-1}) = \mathrm{im}(\partial_n)$, the latter being equal to $(\mathbb{Z}_2 \times \mathbb{Z}_2)/\ker(\partial_n)$. Employing the exactness of the sequence again, we have that p_n is an injection and $\ker(\partial_n) = \mathrm{im}(p_n) = \mathbb{Z}_2$. Combining these statements, we see that $\mathrm{im}(\partial_n) = (\mathbb{Z}_2 \times \mathbb{Z}_2)/\mathbb{Z}_2 = \mathbb{Z}_2$.

Since $\ker(i_{n-1}) \subseteq H_{n-1}(L_{pq};\mathbb{Z}_2)$ and $\ker(i_{n-1}) = \mathbb{Z}_2$, we get the first bound of the first part of the statement of the theorem. On the other hand, since i_{n-1} is a map to $H_{n-1}(M;\mathbb{Z}_2)$, this proves the second bound.

Now if L' is any proper subset of L_{pq} with inclusion map $i : L' \to M$, then since L_{pq} is minimally separating, L' is not separating. Consequently, $M - L'$ has only a single component, so $H^0(M - L';\mathbb{Z}_2) = \mathbb{Z}_2$. The relative homology sequence of the pair (M, L') combined with Lefschetz Duality as before yields the exact sequence

$$\cdots \longrightarrow 0 \xrightarrow{i_n} \mathbb{Z}_2 \xrightarrow{p_n} \mathbb{Z}_2$$
$$\xrightarrow{\partial_n} H_{n-1}(L';\mathbb{Z}_2) \xrightarrow{i_{n-1}} H_{n-1}(M;\mathbb{Z}_2) \xrightarrow{p_{n-1}} \cdots .$$

So p_n is injective, from which it follows that $\ker(\partial_n) = \mathbb{Z}_2$ and $\mathrm{im}(\partial_n) = 0$. Thus the injectivity of $i_{n-1} : H_{n-1}(L';\mathbb{Z}_2) \to H_{n-1}(M;\mathbb{Z}_2)$ follows. □

This theorem implies, in particular, that mediatrices on compact, connected Riemannian manifolds are never contractible topological spaces. We also have the following immediate corollary to the theorem in the case where the lower and upper bounds are equal.

Corollary 4.4. *For any mediatrix L_{pq} on a compact, connected Riemannian manifold M for which $H_1(M;\mathbb{Z}_2) = 0$ (or, equivalently, by Poincaré Duality, for which $H_{n-1}(M;\mathbb{Z}_2) = 0$), the dimension of $H_{n-1}(L_{pq};\mathbb{Z}_2)$ equals 1.* □

As a further example of such techniques, we examine mediatrices L_{pq} on any manifold M whose homology is isomorphic to that of S^n. In this case, Corollary 4.4 implies that $H_{n-1}(L_{pq};\mathbb{Z}_2) = \mathbb{Z}_2$, but more information can also be read from the long exact sequence of the pair (M, L_{pq}). Using that

$$H_k(S^n;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } k = 0, n \\ 0 & \text{otherwise} \end{cases},$$

the exact sequence becomes

$$\cdots \longrightarrow H_k(L_{pq};\mathbb{Z}_2) \xrightarrow{i_k} 0 \xrightarrow{p_n} H_k(M, L_{pq};\mathbb{Z}_2)$$
$$\xrightarrow{\partial_k} H_{k-1}(L_{pq};\mathbb{Z}_2) \xrightarrow{i_{k-1}} 0 \xrightarrow{p_{k-1}} \cdots ,$$

where $0 < k < n$. From this exact sequence, we obtain immediately that

$$H_{k-1}(L_{pq};\mathbb{Z}_2) \;\cong\; H_k(M, L_{pq};\mathbb{Z}_2).$$

Again by Lefschetz Duality, we have that $H_k(M, L_{pq};\mathbb{Z}_2) \cong H^{n-k}(M - L_{pq};\mathbb{Z}_2) \cong H_{n-k}(M - L_{pq};\mathbb{Z}_2)$, so mediatrices L_{pq} in M satisfy

$$H_{k-1}(L_{pq};\mathbb{Z}_2) \cong H_{n-k}(M - L_{pq};\mathbb{Z}_2)$$

for $0 < k < n$. □

To close this section, we observe that the only ingredients used in Theorem 4.2 and Lemma 4.1 are that L is a set in a compact, connected, Riemannian manifold M that minimally separates M into 2 components.

Remark 4.5. Thus the second part of Theorem 4.2 restricted to $n = 2$ states in essence that the maximum number of "circular" cuts which do not disconnect the surface is equal to the number of generators of the first homology group with coefficients in $\mathbb{Z}_2$. This result appears in the book by Seifert and Threlfall [13] (Section 41). However, their proof (whose first German edition appeared in 1934) proceeds along much more classical lines.

Theorem 4.2, of course, generalizes this to arbitrary dimension. Another natural generalization is given in the following corollary to the proof of the first part of the theorem.

Corollary 4.6. *Let M be a compact, connected, n-dimensional Riemannian manifold whose distance function is that associated with the Riemannian metric. Let L be a set that* minimally separates M into $r > 1$ connected components *(i.e., if L' is any proper subset of L, then $M - L'$ cannot be written as the union of r connected components). Then*

$$r - 1 \leq \dim H_{n-1}(L_{pq}; \mathbb{Z}_2) \leq \dim H_{n-1}(M; \mathbb{Z}_2) + r - 1 \quad .$$

Proof. The proof of the first of Theorem 4.2 proceeds as before except that now we have $H^0(M - L; \mathbb{Z}_2) = (\mathbb{Z}_2)^r$. Using Lefschetz Duality as before, followed by the same computation, one arrives at the inequalities. □

We remark that the generalization of the second part of the theorem is not much harder and states that in this case we have that if $i : L' \to M$ is the inclusion map, then

$$\dim(\ker(i_{n-1})) \leq r - 2 \quad .$$

As an interesting example where $r = 3$ consider a "Lakes of Wada" construction on the flat torus (see also Example 3.1).

5. Concluding remarks

The criteria shown in Section 2 to be sufficient for a space to be a Brillouin space give us a large class of Brillouin spaces in which to work. In such spaces, mediatrices are minimally separating, which was shown in Section 4 to have implications for the topology of the mediatrices. For example, in a given 2-dimensional compact, connected, Riemannian manifold, Theorem 4.2 implies that there are only finitely many homotopy types of mediatrices. A natural question is whether one can find a complete topological (up to homeomorphism) classification of mediatrices on such a manifold M, while allowing the metric to vary. In a forthcoming work [15], the authors investigate this question in the case of 2-manifolds.

A classification of mediatrices in higher-dimensional manifolds seems at present still to be elusive, however. Mediatrices appear to take on a considerable variety of shapes in such spaces.

Separating sets occur in a wide range of contexts. For example, let us suppose that a dynamical system (i.e., a smooth map $f : M \to M$) is defined on a connected

Riemannian manifold M. The system may have $r \in \mathbb{N}$ attractors A_1 through A_r. The open set $B(A_i)$ of initial conditions that under iteration tend to the attractor A_i is called the basin of attraction of A_i. The complement $M - \cup_i B(A_i)$ is a closed set called the basin boundary (see [5]). In general, it is not clear that such a set *minimally* separates into r components. In particular, one single basin may already have infinitely many components.

A special case occurs in dissipative twist maps of the cylinder that admit a so-called *trapping region*. (In this case the basin boundary is really the boundary between the "basins of repulsion".) With certain additional assumptions, one can prove that in this case each basin is connected, and from this one can prove as in [14] that there exists an invariant minimally separating set. This set may be highly fractal. In general one can ask the question if bifurcations are possible where the homology of such invariant sets changes. (This is not the case in the present example).

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