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Irrational rotation numbers

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Abstract. For a general class of one-parameter families of 'flat spot' circle maps (such as non-decreasing truncations of non-invertible circle maps), we prove the following facts. The set of parameter values where the rotation number is irrational has Hausdorff dimension zero. Each recurrent set with irrational rotation number has Hausdorff dimension zero. Moreover, the closure of their union has Lebesgue measure zero.

Our results are less general than the ones obtained recently by Swiatek; however, they are stronger and the proofs are much simpler.

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1. Introduction and results

We are interested in a certain class of one-parameter families of circle maps f_t. For each t, f_t has a 'flat spot', i.e. there is an open set U such that for all $x \in U$, $\mathrm{d}/\mathrm{d}x(f_t(x)) = 0$ and in a neighbourhood of $S^1 - U$, f_t can be extended to a C^1 map with $\log[\mathrm{d}/\mathrm{d}x(f_t(x))]$ of bounded variation. In particular, we will denote the following requirement by the derivative condition: the one-sided derivative on the boundary of U is bounded away from zero. Although the derivative condition is a strong condition, there are families of maps that satisfy it naturally, for example the truncations of non-invertible circle maps; see figure 1. Apart from this, we require that the parameter dependence be continuous, non-decreasing and increasing in at least one point. More precise definitions are given in §2.

Thus defined, these families of circle maps have the property that each map has a well defined and unique rotation number $\rho(f_t) \in [0, 1)$ assigned to it (see Nitecki 1971, Herman 1979). Moreover, the map $t \to \rho(f_t)$ is continuous, strictly increasing where $\rho(f_t)$ is irrational, and constant in a closed interval ('locking interval') when $\rho(f_t)$ is rational (Boyland 1983, Herman 1979). An example of such staircase-like functions is given in figure 2. The set of parameters where ρ is irrational will be denoted by A.

The dynamics of f_t admits two cases. If t is in one of the locking intervals, then its non-wandering set Ω_t consists of periodic orbits. If $t \in A$, then the non-wandering set Ω_t consists of a unique minimal set and the orbits in the non-wandering set are well ordered.

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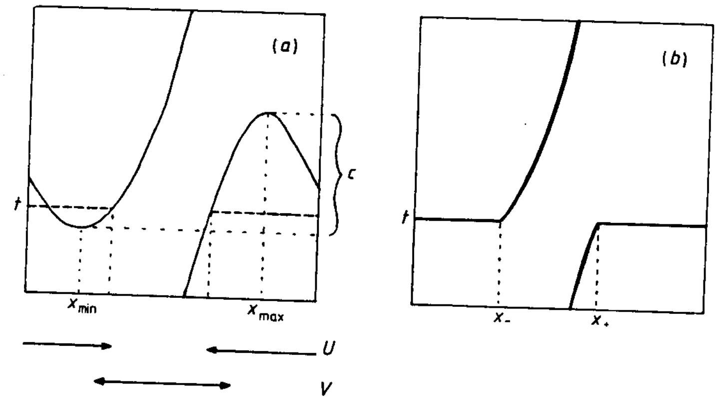

Figure 1. (*a*) A picture of a bimodal map from which a truncated family can be derived. (*b*) The flat spot map. This map is an example of a truncated map.

We are interested in the measure theoretic questions: how often does irrational behaviour occur and how much space do the minimal sets occupy in the circle?

In §3, we give a characterisation of Ω_t, for $t \in A$ (i.e: if $\rho(f_t)$ is irrational), namely $\Omega_t = S^1 \setminus \bigcup_{i=0}^{\infty} f^{-i}(U)$.

In the next section, we use a theorem of Mañé (1985) to prove that for each $t \in A$ and for all $\gamma > 1$, there is an $m > 0$ such that for all x with $f^i(x) \in S^1 \setminus U$ for $i \leq m-1$: $\mathrm{d}/\mathrm{d}x(f_t^m(x)) \geq \gamma > 1$.

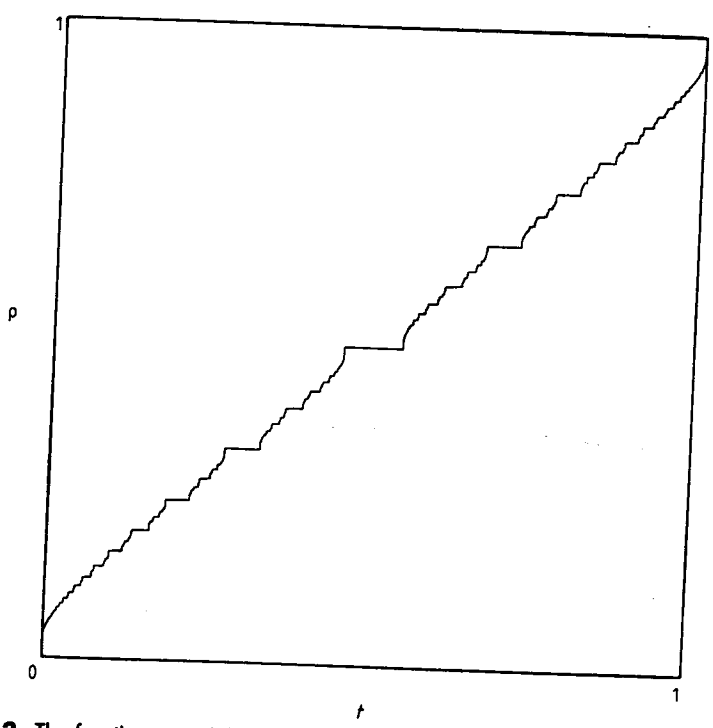

Figure 2. The function $t \to \rho(f_t)$. Here, f_t is a piecewise linear family, its slope in the gap being zero and outside 1.2. All details of this curve can be described by a very simple recursive structure.

When the above inequality holds, we will say that f_t has the property of (γ, m) expansivity. We will prove that the estimates for γ and m are locally uniform in t.

The Lebesgue measure will be denoted by μ, the Hausdorff dimension of a set S by $HD(S)$. Define Γ as follows:

$$\Gamma = \bigcup_{t\in A} \Omega_t.$$

In §5 it is proved that $\mu(A)=0$, in §6: $HD(A)=0$, in §7: $\mu(\Gamma)=0$ and in §8: $HD(\Omega_t)=0$.

A special case of the results of §§5 and 6 (namely, f_t being $(\gamma, 1)$ expansive for all t) was proved by Boyd (1985). Our reasoning in §5 is different from his. We remark here that our way of arriving at the result of §6 is to combine Mañé's result with Boyd's reasoning. The problem of matching them up is solved by the uniform expansitivity. Swiatek (1986) has proved a generalisation of the main result of §5 (he does not need the derivative condition; he needs another condition, but that is a merely technical one, see §5). However, his methods are very different from ours. The main thrust of our exposition is that these results are now elementary. (In Veerman (1986, 1987) exact expressions for the lengths of the locking intervals were derived for families of piecewise linear circle maps with flat spots.)

These results complement the classical theorem of Arnold (1965) and Herman (1977), which asserts that if the f_t are diffeomorphisms of the circle, then $\mu(A)>0$. For 'critical' families, homeomorphisms of the circle with a cubic inflexion point, numerical work indicates (Jensen *et al* 1984) that the Hausdorff dimension of A is 0.87, . . . , independent of the family. The quantity that distinguishes the three cases (diffeomorphic, critical and non-invertible) would then be the Hausdorff dimension of A. We treat here part of the non-invertible case (the other part being the elimination of the derivative condition).

2. Definitions

Let f_t be a one-parameter family of circle maps satisfying the following requirements (from now on we will write D for d/dx and Δ for d/dt).

(i) f_t has degree one and preserves orientation.

(ii) For each t, there is an open interval $U_t \subset S^1$ containing the point $x=0$ such that $Df_t(x)=0$ for x in U_t, and f_t extends to a C^1 on an open neighbourhood of S^1-U_t with $\log Df_t(x)$ of bounded variation.

(iii) The function $f_t(x)$ is C^0 (as a function of (x, t)).

(iv) For all $x\in s^1-\delta U_t$, $f_t(x)$ is C^1 (as a function of t), $Df_t(x)$ is C^0 (as a function of t).

(v) As a function of t, $f_t(x)$ is non-decreasing and for $x\in U_t$, we have $\Delta f_t(x)>0$. In order to keep the notation reasonably transparent, we will refer to the derivative $\lim_{y\to x}[f(y)-f(x)]/(y-x)$, for both x and y in $S^1\setminus U_t$ (note that this is a one-sided derivative on ∂U_t) simply as $Df(x)$. We will denote the points in ∂U_t by $x_-(t)$ and $x_+(t)$.

For later reference, we define the 'truncated family' depicted in figure 1. Let g be a bimodal c^2 map of degree one and fix a lift G. Denote the local minima and maxima of G by $x_{\min}+i$ and $x_{\max}+i$, where i runs through the integers, such that $x_{\min}<x_{\max}<x_{\min}+1$. Choose an interval $U=[x_-, x_+]\subset[x_{\min}, x_{\max}]$ such that

$G(x_-)=G(x_+)$ and define a parametrisation as follows:

$$t=(G(x_-)-G(x_{\min}))/(G(x_{\max})-G(x_{\min})).$$

Then outside U_t, we set $f_t(x)=g(x)$, and inside U_t, we set $f_t(x)=g(x_-)=g(x_+)$. It is clear from this definition that the truncated family satisfies our assumption, provided we consider a closed parameter interval contained in $(0, 1)$.

As is usual, we will define the lift of a map by an upper case symbol. We allow only lifts for which $F(0)\in[0, 1)$. The iterates $f^n(x)$ will be denoted by x_n.

3. $\Omega_t=S^1\backslash\bigcup_{i=0}^{\infty}f^{-i}(U_t)$

In this section we prove that if $\rho(f_t)$ is irrational, then the non-wandering set Ω_t of f_t is equal to the complement of the union of the inverse images of the flat spot. This results from a reasoning which in the case of C^2 diffeomorphisms leads to Denjoy's celebrated theorem that $\Omega_t=S^1$. Since this reasoning is almost identical to the classical reasoning, we only remark on those points where differences occur. For the course of this section, we fix t such that $\rho(t)$ is irrational and leave out the subscript.

A homterval J is an open interval such that $f^n(J)$ are disjoint and f^n carries J to $f^n(J)$ homeomorphically.

Lemma 3.1. If $f^n: J\to F^n(J)$ is homeomorphic for all n, then the $f^n(J)$ are pairwise disjoint (J is a homterval).

Proof. If $f^{n+k}(J)\cap f^n(J)\neq\varnothing$ for some n and k, then for all p, $f^{n+pk}(J)\cap f^n(J)\neq\varnothing$. One verifies easily that $K=\bigcup_{p=0}^{\infty}f^{n+pk}(J)$ cannot contain U and that $f^k(K)\subset K$. Then K contains a periodic point which is incompatible with f having an irrational rotation number (Boyland 1983, Nitecki 1971). □

We can rephrase Denjoy's classical result (see Nitecki 1971, Arnold 1983).

Proposition 3.2 (*'Denjoy'*). f has no homtervals.

Proof. The proof is entirely similar to the proof of the classical result (see Nitecki 1971). First, one establishes the existence of a subinterval H of the homterval such that H is a homterval for both f and f^{-1}. Then one uses the fact that $\log \mathrm{d}f/\mathrm{d}x$ is of bounded variation on the set consisting of the full orbit of the alleged homterval H to contradict the existence of such an interval. The only difference with the classical reasoning is that, in our case, $\log \mathrm{d}f/\mathrm{d}x$ is not of bounded variation on the entire circle. □

Corollary 3.3. The set $\bigcup_{i\geqslant 0}^{\infty}f^{-i}(U)$ is dense in S^1.

Proof. By lemma 3.1, every interval that is not a homterval has a forward image intersecting U_t. The previous proposition says that there are no homtervals.

Theorem 3.4. Let Ω be the non-wandering set of f. Then $\Omega=S^1\backslash\bigcup_{i\geqslant 0}f^{-i}(U)$.

Proof. The inclusion $\Omega\subseteq S^1\backslash\bigcup_{i\geqslant 0}f^{-i}(U)$ is evident. So, we have to prove that each point in $S^1\backslash\bigcup_{i\geqslant 0}f^{-i}(U)$ is non-wandering.

It is clear that $S^1 \backslash \bigcup_{i \geqslant 0} f^{-i}(U)$ contains no isolated points, because such a point would be on the boundary of two inverse images of U and therefore would imply the existence of a periodic orbit. In particular, x_- and x_+ (on the boundary of U) are non-wandering.

By corollary 3.3, the inverse images of x_- and x_+ are dense in $S^1 \backslash \bigcup_{i \geqslant 0} f^{-i}(U)$. Because f is order preserving, their forward images are dense in $S^1 \backslash \bigcup_{i \geqslant 0} f^{-i}(U)$. Thus every point in $S^1 \backslash \bigcup_{i \geqslant 0} f^{-i}(U)$ has a dense orbit (again by the property of being order preserving). □

4. Expansivity

In this section, we formulate and prove results concerning hyperbolicity. The first proposition is a powerful result by Mañé (1985). Theorems 4.2 and 4.3 extend the result somewhat, so that estimates can be given. In this section, we fix τ so that $\rho(\tau)$ is irrational.

Proposition 4.1 (*Mañé 1985*). Let f be C^2 (on the circle or unit interval) and A a compact invariant set, not containing critical points, sinks or non-hyperbolic periodic points. Then either A is the circle or A is a hyperbolic expanding set.

With theorem 3.4, it now follows immediately that for all $\eta > 1$ there is a $k > 0$ such that for all $x \in \Omega_\tau$, we have

$$\mathrm{D}f_\tau^k(x) \geqslant \eta > 1.$$

Remark. If there is a compact invariant set Ω_τ satisfying the above inequality, then there is a $\theta > 1$ and a $C > 0$ such that:

$$\mathrm{D}f_\tau^n(x) \geqslant C\theta^n \qquad \text{uniformly in } \Omega_\tau \text{ and } n > 0.$$

Proof. Choose $C = [\min_{\Omega_\tau} \mathrm{D}f_\tau(x)]^{k-1}$ and $\theta = \eta^{1/k}$. □

From now on, we will denote the closed set $S^1 - \bigcup_{j=0}^{m-1} f_t^{-j}(U_t)$ by $E_m(t)$. Note that these sets have a partial ordering given by

$$\ldots E_{n+1}(t) \subset E_n(t) \ldots \subset E_0(t).$$

Theorem 4.2 ((γ, m) *expansivity*). For all $\gamma > 1$ there exist $m \geqslant 1$ such that for all $x \in E_m(\tau)$ we have $\mathrm{D}f_\tau^m(x) \geqslant \gamma > 1$.

Proof. According to theorem 3.4, $\lim_{m \to \infty} \mathrm{Hdist}(E_m(\tau), \Omega_\tau) = 0$, where Hdist means the Hausdorff distance between two sets (see Falconer 1985). Let k and η be the constants for which the consequence of proposition 4.1 holds and let n and p be positive integers. On $E_m(\tau)$, the derivative $\mathrm{D}f_\tau^k(x)$, $k < m$, is continuous, so that for p sufficiently big, Ω_τ is a sufficiently dense ε-net in $E_{pk}(\tau)$ so that if $x \in E_{pk}(\tau)$, and $y \in \Omega_\tau$ sufficiently close to x, then

$$|\mathrm{D}f_\tau^k(x) - \mathrm{D}f_\tau^k(y)| < \tfrac{1}{4}(\eta - 1).$$

Now, choose $n > p$ such that

$$\left(\inf_{x \in E_k(\tau)} \{\mathrm{D}f_\tau^k(x)\}\right)^p [1 + \tfrac{3}{4}(\eta - 1)]^{n-p} > [1 + \tfrac{1}{2}(\eta - 1)]^n.$$

Now,

$$Df_\tau^{nk}(x_0)|_{E_{nk}} = Df_\tau^k(x_{(n-1)k}) \dots Df_\tau^k(x_0)$$

for $x_0 \in E_{nk}(\tau)$, and noting that $x_{ik} \in E_{(n-i)k}(\tau)$, the theorem follows with $m = nk$ and $\gamma = \frac{1}{2}(1+\eta)$. □

Theorem 4.3 (uniform (γ, m) expansivity). If, for $t = \tau$, f_τ is (γ, m) expansive, then there is an $\varepsilon > 0$ such that for each t in an ε neighbourhood of τ, f_t is $(\frac{1}{2}(1+\gamma), m)$ expansive.

Proof. As a function of t, $f_t^{-1}(x)$ is continuous except in $X(t) = f_t(x_-)$. Since $\rho(\tau)$ is irrational, $f_\tau^{-i}(U_\tau)$, $i \geq 1$, does not touch or contain $X(\tau)$. Therefore, the map $t \to E_m(t)$ is continuous at $t = \tau$ (with the Hausdorff topology on sets). The theorem then follows from the continuity of $Df_t^m(x)$ as function of (x, t). □

5. $\mu(A) = 0$

In this section we prove that the set of parameters such that $\rho(f_t)$ is irrational, has zero Lebesgue measure. A slight restriction of this also follows from results by Swiatek (1986, 1988) (he needs, for technical reasons, $\Delta f_t > 0$ on the entire circle, not just on the flat spot). We include the reasoning here, because our methods are vastly different from Swiatek's and because we need the result in the next section.

Let $x_i = f_t^i(x_-(t))$ and recall that F is a lift of f and that Δ stands for d/dt. We will write $\Delta f_t(U_t)$ for $\Delta f_t(x)|_{x \in U_t}$.

Lemma 5.1. $\Delta[f_t^k(x_i(t))] \geq \Delta f_t(U_t) \prod_{i=1}^{k-1} Df_t(x_i)$.

Proof.

$$\Delta[f_t^k(x_-(t))] \geq \Delta[f_t \circ f_t^{k-1}(x_-(t))] = (\partial f_t/\partial t)(x_{k-1}) + Df_t(x_{k-1})\Delta[f_t^{k-1}(x_-(t))].$$

The partial derivative is certainly ≥ 0. By induction one then obtains

$$\Delta[f_t^k(x_-(t))] \geq [f_t(x_-(t))] \prod_{i=1}^{k-1} Df_t(x_i).$$

The lemma follows from the equality

$$\Delta f_t(U_t) = \Delta[f_t(x_-(t))].$$

□

Define a monotone function $h_n(t)$ by

$$h_n(t) = \frac{1}{n} F_t^n(x_-(t)).$$

Since $f_t^n(x_-(t))$ is not in $\bar{U}_t$ for $t \in A$, it follows that h_n is C^1 in some neighbourhood of A. Define the sets $K_{n,N}$ as

$$K_{n,N} = \{t \in A \mid \Delta h_n(t) \geq N\}.$$

Lemma 5.2. $\lim_{n \to \infty} \mu(K_{n,N}) \geq \mu(A)$.

Proof. According to theorem 4.3, there is an open countable cover $\{T_i\}$ of A, such that for each $t \in T_i$, f_t is (γ, m_i) expansive. We can thin the cover out, so that $\lim_{k\to\infty} \mu(\bigcup_{i=k+1} T_i) = 0$. Now, consider the set

$$A_k = \bigcup_{i=0}^{k} T_i \cap A.$$

It is clear that $\lim_{k\to\infty} \mu(A_k) - \mu(A) = 0$. We prove that if n is sufficiently large, then $A_k \subseteq K_{n,N}$.

Let $M = \prod_{i=1}^{k} m_i$, and $Q = \mathrm{int}(n/M)$. For all $t \in A_k$, f is (γ, M) expansive. So, by lemma 5.1 (with $n > m$) we have

$$\begin{aligned}\Delta h_n(t) &\geqslant \frac{1}{n} \Delta f_t(U_t) \prod_{i=1}^{n-1} \mathrm{D} f_t(x_i)\\ &= \frac{1}{n} \Delta f_t(U_t) \prod_{i=0}^{Q} \mathrm{D} f_t^M(x_{iM+1}) \prod_{i=0}^{s-1} \mathrm{D} f_t(X_{M-s+i}) \qquad \text{for some } S < M\\ &\geqslant \frac{1}{n} \Delta f_t(U_t) \gamma^Q (\min \mathrm{D} f_t(x))^{M-1}\\ &\geqslant N \qquad \text{for } Q \text{ large enough.}\end{aligned}$$

□

Theorem 5.3. $\mu(A) = 0$.

Proof. The function $h_n(t)$ is C^1 on an open neighbourhood of $K_{n,N}$ with derivative greater then or equal to N. Its range is bounded from below by $\min_t(1/n)F_t^n(x_-(t))$ and from above by $\max_t(1/n)F_t^n(x_-(t))$. These sequences converge to $\min_t \rho(f_t)$ and $\max_t \rho(f_t)$, respectively, and are therefore uniformly (in n) bounded. Thus, for all n, the range of $h_n(t)$ is contained in some interval of length L. It follows that $\mu(K_{n,N}) \leqslant L/N$, so that

$$\lim_{N} \mu(K_{n,N}) = 0.$$

Combining this with lemma 5.2 yields the result. □

6. *HD*(*A*) = 0

We prove that the Hausdorff dimension of A is zero. In order to do this, we make use of uniform expansivity and a theorem of Besicovitch and Taylor (1954). For definitions and properties of the Hausdorff dimension and Hausdorff measure, we refer to Falconer (1985). We assume in this section that $\rho(f_\tau)$ is irrational. We remark that the proof of proposition 6.3 is essentially due to Boyd (1985).

Slightly reformulated, the theorem of Besicovitch and Taylor says the following.

Proposition 6.1 (*Besicovitch and Taylor 1954*). Let $A \subset [0, 1]$ be a set of zero measure and which is the complement of countably many disjoint intervals of length r_n. Suppose that r_n can be bounded from above by s_n. Then

$$HD(A) \leqslant \inf\{\beta \mid \sum s_n^\beta < \infty\}.$$

Let us denote the parameter interval for which $\rho(f_t)$ is equal to p/q by $I_{p/q}$. If t is in the interior of $I_{p/q}$, then f_t has at least one stable q periodic orbit (see Herman 1979, Boyland 1983). Let T denote an interval where f_t is (γ, m) expansive. Finally, define N as the smallest integer such that

$$\gamma^N\Big(\min_{t,x} \mathrm{D}f_t(x)\Big)^{m-1} > 1.$$

Lemma 6.2. For all p/q such that $I_{p/q}$ is contained in T and $q > Nm$, there is a C with $\mu(I_{p/q}) \leqslant C/\gamma^n$, where $n = \mathrm{int}(q/m)$.

Proof. By expansivity,

$$\mathrm{D}f_t^q(x) \geqslant \gamma^n\Big(\min_{t,x} \mathrm{D}f_t(x)\Big)^{m-1} \equiv \eta$$

so that, for $t \in I_{p/q}$ and under the assumptions of the lemma, f_t is (η, q) expansive. Thus stable periodic orbits have to intersect U_t and we can characterise $I_{p/q}$ by

$$t \in I_{p/q} \leftrightarrow \begin{cases} F_t^i(x_-(t)) \notin \bar{U}_t - p & 0 < i < q \\ F_t^q(x_-(t)) \in \bar{U}_t + p. & \end{cases}$$

According to lemma 5.1,

$$\Delta(f_t^q(x_-(t)) \geqslant \Delta f_t(U_t) \prod_{i=1}^{n-1} \mathrm{D}f_t(x_i) \geqslant \gamma^n \Delta f_t(U_t)\Big(\min_{t,x} \mathrm{D}f_t(x)\Big)^{m-1}.$$

Because each U_t contains the point $x = 0$, it is easy to see that $\cup_{t\in I_{p/q}} \bar{U}_t$ is contained in an interval $S_{p/q}$ of length less than 2. The lemma is then implied by

$$\mu(I_{p/q}) < \frac{\mu(S_{p/q})}{\min_t \Delta[f_t^q(x_-(t))]}.$$

□

Proposition 6.3. $HD(A \cap T) = 0$.

Proof. According to theorem 5.3 the intervals $I_{p/q}$ such that $I_{p/q} \cap T \neq \varnothing$ have full measure in T. For all $\beta > 0$, we have (Σ^* is the sum over pairs (p, q) of relative primes)

$$\begin{aligned}\sum{}^* [\mu(I_{p/q} \cap T)]^\beta &\leqslant \sum_{q\leqslant Nm}^* [\mu(I_{p/q} \cap T)]^\beta + \sum_{q>Nm}^* C^\beta \gamma^{-\beta n} \\ &\leqslant \text{finite part } + C^\beta \sum_{q>Nm}^* \phi(q)\gamma^{-\beta n}\end{aligned}$$

where $\phi(q) < q$ is Euler's phi function counting the number of relative primes to q. Since the sum is finite for every $\beta > 0$, we obtain the proposition as a consequence of proposition 6.1.

□

The main result of this section is the following theorem.

Theorem 6.4. $HD(A) = 0$.

Proof. Choose a countable covering $\{T_i\}_{i=1}^\infty$ of A such that f_t is (γ, m_i) expansive for $t \in T_i$. For each T_i, we apply the previous proposition.

□

7. $\mu(\Gamma)=0$ for the bimodal family

In this section we specialise the discussion to a truncated family derived, as described in §2, from a bimodal map g. It is proved that the measure of the union Γ of all order-preserving non-wandering minimal sets of g with irrational rotation number, is zero.

Lemma 7.1. Every order-preserving minimal set S with irrational rotation number is equal (as set) to the non-wandering set Ω_t of f_t for some t.

Proof. Clearly $G(S)$ is contained in an interval of length one. So, either the lemma is true or S contains a point x that lies under the orientation-reversing part of the graph of G. In the latter case, however, since S is minimal, there is a point $y \in S$ so close to x that G is orientation reversing on an interval containing both x and y. This is a contradiction. □

Theorem 7.2. Let g be a bimodal map such that its endpoints have rational rotation number. Then $\mu(\Gamma)=0$.

Proof. $\{T_i\}_{i=1}^{\infty}$ is a covering of A as defined in §6. For every $t \in T_i$, f_t is (γ, m_i) expansive in an open neighbourhood N_i of $\bigcup_{y \in T_i} \Omega_{t_i}$. Therefore g^{m_i} restricted to N_i is expanding and restricted to $\bigcup_{t \in T_i} \{\Omega_t \backslash x_-(t)\} \subset N_i$ it is also a bijection onto its image which is $\bigcup_{t \in T_i} \Omega_t$. One sees easily that

$$\mu\Big(\bigcup_{t \in T_i} \Omega_t \backslash x_-(t)\Big) = \mu\Big(\bigcup_{t \in T_i} \Omega_t\Big).$$

(This is implied by theorem 5.3 and

$$\mu\Big(\bigcup_{t \in T_i} x_-(t)\Big) < \frac{\mathrm{d}x_-(t)}{\mathrm{d}t}\,\mu(A \cup T_i)$$

and $\mathrm{d}x_-(t)/\mathrm{d}t > 0$ for the family discussed in this section.) The theorem follows by noting that the covering is a countable one. □

Remark. Swiatek (1986) has proved that the condition in the above theorem is Lebesgue almost always satisfied in a smooth one-parameter family. If the condition is not satisfied, the result may still be true, but there is no proof of this yet.

8. $HD(\Omega_t)=0$

For families of maps satisfying the general requirements (i)–(v) of §2, we prove that the Hausdorff dimension of Ω_t is zero if $t \in A$. Since this is essentially a repetition of the techniques used in the above, it will suffice to merely give an outline here.

Theorem 8.1. For $t \in A$, $HD(\Omega_t)=0$.

Proof. We know that $\bigcup_{i=0}^{\infty} f^{-1}(U_t)$ is the complement of Ω_t. Moreover, these 'gaps in Ω_t' have full measure, as follows from the fact that f_t is expansive. The lengths of these gaps thus decreases exponentially. Proposition 6.1 then immediately gives the result. □

Remark. The above theorem deals with the irrational case. In the case that $\rho(f_t)$ is rational, it could occur that Ω_t contains open intervals (if $Df_t^q(x)=1$). Generically, however, Ω_t for rational rotation number, is finite (see Nitecki 1971). In particular, if f_t is expansive there are at most two periodic orbits (one stable and one unstable).

9. Concluding remarks

The fact that expansitivity can be made uniform in the parameter means that the family for small parameter intervals is similar to piecewise linear flat spot families. In figure 2, we have drawn the function $t \to \rho(f_t)$ for such a family, which can be generated by a very simple recursive procedure (see Veerman 1986, 1987). If zero derivatives are admitted, they may annihilate all derivatives of subsequent iterates, and so no iterate of the map resembles a piecewise linear flat spot map.

The result of §7 can be sharpened. The dimension of the union of the order-preserving sets is zero. This will be elaborated in a forthcoming preprint (Veerman 1989).

The assumption that U_t is a single open interval is not essential. If U_t consists of at most finitely many intervals (one of them always containing $x=0$) our results generalise (the proof of theorem 3.4 has to be modified). An example of such a family is f_t^q, where f_t satisfies the assumption of §2. We have not taken this case into account, in order not to clog up the notation.

As a last remark, we note that the results in our paper can be used to extend the ones in Boyd (1985) concerning Cherry flows. A Cherry flow is a vector field on the torus with a saddle and a sink, whose first return map is a map of the circle with irrational rotation number.

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