Physica A 295 (2001) 261–267

PHYSICA A

www.elsevier.com/locate/physa

Chaotic behavior in a model for grain dynamics

G.L. Vasconcelos[a,*], F.V. Cunha-Jr.[a], J.J.P. Veerman[b,c]

[a] *Departamento de Física, Laboratório de Física Teórica e Computacional, Universidade Federal de Pernambuco, 50670-901 Recife, Brazil*

[b] *Mathematics Department, Queens College-CUNY, Flushing, NY 11367, USA*

[c] *Mathematical Sciences, Portland State University, Portland, OR 97207, USA*

Abstract

A simple model is presented for the motion of a grain down a rough inclined surface. Considering that the surface has a periodic profile and adopting a simple collision law, we arrive at a model in which the dynamics is described by a three-dimensional map. As the surface inclination increases, this map exhibits a transition from a regime of bounded velocity to accelerated motion. In the region of bounded velocity, the original 3D map can be reduced to an effective one-dimensional map that shows several dynamical features (stable fixed points, periodic orbits, and chaotic behavior). A bifurcation diagram for the 1D map is presented. © 2001 Elsevier Science B.V. All rights reserved.

PACS: 45.70.−n; 83.70.−f; 45.50.−j

Keywords: Grain dynamics; Chaos; Granular flow

1. Introduction

In recent years, the flow of granular material on rough inclined surfaces has been a topic of considerable attention within the physics community (see, for example, Ref. [1]). In spite of this research effort, the actual grain dynamics during such flows is still poorly characterized [2]. Given this scenario, the study of simple models (see, e.g., Ref. [3]) that are exactly solvable is of great interest since such endeavor may provide useful insights into the nature of grain dynamics.

In this vein, we have recently introduced [4,5] a class of models for the gravity-driven motion of a single grain down a rough inclined surface. In order to render our models analytically tractable, several simplifying assumptions were made: (i) the rough surface

* Corresponding author. Tel.: +55-81-271-8450; fax: +55-81-271-0359.

E-mail address: giovani@lftc.ufpe.br (G.L. Vasconcelos).

PII: S0378-4371(01)00085-1

was supposed to have a simple geometrical profile; (ii) the grain size was neglected and so the grain was treated as a point particle; (iii) a simple restitution law was adopted for the collisions between the grain and the rough surface. More specifically, this collision law was given by $v_n' = e_n v_n$ and $v_t' = e_t C(v_t, v_n)$, where the prime (unprimed) denotes the velocity immediately after (before) the collision, e_n and e_n are constants, and $C(x, y)$ is a homogeneous function of degree 1 [5]. For simplicity, we also set $e_n = 0$. With these assumptions the model could be described in terms of one-dimensional maps whose properties were studied in great detail. First we analyzed the simpler case in which $C(v_t, v_n) = v_t$ whose main results were summarized in a phase diagram in parameter space showing all the possible dynamical regimes of the system [4]. The grain dynamics predicted in this case was found to be in qualitative agreement with what has been observed, for instance, in experiments [6] on the motion of a ball down a rough inclined plane. Subsequently [5] we showed that the behavior of this class of models was robust in the sense that for any physically reasonable choice of $C(v_t, v_n)$ (and $e_n = 0$) the nature of the phase diagram was preserved. This is an important result, since tangential restitution law for binary collisions are not very well known experimentally [7].

In this paper, we present a generalization of our previous model that includes the case of a nonzero normal restitution coefficient e_n. As will be seen below, in this case the model is described by a three-dimensional map that displays a much richer dynamics which includes some of the regimes seen in the 1D map as well as additional features.

2. The model

The model we consider is shown in Fig. 1. The rough surface is assumed to have a simple staircase shape whose steps have height a and length b. For convenience, we choose a system of coordinates such that the step plateaus are aligned with the

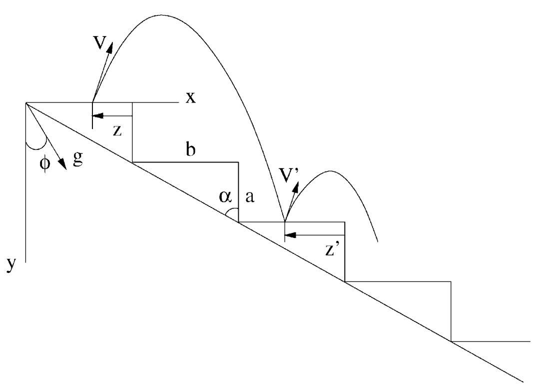

Fig. 1. Model for the gravity-driven motion of a particle on a rough inclined surface.

x-axis and the direction of the acceleration of gravity $\mathbf{g}$ makes an angle ϕ with the y-axis. A particle, thrown on the top of the 'staircase' with a given initial velocity, moves downward through a succession of collisions and ballistic flights, as indicated in Fig. 1. We assume, as already mentioned, that the collision law is characterized by two (constant) coefficients of restitution, e_n and e_t, which represent the respective reduction factors for the normal and tangential velocity components after a collision.

Let $\mathbf{V}=(U,V)$ denote the components of the particle velocity parallel and perpendicular to the collision plane after a given collision and let Z be the distance from the collision point to the edge of the ramp where the flight started; see Fig. 1. In what follows, it is convenient to introduce dimensionless quantities defined by

$$u=\frac{U}{\sqrt{2ag\cos\phi}}, \quad v=\frac{V}{\sqrt{2ag\cos\phi}}, \quad z=\frac{Z}{a}\,. \tag{1}$$

Using the kinematics of ballistic motion and the collision law mentioned above, one can show [8] that the velocity $\mathbf{v}'=(u',v')$ after the next collision and the corresponding coordinate z' of the new collision point are given by the following three-dimensional map:

$$u'=e_t(u+tv+t\sqrt{n+v^2})\,, \tag{2}$$

$$v'=e_n\sqrt{n+v^2}\,, \tag{3}$$

$$z'=z+n(\tau-t)-2(u+tv)(v+\sqrt{n+v^2})\,, \tag{4}$$

where $t=\tan\phi$, $\tau=b/a$, and n, referred to as the 'jump number', is the number of steps the particle has fallen during the flight and corresponds to the smallest non-negative integer such that $z'\geqslant 0$. We observe that this last condition implies that n must satisfy the following requirement:

$$\sqrt{v^2+n}\geqslant\frac{u+tv+\sqrt{(u+\tau v)^2-z(\tau-t)}}{\tau-t}>\sqrt{v^2+n-1}\,. \tag{5}$$

We also note that if the argument of the square root appearing in Eq. (5) is negative then n should be taken zero, in which cases the particle lands on the same step where the flight started.

3. Bounded vs. unbounded orbits

In this section, we investigate under which circumstances the map given in Eqs. (2–4) has orbits for which the velocity grows without bound. In order to do this, we shall examine the behavior of this map for large initial velocities. Since the ensuing flights in this case will be very long we can take the continuum limit for the jump number n and consider for simplicity that $z=0$.[1] Thus, assuming that both u

[1] A more rigorous analysis can be carried out [8], leading to the same results.

and v are very large and setting $z=0$ in Eq. (5), we obtain that n is approximately given by

$$\sqrt{v^2+n} \approx \frac{2u+(\tau+t)v}{\tau-t} . \tag{6}$$

Substituting this into Eqs. (2) and (3) we obtain a linear map

$$u' \approx \left(\frac{e_t}{\tau-t}\right)[(\tau+t)u+2t\tau v] , \tag{7}$$

$$v' \approx \left(\frac{e_n}{\tau-t}\right)[2u+(\tau+t)v] . \tag{8}$$

It should be emphasized however that this linear behavior holds only for very large velocities—for smaller velocities the map is highly nonlinear. The linear map above will predict growth if its derivative matrix has at least one eigenvalue (Floquet multiplier) greater than unity. One can readily show that this will be the case if and only if $t > t_\infty$, where

$$t_\infty = \tau\frac{(1-e_t)(1-e_n)}{(1+e_t)(1+e_n)} . \tag{9}$$

Thus for $t < t_\infty$ the orbits are always bounded, whereas for $t > t_\infty$ unbounded orbits exist. The region of bounded motion is characterized by the fact that the orbits have halting points, that is, the particle eventually reaches a step where it will execute an infinite sequence of bounces, with ever decreasing velocity, until finally coming to a halt. After the particle has stopped the motion must be restarted in some fashion, as we will see shortly. But before doing that we shall first discuss the halting condition.

Let us denote by u_0 and v_0 the velocity components after the very first collision on a given step and by z_0 the coordinate of this first collision point. One can show [8] that if the particle were to stop on this step then the halting point would lie at a distance d from the first point of impact, with d being given by

$$d = 4\frac{(1+e_te_n)tv_0^2+(1-e_n^2)u_0v_0}{(1-e_te_n)(1-e_n^2)} . \tag{10}$$

Thus if $d < z_0$ then the particle will indeed stop on this step, otherwise it will surely reach a step below. We can now easily determine a bound on the inclination t for which halting orbits exist. First, imagine that the particle falls off the edge of a ramp with zero initial velocity. Such a particle will then hit the step immediately below at a point $z_0 = \tau - t$ and bounce off with a velocity $\mathbf{v}_0 = (te_t, e_n)$, as follows from Eqs. (2) and (3) with $u=v=0$ and $n=1$. Substituting this $\mathbf{v}_0$ into Eq. (10) we obtain the corresponding halting distance d_0. The particle will thus come to a halt on the first step provided that $d_0 < \tau - t$. Proceeding along these lines, one can then prove [8] that there exist halting orbits if and only if $t < t_s$, where

$$t_s = \frac{\tau(1-e_te_n)(1-e_n^2)}{1+3e_n(e_t+e_n)+e_te_n^3} . \tag{11}$$

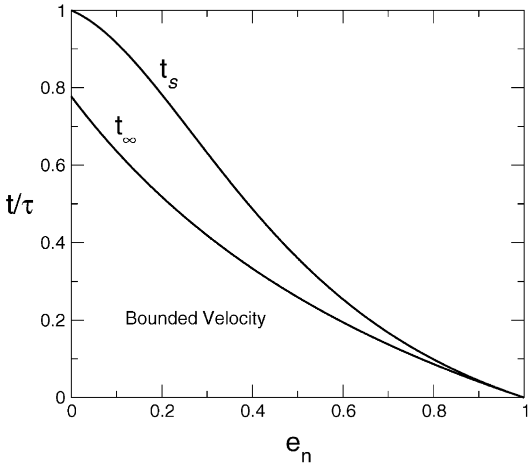

Fig. 2. Phase diagram in the plane $(t/\tau, e_n)$ for $e_t = 0.125$. Below the line t_∞ the orbit is always bounded, whereas above the line t_s no halting orbits exists and the particle accelerates; between these two lines bounded and unbounded orbits may co-exist (see text).

In Fig. 2 we show a plot of both t_∞ and t_s as a function of e_n for $e_t = 0.125$. For $t > 0$ and below the line corresponding to t_∞ the motion is always bounded, as we have shown above. (Note that for $t < 0$ the motion is trivial: regardless of its initial velocity the particle will after a while get stuck on a local minimum of the gravitational potential.) Above the line t_s there are no halting orbits and we conjecture that the orbits are always unbounded (our simulations support this conclusion). In the intermediate region $t_\infty < t < t_s$ the map has a somewhat more complex phase diagram. For instance, if t is only slightly greater than t_∞ then bounded and unbounded orbits may co-exist: for small initial velocity the orbit will remain bounded (with halting points), whereas for sufficiently large initial velocity the particle accelerates. It is also possible to find bounded orbits where the particle never comes to a halt [9].

4. The one-dimensional map

We shall now study in more detail the region of bounded motion ($0 < t < t_\infty$) in which we always have halting orbits. After the particle has momentarily stopped we need to specify how motion proceeds. We consider for simplicity that upon coming to a halt the particle will slide frictionlessly (with constant acceleration $g \sin \phi$) until reaching the edge of the ramp, at which point it takes off with a certain velocity $\mathbf{v} = (u, 0)$. Upon making a succession of collisions and flights the particle will come again to halt on another step. Let us then denote by z_s the first point of collision on this step. Recalling that the particle traverses a distance d from this first collision point to the halting point, we have that the particle will slide with a constant acceleration $\frac{1}{2}t$

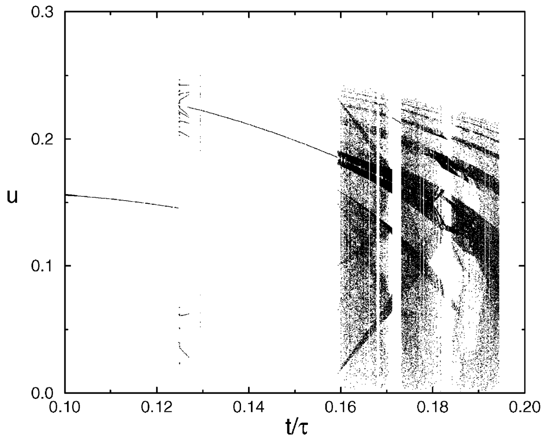

Fig. 3. Bifurcation diagram for the one-dimensional map with $e_n = 0.6$ and $e_t = 0.125$.

(in our dimensionless units) over a distance $z_s - d$ and then fly off with a new velocity $\mathbf{v}' = (u', 0)$, where $u' = \sqrt{t(z_s - d)}$.

It is clear from the discussion above that for $t < t_\infty$ the dynamics of the model can be described in terms of a one-dimensional map $u' = f(u) \equiv \sqrt{t(z_s - d)}$, since for fixed parameters both z_s and d depend only on u. The mapping function $f(u)$ is a piecewise continuous function with infinitely many branches [8]. If we define the total jump number N as the number of steps the particle has fallen between two halting points, then $f(u)$ can be divided into families of branches, with each family corresponding to a specific N. (Members of a given N-family are labelled by additional $N - 1$ indexes [8].) Unfortunately, $f(u)$ cannot be expressed in closed form, except for a few cases such as $N = 1$ and $N = 2$.

We have studied the map $u' = f(u)$ both analytically [8] and numerically [9] and have found that it displays several distinct behaviors as the inclination parameter t increases. Here for want of space we shall only outline the map main features. (A fuller description will be left for a forthcoming publication [8].) First, for small $t > 0$ the mapping function $f(u)$ always has a stable fixed point with $N = 1$. For intermediate inclinations and for some values of the parameters e_n and e_t, stable fixed points with $N = 2$ (and higher) may also appear. If the fixed point $N = 1$ has already become unstable when the new fixed point is born then the latter is the sole attractor, otherwise there are co-existing stable fixed points. As we increase t further, the fixed points all go unstable and the dynamics becomes chaotic. In Fig. 3 we show a bifurcation diagram for $e_n = 0.6$ and $e_t = 0.125$. The first branch in Fig. 3 corresponds to the fixed point with $N = 1$ while the second one in the middle is a fixed point with $N = 2$. One can also see that periodic orbits of quite high period appear between these two fixed points. Also noticeable in Fig. 3 is a window within the chaotic region corresponding to a stable fixed point with $N = 3$.

5. Conclusions

We have presented a simple model for the gravity-driven motion of a grain down a rough inclined surface. For small inclinations, the model has a stable fixed point that is akin to the regime of steady granular flow observed in experiments [2,6]. For higher inclinations, the model displays an unsteady (chaotic) behavior where the velocity remains finite but fluctuates considerably (such a phase corresponds, for instance, to the jump regime seen in experiments [6] on a sphere moving down a rough plane). For even higher inclinations the particle accelerates indefinitely. Several issues remain to be investigated such as the nature of the fluctuations in the chaotic regime and a better characterization of the transition from bounded to unbounded motion.

References

[1] H.J. Herrmann, J.-P. Hovi, S. Luding (Eds.), in: Physics of dry granular media, Nato ASI Series E: Applied Sciences, Vol. 350, Kluwer Academic, Dordrecht, 1998.
[2] P.-A. Lemieux, D.J. Durian, Phys. Rev. Lett. 85 (2000) 4273.
[3] A. Valance, D. Bideau, Phys. Rev. E 57 (1998) 1886.
[4] G.L. Vasconcelos, J.J.P. Veerman, Phys. Rev. E 59 (1999) 5641.
[5] G.L. Vasconcelos, J.J.P. Veerman, Physica A 271 (1999) 251.
[6] G.H. Ristow, F.-X. Riguidel, D. Bideau, J. Phys. I (France) 4 (1994) 1161.
[7] S.F. Foerster, M.Y. Lounge, H. Chang, K. Allia, Phys. Fluids 6 (1994) 1108.
[8] G.L. Vasconcelos, F.V. Cunha-Jr., J.J.P. Veerman, in preparation.
[9] S.J. Baker, M.Sc. Thesis, Georgia Institute of Technology, 1999.