STRANGE ATTRACTORS IN DISSIPATIVE MAPS WITH ONE ANGULAR VARIABLE

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ABSTRACT

It is often suggested that strange attractors occur abundantly in nature. In this review we consider a possible definition of a strange attractor and apply it to a gengralized version of the Birkhoff attractor (from now on referred to as Birkhoff set) of a dissipative map from $S^1 \times R^{n-1}$ to itself. We specialize this study to n = 2 and n = 3.

The first main result is that if there is a "rotationally" strange attractor, for n = 2, then the Birkhoff set is contained in it. It is pointed out, however, that not all irregular behavior is contained in the Birkhoff set.

Properties of accessible points are discussed and the importance of the concept of rotation number and the twist condition are reviewed.

Examples of Birkhoff sets in three dimensions are given. For a three-dimensional version of the standard map we prove that order preserving orbits persist for small enough coupling.

Note: In this manuscript we have used a "c" for the set theoretic inclusion, and a "Λ" for set theoretic intersection.

I INTRODUCTION

In some cases, attractors can be easily described by almost any definition that intuitively captures the meaning of that word. Such cases are, in dissipative systems, simple sinks or attracting curves. However, physical reality indicates, that the objects we wish to call strange attractors in, for instance, dissipative fluid flows can be objects much more complicated than these, moreover with complicated dynamics governing the motion on the attractor. (Note that the motion on complicated sets can be simple, as on Aubry Mather sets in two-dimensional area-preserving twist maps. The opposite is also possible as on the unit circle in the complex plane under the transformation $z \to z^2$.

In physical systems, one would like to think about the object in phase space, and its dynamics, that one observes after transients have died out (Eckmann and Ruelle, 1985). There is little or no general theory about such objects.

Our approach in this work, is to study the simplest (non-trivial) cases. The consequence is, of course, that results might be difficult to generalize.

Problems arise from the outset. It is not clear what the ideal definition is (Milnor, 1985, Eckmann and Ruelle, 1985). Furthermore, with any reasonably precise definition, it appears difficult to establish existence of strange attractors.

We will illustrate this in section 3 by applying a 'reasonable' definition to a two-dimensional twist map, making use of strong mathematical results that exist in this case. The "Birkhoff attractor" (definition given below) cannot be proven to be equal to a (rotationally) strange attractor. To avoid confusion, we will therefore denote this set from now on by the Birkhoff set or β.

It will result, however, that there is at most one strange sttractor, that the Birkhoff set is unique, and that the latter is contained in the strange attractor if there is one. We will therefore review some of the theorems known about Birkhoff sets. Some of the results referred to are joint work of Martin Casdagli and the author.

In section 4, we propose a generalization of the Birkhoff set to three dimensions. We will prove that this object is non-vacuous. We will reflect on what properties of the two dimensional case carry over to three dimensions.

Sections 2 and 3 will partly have the function of a review of this subject, although it should be pointed out that our discussion of Birkhoff sets is done in much more general terms than the treatments given by other workers. Not all results mentioned are original work of the author, references will be mentioned.

II SEPARATING SETS

We consider C^2 diffeomorphisms of $S^1 \times R \times R^{n-2}$ (we will restrict this later to $n = 2$ and $n = 3$) to itself. Denote the angular coordinate by θ and the others by r and z_i, respectively. In addition f satisfies the following:

- f is uniformly dissipative: $0 < |Df| \leq \lambda < 1$,
- there exists an trapping strip $s = S^1 \times [-M, M] \times R^{n-1}$ such that $f(s) \subset \overset{\circ}{s}$,
- there exists a closed and bounded trapping region $t = S^1 \times [-M, M]^{n-1}$ such that $f(t) \subset \overset{\circ}{t}$,
- Any invariant separating set (a definition will be given below) intersected with t has zero Lebesgue measure.

Examples of such diffeomorphisms are given in sections 3 and 4. Note that the second and third requirement coincide if $n = 2$. In that case, the last requirement is trivial. In this work, we will restrict ourselves to the study of separating sets contained in some fixed trapping strip s.

To each point in t we assign a rotation number iff the following limit exists:
$$\rho(\theta, r, z) = \lim_{n\to\infty} \frac{F^n(\theta, r, z)}{n} .$$
(Similarly, a backward rotation number can be defined when $n \to -\infty$.) An orbit is called well-ordered if f preserves the cyclic order on its projection to the θ-axis.

Here is the definition of attractor that we will work with. A compact set α that satisfies $f(\alpha) = \alpha$ is called :

- attracting: if there is a neighborhood u of α such that $\alpha = \bigcap_{i=0}^{\infty} f^i(u)$
- attractor: if α is attracting and contains a point whose forward orbit is dense in α.
- strange attractor: if α is an attractor and contains at least two orbits with different rational rotation numbers.

Note that our definitions are not precisely the same as the ones in Casdagli (1988). In the proof of the main result there is a little extra work that we do in the following lemma:

Lemma 2.1: If α is attracting, then every open neighborhood v of α contains a neighborhood u' of α, such that:

$$\text{i)}: \quad \bigcap_{i=0}^{\infty} f^i(u') = \alpha$$

$$\text{ii)}: \quad f(u') \subset u' \quad .$$

Proof: Choose a neighborhood g of α with $g \subset v$ and let $u' = g \cap u$. Then

$$\bigcap_{i=0}^{\infty} f^i(u') \subseteq \bigcap_{i=0}^{\infty} f^i(u) = \alpha$$

and $$\alpha = \bigcap_{i=0}^{\infty} f^i(\alpha) \subseteq \bigcap_{i=0}^{\infty} f^i(u') \ ,$$

which proves i). To prove ii), observe that (Milnor, 1985b) for every neighborhood n, there is an n_0 such that for all $m > n_0$, $f^m(u) \subset n$. So take n to be u and define $u' = \bigcap_{i=0}^{m} f^i(u)$. Then

$$f(u') = \bigcap_{i=0}^{m} f^{i+1}(u) \quad \subset \bigcap_{i=1}^{m} f^i(u) \cap u \ ,$$

which proves ii). □

We remark that this definition of a strange sttractor is a dynamical one. The dynamics on α has to be strange, not necessarily its geometry. The unit circle is a strange attractor for the following (non-invertible) map on the cylinder:

$$f: \begin{array}{l} \theta' = 2\theta \\ r' = \lambda(r-1) \quad , \quad 0 < \lambda < 1/2 \end{array}$$

Back to diffeomorphisms, note that a two-dimensional twist map of the cylinder can be made to contract to any smooth invariant curve c. Then c, of course, is attracting. Suppose points on c have irrational rotation number and that the non-wandering set of f restricted to c is a Cantor set s (as in Denjoy counterexamples). It is easy to see that c is also attracting, but it is not an attractor.

We proceed with the definition of the Birkhoff set β. An invariant set λ contained in s is called separating, if $s\backslash\lambda$ has at least two connected components, one containing $r = +M$, the other containing $r = -M$. It is clear, that there is such a separating set, namely:

<u>Lemma 2.2</u>: The set $\lambda = \bigwedge_{i=0}^{\infty} f^i(s)$ is separating.

<u>Proof</u>: First of all, we note that $\bigwedge_{i=0}^{\infty} f^i(s)$ is closed, so that its complement is open. Thus, the connectedness of this complement is equivalent to the arcwise connectedness.

By definition of s, f(s) is contained in s. Let k be a closed connected set which is contained in f(s) and contains $f^2(s)$, so that all its iterates are separating. Let γ be a curve connecting $r = -M$ with $r = +M$. Denote the part of $\gamma \backslash (f^n(k) \wedge \gamma)$ that maps to $r > M$ ($r < M$) under f^{-1} by γ^n_{ext} (γ^n_{int}). Then, for all $n,m > 0$: $\gamma^n_{ext} \wedge \gamma^m_{int} = \phi$, and γ^n_{ext} , γ^n_{int} are open. Let

$$\gamma^\infty_{ext} = \bigcup_n \gamma^n_{ext} \quad \text{and} \quad \gamma^\infty_{int} = \bigcup_n \gamma^n_{int} .$$

Then γ^∞_{int} and γ^∞_{ext} are open and disjoint in γ. Hence γ contains a point that belongs to neither. □

<u>Remark</u>: Any invariant set in t is contained in λ.

Denote the two components of $s\backslash\lambda$ by λ_{ext} and λ_{int}, respectively. Generalizing the definitions of Le Calvez (1986) and Casdagli (1988) we define the Birkhoff set β:

$$\beta_s = \overline{\lambda_{int}} \wedge \overline{\lambda_{ext}}$$

$$\beta = \beta_s \wedge t \tag{2.1}$$

In other words, we can characterize β as those points in t for which every neighborhood contains points that under f^{-1} escape from the trapping strip s past $r = -M$ as well as points that escape past $r = +M$.

<u>Lemma 2.3</u>: β_s is non-empty and separating.

Proof: If β_s is not separating (or if it is empty), then, using the conventions of the last proof, there is a curve γ such that $\gamma \wedge \beta_s = \phi$.

Now, γ lies entirely in $\bar{\lambda}^{---}_{int} \cup \bar{\lambda}^{---}_{ext}$, because if not, then one constructs an open set contained in λ violating the requirement that invariant separating sets have measure zero. Both sets $\gamma \cup \bar{\lambda}^{---}_{int}$ and $\gamma \cup \bar{\lambda}^{---}_{ext}$ are closed in γ and their complement is empty, therefore they intersect in a non-empty set. □

It is clear that $s \backslash \beta_s$ consists of exactly two components (the complement of β_s cannot have bounded invariant components because volumes are contracted), which, using similar notation as before, will be written as $\beta_{s,int}$ and $\beta_{s,ext}$. We have the following theorem of which a two dimensional version was stated in Le Calvez [1986]:

Theorem 2.4: i) $\beta_s = \delta\beta_{s,int} = \delta\beta_{s,ext}$,

ii) β_s is the unique smallest closed invariant separating set contained in s.

Proof: i): Any $x \in \delta\beta_{s,int}$ cannot lie in $\beta_{s,ext}$ or $\beta_{s,int}$, therefore by definition of $\beta_{s,ext}$ or $\beta_{s,int}$ it lies in β_s.

Vice versa, according to (2.1), any $x \in \beta_s$ lies in $\delta\lambda_{ext} \wedge \delta\lambda_{int}$. But $\delta\lambda_{ext}$ contains both $\delta\beta_{s,ext}$ (by the definition of β) and $\delta\beta_{s,int}$ (by the definition of β and the fact that $\delta\beta_{s,int} \subset \beta_s$).

ii): Suppose β'_s is a different Birkhoff set in s. Note that to be invariant, β'_s must be contained in $\bigwedge_{i=0}^{\infty} f^i(s)$. Then β_s has points, say, in $\beta'_{s,int}$. Therefore $\beta_{s,ext}$ and $\beta'_{s,int}$ have at least one open set contained in s in their intersection. The union of such sets together with β_s and β'_s is an invariant separating set in s contained in $\bigwedge_{i=0}^{\infty} f^i(s)$. This contradicts the requirement that the measure of such sets be zero. □

We end this section with two remarks. The first one is that β or β_s are not necessarily attractors, as the example with the invariant curve, given earlier, shows. The second is that λ, as we defined it here, is in fact the basin boundary between two sinks (at $r = +\infty$ and $r = -\infty$) of f^{-1}.

III THE TWO DIMENSIONAL CASE

In this section, we describe some of the main features of the Birkhoff set in two dimensions. In particular, we relate it to the existence of a strange attractor. In this case, we can identify s and t, and β_s and β.

We now consider C^2 diffeomorphisms of the cylinder $S^1 \times R$ to itself. Projections to these coordinates are π_1, resp. π_2.

An example of a diffeomorphism that satisfies the requirements of section 2 is the dissipative $(0 < \lambda < 1\ ,\ k > 0)$ standard map f with lift:

$$\theta' = \theta + \omega + \lambda r - \frac{k}{2\pi} \sin(2\pi\theta)$$

$$r' = \lambda r - \frac{k}{2\pi} \sin(2\pi\theta)$$

It is easy to check that $S^1 \times [-M,M]$ is a trapping region iff $M > \frac{k}{2\pi(1-\lambda)}$.

Note that the determinant of Df equals λ.

Whenever necessary we will work with the lift F of f to R^2. In this case we will generally use capitals to avoid confusion. Note that F commutes with the unit translation R in the θ direction.

The following proposition results from joint work of Casdagli (see Casdagli (1988)) and the author. it is in fact a disclaimer: Not all chaotic behavior (here meaning transversal homoclinic points) is necessarily contained in β !

<u>Proposition 3.1</u> (Casdagli, Veerman): There exists a dissipative twist map, with a hyperbolic fixed point x, such that:

-the invariant manifolds intersect transversally, in a homotopically non-trivial way.

-x is not in β.

<u>Proof</u>: One starts with a conservative map of the cylinder having small nonlinearity and a hyperbolic fixed point, such as the above standard map for $\lambda = 1$. Theorems of Mather (1986) and Katok (1982) insure that there are heteroclinic orbits. If the intersections are not transversal, one can make them transversal by applying an arbitrarily (C^r-) small (area preserving) perturbation (Robinson, 1970). The nonlinearity, k, is small, so that by the KAM theorem there are many invariant curves left. Now, compose f with a contraction to r = c in the r direction. Here, c is chosen so that between r = c and r = x there is at least one invariant curve Γ in the area preserving case. The contraction can be chosen so small that existence and hyperbolicity of x and transversality of its invariant manifolds are conserved, and so that x does not cross Γ. Then the region between Γ and r = M (M sufficiently large) is a trapping region not containing x. □

The following proposition is one of the main results of this section. It is based on a lemma of Casdagli (1988) and joint work of Casdagli and the author.

<u>Theorem 3.2</u> **(Casdagli, Veerman): If there exists a strange attractor α, then**

i): $\beta \subseteq \alpha$

ii): If β is attracting, then $\alpha = \beta$

iii): If $\alpha \neq \beta$, then either $\alpha\backslash\beta \subset \beta_{int}$ or $\alpha\backslash\beta \subset \beta_{ext}$.

<u>Proof</u>: i): By definition α is invariant and compact and we only have to show that it is separating (see theorem 2.4). It is enough to prove that the lift A of α is connected (A commutes with the unit translation $R : (\theta,r) \to (\theta+1,r)$). In the following, we suppose that this is not the case.

By translation invariance, A falls apart in compact disjoint sets A_i. The index i will be used to express translation invariance. Since two closed sets must be a non-zero distance apart, there is a disjoint covering V_i of A_i such that $V_i = R^i V_0$. Then, by lemma 2.1, $\cup V_i$ contains an open disjoint subcovering $\cup U_i$ of A_i with:

$$U_i = R^i U_0$$

$$f(u) \subset u$$

(u is the projection of U_i on the cylinder). Furthermore, because each A_i is compact, U_i has at most finitely many components U_{ij}. It follows directly that there exist p, q, i, and j such that

$$F^q R^{-p}(U_{ij}) \subset U_{ij}. \tag{3.1}$$

By translation symmetry this holds for all i. Also, since α has a dense orbit, $f^l(u_j)$ for l in {0,...,q} runs through all components of u. Therefore, these components form a periodic "orbit" under f. So, relation (3.1) holds for all components, and all rotation numbers are equal to p/q, which contradicts assumptions on α.

ii): This follows from the definition of attracting and lemma 2.1. There is an open neighborhood n of β whose forward images converge to β. Since α is invariant ($f(\alpha)=\alpha$), $\alpha\backslash\beta$ is empty or is contained in the complement of n. An orbit that is dense in β cannot approximate $\alpha\backslash\beta$.

iii): This is proven by first observing that β_{int} and β_{ext} are invariant, and then, that no orbit can arbitrarily well approximate points on the two sides of β simultaneously. □

This proposition does not establish the existence of a strange attractor. This illustrates the point made in the introduction, that with a reasonably precise definition, existence of strange attractors is hard to establish. The easiest case seems to be when β is attracting. There one needs to establish the existence of a dense orbit in β. These questions are still open, as far as we know. (Note that λ is attracting by definition.)

<u>Corollary 3.3</u> (MacKay, personal communication): f has at most one strange attractor in t.

<u>Proof</u>: Attractors are disjoint and Birkhoff sets are unique in t (lemma 2.4).□

We now change to another main theme of this exposition, namely rotation numbers and well-ordered orbits. First, we define accessible points of β. These are the points of β that can be joined to $r = +/-M$ by a finite curve not intersecting β. The points that are accessible with a vertical line will be denoted by β_+ and β_-.

In addition to the requirements of section 2, we demand the following:

- f is uniformly twist: $\frac{\partial\theta'}{\partial r} > 0$ where $\theta' = \pi_1 f(\theta, r)$.

A curve γ is called 'of positive tilt' if its integrated angle with a vertical oriented upward is always positive (not necessarily smaller than 2π, see Le Calvez 1986b for more details). An example is any forward iterate of a vertical line under f. From the twist property of f one can prove the following statements.

<u>Lemma 3.4</u> (Le Calvez, 1986): Every point of β_{int} and every point of β accessible from below, is accessible with a curve of positive tilt starting from $r = -M$. Similarly for points accessible from above.

<u>Lemma 3.5</u> (Birkhoff, 1932): f^{-1} restricted to $\beta_{+/-}$ is a non-decreasing circle map, with at most countably many discontinuities. Discontinuities occur if $\rho(\beta_-) \neq \rho(\beta_+)$.

Here, $\rho(\beta_{-/+})$ is minus the rotation number of $\beta_{-/+}$ under f^{-1}, which, one knows by virtue of lemma 3.5, exists.

<u>Proposition 3.6</u> (Le Calvez, 1986): The Birkhoff set has well ordered orbits of all rotation numbers in the interval $[\rho(\beta_{-}), \rho(\beta_{+})]$.

These three statements seem to indicate that there is a parallel between the dynamics on the Birkhoff set and that on the circle under a non-invertible circle map. This parallel breaks down when one considers the following counter-example.

<u>Lemma 3.7</u> (Le Calvez, 1986): There is a continuous one-parameter family of maps f_{λ}, such that $\rho_{\lambda}(\beta_{-})$ has discontinuities.

It is not clear how often these discontinuities arise, since Le Calvez' proof is partly constructive and partly based on contradiction. It is worth noting, that under some condition (absence of invariant curves ?), ρ_{λ} might be continuous. These ideas have been pursued in Casdagli (1988).

IV BIRKHOFF SETS IN THREE DIMENSIONS

In this section, we study examples of Birkhoff sets in three dimensions. The fact that we have only one angular coordinate makes that properties of rotation numbers are somewhat easier to study than, say, in the case of two coupled dissipative standard maps.

We consider a two-dimensional twist map coupled to a third dimension, in such a way that we obtain a C^1 diffeomorphism (which could be a model for a four-dimensional dissipative flow). Rather than stating abstract requirements, we give an example:

$$f : \quad \begin{aligned} \theta' &= \left\{\theta + \omega + \lambda r - \frac{k}{2\pi} \sin(2\pi\theta) + \epsilon z\right\} \bmod 1 \\ r' &= \lambda r - \frac{k}{2\pi} \sin(2\pi\theta) \\ z' &= (1 - \epsilon)z + \epsilon(\lambda r - \frac{k}{2\pi} \sin(2\pi\theta)) \end{aligned}$$

The determinant of f now equals $\lambda(1 - \epsilon)$. By considering first the r-equation and then the z-equation, it is easy to see that, for any $M > k/2\pi(1 - \lambda)$, there is a trapping strip $s = S^1 \times [-M,M] \times R$ and a trapping region $t = S^1 \times [-M,M] \times [-M,M]$. The validity of the assumption that $\lambda \wedge t$ has measure zero will be proven in lemma 4.1.

Now, observe that although $s \backslash \beta_s$ consists of precisely two components, the same is not necessarily true for $t \backslash \beta$. The reason is that there might be tongues emanating from the large z region protruding into t. This situation is depicted in figure 4.1. This would not be possible if β_s were a graph for

$|z| > M$. We will assume that this is so (otherwise one can adapt the definitions).

<u>Lemma 4.1</u>: The Lebesgue measure of $\lambda \wedge t$, $\mu(\lambda \wedge t)$, is zero.

<u>Proof</u>: In the two dimensional case, this is obvious. However, here we do not necessarily have that $f(\lambda \wedge t) = \lambda \wedge t$.

Define the half-strips s_0^+ and s_0^-:

$$s_0^+ = \{ \vec{x} \mid |r| \le M,\ z > +M \},$$

$$s_0^- = \{ \vec{x} \mid |r| \le M,\ z < -M \},$$

and define for $i > 0$:

$$s_i^{+(-)} = \{ \vec{x} \in \beta \mid f^{-j}(\vec{x}) \in s_0^{+(-)} \text{ for all } j \ge i \text{ and not for } j < i \}$$

These regions are disjoint, because f is invertible. To prove that for each i, $\mu(s_i^{+(-)}) = 0$, just suppose that it is equal to some $\epsilon > 0$. Then, under backward iterates of f:

$$\lim_{n\to\infty} \frac{\mu\left\{f^{-(n+1)}(s_i^{+(-)})\right\}}{\mu\left\{f^{-n}(s_i^{+(-)})\right\}} = \frac{1}{\lambda(1 - \epsilon)} = |Df|^{-1} \quad . \tag{4.1}$$

On the other hand, since r remains bounded between -M and +M and z grows asymptotically at most as $1/(1 - \epsilon)$, one also has:

$$\lim_{n\to\infty} \frac{\mu\left\{f^{-(n+1)}(s_i^{+(-)})\right\}}{\mu\left\{f^{-n}(s_i^{+(-)})\right\}} = \frac{1}{1 - \epsilon} \quad . \tag{4.2}$$

Clearly, (4.1) and (4.2) are contradictory, unless $\mu(s_i^{+(-)}) = 0$. Now, β consists of a countable union of s_i's plus a set that never leaves t under f^{-1}. Therefore, $\mu(\lambda \wedge t) = 0$. □

If $\epsilon = 0$, the three dimensional space is foliated by invariant surfaces z = constant on the dynamics as discussed in the previous section. Suppose one projects to the z = 0 plane. In what sense does this projection resemble the ϵ = 0 case, if ϵ is positive but small ? The following proposition gives a clue.

<u>Proposition 4.2</u>: Let γ be a q periodic orbit in t for $\epsilon = 0$, which is hyperbolic in the two dimensional system (z = 0). There exist $\epsilon(\gamma) > 0$, such that for all $\epsilon < \epsilon(\gamma)$, there exists a q periodic orbit γ_ϵ whose projection (in the z = 0 plane) is close to γ.

<u>Proof</u>: Denote the projection to the z = 0 plane by π^3. Define

$$H(\epsilon,z;\theta,r) = \pi^3 f_\epsilon^q(\theta,r,z) - (\theta,r).$$

Then, $$H(0,z_0;\theta_0,r_0) = 0 \text{ , for all } z_0 \text{ in R.}$$

Let $\tilde{f}$ be the two dimensional projection of f when $\epsilon = 0$. Since $\tilde{f}^q$ has a hyperbolic fixed point at (θ_0,r_0), we can do a linear coordinate transformation that diagonalizes $D\tilde{f}$ Call the eigenvalues μ_1 and μ_2. Since the determinant is an invariant, one gets:

$$|\partial_{\theta,r} H(0,z_0;\theta_0,r_0)| = (\mu_1 - 1)(\mu_2 - 1),$$

which is not equal to zero. Then the implicit function theorem applies. For each z_0, there are open neighborhoods A of $(\epsilon=0, z=z_0)$ and B of $(\theta=\theta_0, r=r_0)$ such that for (ϵ, z) in A

$$H(\epsilon,z;\theta(\epsilon,z),r(\epsilon,z)) = 0$$

where $\qquad\qquad \theta(\epsilon,z),\ r(\epsilon,z)$

are C^1 functions and unique. But we know there is a solution for each pair $(0,z_0)$ and so, for constant ϵ there is a C^1 curve c of solutions as in figure 4.2. This curve can be parametrized by its z-coordinate. It is the locus of points in t which, under q iterates of f, don't change their θ- and r-coordinates.

Define $\Delta(z)$ as the z-coordinate of $f^q(\theta(c(z)),r(c(z)),z)$. Since we consider points in the region $S^1 \times [-M,M] \times R$, one can deduce from the defintion of f, that for large enough z, $\Delta(z) < z$. By the same token, for z negative with large enough modulus, one has $\Delta(z) > z$. So, somewhere in between $\Delta(z)$ equals z, which is the locus of the fixed point of f^q. $\square$

This proposition implies that all orbits, including the ones that are well ordered, up to a certain period, survive for ϵ small enough, and stay in roughly the same location, and well ordered.

In the following result we use the notation of the last proposition.

Theorem 4.3: If, for $\epsilon = 0$, there exists a strange attractor α, then for each hyperbolic periodic orbit in α there exists an $\eta(\gamma)$ such that for all $\epsilon < \eta(\gamma)$:

i) the projections of γ and γ_ϵ are close,

ii) if $\gamma \in \beta$, then $\gamma_\epsilon \in \beta$.

Proof: The first part of this theorem is just the previous proposition.

For the second part, observe that $\gamma_\epsilon \in \beta$ is equivalent to the statement that γ_ϵ has a stable branch extending into $r < -M$ and one into $r > M$.

Since α cannot contain sinks, we have that, at $\epsilon = 0$, one eigenvalue is smaller than 1, one is greater than 1 (hyperbolicity), and one equals 1. One then proves easily that the eigenvalues and eigenvectors of Df_ϵ^q are continuous at $\epsilon = 0$. Therefore, for ϵ small, the local invariant manifolds change continuously as a function of ϵ. The inverse iterates of the local stable manifolds are also continuous. So, if, for $\epsilon = 0$, stable branches are not contained in s, then for ϵ small enough, they are not bounded by $|r| = M$ either. □

Here starts the speculative part of this review. What we are really after, are answers to the questions: What sort of notions can be generalized from two to three dimensions, and what new phenomena come into existence ? The results seem to indicate that the projection of β changes very little as long as ϵ is small. One would expect, then, that the limit set $\omega(\beta)$ changes its appearance little for small ϵ. For $\epsilon = 0$, if $\beta = \alpha$, then $\beta = \omega(\beta)$. Does that mean that the strange attractor for small ϵ has the same appearance ? This similarity of appearance might break down either at high periods, to be seen by magnifying, or else at some finite value for ϵ. In the latter case one might expect some transition to more complicated behavior, from chaotic 'in two dimensions' to chaotic 'in three dimensions'. It is possible that for instance the Hausdorff dimension of $\omega(\beta)$ changes suddenly. In the former case, maybe more plausible, there might still be cross-over phenomena.

Whatever the case may be, we have to take into account, that we proved existence of at least one γ_ϵ for each hyperbolic γ_0 (for small enough ϵ). But in fact, it does not appear unlikely that there are many more, though typically finitely many. In this sense, behavior is somewhat more complicated than in the $\epsilon = 0$ case.

Is it possible to define a notion of vertical accessibility, with which vertically accessible points under f^{-1} are well ordered ? Then one could define a rotation interval $[\rho(\beta_-), \rho(\beta_+)]$. Are the boundaries of this interval always or almost always (see lemma 3.7) phase-locked ? Do there exist (possibly well ordered) orbits for all rotation numbers in that interval (as in proposition 3.6) ? It is almost clear that lemma 3.4 cannot be generalized. By writing down the partial derivatives of f, one notes that a vector $\vec{v}$ whose projection under π^3 has no component in the θ direction, does not necessarily have an image under Df whose projection makes a positive angle with the vertical. So, what remains of the advantages of twist ?

Finally, the question we started with, of course, is: What is the connection between α and β, or α and $\omega(\beta)$? Theorem 3.2 cannot be extended easily. The reasoning of its proof yields the following:

Proposition 4.4: If there is a strange attractor, it contains a connected, homotopically nontrivial set.

Proof: As theorem 3.2. $\square$

However, it is not clear, whether β (or $\omega(\beta)$) contains (or is) the unique smallest invariant connected homotopically nontrivial set. Still, we speculate that something like proposition 3.2 will hold with β replaced with $\omega(\beta)$. It is our also our suspicion, that for ϵ small enough, 'strangeness' can be defined as in section 2.

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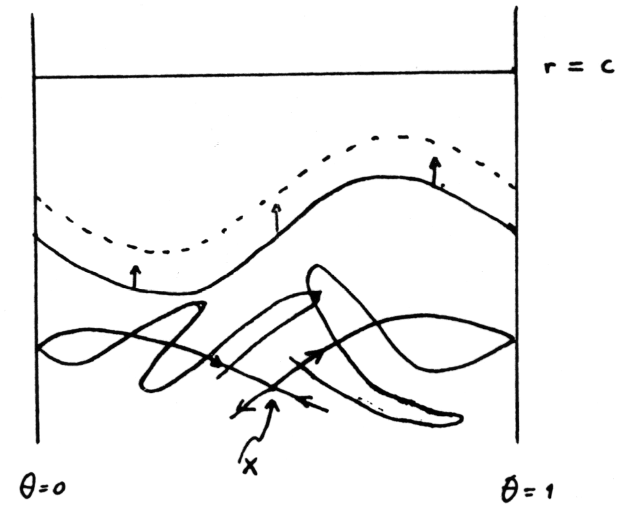

Figure 3.1: The invariant curve Γ of a conservative twist map becomes the boundary of a trapping region if dissipation is added.

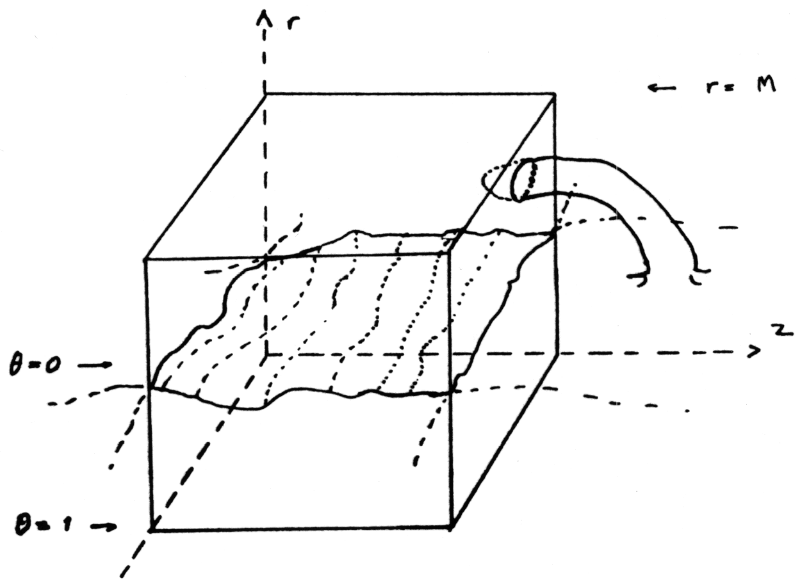

Figure 4.1: If a tongue emanates from the large z region into t, then it is possible that t\β separates t in three components.

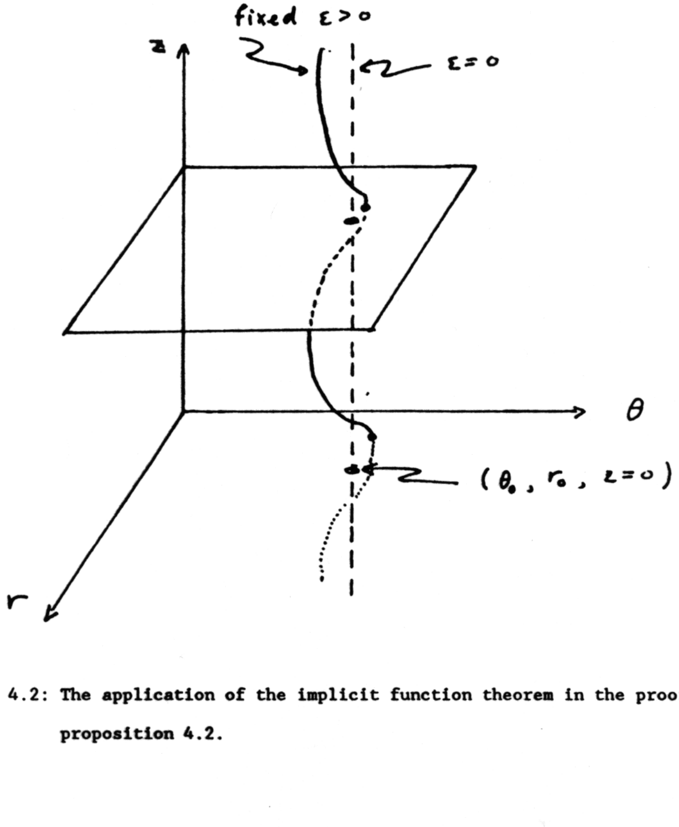

Figure 4.2: The application of the implicit function theorem in the proof of proposition 4.2.