Dynamics of particles in a vertical rough channel

Rogério L. Costa[1], J. J. P. Veerman[2] and G. L. Vasconcelos[1](*)

[1] *Laboratório de Física Teórica e Computacional, Departamento de Física, Universidade Federal de Pernambuco, 50670-901, Recife, Brazil.*
[2] *Mathematical Sciences, Portland State University, Portland, OR 97207, USA.*

PACS. 45.70.-n – Granular systems.
PACS. 83.80.Fg – Granular solids.
PACS. 05.45.-a – Nonlinear dynamics and nonlinear dynamical systems.

Abstract. – A simple model is presented for the gravity-driven motion of a particle in a two-dimensional vertical channel with rough walls, where the dynamics is described by a 2D nonlinear mapping. It is shown that if the collisions with the channel walls are inelastic then the particle reaches a steady state where it falls with a constant average velocity. If the collisions are elastic, then the dynamics is governed by a 2D area-preserving mapping that exhibits a complex behavior in phase space. The model is then extended to include the case of several vertical plates falling under gravity inside a channel, where a steady state is reached with a parabolic velocity profile across the channel.

The gravity-driven motion of grains in a confined geometry, such as granular flows in a hopper, is not only of practical importance to many technological process but also of great scientific interest. In fact, a characterization of the full range of grain dynamics during such flows remains a challenge, both experimentally [1,2] and theoretically [3]. From a theoretician's viewpoint, perhaps the simplest approach to tackle such difficult problem is to treat the grains as non-interacting particles and study the corresponding particle dynamics in the geometry of interest. Single-particle models have, indeed, be used with some success to describe the grain dynamics during gravity-driven granular flows on an inclined rough surface [4–8].

In this Letter we present a simple model for the gravity-driven motion of a single grain inside a two-dimensional vertical channel. In our model, the grain is treated as a point particle and moves downward through a sequence of ballistic flights and inelastic collisions with the channel walls, which may be either smooth or 'rough' (in a sense to be made more precise below). It is shown that when the walls are rough the particle will in general reach a steady state where it falls with a constant average velocity, which can be computed analytically in terms of the model parameters (the coefficients of restitution and the roughness parameter). When the walls are smooth a steady state is still possible in the particular case that the collisions are elastic with respect to the normal velocity component but inelastic regarding the tangential velocity, otherwise the particle accelerates. We also briefly discuss the situation when the collisions are elastic (and the walls rough), in which case the system is described by a

(*) E-mail: giovani@lftc.ufpe.br

2D area-preserving mapping that exhibits a complex dynamics with islands of near-integrable curves surrounded by a sea of chaotic orbits. The case where the wall roughness parameter is allowed to vary randomly is also considered and it is seen that the mean downward velocity increases as the degree of irregularity increases.

We shall also briefly present an extension of our single-particle where we consider the motion of N vertical plates falling under gravity inside a channel with smooth walls. As the plates move downward they collide inelastically with their neighbors (or with the channel walls in the case of the leftmost and rightmost plates). Eventually a steady state is reached where the velocity distribution across the channel assumes a parabolic profile. The velocity fluctuation, on the other hand, is minimum at the central region and displays a peak near the walls—a behavior also seen in 2D simulations of gravity-driven granular flow in a tube [3].

The model we consider first is illustrated in Fig. 1. We imagine a grain particle moving under gravity inside a two-dimensional vertical channel formed by two parallel rough walls placed a distance L apart of one another. The particle is launched at the top of the channel with a given initial velocity that we assume has a nonzero horizontal component, otherwise the motion would be trivial. The particle then moves inside the channel through a succession of ballistic flights and inelastic collisions with the rough walls. For simplicity, the roughness of the channel walls is represented by extensionless facets, the so-called 'microfacets' [4, 6], which are attached to the walls forming an angle α with the vertical; see Fig. 1. We assume that after a collision with a microfacet the particle velocity changes according to the following simple rule:

$$v_t' = e_t v_t, \tag{1}$$

$$v_n' = -e_n v_n, \tag{2}$$

where v_t and v_n are the velocity components tangential and normal to the microfacet, respectively, with prime denoting post-collisional velocities, and e_t and e_n are the corresponding tangential and normal coefficients of restitution, taking values in the interval $(0, 1]$.

Let us introduce a system of coordinates where the y axis is along the left wall and the origin is placed at an arbitrary position; see Fig. 1. Thus, at collisions with a microfacet on the left wall, the transformation from the x–y velocity components, $\mathbf{v} = (v_x, v_y)$, to the velocity components tangential and normal to the microfacet, $\mathbf{v} = (v_t, v_n)$, is enacted by a clockwise rotation of $\pi/2 - \alpha$, which in matrix notation reads

$$\begin{pmatrix} v_t \\ v_n \end{pmatrix} = \begin{pmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}. \tag{3}$$

For collisions with the microfacets on the right wall, however, it is more convenient to work with a system of axes x' and y' that are mirror images of the axes x and y; see Fig. 1. (Similar definition applies to the normal and tangential directions n' and t'.) The advantage of this choice is that the transformation from the velocity components $(v_{x'}, v_{y'})$ to $(v_{t'}, v_{n'})$ is given by exactly the same relation shown in (3) and hence we need to make no distinction between right and left walls. Accordingly, we will drop the prime notation for the coordinates at the right wall, with the understanding that the velocity components before and after any given collision will be written in the local system of coordinates attached to that particular wall.

Let us now denote by $\mathbf{v} = (u, v)$ the particle x-y velocity components after the last collision with a given wall. The particle then undergoes a ballistic flight during the time $t = L/u$, until colliding with the opposite wall. The particle velocity components $\mathbf{v_c} = (u_c, v_c)$ just before this new collision thus read

$$u_c = -u, \tag{4}$$

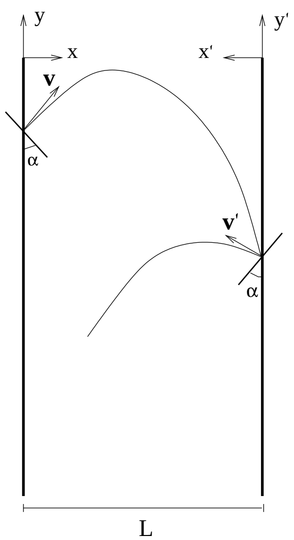

Fig. 1

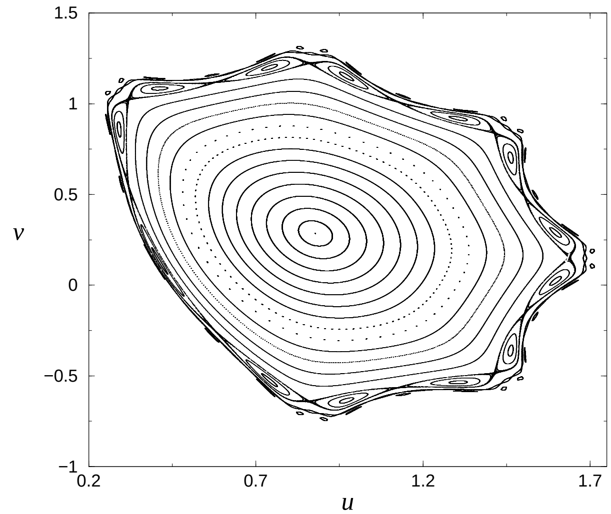

Fig. 2

Fig. 1 – Model for a single grain moving in a 2D vertical rough channel.

Fig. 2 – Trajectories in the (u, v) plane for $e_n = e_t = 1$ and $\alpha = 18°$.

$$v_c \;=\; v - \frac{1}{u}. \tag{5}$$

Here we have applied a coordinate transformation $u \to u/\sqrt{gL}$ and $v \to v/\sqrt{gL}$, so that the quantities in (4) and (5) are all dimensionless. If we now express the velocity $\mathbf{v}_c$ in the rotated frame via (3), apply the collision rule (1)–(2), and then rotate back to the x-y system of coordinates, we can readily obtain the new post-collisional velocity $\mathbf{v}' = (u', v')$. Performing this calculation we obtain the following two-dimensional mapping

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = F \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} au - bv + \frac{b}{u} \\ \\ bu + cv - \frac{c}{u} \end{pmatrix}, \tag{6}$$

where the coefficients a, b, and c are given by

$$a = e_n \cos^2\alpha - e_t \sin^2\alpha, \quad b = (e_n + e_t)\sin\alpha\cos\alpha, \quad c = e_t\cos^2\alpha - e_n\sin^2\alpha. \tag{7}$$

We note, for later use, that during a ballistic flight the velocity component v_y changes linearly in time, so that the average vertical velocity $V \equiv < v_y >$ between two consecutive collisions equals the arithmetic mean of the velocities at the beginning and end of the flight, that is, $V = \frac{1}{2}(v + v_c)$, which yields

$$V = v - \frac{1}{2u}. \tag{8}$$

There is a final caveat about the map above. In obtaining the mapping F given in (6), we have implicitly assumed that upon collision the particle bounces back toward the opposite wall, and so we must have $u' > 0$ at all times. Of course, this condition can be violated for an ill-suited initial condition. For example, when $e_n = e_t = e$ (in which case we have specular reflection) such a violation happens if the incoming velocity $\mathbf{v}_c$ makes an angle $\theta > \frac{\pi}{2} - 2\alpha$ above the horizontal. If were to apply (6) blindly to this case, it would mean that the particle would penetrate the wall and reappear on the other side. Such an unphysical situation comes about, of course, because of the extensionless nature of the microfacets and could be averted by imposing an extra condition, say, applying the collision condition once again or simply reverting the sign of u'. We have decided, however, to take a simpler approach: To prevent the problem from occurring we will consider only initial conditions for which an 'unphysical collision' (i.e., $u' < 0$) never happens. As we will see below, for angles in the range $0 < \alpha < \pi/4$, there is always a large (and more physically relevant) region in the (u, v) phase plane where the orbits never violate the condition $u' > 0$. We shall henceforth be concerned only with such physically acceptable orbits. Before discussing the general map (6), however, we shall first consider the especial situation when the channel walls are smooth, i.e., $\alpha = 0$, in which case the dynamics can be solved exactly.

After setting $\alpha = 0$ in (6) we obtain

$$u' = e_n u, \tag{9}$$

$$v' = e_t v - \frac{e_t}{u}. \tag{10}$$

The first equation above has a trivial solution and, after inserting this solution into (10), the second equation can also be solved exactly. One then finds

$$u_k = e_n^k u_0 \tag{11}$$

$$v_k = e_t^k v_0 - \frac{e_n e_t (e_n^{-k} - e_t^k)}{(1 - e_n e_t) u_0}, \tag{12}$$

where (u_0, v_0) is the particle initial velocity and (u_k, v_k) is the velocity after k collisions with the channel walls. From (11) it also follows that the the total elapsed time t_k until the k-th collision is

$$t_k = \sum_{j=0}^{k-1} \frac{1}{u_j} = \frac{e_n (e_n^{-k} - 1)}{(1 - e_n) u_0}. \tag{13}$$

It is now an easy matter to determine the particle long-time dynamics, i.e., for $k \to \infty$. Here there are three situations to consider: i) the case $0 < e_n < 1$, when for $k \to \infty$ one gets $u = \frac{e_n}{(1-e_n)t}$ and $v = -\frac{e_t(1-e_n)}{1-e_n e_t} t$, where we have dropped the k subscripts; ii) the case $e_n = 1$ and $e_t < 1$, for which the particle reaches a *steady state* where $u = u_0$ and $v = -\frac{e_t}{(1-e_t)u_0}$, so that it falls with a constant average velocity $V = \frac{1+e_t}{2(1-e_t)u_0}$; and iii) the case of elastic collisions, $e_n = e_t = 1$, where we find $u = u_0$ and $v = v_0 - t$, thus showing that the particle is effectively in a free fall, as expected. It is worth pointing out that the condition $e_n = 1$ and $e_t = \mu < 1$ for collisions with the walls (and $e_t = 1$ and $e_n = \mu$ for inter-grain collisions) was used in the numerical simulations of gravity-driven granular flows in a 2D channel recently performed by Denniston and Li [3]. There they found that, overall, the column of grains behaves like a solid sliding down the tube with an effective friction force at the walls balancing off the force of gravity, which is precisely the scenario predicted by our model with smooth walls for $e_n = 1$ and $e_t < 1$.

Now we consider the general case $\alpha > 0$. We start our analysis by looking for fixed points of the map (6). Solving the fixed point equations $u' = u = u^*$ and $v' = v = v^*$ yields

$$u^* = \left[\frac{(e_n + e_t) \sin 2\alpha}{2 \left[1 + e_n e_t - (e_n + e_t) \cos 2\alpha \right]} \right]^{1/2} \tag{14}$$

$$v^* = \frac{2e_n e_t + e_n - e_t - (e_n + e_t)\cos 2\alpha}{\sqrt{2(e_n + e_t)\sin 2\alpha[1 + e_n e_t - (e_n + e_t)\cos 2\alpha]}}. \tag{15}$$

The stability of the fixed point is determined by the eigenvalues $\lambda_\pm$ of the Jacobian matrix $\frac{\partial(u',v')}{\partial(u,v)}$ evaluated at (u^*, v^*). We will spare the reader the details of this calculation and simply quote the result for the eigenvalues

$$\lambda_\pm = \frac{1}{2}\left[-\eta \pm \sqrt{\eta^2 - 4e_n e_t}\right], \tag{16}$$

where

$$\eta = 1 + e_n e_t - 2(e_n + e_t)\cos 2\alpha. \tag{17}$$

We thus see that depending on the value of the angle α the eigenvalues can be both real or complex conjugates. One can show, however, that the moduli of the eigenvalues (be they real or complex) are always smaller than unity for $0 < \alpha < \frac{\pi}{4}$ and $e_n e_t \neq 1$. In other words, for angles in the range $0 < \alpha < \frac{\pi}{4}$, the fixed point is an attractor of the dynamics so long as the collisions have some (any) degree of inelasticity. [Numerical simulations support the conjecture that the fixed point is the only attractor in this case.]

At the fixed point, the particle falls downward with a constant average speed V^* which can be easily obtained by inserting (14) and (15) into (8). Upon doing this and performing some simplification one finds

$$V^* = \frac{(1 - e_n)(1 + e_t)}{\sqrt{2\sin 2\alpha(e_n + e_t)[1 + e_n e_t - (e_n + e_t)\cos 2\alpha]}}. \tag{18}$$

We thus see that, except for the particular case $e_n = 1$, the rougher the channel walls (i.e., the greater α), the smaller the average velocity V^*, as one would expect. Note also that if $e_n = 1$ then $V^* = 0$. In this case the fixed point corresponds to the physical situation where the collisions with the microfacets are always frontal, so that the particle remains 'suspended' in the channel in the sense that it moves back and forth between the two walls always retracing the same parabola. If $e_t < 1$ this fixed point is an attractor of the dynamics since $|\lambda_\pm| < 1$, whereas for $e_t = 1$ the fixed point becomes elliptic and a rather complex dynamics emerges, as shown next.

From (6) one can easily verify that if $e_n = e_t = 1$ then the determinant of the Jacobian matrix of F is precisely equal to 1, hence F is an area-preserving mapping in this case. Furthermore, one finds that $\lambda_\pm = e^{\pm i\beta}$, where $\cos\beta = 2\cos 2\alpha - 1$, so that the fixed point is elliptically stable, as already mentioned. A detailed study of the map F with $e_n = e_t = 1$ will be left for a forthcoming publication. Here we simply wish to mention that, as illustrated in Fig. 2, this system exhibits the usual dynamical features of 2D area-preserving nonlinear mappings [9], among which we cite: i) a large region around the elliptic fixed point containing near-integrable curves, the so-called 'KAM curves'; ii) chains of islands of near-integrable curves at whose centers we find periodic orbits of higher period (clearly seen in Fig. 2 are islands associated with a periodic orbit of period $q = 13$); and iii) a 'sea' of chaotic orbits surrounding the islands.

We now wish to discuss the case in which the orientation of the microfacets is allowed to vary randomly from place to place. For simplicity, we will consider the situation where the angle α is distributed uniformly around a given mean value $\overline{\alpha}$. More specifically, we will assume that at each new collision the angle α is chosen according to the following prescription

$$\alpha = \overline{\alpha} + \delta\,(\epsilon - 0.5), \tag{19}$$

where δ is a given number, to be referred to as the 'noise amplitude', and ϵ is a random number uniformly distributed in the interval $[0,1]$. We have found that the particle mean vertical velocity V increases roughly quadratically with the noise amplitude δ: $V - V^* \propto \delta^2$. (Recall that V^* is the average downward velocity in homogeneous case, i.e., for $\delta = 0$.) In particular, we find that when $e_n = 1$ the particle falls downward with a small but nonzero velocity for any $\delta > 0$, thus showing that any amount of noise will destroy the 'suspended' stationary state existent in the homogeneous case for $e_n = 1$.

One natural way to extend the model above (which treats the grains as non-interacting particles) to render it more realistic is to consider several grains inside the channel and take into account pair collisions. Here, however, we shall seek a multi-particle model that builds upon our understanding of the single-grain model rather than resort to a full-fledged granular flow simulation [3]. Our starting point in this direction is the observation [1,3] that granular flows in a tube tend to be 'columnar' in the sense that the grains are highly constrained by their neighbors so that there is little transversal motion [3]. In order to mimic the motion of such 'columns of grains' we consider the problem of N parallel vertical plates of width a falling under gravity inside a vertical channel of length L, with $Na < L$. (We shall again work in dimensionless units where $g = L = 1$). Interplate collisions are inelastic but conserve momentum. For simplicity, the collisions between plates as well as between a plate and a channel wall are described by the same set of restitution coefficients e_n and e_t. (Lifting this restriction does not significantly alter the main results.) On the basis of our previous single-particle model (with smooth walls) we know that a steady state can be achieved only if $e_n = 1$ and so we will concern ourselves solely with this case.

Owing to lack of space, we report here only the main features of the multiplate model. More details will be presented elsewhere [10]. A typical result for the velocity distribution (in the stationary regime) across the channel is shown in Fig. 3, where it is plotted both the mean vertical velocity $V_i \equiv < v_y^i >$ and the velocity fluctuation $\Delta V_i \equiv \sqrt{< (v_y^i)^2 > - V_i^2}$ as a function of the plate label $i = 1, ..., N$ for the case $N = 100$, $a = 0.0098$, and $e_t = 0.9$. (In our simulations the plates were initially placed at equal distance from one another and given random velocities, with the statistics being performed after a long transient had elapsed.) In Fig. 3a we see that the velocity profile is parabolic—that is, we have a Poiseuille-like flow inside the channel. This is in contrast with experiments [1,11] and numerical simulations [3] of gravity-driven granular flow in a tube, where the velocity profile is considerably flat across the tube, except for a thin boundary layer. The flattening of the velocity profile in such granular flows thus seems to be a direct consequence of the 'granularity' of the medium, which allows for transfer of momentum between the vertical and horizontal directions during intergrain collisions. (Such possibility is nonexistent in our plate model.) This mechanism would thus tend to render the velocity distribution more uniform across the tube. We also note that the velocity fluctuation in our multiplate model is minimum in the central region and exhibit characteristic peaks near the walls, as shown in Fig. 3b. (Similar behavior was also seen in the 2D granular flow simulations performed by Denniston add Li [3], although there the fluctuation profile is flatter in central region.) Notice, however, that the velocity fluctuation is much smaller than the mean velocity.

In conclusion, we have introduced a class of simple models for the gravity-driven motion of a single grain (or a collection of noninteracting grains) in a two-dimensional channel. The model has the advantage of being analytically treatable so that the condition for the particle to attain a steady state and its velocity in such regime can be worked out exactly. In spite of its simplicity, the model might provide a theoretical framework in which certain aspects of the grain dynamics during granular flows in a tube can be understood. For instance, the

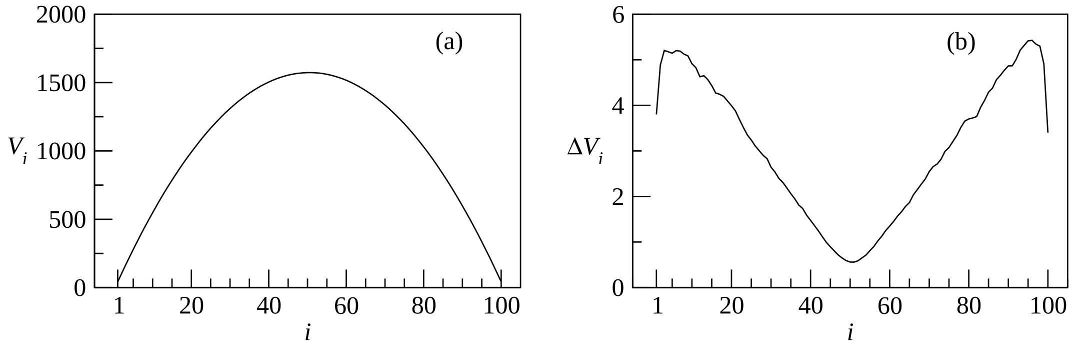

Fig. 3 – Velocity distribution in the multiplate model: (a) mean downward velocity V_i as a function of the position i along the channel and (b) velocity fluctuation ΔV_i as a function of i. Here $N = 100$, $a = 0.0098$, $e_n = 1$, $e_t = 0.9$, and the initial velocities were randomly chosen in the interval $[-1, 1]$.

case when both walls are smooth (with $e_n = 1$ and $e_t < 1$) may describe the overall motion of the column of grains down the tube, whereas the case of a particle in a rough channel might be seen as a simplified model for the actual motion of grains in the tube central region. Similarly, the grain dynamics in the boundary layer near the tube walls can be qualitatively understood in terms of a variant of our model where one wall is smooth and the other one is rough. We have also considered an extension of our model where intergrain collisions are take into account in a rather simplified manner. Here the idea was to model the columns of grains formed in actual gravity-driven granular flows in a tube as vertical plates falling under gravity inside a channel and subjected to inelastic collisions amongst themselves and with the channel walls. In our multiplate model the velocity profile across the channel is parabolic, while the velocity fluctuation is minimum at midchannel and displays a peak near each wall.

* * *

This work was partially supported by the Brazilian agencies CNPq and FINEP, and by Brazil's special program PRONEX. J. J. P. V. would like to thank the Physics Department at UFPE for its hospitality during his stay there in the early stages of this work.

REFERENCES

[1] Menon N. and Durian D., *Science*, **275** (1997) 1920.
[2] Durian D. J., *J. Phys.: Condes. Matter*, **12** (2000) A507.
[3] Denniston C. and Li H., *Phys. Rev. E*, **59** (1999) 3289.
[4] Valance A. and Bideau D., *Phys. Rev. E*, **57** (1998) 1886.
[5] Vasconcelos G. L. and Veerman J. J. P., *Phys. Rev. E*, **59** (1999) 5641; *Physica A*, **271** (1999) 251.
[6] Marconi U. M. B., Conti M. and Vulpiani A., *Europhys. Lett.*, **51** (2000) 685.
[7] Vasconcelos G. L., Cunha-Jr. F. V. and Veerman J. J. P., *Physica A*, **295** (2001) 261.
[8] Veerman J. J. P., Cunha, Jr. F. V. and Vasconcelos G. L., *Physica D*, **168** (2002) 220.
[9] Lichtenberg A. J. and Liberman M. J., *Regular and Stochastic Motion* (Springer-Verlag, New York) 1983.
[10] Costa R.L. and Vasconcelos G. L., in preparation.
[11] Pouliquen O. and Gutfraind R., *Phys. Rev. E*, **53** (1996) 552.