Single-particle model for a granular ratchet

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Abstract

A simple model for a granular ratchet corresponding to a single grain bouncing off a vertically vibrating sawtooth-shaped base is studied. Depending on the model parameters, horizontal transport is observed in both the preferred and unfavoured directions. A phase diagram is presented indicating the regions in parameter space where the different regimes (no current, normal current, and current reversal) occur.

Key words: Granular ratchet, Fluctuation phenomena, Directed transport
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1 Introduction

Ratchet-like motion, whereby directed transport is obtained from nonequilibrium fluctuations, is important in many areas of scientific and technological interest, such as, molecular motors in biology and particle separation on nano- and micro-scales [1]. Recently, the problem of horizontal transport in a granular layer that is vertically vibrated by an asymmetric base has also been considered, both experimentally [2] and computationally [3]. These so-called "granular ratchets" have been shown to exhibit interesting dynamical features akin to those found in Brownian-motor models.

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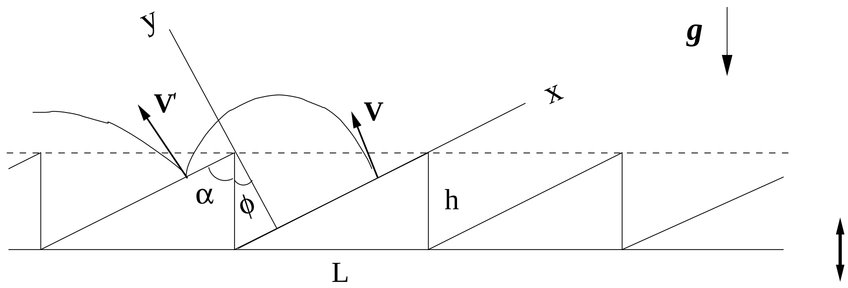

Fig. 1. A simple model for a granular ratchet.

In the present paper we tackle the difficult problem of granular ratchets by considering the dynamics of a *single* grain bouncing off a vertically vibrating sawtooth-shaped base. The model proposed here is a generalization of a previous model studied by some of us [4] for the gravity-driven motion of a particle on an inclined rough surface. In the present case, the 'rough surface' has a sawtooth shape with a horizontal baseline and is, in addition, subjected to a vertical vibration. We find that our single-particle model is able to reproduce much of the characteristic behavior observed in experiments [2] and computer simulations [3] of granular ratchets, such as horizontal transport and current reversals. The main results of the paper are summarized in a phase diagram indicating the regions in parameter space where the different dynamical regimes (no current, normal current, and current reversal) can be observed.

2 The model

Our granular ratchet model consists of a particle bouncing off a vibrating sawtooth-like surface whose 'teeth' have a vertical height h and a horizontal length L, as shown in Fig. 1. A particle is launched with given initial velocity over the vibrating ratchet and then moves around through a succession of ballistic flights and inelastic collisions. We adopt a simple collision rule given by $v_t' = v_t$ and $v_n' = -ev_n$, where v_t and v_n are respectively the velocity components tangential and normal to the collision plane, with prime denoting post-collisional velocities, and e is the coefficient of restitution. We assume furthermore that the sawtooth vibrates vertically with constant speed $V = 2A/\tau$, where A is the vibration amplitude and τ its period. For simplicity, we consider that A (and hence τ) is negligibly small, so that the sawtooth base remains essentially at the same height, which allows us to determine explicitly the successive points of collision between the particle and the ratchet. The drawback here is that the effect of the vibration on the particle velocity is taken into account in a *probabilistic* fashion, as will be discussed later. We believe, however, that the dynamics seen in our model is quite robust and should not be significantly altered if we consider a fully deterministic model.

Suppose now that the particle takes off with velocity $\vec{v} = (u, v)$ from a point (x, y), which can be either on the sawtooth left edge (inclined ramp) or on its right edge (vertical wall). For convenience, we place our system of coordinates in such a way that the x axis is aligned with the inclined ramp and the y axis makes an angle $\phi = \frac{\pi}{2} - \alpha$ with the vertical direction, where $\alpha = \tan^{-1}(L/h)$; see Fig. 1. (By convention, at each new flight we translate our system of coordinates to the tooth from which the particle departs.) Our goal then is to determine the next departure point (x', y') and the corresponding takeoff velocity $\vec{v}' = (u', v')$, so that the model dynamics can be described by a map

$$F : (x, y, u, v) \to (x', y', u', v').$$

To this end, let us introduce the following dimensionless variables [4]

$$X = \frac{x}{b}, \qquad Y = \frac{y}{a}, \qquad T = \sqrt{\frac{gc}{2a}}\, t,$$
$$U = \frac{c}{s}\frac{u}{\sqrt{2gca}}, \quad V = \frac{v}{\sqrt{2gca}}.$$

and the dimensionless roughness parameter $\kappa \equiv s^2/c^2$, where $a = L\cos\alpha$, $b = L\sin\alpha$, $c \equiv \cos\phi$, and $s \equiv \sin\phi$. Hereafter we drop the capital notation with the understanding that we shall work solely with dimensionless quantities.

Let us now define the jump number $n \in \mathbb{Z}$ in such a way that its modulus equals the number of teeth the particle skips during the flight, with the convention that n is positive if the particle moves in the direction of the positive x axis. To calculate n, we must first determine the points (if any) where the particle trajectory crosses the line $x + y = 1$, connecting the tips of the sawteeth (the dashed line in Fig. 1). Let $y_h^{\pm}$ be the y coordinates of these two crossing points (if they exist). An easy calculation shows [5] that

$$y_h^{\pm} = \frac{1 - x + \kappa y}{1 + \kappa} - 2\kappa(u - v)\frac{\kappa u + v \pm \sqrt{(\kappa u + v)^2 - (1 - x - y)(1 + \kappa)}}{(1 + \kappa)^2}.$$

If $y_h^- \notin (0, 1]$ (including the case where there is no real root) then the particle does not cross the line $x + y = 1$ and hence $n = 0$. On the other hand, if $y_h^- \in (0, 1]$ then the jump number is determined by the second crossing point y_h^+. More precisely, $n = \lceil -y_h^+ \rceil$, where $\lceil x \rceil$ denotes the ceiling function (i.e., the smallest integer greater than x). Having thus determined n, we then need to find the point at which the particle hits the surface. Note that if the particle jumps to the right then it can land only on a ramp, but if the jump is to the left then the particle can hit either a ramp or a vertical wall. Let us first consider the case when the particle lands on a ramp.

Collision with a ramp. Let (x_r^*, y_r^*) denote the collision point and (u_r^*, v_r^*) the particle velocity prior to the collision. Using the kinematics of ballistic motion and the fact that $y_r^* = -n$, one obtains after a simple calculation that

$$
\begin{aligned}
u_r^* &= u - v + \sqrt{v^2 + n + y}, \\
v_r^* &= -\sqrt{v^2 + n + y}, \\
x_r^* &= x - \kappa(n + y) + 2\kappa(u - v)\left(v + \sqrt{v^2 + n + y}\right).
\end{aligned}
\tag{1}
$$

If the particle jumps to the left (this happens when $u - v < 0$), the condition $n - x_r^* \leq \kappa$ must also be satisfied, otherwise the particle would have hit a vertical wall (this case will be treated below). On the assumption that the particle lands on the n-th ramp, the new departure point (x', y') and the new tangential velocity u' after the collision will then be

$$
x' = x_r^* - n, \quad y' = 0, \quad u' = u_r^*. \tag{2}
$$

Let us now calculate the normal velocity component v' after the collision. First note that if $|v^*| \leq \tilde{V} \equiv cV$ (here we have dropped the subscript from v_r^*), then the particle can never overtake the vibrating ramp and hence the collision will surely happen when the sawtooth is moving upwards. On the other hand, if $|v^*| > \tilde{V}$ then the particle can hit the ramp either in its upward or downward motion. Furthermore, if the collision takes place during the downward motion, then depending on the particle velocity after the first collision there might be enough time for the sawtooth to revert its motion and hit the particle a second time. Under the assumption that the vibration amplitude is negligibly small, one can calculate the probabilities for all four possible cases. Here we shall simply quote the final result and refer the interested reader to Ref. [6] for details of the calculation. If we define the parameters $r = |v^*|/\tilde{V}$, $P_1 = (1 + r^{-1})/2$, and $P_2 = 1 - (1 + e^{-1})/r$, then the outgoing velocity v' and the corresponding probability p are as follows:

i) if $r \in (0, 1]$ then $v' = -ev^* + (1 + e)\tilde{V}, \quad p = 1;$

ii) if $r \in (1, 1 + e^{-1}]$ then

$$
\begin{aligned}
v' &= -ev^* + (1 + e)\tilde{V}, \quad p = P_1, \\
v' &= e^2 v^* + (1 + e)^2 \tilde{V}, \quad p = 1 - P_1;
\end{aligned}
$$

iii) if $r \in (1 + e^{-1}, 1 + 2e^{-1})$ then

$$
\begin{aligned}
v' &= -ev^* + (1 + e)\tilde{V}, \quad p = P_1, \\
v' &= -ev^* - (1 + e)\tilde{V}, \quad p = P_2, \\
v' &= e^2 v^* + (1 + e)^2 \tilde{V}, \quad p = 1 - P_1 - P_2;
\end{aligned}
$$

iv) if $r \geq 1 + 2e^{-1}$ then

$$v' = -ev^* + (1+e)\tilde{V}, \quad p = P_1,$$
$$v' = -ev^* - (1+e)\tilde{V}, \quad p = 1 - P_1.$$

This probabilistic collision rule together with (2) thus yield the map F for the case when the collision occurs with a ramp. Next we consider the case when the particle hits a vertical wall (i.e., $u - v < 0$ and $n - x_r^* > \kappa$).

Collision with a vertical wall. Here we first need to determine the actual collision point (x_w^*, y_w^*) with the vertical wall and the particle velocity (u_w^*, v_w^*) just before such collision. It is easy to show [5] that in this case we have

$$\begin{aligned} x_w^* &= x_0 + 2\kappa u t^* - \kappa t^{*2}, \qquad y_w^* = 2vt^* - t^{*2}, \\ u_w^* &= u - t^*, \qquad v_w^* = v - t^*, \end{aligned} \tag{3}$$

where

$$t^* = \frac{\kappa(n-1) + n - x_0}{2\kappa(u-v)} \quad . \tag{4}$$

According to our convention, the new departure point (x', y') is then given by

$$x' = x_w^* - n, \tag{5}$$
$$y' = y_w^* + n. \tag{6}$$

Now, to determine the velocity (u', v') after the collision we simply apply the collision rule, $v_t' = v_t$ and $v_n' = -ev_n$, since the vibration has no effect on the outgoing velocity in this case. To do that, we must first obtain (v_t, v_n) in terms of (u_w^*, v_w^*), apply the above collision rule, and then return to the (u', v') coordinates. Performing this calculation (details omitted), we find

$$u' = (s^2 - ec^2)u_w^* + c^2(1+e)v_w^*, \tag{7}$$
$$v' = s^2(1+e)u_w^* + (c^2 - es^2)v_w^*. \tag{8}$$

Equations (5)–(8) give the map F when the collision takes place with a vertical wall, thus completing the mathematical formulation of our model.

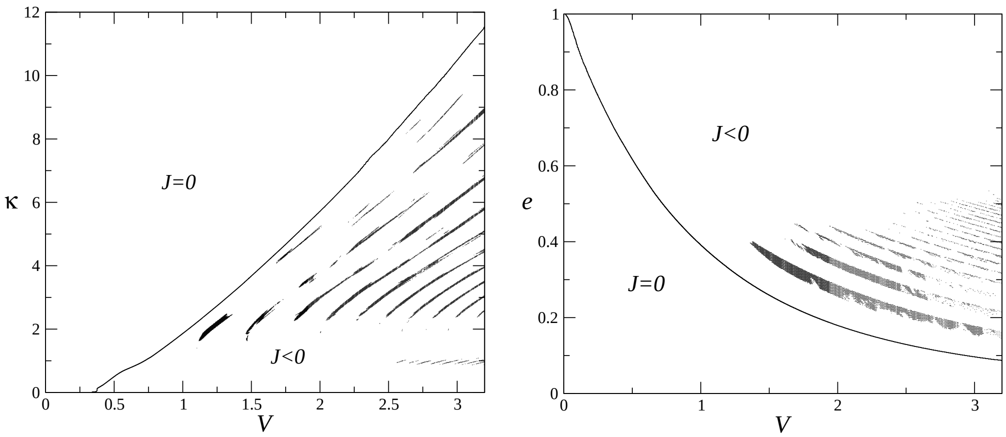

Fig. 2. Two-dimensional phase diagrams for the planes: (a) $(V, \kappa, e = 0.5)$ and (b) $(V, \kappa = 1, e)$. The regions with $J = 0$ and $J < 0$ are separated by a critical line, and the dark regions correspond to points where current reversal ($J > 0$) occurs.

3 Phase diagram

Although the map F found above cannot be studied analytically owing to the presence of a stochastic term, its long-term dynamics can be easily investigated on the computer. To this end, let us define the current J in our model as the particle average horizontal velocity: $J \equiv \langle v_x \rangle$, where the statistics is taken over a large number of collisions. In the present paper, we are mainly interested in mapping out the parameter space (V, κ, e) according to which dynamical regime ($J = 0$, $J < 0$, or $J > 0$) is observed at a given point. For illustration purposes, it is best however to look at 2D slices of the 3D parameter-space and construct the corresponding 2D phase diagrams. Two such phase diagrams are shown in Fig. 2 for the planes $(V, \kappa, e = 0.5)$ and $(V, \kappa = 1, e)$.

With the help of Fig. 2, the generic behavior of our ratchet model can be summarized, as follows. For given values of κ and e, we see that for sufficiently small V there is no net current, i.e., $J = 0$, meaning that the particle remains confined in one of the "potential wells" of the sawtooth profile. Then, as V increases past a critical value $V_c(\kappa, e)$, we observe a transition to a regime with $J < 0$, where the particle, on average, moves horizontally in the direction favoured by the ratchet asymmetry, namely, to the left. In the two-dimensional phase diagrams of Fig. 2, the critical surface $V_c(\kappa, e)$ appears as a critical line separating the region with $J = 0$ from that with $J < 0$. Furthermore, for certain values of V, κ and e, corresponding to the dark regions in Fig. 2, motion in the 'unfavoured' direction (i.e., with $J > 0$) can occur. Notice that these current-reversal regions appear as band-like structures, a fact that can be traced back to the periodic nature of the ratchet profile [5]. It is important to note, however, that current reversals can occur only for sufficiently large κ and sufficiently small e. This is clearly seen in Fig. 2, where there is no current reversal neither for $\kappa < 0.8$ in Fig. 2a nor for $e > 0.6$ in Fig. 2b.

4 Conclusions

We have studied a simple model for a granular ratchet corresponding to a single grain bouncing off a vertically vibrating sawtooth-shaped base. We observe that for small shaking velocity V, the particle remains confined between two teeth, with no net motion, whereas for larger V a nonzero horizontal current is established usually in the direction favoured by the sawtooth asymmetry, but current reversals also occur rather regularly. This complex behavior is summarized in the phase diagram presented in Fig. 2, which shows the regions in the parameter space (V, κ, e) where each of the three possible regimes (no current, normal current, and current reversal) can occur. Our single-particle model is thus able to reproduce the qualitative behavior observed in experiments [2] and computer simulations [3] of granular ratchets. In particular, our model shows that for current reversal to occur two key ingredients are necessary: i) a sufficiently rough vibrating surface and ii) sufficiently large dissipation at collisions [7]. Further investigation concerning the nature of the 'phase transition' to the nonzero-current regime as well as a more detailed study of the current-reversal mechanism will be presented in a forthcoming publication.

Acknowledgements

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