On a Conjecture of Furstenberg

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1 Estimation of the Hausdorff Dimension

1.1 Statement of the problem

Consider and iterated function system Ψ_t given by three generators:

$$\begin{aligned}\psi_0(x) &= \frac{x}{3}\,,\\ \psi_1(x) &= \frac{x+1}{3}\,,\end{aligned}$$

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$$\psi_t(x) = \frac{x+t}{3}$$

where $t \in \mathbf{R}$ is a fixed parameter.

By [1], for every t there is a unique compact set Z_t which is invariant under Ψ_t and such that the orbit of any compact set under Ψ_t converges to Z_t in the Hausdorff metric. An elementary intepretation of Z_t is as the set of number which can be represented by generally infinite expressions in base 3 which use digits $0, 1, t$.

In this paper we are proving the following:

Theorem 1 *For every t irrational, $HD(Z_t) \geq 1 - \frac{\log(5/3)}{2\log 3} > 0.767$.*

Since the Hausdorff dimension in an affine invariant, from now we will assume without loss of generality that $|t| \leq 1$. Theorem 1 will be derived from a technical Theorem 2 which is stated later.

Conjectures of Furstenberg. Let's quote three related conjectures of Furstenberg.

Conjecture 1 *For every t irratonal, $HD(Z_t) = 1$.*

Let W be the limit set of the iterated function systems in $\mathbf{R}^2$ which is generated by $x \to x/3$, $x \to [x + (1, 0)]/3$ and $x \to [x + (0, 1)]/3$.

Conjecture 2 *For every t irrational almost every $\beta \in \mathbf{R}$ the line $v = tu + \beta$ intersects W along a set with Hausdorff dimension 0.*

Let T denote the operator

$$Tf(x) = \frac{1}{3}\left[f(x) + f(x-1) + f(x-t)\right]$$

acting on the space of continuous functions with compact support.

Conjecture 3 *For every t irrational the spectral radius of the adjoint T^* is equal to 1.*

Historical remarks. Theorem 1 is a step towards proving Conjecture 1. Conjecture 1 was the subject of work by several authors. One should mention [3] where it was established that for almost every t, both in the topological and category sense, $HD(Z_t) = 1$, and that $|Z_t| = 0$ (Lebesgue measure) for every t irrational, see also [4]. In [2] a study of the continuity properties of the function $t \to HD(Z_t)$ was undertaken, while [6] contains numerical data mostly in support of Conjecture 1.

1.2 Energy estimate

Given a postive probablistic measure μ on $\mathbf{R}$ and $\alpha \geq 0$, we define its *energy integral*

$$I_\alpha(\mu) := \int \frac{d\mu(x)\, d\mu(y)}{|x - y|^\alpha} .$$

For a Borel set $Z \subset \mathbf{R}$ consider the set $A(Z)$ which consists of those $\alpha \geq 0$ for which there exists a Radon measure μ_α supported on Z and $I_\alpha(\mu_\alpha) < \infty$. It is known that $HD(Z) = \sup A(Z)$, see [5]. Hence, each time we get a measure μ supported on Z and $I_\alpha(\mu) < \infty$, we have bounded the Hausdorff dimension of Z by α from below.

Natural measures. We will work with a concrete measure μ^t supported on Z_t. Consider a sequence of measures

$$\mu_0^t = \delta_0 \ , \mu_1^t = \frac{1}{3}(\delta_0 + \delta_1 + \delta_t)$$

and

$$\mu_n^t = \mu_1^t * (\psi_{0*}\mu_1^t) * \cdots * (\psi_{0*}^{n-1}\mu_1^t)$$

for $n > 0$. The choice of equal weighting of the measures transfered by all generators was of course arbitrary. One easily verifies that

$$\mu_n^t = \mu_k^t * \phi_{0*}^k \mu_{n-k}^t$$

for $0 \leq k \leq n$. Hence, measures μ_n^t converge weakly to μ^t which is supported on the interval $[0, 1/2]$.

Estimates. Let us begin to estimate the energy integral. Let $0 < \alpha < 1$. For $n \geq 0$ denote $L_n := \{(x,y) :\ 2 \cdot 3^{-n} < |x-y| \leq 2 \cdot 3^{1-n}$.

$$I_\alpha(\mu^t) = \int \frac{d\mu^t(x)\, d\mu^t(y)}{|x-y|^\alpha} = \sum_{n=1}^{\infty} \int_{L_n} \frac{d\mu^t(x)\, d\mu^t(y)}{|x-y|^\alpha} \ .$$

Since $\mu^t = \mu_n^t * \psi_{0*}^n \mu^t$ and the support of $\psi_{0*}^n \mu^t$ is contained in $[-3^{-n}/2, 3^{-n}/2]$, we can write

$$\int_{L_n} \frac{d\mu^t(x)\, d\mu^t(y)}{|x-y|^\alpha} = \int_{L_n} |x-y|^{-\alpha} * h(x) * h(y)\, d\mu_n^t(x)\, d\mu_n^t(y)$$

where h is a non-negative function with total mass 1 and support contained in $[-3^{-2}/2, 3^{-n}/2]$. Because of that, for $(x,y) \in L_n$ we get

$$|x-y|^{-\alpha} * h(x) * h(y) \leq 3^{na}$$

and

$$I_\alpha(\mu^t) \leq \sum_{n=1}^{\infty} 3^{n\alpha} \int_{L_n} d\mu_n^t(x)\, d\mu_n^t(y) \ . \qquad (1)$$

Denote

$$s(n,\beta) = 3^n \int \chi_{(-3^{-n}/2, 3^{-n}/2]}(x-y-\beta)\, d\mu_n^t(x)\, d\mu_n^t(y) = \qquad (2)$$

$$3^n \int \chi_{(-3^{-n}/2, 3^{-n}/2]}(z-a)\, d(\mu_n^t * (\mu_n^t)')(z)$$

where the apostrophe means the measure transported by the map $x \to -x$. Then

$$3^{n\alpha} \int_{L_n} d\mu_n^t(x)\, d\mu_n^t(y) \leq$$

$$\frac{s(n, \frac{5}{2}3^{-n}) + \cdots + s(n, \frac{11}{2}3^{-n}) + s(n, -\frac{5}{2}3^{-n}) + \cdots + s(n, -\frac{11}{2}3^{-n})}{3^{n(1-\alpha)}} \ .$$

If we write $S_n = \sup_{\beta \in \mathbf{R}} s(n,\beta)$, then we get from estimate (1) that

$$I_\alpha(\mu^t) \leq 8 \sum_{n=1}^{\infty} 3^{n(\alpha-1)} S_n \ . \qquad (3)$$

So the task is reduced to estimating the exponential rate of increase for S_n.

1.3 Projection measure

Consider a measure ν_1 is $\mathbf{R}^2$ defined by

$$\nu_1 = \frac{1}{3}(\delta_{(0,0)} + \delta_{(0,1/3)} + \delta_{(1/3,0)}) \ .$$

If ψ denotes the homothety with scale 1/3, then we define

$$\nu_n = \nu_1 * (\psi_* \nu_1) \cdots * (\psi_*^{n-1} \nu_1) \ .$$

If $\pi_t : \mathbf{R}^2 \to \mathbf{R}$ denotes the linear projection given by $\pi_t(u,v) = tu + v$, then we have $\mu_n^t = \pi_{t*}\nu_n$. Hence

$$\mu_n^t * (\mu_n^t)' = \pi_{t*} \left[\nu_1 * \nu_1' * \psi_*(\nu_1 * \nu_1') * \cdots * \psi_*^{n-1}(\nu_1 * \nu_1') \right] \ .$$

Measure $\nu_1 * (\nu_1)'$ is obtained explicitly and equals

$$\frac{1}{9} \sum_{k,\ell=-1,0,1} b(k,\ell) \delta_{(k/3,\ell/3)} \tag{4}$$

where $b(k,\ell) = 1$ if $k \neq \ell$, 3 if $k = \ell = 0$ and 0 otherwise. Function b extends to $\mathbf{Z} \times \mathbf{Z}$ by $b(k_1,\ell_1) = b(k,\ell)$ where $k,\ell = -1,0,1$ and $k - k_1, \ell - \ell_1 \in 3\mathbf{Z}$.

From the defining formula (2),

$$s(n,\beta) = 3^n \int \chi_{(-3^{-n}/2, 3^{-n}/2]}(z-\beta) \, d(\mu_n^t * (\mu_n^t)')(z) =$$

$$= 3^n \sum_{k,\ell \in \mathbf{Z}} (\nu_n * \nu_n')(k3^{-n}, \ell 3^{-n}) \chi_{(-3^{-n}/2, 3^{-n}/2]}(tk3^{-n} + \ell 3^{-n} - \beta) \ .$$

For every $k \in \mathbf{Z}$ and β, a non-zero contribution is obtained only when $\ell = \ell_{n,\beta}(k) := \langle 3^n(\beta - kt3^{-n}) \rangle$ where $\langle x \rangle$ is the integer characterized by the condition $-1/2 < x - \langle x \rangle \leq 1/2$. If we also introduce the notation $k_n(x) = \langle 3^n x \rangle$, then we can write

$$s(n,\beta) = 3^n \sum_{k \in \mathbf{Z}} (\nu_n * \nu_n')(k3^{-n}, \ell_{n,\beta}(k) 3^{-n}) =$$

$$= 9^n \int_{-\infty}^{+\infty} (\nu_n * \nu_n') \left(k_n(x) 3^{-n}, k_n(\beta - t k_n(x)) 3^{-n} \right) \, dx \ .$$

Define $b_i(u,v) = b(< 3^i u >, < 3^i v >)$. Then we can write for $-\frac{3^n-1}{2} \leq k, \ell \leq \frac{3^n-1}{2}$ that

$$(\nu_n * \nu_n')(k3^{-n}, \ell 3^{-n}) = 9^{-n} \prod_{i=1}^{n} b_i(k3^{-n}, \ell 3^{-n}) \ .$$

If k, ℓ are outside that range, then $(\nu_n * \nu_n')(k3^{-n}, \ell 3^{-n}) = 0$. Hence we can write

$$s(n,\beta) = \int_{-1/2}^{1/2} \prod_{i=1}^{n} B_i(x, \beta - tk_n(x))\, dx$$

where functions $B_i : T^2 \to \mathbf{R}$ are defined below.

Definition 1 *If $(x,y) \in T^2$ and $i > 0$, then*

$$B_i(x,y) := b(k_i(x), k_i(y)) \ .$$

2 Averaging Estimates

We will denote $T^1 := (-1/2, 1/2]$ and $T^2 := T^1 \times T^1$ and think of identifying pieces of the boundary so that tori are obtained. Let $\pi(x) := x'$ where $x' \in T^1$ and $x - x' \in \mathbf{Z}$. Recall that $k_n(x) = \langle 3^n x \rangle$.

2.1 Partitions related to base 3 expansions

It will be useful to think of the circle T^1 with the Lebesgue measure as a probabilistic space.

Definition 2 *Say that an interval $I \subset T^1$ is a* basic interval *of order n, $n \geq 0$, if the transformation $x \to \pi(3^n x)$ maps I onto T^1 with degree 1.*

For example, interval $(-\frac{1}{6}, \frac{1}{6}]$ is basic of order 1. Let $\mathcal{P}_n$ denote the partition of T^1 into basic intervals of order n.

Now let $\phi : \mathbf{R} \to \mathbf{R}$. For a positive integer n, define $\phi_n : T^1 \to T^1$ by

$$\phi_n(x) := \phi(3^{-n} k_n(x)) \ . \tag{5}$$

So, ϕ_n is a $\mathcal{P}_n$-measurable approximation of ϕ.

Lemma 1 *Let $q \geq 0$ and $n > 0$ and $\phi(x) = tx + t_0$. Then for every $x \in T^1$*

$$\phi_n(\pi(3^q x)) + T(x) - 3^q \phi_{n+q}(x)$$

is an integer, where $T(x) = k_q(x)t + (3^q - 1)t_0$.

Proof:
The expression which is to be shown to yield an integer is measurable with respect to $\mathcal{P}_{n+q}$. It suffices to prove the claim for $x = (3^n J + j)3^{-n-q}$ with integers J and j ranging over $[-\frac{3^q-1}{2}, \frac{3^q-1}{2}]$ and $[-\frac{3^n-1}{2}, \frac{3^n-1}{2}]$, respectively. For x in such a form

$$3^q \phi_{n+q}(x) = 3^q(xt + t_0) = Jt + jt3^{-n} + 3^q t_0 \ .$$

On the other hand,

$$\phi_n(\pi(3^q x)) = \phi_n(\pi(J + j3^{-n})) = \phi_n(j3^{-n}) = \pi(jt3^{-n} + t_0) \ .$$

Finally, $k_q(x) = J$ and

$$T(x) = Jt + (3^q - 1)t_0$$

which implies the claim.

QED

Lemmas about circle rotations. Let $R_t : T^1 \to T^1$ be defined by $R_t(x) = \pi(x + t)$.

Definition 3 *Define the set $U(t, K) \subset \mathbf{N}$ by the following requirement: $m \in U(t, K)$ if and only if for every $x \in T^1$ and every J which is a sub-arc of T^1 with length 3^{-m}, the set*

$$\{R_t^p(x) : \ p = 0, 1, \cdots, 3^m - 1\} \cap J$$

has no more than K elements.

Thus, for $m \in U(t, K)$ the first 3^m points of any orbit are uniformly spread out, in the sense that no “lumps” are formed.

Lemma 2 *For every t irrational the set $U(t, 6)$ is infinite.*

Proof:
Let q be a closest return time for the rotation $x \to x+t \mod 1$. Then the orbit $x, \cdots, R_t^{q-1}(x)$ cuts the circle into pieces of two sizes and the shorter ones are never adjacent. Hence, any arc of length not exceeding $1/q$ may contain at most two points of the orbit. If m is chosen so that $3^{m-1} < q \leq 3^m$, the the orbit $x, \cdots, R_t^{3^m-1}(x)$ can be covered by three orbits of length q. Thus, no arc of length 3^{-m} contains more than 6 points.

QED

2.2 Averages along graphs

Definition 4 *Suppose that* $F : T^2 \to \mathbf{R}$ *and* $g : T^1 \to \mathbf{R}$ *are given. Then we can form a function* $F_g : T^1 \to \mathbf{R}$ *by the following formula:* $F_g(x) = F(x, g(x))$.

The general type of the problem we will consider is as follows. We wish to average F_g along basic intervals, which corresponds to taking conditional expectations with respect to partitions $\mathcal{P}_n$. The problem is under what assumptions these averages can be estimated in terms of the average of F over T^2.

Proposition 1 *Consider* $\phi(x) = tx + t_0$ *and choose* $N > 0$ *and* K *so that* $N \in U(t, K)$.

For every $n \geq 0$ *and every set* $A \subset T^2$ *which is measurable with respect to* $\mathcal{P}_N \times \mathcal{P}_N$ *we consider the function* $F : T^2 \to \mathbf{R}$ *given by* $F^{(n)}(x, y) = \chi_A(\pi(3^{n+N}x), \pi(3^{n+N}y))$.

Then,

$$E(F^{(n)}_{\phi_{n+2N}} | \mathcal{P}_n)(x) \leq K \int_{T^2} \chi_A \, d\lambda_2$$

for every $x \in T^1$*, using the notation of Definition 4.*

Proof:
Choose an interval $I \in \mathcal{P}_n$. Observe first that without loss of generality $n = 0$. Indeed, for $n \geq 0$ the interval I can be parameterized by a variable $x' = \pi(3^n x)$ which runs over T^1. We get $\pi(3^N 3^n x) = \pi(3^N x')$ and, by Lemma 1,

$$\pi(3^{n+N}\phi_{n+2N}(x)) = \pi(3^N(\phi + T(x))_{2N}(x'))$$

with $T(x)$ constant and equal to $T(I)$ on I. Thus for every $x \in I$

$$E(F^{(n)}_{\phi_{n+2N}} | \mathcal{P}_n)(x) = E(F^{(0)}_{\phi_{2N}+T(I)})$$

which leads to the initial problem with $n = 0$ and t_0 increased by $T(I)$. Since the claim is supposed to be valid for every t_0, the reduction is complete.

We will write F for $F^{(0)}$ and ϕ for $\phi + T(I)$.

$$F_{\phi_{2N}}(x) = \chi_A(\pi(3^N x), \pi(3^N \phi_{2N}(x))) =$$

$$= \chi_A \left[\pi(3^N x), \pi(\phi_N(\pi(3^N x)) + T(x))\right]$$

by Lemma 1 used with $n = q = N$. If we write $x = J3^{-N} + j3^{-2N}$ with J, j both integers from the range $[-\frac{3^N-1}{2}, \frac{3^N-1}{2}]$, we get $\pi(3^N x) = j3^{-N}$ and $T(x) = Jt + T_0$ where T_0 is a constant. We can then write

$$E(F_{\phi_{2N}}) = E\left[\chi_A(\pi(3^N x), \pi(\phi_N(\pi(3^N x)) + T(x))\right] =$$

$$= 3^{-2N} \sum_{j=-\frac{3^N-1}{2}}^{\frac{3^N-1}{2}} \sum_{J=-\frac{3^N-1}{2}}^{\frac{3^N-1}{2}} \chi_A(j3^{-N}, \pi(Jt + \phi(j3^{-n}) + T_0)) \, .$$

For j fixed, points $\pi(Jt + \phi(j3^{-n}) + T_0)$ form an orbit of the rotation R_t of length 3^N. By the hypothesis of the Lemma, each square of the partition $\mathcal{P}_N \times \mathcal{P}_N$ contains no more than K points in the form $(j3^{-N}, \pi(Jt + \phi(j3^{-n}) + T_0)))$. Hence,

$$3^{-2N} \sum_{J=-\frac{3^N-1}{2}}^{\frac{3^N-1}{2}} \sum_{j=-\frac{3^N-1}{2}}^{\frac{3^N-1}{2}} \chi_A(j3^{-N}, \pi(Jt+\phi(j3^{-n})+T_0)) \leq K \int_{T^2} \chi_A \, d\lambda_2.$$

QED

Lemma 3 *For every* $t \in \mathbf{R}$, $m \geq n \geq 0$, *if* $\phi(x) = tx$, $t_0 \in \mathbf{R}$, *and* $F : T^2 \to [0, \infty)$ *is measurable with respect to* $\mathcal{P}_n \times \mathcal{P}_n$, *then*

$$F(\pi(x), \pi(\phi_m(x) + t_0)) \leq \sum_{2|i|<|t|+2} F(\pi(x), \pi(\phi_n(x) + t_0 + i3^{-n}))$$

where i *runs through integer values only.*

Proof:
Estimate $|\phi_m(x) - \phi_n(x)| \leq |t|\frac{3^{-n}}{2}$. Thus, for every x we can choose $\tau(x)$ in the form $i3^{-n}$, where i is an integer and $-|t|/2 - 1 < i < |t|/2 + 1$ so that $\pi(\phi_n(x) + t_0 + \tau(x))$ and $\pi(\phi_m(x) + t_0)$ belong to the same element of $\mathcal{P}_n$, and so

$$F(\pi(x), \pi(\phi_m(x) + t_0)) = F(\pi(x), \pi(\phi_n(x) + t_0 + \tau(x))) \ .$$

Now $\tau(x)$ only takes values in the set $i3^{-n}$ with $|i| < |t|/2 + 1$ and so the lemma follows.

QED

Proposition 2 *Let $t \in \mathbf{R}$, $K > 0$, $n \geq 0$ and $N \in U(t, K)$, see Definition 3. Denote $\phi(x) = tx$. Let $F : T^2 \to [0, \infty)$ be measurable with respect to $\mathcal{P}_n \times \mathcal{P}_n$. Suppose that for a fixed I and every $t_1 \in \mathbf{R}$,*

$$\int_{T^1} F_{\phi_n + t_1}(x)\, dx \leq I \ ,$$

se Definition 4 for the explanation of the notation.

Now $G : T^2 \to [0, \infty)$ is measurable with respect to $\mathcal{P}_N \times \mathcal{P}_N$, $N > 0$. Define $\tilde{G}(x, y) = G(\pi(3^{n+N} x), \pi(3^{n+N} y))$.

Then, for every choice of t and K and every $t_0 \in \mathbf{R}$

$$\int_{T^1} F_{\phi_{n+2N} + t_0}(x)\, \tilde{G}_{\phi_{n+2N} + t_0}(x)\, dx \leq$$

$$K(|t| + 3) I \int_{T^2} G\, d\lambda_2 \ .$$

Proof:
Fix some $t_1 \in \mathbf{R}$. Function $F_{\phi_n + t_1}$ is measurable with respect to $\mathcal{P}_n$. By Proposition 1,

$$E(\tilde{G}_{\phi_{n+2N} + t_0} | \mathcal{P}_n)(x) \leq K I_G$$

for every x where $I_G := \int_{T^2} G\, d\lambda_2$. Now,

$$\int_{T^1} F_{\phi_n + t_1}(x)\, \tilde{G}_{\phi_{n+2N} + t_0}(x)\, dx = \int_{T^1} E(F_{\phi_n + t_1}\, \tilde{G}_{\phi_{n+2N} + t_0} | \mathcal{P}_n)(x)\, dx \leq \tag{6}$$

$$\leq KI_G \int_{T^1} F_{\phi_n+t_1}(x)\,dx \leq KI_G I$$

by the hypothesis of Proposition 2.
By Lemma 3

$$F_{\phi_{n+2N}+t_0}(x) = F(x, \pi(\phi_{n+2N}(x) + t_0)) \leq$$

$$\leq \sum_{j\in(-1-|t|/2,|t|/2+1)} F_{\phi_n+t_0+j3^{-n}}(x) \ .$$

If we use estimate (6) for all $t_1 = t_0 + j3^{-n}$, we get

$$\int_{T^1} F_{\phi_{n+2N}+t_0}(x)\ \tilde{G}_{\phi_{n+2N}+t_0}(x)\,dx \leq$$

$$\leq KI_G(|t|+3)I \ .$$

QED

2.3 Averages of products

Theorem 2 *Fix t irrational and let $\phi(x) = tx + t_0$. Then for every $\lambda > \sqrt{5/3}$ and $t_0 \in \mathbf{R}$ we have*

$$\lim_{m\to\infty} \left[\lambda^{-m} \int_{T^1} \prod_{i=1}^{m} B_i(x, \phi_m(x))\,dx\right] = 0 \ .$$

Recall that functions B_i are given by Definition 1. Since

$$s(m,\beta) = \int_{T^1} \prod_{i=1}^{m} B_i(x, \phi_m(x))$$

with $\phi(x) = \beta - tx$, Theorem 2 implies that $\lambda^{-m} S_m \to 0$. By estimate (3), $I_\alpha(\mu^t) < \infty$ provided that $3^{1-\alpha} > \sqrt{5/3}$ and Theorem 1 follows.

Hölder estimate. From Lemma 2, see that $U(t,6)$ is infinite and choose $N \in U(t,6)$. Let $J_{k,0}$ denote the set of integers i which belong to $(2(j-1)N,(2j-1)N]$ for some $j = 1,\cdots,k$ and $J_{k,1}$ be the complement of $J_{k,0}$ in the set $1,2,\cdots,2kN$. Define $P_{k,0}(x,y) = \prod_{i\in J_{k,0}} B_i(x,y)$ and $P_{k,1}(x,y) = \prod_{i\in J_{k,1}} B_i(x,y)$. Then

$$\prod_{i=1}^{2kN} B_i(x,\phi_{2kN}(x)) = P_{k,0}(x,\phi_{2kN}(x))P_{k,1}(x,\phi_{2kN}(x)) \ .$$

Our approach is to apply the Hölder inequality to this product. It is easier to estimate the second norm of $P_{k,1}(x,(\phi+t_1)_{2kN})$ with $t_1 \in \mathbf{R}$. Using Proposition 2 with $n = 2(k-1)N + N$, $F := P^2_{k-1,1}$ and $G(x,y) = \prod_{i=1}^N B_i^2(x,y)$, we get

$$\|P_{k,1}(x,(\phi+t_1)_{2kN})\| \le 6(|t|+3)I\int_{T^2} G\,d\lambda_2$$

where I is an upper estimate for $\|P_{k-1,1}(x,(\phi+t_1)_{2(k-1)N}\|$ for any $t_1 \in \mathbf{R}$. Note that $\int_{T^2} G\,d\lambda_2 = (5/3)^N$ and hence one gets by induction starting with $P_{0,1} \equiv 1$ that

$$\|P_{k,1}(x,\phi_{2kN}(x))\|_2^2 \le K_1^k (5/3)^{kN} \ .$$

The same method is used to estimate the second norm of

$$P_{k,0}(x,(\phi+t_1)_{(2k-1)N}(x)) \ .$$

This time, the induction starts with $\|P_{1,0}(x,(\phi+t_1)_N(x))\|_2^2 \le 3^N$ since 3^N is the maximum. Thus,

$$\|P_{k,0}(x,(\phi+t_1)_{(2k-1)N}(x))\|_2^2 \le 3^N K_1^{k-1}(5/3)^{(k-1)N} \ .$$

Using Lemma 3 and applying the previous estimate for

$$t_1 = j3^{-2(k-1)N} \ , \ 2|j| < |t| \ ,$$

we get

$$\|P_{k,0}(x,\phi_{2kN}(x))\|_2^2 \le (|t|+3)\|P_{k,0}(x,\phi_{2(k-1)N}(x))\|_2^2 \le$$
$$\le (|t|+1)3^N K_1^{k-1}(5/3)^{2(k-1)N} \ .$$

By Hölder's inequality,

$$\lambda^{-2kN} \int_{T^1} \prod_{i=1}^{2kN} B_i(x, \phi_{2kN}(x))\, dx \leq \sqrt{3^N(|t|+3)} \left[\frac{5}{3} \frac{\sqrt[N]{K_1}}{\lambda^2} \right]^{kN} .$$

If $\lambda > \sqrt{5/3}$ then N can be chosen so large that

$$\frac{5}{3} \frac{\sqrt[N]{K_1}}{\lambda^2} < 1 .$$

Then

$$\lim_{k \to \infty} \left[\lambda^{-2kN} \int_{T^1} \prod_{i=1}^{2kN} B_i(x, \phi_{2kN}(x))\, dx \right] = 0 . \tag{7}$$

Any $j > 0$ can be represented as $2k_jN + j_0$ with $j_0 < 2N$. Then

$$\prod_{i=1}^{j} B_i(x, \phi_j(x)) \leq 3^{j_0} \prod_{i=1}^{2k_jN} B_i(x, \phi_j(x)) .$$

Again using Lemma 3 and the fact that we estimate

$$\int_{T^1} \prod_{i=1}^{2k_jN} B_i(x, \phi_j(x))\, dx \leq (|t|+1) \int_{T^1} \prod_{i=1}^{2k_jN} B_i(x, (\phi + t_1)_{2k_jN}(x))\, dx$$

where t_1 was chosen to attain the supremum of the integral on the rihgt-hand side. Hence, for any $j > 0$,

$$\int_{T^1} \prod_{i=1}^{j} B_i(x, \phi_j(x))\, dx \leq 9^N(|t|+1) \int_{T^1} \prod_{i=1}^{2k_jN} B_i(x, (\phi + t_1)_{2k_jN}(x))\, dx$$

and Theorem 2 follows from this together with assertion (7). Notice that (7) holds for any t_0, in particular one can set $t_0 := t_0 + t_1$ in that estimate.

References

[1] Hutchinson, J.E.: *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713-747

[2] Jonker, L.B. & Veerman, J.J.P.: *Semicontinuity of Dimension and Measure of Locally Scaling Fractals*, preprint, Stony Brook IMS Series, (1997) #2

[3] Kenyon, R.: *Projecting the one-dimensional Sierpinski gasket*, Israel J. Math. **97** (1997), 221-238

[4] Lagarias, J.C. & Wang, Y.: *Tiling the line with translates of one tile*, Invent. Math. **124** (1996), 341-365

[5] Mattila, P.: *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability.*, Cambrdge Studies in Advanced Mathematics vol. 44, Cambridge University Press (1995)

[6] Veerman, J.J.P. & Stošić, B.D.: *On the Dimensions of Certain Incommensurably Constructed Sets*, Experim. Math. **9** (2000), 413-425