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Dynamical Systems and Ergodic Theory

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September 12.

A C^2 measure preserving map of a manifold M to itself is said to be hyperbolic if we can foliate the manifold by stable manifolds and by unstable manifolds (intersecting transversally). In general such systems are ergodic. The main ingredient in the proof of this statement is the fact that the holonomy maps have uniformly continuous Radon-Nikodym derivative.

The stable holonomy map is just the map that projects one stable manifold to another along the unstable manifold. Similarly for the unstable holonomy map. It turns out that these maps (in dimension greater than 2) are not generally differentiable, but only α-Hölder continuous. However, the Radon-Nikodym derivative of the map h,

$$\lim_{r \to \infty} \frac{\text{volume}(h(B_r(x)))}{\text{volume}(B_r(x))}$$

where $B_r(x)$ is a ball of radius r centered in x, does exist and is continuous.

Work in progress extends the ergodicity for maps that are products of hyperbolic and elliptic maps.

Symplectic Geometry

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September 19.

A symplectic form ω assigns an area to the parallellepipid formed by two tangentvectors to a even dimensional manifold M^{2n}. At any given point of M^{2n}, it is an anti-symmetric and bilinear function on the tangentspace. Darboux' theorem says that locally the form can be written as

$$\omega = dp_1 \wedge dq_1 + \cdots dp_n \wedge dq_n \quad .$$

The group of symplectomorphisms, that is diffeomorphisms from M^{2n} to itself that preserve the form ω, is large. For each smooth function from M^{2n} to $\mathbb{R}$ we can construct a Hamiltonian flow (a smooth one parameter family of symplectomorphisms). Thus the group of symplectomorphisms is infinite dimensional. For comparison, note that the group of isometries of $\mathbb{R}^n$ (restricted to orientation preserving) consists only of translations and rotations.

Work of Gromov and others has recently led to some understanding of the global properties of the group So suppose that in R^{2n}we have a ball of radius greater than 1. It was recently proved that it is impossible to move this ball, using a smooth one parameter family of symplectomorphisms, through a $2n-1$-dimensional 'wall with a unit hole'.

A Lagrangian torus is a torus whose tangent-planes have the property that Borel sets contained in them have zero (symplectic) area. An example in $\mathbb{R}^4$ is the product of the unit circle in the (p_1, q_1) plane and the unit circle in the (p_2, q_2) plane. The disks contained in these circles are called spanning disks. They have non-zero area in this example (the standard structure). An important result shows that $\mathbb{R}^4$ admits a symplectic structure not equivalent to the usual one (given above). More precisely, there exists a symplectic structure that admits a Lagrangian torus all whose spanning disks have zero (symplectic) area.

Moduli Spaces

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September 25.

A differentiable manifold has a conformal structure if all coordinate transformations between two charts in the atlas of the manifold are conformal. The space of all conformal structures for a given topological manifold is called the moduli space for the manifold. In string theory, the evolution of a string sweeps out a two-dimensional surface. Path-integrals here can be interpreted as integrals over moduli space of such surfaces.

The moduli space of the torus is virtually the only (non-trivial) example that can be understood with relative ease. The uniformization theorem implies that any smooth torus is conformally equivalent to a quotient of $\mathbb{R}^2$ by $A\mathbb{Z}^2$, where A is a non-singular 2×2 matrix. In fact, since rotation and dilatation do not change the conformal class, we can take $(1, 0)$ for the first and a vector (a, b) in the upper half plane for the second column. Two vectors in the upper half plane define the same conformal class if and only if they are related by a transformation in PSL(2,$\mathbb{Z}$), the linear transformations with unit determinant and integer entries. So the moduli space is equal to a fundamental domain of PSL(2,$\mathbb{Z}$) in the upper half plane. In fact, one can choose the fundamental domain in such a way that the 'unit cell' of the lattice $A\mathbb{Z}^2$ defining the torus has the property that both its diagonals are longer than each of its faces.

The moduli space of the 2-holed 'torus' is only partially understood. The special choice of the fundamental domain described above can be re-interpreted as a construction involving 'shortest non-separating linked geodesics'. This leads to a geometrical description of the 6-dimensional moduli space. But it is unclear what the natural metric is.

Geometric Integration on Non-smooth Sets

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October 3.

A chain $\sum t_i A_i$, $t_i \in \mathbb{R}$, is a linear sum of oriented polyhedra (or simplices) A_i. One can think of the addition as the algebraic expression of the geometric notion of union. Multiplication by -1 corresponds to reversing the orientation. Formally we can multiply and sum, so the space of chains is actually a vectorspace over $\mathbb{R}$. Given an appropriate norm (we discuss Whitney's 'flat' norm), one can complete this space in such a fashion that fractal sets are contained in it. That is to say: if, with respect to the norm, we have a Cauchy series of chains $\{A_k\}$ converging to some, perhaps fractal, set, we include its limit-point A in the space.

Now one can define the integration $\int_A \omega$ of a p-form ω over a chain A, and, what is more, if for a sequence of chains we have that $\lim A_k = A$, then it is easy to show that

$$\lim_k \int_{A_k} \omega = \int_A \omega$$

converges. This opens up a way of defining integration over fractal sets. It turns out that the properties of these intgrals are very good: the classical theorems, such as Stokes' theorem

$$\int_A d\omega = \int_{\delta A} \omega \quad ,$$

hold, and on smooth sets it coincides with the usual integral. In a similar vein one can define a geometric generalization of the Laplacian using the Hodge star operator $\star$ (see Sullivan's abstract): $\triangle = d\star d\star + \star d\star d$.

The Cost of Computation

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October 9.

We discuss some basic issues related to complexity of scientific computation and illustrate them with an analysis of the cost of solving systems of polynomial equations by continuation methods. Among other things, we give a simple homotopy proof of Bezout's theorem (counting the number of complex solutions, with multiplicity, of simultaneous polynomial equations). On the average we find that systems can be approximately solved in polynomial time in the input data. This result is somewhat surprising in view of the fact that algebraic methods for the problem are intrinsically of exponential size in the input. This is joint work with Steve Smale.

Front Tracking for Hyperbolic P D E's.

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October 16.

In this talk we present results from our three dimensional simulation of etching and deposition processes. Input to these simulations is a continuum model for the dynamics of the chip's surface, which leads to the solution of a Hamilton-Jacobi problem.

Our simulation can address various issues. Do sharp two and three dimensional features in the chip's design, corners and edges, remain sharp or do they round off? Does resputtering create voids? We show the extent to which the answers to questions depend on chip geometry and the details of the continuum models. We present the class of models we work with, briefly describe our front tracking algorithm, and present numerical results for various models.

Non-equilibrium Statistical Mechanics and Dynamical Systems

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October 23.

A diffeomorphism f of a compact manifold M to itself is a model for statistical mechanics: M is the phase space and f models, say, the Poincaré map of the equations of motion of the system. Suppose that M is foliated by stable and unstable manifolds with respect to f (f is Anosov). A smooth (with respect to Lebesgue measure) distribution of initial conditions will evolve to a unique limit, the so-called SRB measure. If the statistical system is in equilibrium then f preserves volume (in fact, is symplectic) and the SRB measure is just the Lebesgue measure on M.

For stationary states in non-equilibrium statistical mechanics one can think of the following system. A long thin volume contains two gases A and B. We maintain concentration gradients artificially: at one end pump out A and inject B, at the other end do the opposite. Thus stationary fluxes are maintained. These fluxes would heat up the gases if the heat or entropy was not siphoned away. For this reason, we now drop the requirement that f preserves volume (although the total volume remains constant), but we maintain the hyperbolicity assumption. Now the SRB measure lives on an attractor (essentially the closure of an unstable manifold) contained in but not equal to M. Its density at a given point of the attractor is proportional to the sum of the inverses of the positive Lyapunov exponents.

When we look at variations of f, the attractor varies Hölder continuously. By considering the SRB measures as acting on differentiable functions (via Riesz' theorem) with the C^1 topology, one proves that these measures are smooth functions of f. It also becomes possible to derive certain symmetry relations between the fluxes in physical systems as the example mentioned before (the Onsager relations).

Morse Theory

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October 30.

A Morse function f on an n-dimensional manifold M is a (real) function whose Hessian at any critical point is non-degenerate. Its index at p is the number of negative eigenvalues of the Hessian at p. A Morse function f has the property that if the pre-image $M_{[a,b]}$ contains a unique critical point of index k, then there is a k-dimensional cell e^k whose boundary can be glued to $f^{-1}(a)$ with the property that there is a deformation retraction of $M_{[a,b]}$ onto $f^{-1}(a) \cup e^k$. The singularities that can occur in f are thus related to the topology of M. The following formulation of Morse theory gives a natural choice for these cells.

After a choice of Riemannian metric, the Morse function just considered defines a gradient flow ϕ_t. Such a flow is called Morse-Stokes if it satisfies various other conditions as well. First, ϕ_t must be a finite volume flow. Second, the stable and unstable manifolds must have finite volume in their dimension. Finally, if there is a stable manifold flowing from p to q, then the index of q must be greater than the index of p. One can show that for every Morse function, there is a Riemannnian metric such that the induced flow is Morse-Stokes.

The idea of the new formulation is to let a Morse-Stokes flow ϕ_t act on k-forms by pushing them forward along the flow. A k-form α can be interpreted as a linear functional $[\alpha]$ on $(n-k)$-forms β by setting

$$[\alpha](\beta) = \int_M \alpha \wedge \beta \quad .$$

These functionals are called k-currents. Now $\lim_{t\to\infty} \phi_t^*(\alpha)$ exists in the weak limit (that is: as a current). This limit is, by dynamics, a linear combination of the currents given by integration over the stable manifolds. Now it can be shown that the map P and the inclusion $\alpha \to [\alpha]$ are chain-homotopic, from which all basic results of Morse theory follow. In addition, the k-cells of classical Morse theory are now naturally interpreted as stable manifolds. If the direction of time is changed in this formalism, the cells will of course consist of unstable manifolds.

Riemannian Geometry

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November 7.

Let M be a manifold with a metric d and F a partition of M into disjoint submanifolds of codimension greater than zero whose union covers M. The fibration F (its connected sets are called fibers), is called metric if mutual distance is preserved along the fibers, that is: given any two fibers A and B, a point $a \in A$, and the point $b(a) \in B$ closest to it, then the distance $d(a, b(a))$ is independent of a. Familiar examples are the fibration of $\mathbb{R}^2$ by parallel lines or by concentric circles. The latter fibration has a singularity at the origin (the fiber at the origin has a different codimension), because the radius of curvature of nearby fibers tends to zero.

Historically, the subject arose with Elie Cartan who studied the ways in which S^n could be smoothly partitioned into codimension k surfaces. The example is the Hopf fibration. There is also a connection with the study of isometries. If a space admits a group G of isometries then its orbits Gx are the fibers of a metric fibration.

One can show that for a given manifold M with a metric d, and a codimension one submanifold A (with radius of curvature bounded away from zero), there is a local metric fibration with A as one of its fibers. However, the problem of characterizing global metric fibrations is open, even for $\mathbb{R}^2$. For a generic metric on M, there exist no metric fibrations of codimension greater than one.

Geometric Structures and the Navier-Stokes Equation

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November 12.

The incompressible, inviscid Navier-Stokes equation in $\mathbb{R}^3$ takes the form of $\dot{\omega} = L_v \omega$, where $v(x,t)$ is the vector-field corresponding to the instantaneous velocity of the fluid, ω is the curl of this velocity (the vorticity), and L_v the directional derivative along the velocity. This equation expresses the fact that the vorticity is pushed forward by the fluid ("frozen in the fluid" in Kelvin's words). Incompressibility implies (via the continuity equation) that the divergence of v is zero. With one additional assumption (the velocity field integrated over $\mathbb{R}^3$ is zero), there is a unique inverse for $v = \operatorname{curl} \omega$. The equation for the vorticity thus becomes:

$$\dot{\omega} = L_{\operatorname{curl}^{-1}\omega} \omega \quad .$$

The only additional restraint is that total energy ($\int_M |v|^2$) is conserved.

The language of forms is used to give this equation as well as its finite-dimensional discretization a natural form. The star operation $\star$ is a linear map from k-forms to $(n-k)$-forms, such that the wedge product of the two is a multiple of the standard volume. Take $M = \mathbb{R}^3$ as the underlying manifold. Using the usual metric on M, functions on M can be identified with 0-forms and (via $\star$) 3-forms, and vector-fields with 1-forms and (via $\star$) 2-forms. For instance $(v_1, 0, 0) \to v_1\, dx$ or $\to v_1\, dy \wedge dz$. In the language of forms, all differential operators (like curl) now correspond to the exterior derivative d. Thus, the curl corresponds to the exterior derivative on 1-forms.

Discretization is done by replacing M with a finite triangulation. The k-forms become k-cochains, functions on simplicial k-chains and the exterior derivative becomes the co-boundary operator δ. The directional derivative with respect to x, L_x, becomes $\delta\, i_x + i_x \delta$, where i_x is the contraction operation (from k-cochains to $(k-1)$-cochains). The equation can now be cast as a differential equation for a 2-cochain c (formerly the vorticity). Noting that c is the co-boundary ("curl") of something and thus $\delta c = 0$:

$$\dot{c} \quad = \quad \delta\, i_{\delta^{-1}c} c$$

$$\text{with } \delta(\star\delta^{-1}c) = 0 \quad \text{and} \quad \sum_{\text{edges}} |\delta^{-1}c|^2 = 0 \quad .$$

The first of these two conditions is a local condition, essentially Kirchhoff's law, that the sum of positive contributions at any vertex equals the sum of negative contributions. The second corresponds to the conservation of total energy.

Topological Entropy

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November 21.

Suppose f is a map of a metric space X into itself and $\{E\} = \cup_{i=1}^{n} E_i$ is a partition of X. Let $N(t)$ be the number of distinct sequences $E_{r_1} E_{r_2} \cdots E_{r_t}$ of length t such that there is a point $x_0 \in X$ with $f^i(x_0) \in E_{r_i}$. The entropy $h(f, \{E\})$ of f with respect to $\{E\}$ is $\lim_{t \to \infty} \frac{1}{t} \ln N(t)$. The topological entropy $h(t)$ of f is given by the supremum of $h(f, \{E\})$ over all finite partitions $\{E\}$. A Markov partition is a partition such that the image of any of the sets E_i covers any other set E_j either entirely or not at all (modulo sets of measure zero). In this case, the entropy of f equals the logarithm of the largest eigenvalue of the associated transition matrix. If one has no such partition at hand, then calculation of the entropy may pose problems.

As an example, it turns out that one can show that there exists no algorithm that will calculate the topological entropy for one-dimensional cellular automata on $\{0, 1\}^{\mathbb{Z}}$. Nonetheless, topological entropy has some good properties. In the family of maps $f_\lambda(x) = \lambda x(1 - x)$, and $\lambda \in (0, 4]$, it is a weakly monotone (non-decreasing) function of λ. (This is a slightly weaker property than the monotonicity of the kneading-sequences.) In work with Heckmann and Tresser, this was extended by proving that for certain 2-parameter families of cubic maps, the set of parameters for which the entropy is smaller than or equal to a certain number is a simply connected set.

Critical Circle Maps

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December 4.

This is an outline of work in progress aiming to prove uniqueness of the renormalization fixed point for critical one-dimensional dynamical systems with golden mean like combinatorics and using real variable methods only. The questions of existence of a fixed point of renormalization and hyperbolicity of renormalization are not addressed. The role of the combinatorics is minor, and generalizations to more complicated cominatorics appear straightforward.

Assume that there are two critical circle maps f_0 and f_1 that are distinct fixed points under remnormalization and whose criticality is of the same type (locally equal to $\text{sign}(x)|x|^p$ for some $p > 1$). Assume that outside a neighborhood of the critical point they are diffeomorphisms onto their image and that their derivative is Zygmund, that is, $\phi \equiv \ln Df$ satisfies:

$$|\phi(x + t) + \phi(x - t) - 2\phi(x)| = \mathcal{O}(t) \quad .$$

One now constructs a curve $\{f_t\}_{t\in[0,\infty)}$ of infinitely renormalized maps through f_0 and f_1, and shows that as t goes to infinity, some scalings diverge. This contradicts Sullivan's a priori bounds theorem, with the conclusion that f_0 and f_1 cannot be distinct.

The strategy to construct the curve of renormalized maps is the following. The maps f_0 and f_1 are conjugate by a homeomorphism h_1. By a theorem of Thurston, h_1 can then be generated up to a Möbius transformation by an earthquake (defined by a lamination of the Poincaré disk and a weight for each lamina). Now let h_t be the conjugacy obtained from h_1 by multiplying all weights by t and define $f_t = h_t f_0 h_t^{-1}$.

The conditions on f_0 and f_1 imply that h_1 must be quasi-symmetric, that is, there is an $M > 0$ with

$$\sup_{x,s} | \ln \left| \frac{h_1(x+s) - h_1(x)}{h_1(x) - h_1(x-s)} \right| | = \ln M \quad .$$

Let D be a disk of radius smaller than $\frac{1}{2}\ln 3$. Then the geodesics passing through D determine two intervals I and J on the circumference. The measure μ_h of h associated to the geodesics passing through D is thus a measure on pairs of intervals. It turns out out that h_1 is quasi-symmetric if and only if $\mu_h(I,J)$ is bounded. Clearly, h_t is quasi-symmetric as well.

The functions f_t must all have the same type of criticality. This is equivalent to

$$f_t^{-1} f_1 = (f_t^{-1} h_1 f_0 h_t^{-1})(h_t h_1^{-1})$$

being Zygmund differentiable. To prove that this is so, one uses the following criterion. A map u is Zygmund differentiable if and only if its associated measure μ_u satisfies

$$\mu_u(I,J) \leq C\delta(D) \quad ,$$

where $\delta(D)$ is the (Euclidean) distance from the disk D to the boundary of the unit disk. This holds for $f_t^{-1} h_1 f_0 h_t^{-1}$. Work in progress concentrates on proving this for $h_t h_1^{-1}$ and proving that f_t is infinitely renormalized.

Partial Differential Equations with Random Forces

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December 12.

The one-dimensional Navier-Stokes equation without the pressure term and with zero viscosity takes the form of Burger's equation:

$$\partial_t u + \frac{1}{2}\partial_x(u^2) = \partial_x F(x,t) \quad .$$

The right hand side is a forcing term. It consists of a random excitation of finitely modes:

$$F(x,t) = \sum_{|k|\leq K} B_k(t)\cos 2\pi kx \quad ,$$

where the $B_k(t)$ are random variables. The solutions of this equation are real functions and correspond to the velocity field of the fluid.

Physical solutions of this system contain discontinuities in the velocity field, or shock waves. These travel along paths $x(t)$ satisfying the following variational principle. Let A be the functional on paths

$$A(x) = \int_{-\infty}^{0} (\frac{1}{2}(\frac{dx}{d\tau})^2 + F(x,\tau))\, d\tau \quad .$$

The path x of a shock wave (as long as it does not collide with other shocks) is given by a minimum of A, that is: for all local perturbations x' of x, the difference $A(x') - A(x)$ is positive. Any extremum of the functional satisfies the Euler-Lagrange equation, which, in this case, reduces to that of a particle moving in a potential $F(x,t)$:

$$\frac{d^2x}{dt^2} = \partial_x F(x,t) \quad .$$

Now $v(x,t) = \frac{dx}{dt}(x,t)$ solves the Burger's equation. So starting with a distribution of shocks the velocity field along which the shocks travel solves the equation.

When shocks collide, they must have different velocities (by uniqueness of solutions). Assuming that the 1-dimensional configuration space is a circle, one shows that minima of A cannot cross. So the shock waves meet with different velocities and after collision continue as a single shock wave. In such a collision, momentum is conserved but energy is decreased. Since the likelihood of collisions is exponential in time, the result is that the shocks observed at $t = T$ originated at $t = 0$ from a set located in a set of exponential (in T) small measure on the circle. The dynamics on this set, then is expanding. The dissipation resulting from the shock collisions, forces the dynamics onto an unstable manifold.

Orbit Focussing in Physics, Geometry, and Arithmetic

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December 19.

When initial position and velocity are given, then a (Lipshitz) second order ordinary differential equation has a unique solution. In many applications (for instance Feynman quantization) a different problem arises. Given two endpoints, how many solutions connects the two points in a given time? To simplify, fix one of the endpoints, and suppose the equation is autonomous. As an example, consider the equation $\frac{d^2x}{dt^2} = -\sin x$ in $\mathbb{R}$. A point $(x_0, t_0) \in \mathbb{R}^2$ is in σ_k if there are k solutions $x(t)$ of the equation with $x(0) = 0$ and $x(t_0) = x_0$. For this particular system this sigma decomposition can be worked out.

Let M be a complete Riemannian manifold and $TM = \cup_{x\in M} T_xM$ its tangent bundle. There is a natural map $\exp_x : T_xM \to M$ projecting points of T_xM to M. The sigma decomposition of T_xM, the tangent plane with base-point x, is given by the partition of the tangent space T_xM in sets $\Sigma_{x,k}$ with the following property. The point $v \in T_xM$ is in $\Sigma_{x,k}$ if there are exactly k geodesics of length $|v|$ on M connecting x to the point $\exp_x(v)$ in M. The sigma decomposition of the tangent bundle is given by

$\Sigma_k = \cup_{x \in M} \Sigma_{x,k} \subset TM$. When M is an analytic manifold, then the sigma decomposition is an analytic Whitney stratification.

One of the simplest examples is the flat torus $\mathbb{R}^2/\mathbb{Z}^2$. Since the metric in this manifold is constant, all tangent spaces have the same decomposition (independent of the base-point). So it is sufficient to study the one with the origin as its base-point. In the covering space $\mathbb{R}^2$, one asks the question, given a point x at a distance $|x|$ from the origin, how many points in $\mathbb{Z}^2$ have the same distance it? Suppose the set of points equidistant to 0 and to $k \in \mathbb{Z}^2$ is called L_{0k}. Then x is in σ_k if it lies in k equidistant sets. This decomposition is thus equivalent to the classical decomposition of $\mathbb{R}^2$ in Brillouin zones relative to the lattice $\mathbb{Z}^2$. The decomposition of $\mathbb{R}^2/A\mathbb{Z}^2$, where A is a non-negative invertible symmetric matrix leads to new results about the number of solutions of the Diophantine equation: given $N \in \mathbb{Z}$, solve $x^t A x = N$ in $x \in \mathbb{Z}^2$.