A note on lattice chains and Delannoy numbers

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Abstract. Fix nonnegative integers $n_1, \ldots, n_d$ and let L denote the lattice of integer points $(a_1, \ldots, a_d) \in \mathbb{Z}^d$ satisfying $0 \le a_i \le n_i$ for $1 \le i \le d$. Let L be partially ordered by the usual dominance ordering. In this paper we offer combinatorial derivations of a number of results concerning chains in L. In particular, the results obtained are established without recourse to generating functions or recurrence relations. We begin with an elementary derivation of the number of chains in L of a given size, from which one can deduce the classical expression for the total number of chains in L. Then we derive a second, alternative, expression for the total number of chains in L when $d = 2$. Setting $n_1 = n_2$ in this expression yields a new proof of a result of Stanley [7] relating the total number of chains to the central Delannoy numbers. We also conjecture a generalization of Stanley's result to higher dimensions.

1 Introduction

Fix nonnegative integers $n_1, \ldots, n_d$ and let L denote the lattice of integer points $(a_1, \ldots, a_d) \in \mathbb{Z}^d$ satisfying $0 \le a_i \le n_i$ for $1 \le i \le d$. Partially order L by setting $(a_1, \ldots, a_d) \preceq (b_1, \ldots, b_d)$ whenever $a_i \le b_i$ for each i $(1 \le i \le d)$. In various contexts [2, 3, 5], the number of chains in L of a given size has been computed using either recurrence relations or generating functions. Summing this expression over all possible sizes, one obtains an expression for the total number of chains L. In the case when the dimension $d = 2$ and the lattice L is a square (so that n_1, n_2 share a common value n), an alternative expression for this quantity was given by Stanley [7]. In particular, he used generating functions to establish that the total number of chains in L equals $2^{n+1} d_n$, where d_n denotes the n^{th} Delannoy number. In [8], a bijective proof of Stanley's result is given. The bijection given there is the composition of five combinatorially defined bijections, perhaps a testament to its subtlety.

In this paper we begin with an elementary derivation of the number of chains in L of a given size using inclusion/exclusion. We then derive a formula for the total number of chains in L when $d = 2$. Setting $n_1 = n_2$ in this expression yields a new proof of Stanley's result. We conclude with a few remarks on the hypergeometric form of the expressions derived, and finally, we conjecture a generalization of Stanley's result to higher dimensions.

2 Lattice chains

Fix nonnegative integers $n_1, \ldots, n_d$. Let $L = L(n_1, \ldots, n_d)$ denote the lattice of integer points $(a_1, \ldots, a_d) \in \mathbb{Z}^d$ satisfying $0 \le a_i \le n_i$ for $1 \le i \le d$. Recall L is partially ordered by the dominance relation, defined as follows. Given $\mathbf{a}, \mathbf{b} \in L$ with $\mathbf{a} = (a_1, \ldots, a_d)$ and $\mathbf{b} = (b_1, \ldots, b_d)$, we say $\mathbf{a} \preceq \mathbf{b}$ whenever $a_i \le b_i$ for $1 \le i \le d$.

By a *chain* in L we mean a subset of L that is totally ordered by $\preceq$. A k-*chain* is a chain with k elements. Define $k_{max} = n_1 + \cdots + n_d + 1$ and observe that k_{max} is the maximum number of elements of a chain in L.

Let $C = C(n_1, \ldots n_d)$ denote the set of chains in L, and for each integer k, let $C_k = C_k(n_1, \ldots n_d)$ denote the set of k-chains in L. These sets have been studied in the contexts of subsets of multi-sets and partitions of a set [2, 3, 5]. In the next two sections we study expressions for $|C_k|$ and $|C|$.

2.1 Counting k-chains

One obvious way to obtain an expression for $|C|$ is to sum $|C_k|$ over all k. This requires us to first find an expression for $|C_k|$. And indeed, a simple expression for $|C_k|$ is not difficult to derive, and has been computed in several places [3, 5] for the special case $n_i = 1$ for all i, and, in [2], for the general case. Each of these derivations proceeds either by solving an appropriate recurrence or through the use of generating functions. In this section we offer a direct counting argument for $|C_k|$ using the principle of inclusion/exclusion.

Theorem 1 *[2] Fix $n_1, \ldots, n_d \in \mathbb{Z}^{\geq 0}$ and set $k_{max} = 1 + \sum_1^d n_i$. Then for any integer k $(0 \leq k \leq k_{max})$,*

$$|C_k(n_1, \ldots, n_d)| = \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} \prod_{i=1}^{d} \binom{n_i + k - r}{n_i}.$$

□

Our proof begins with Lemma 1, which counts the number of sequences $\langle \mathbf{a}_1, \ldots, \mathbf{a}_k \rangle$ in L satisfying $\mathbf{a}_1 \preceq \cdots \preceq \mathbf{a}_k$. Since such sequences allow duplicate entries, while chains do not, Lemma 1 does not directly compute $|C_k|$.

Lemma 1 *With the notation of Theorem 1, fix any integer k $(0 \leq k \leq k_{max})$. Let S_k denote the set of all sequences $\langle \mathbf{a}_1, \ldots, \mathbf{a}_k \rangle$ in L satisfying $\mathbf{a}_1 \preceq \cdots \preceq \mathbf{a}_k$. Then*

$$|S_k| = \prod_{i=1}^{d} \binom{n_i + k}{n_i}. \tag{1}$$

Proof. Consider a sequence $\langle \mathbf{a}_1, \ldots, \mathbf{a}_k \rangle$ in L, where $\mathbf{a}_j = (a_{j1}, \ldots, a_{jd})$ for $(1 \leq j \leq k)$. This sequence belongs to S_k if and only if for each i $(1 \leq i \leq d)$

$$0 \leq a_{1i} \leq \cdots \leq a_{ki} \leq n_i. \tag{2}$$

The number of integer sequences $a_{1i}, \ldots, a_{ki}$ satisfying (2) is given by $\binom{n_i+k}{n_i}$. Multiplying these factors together as i ranges from 1 to d, we obtain the result. □

As discussed above, S_k includes sequences with repeated elements. So we will apply inclusion/exclusion to obtain $|C_k|$. The next lemma considers the sets to be excluded.

Lemma 2 *With the notation of Theorem 1, fix any integer k $(0 \leq k \leq k_{max})$, and let S_k be as in Lemma 1. For each $1 \leq i \leq k-1$, let*

$$S_k(i) = \{\langle \mathbf{a}_1, \ldots, \mathbf{a}_k \rangle \in S_k \,|\, \mathbf{a}_i = \mathbf{a}_{i+1}\}.$$

Then for any integers $i_i, \ldots, i_r$ such that $1 \leq i_i < \cdots < i_r \leq k-1$, we have

$$|S_k(i_1) \cap S_k(i_2) \cdots \cap S_k(i_r)| = \prod_{i=1}^{d} \binom{n_i + k - r}{n_i}.$$

Proof. Fix integers $i_i, \ldots, i_r$ such that $1 \leq i_i < \cdots < i_r \leq k-1$. Each sequence $\mathbf{a} \in S_k(i_1) \cap S_k(i_2) \cdots \cap S_k(i_r)$ satisfies $\mathbf{a}_{i_j} = \mathbf{a}_{i_j+1}$ for $1 \leq j \leq r$. Such a sequence corresponds naturally to a sequence in S_{k-r} by deleting the r terms $\mathbf{a}_{i_j}$ for $1 \leq j \leq r$. Replacing k by $k-r$ in Lemma 1 counts $|S_{k-r}|$. The result follows. □

We now prove the theorem.

Proof of Theorem 1. Observe $|C_k| = \left|S_k \setminus \bigcup_{i=1}^{k-1} S_k(i)\right|$. By the principle of inclusion/exclusion,

$$\begin{aligned} |C_k| &= \sum_{r=0}^{k-1} (-1)^r \sum_{i_1<\cdots<i_r} |S_k(i_1) \cap \cdots \cap S_k(i_r)| \\ &= \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} \prod_{i=1}^{d} \binom{n_i+k-r}{n_i}. \qquad \square \end{aligned}$$

2.2 Counting the total number of chains

In this section we compute $|C|$, the total number of chains in L. One expression is easily obtained using the result of the previous section. Indeed, recall that a chain in L has at most $k_{max} = n_1 + \cdots + n_d + 1$ elements. It follows that

$$|C(n_1, \ldots, n_d)| = \sum_{k=0}^{k_{max}} |C_k(n_1, \ldots, n_d)|. \tag{3}$$

In the special case when $d = 2$ and the lattice L is a square (so that $n_1 = n_2$), Stanley [7, Section 6.3] used generating functions to find an alternative expression for this quantity, which we will obtain as Corollary 4 below. In this section, however, we begin with a combinatorial derivation of a slight generalization of Stanley's result; in particular, we count the total number of chains for the case $d = 2$ without the assumption that $n_1 = n_2$. Central to our proof is the notion of a y-*strict* chain: a chain in which no two elements have the same y-coordinate (ie., 2nd coordinate). We obtain the following theorem.

Theorem 2 *Let n_1 and n_2 be nonnegative integers. Then*

$$|C(n_1, n_2)| = 2^{n_1+1} \sum_{i=0}^{n_2} \binom{n_1+i}{i} \binom{n_2}{i}. \tag{4}$$

Proof. To each chain ξ in $L(n_1, n_2)$ we associate a pair (A_ξ, ξ') where $A_\xi \subseteq \{0, 1, \ldots, n_1\}$ and ξ' is a y-strict chain in $L(n_1, n_2 - 1)$. If ξ denotes the chain $(x_1, y_1) \preceq (x_2, y_2) \preceq \cdots \preceq (x_k, y_k)$, then we define $A_\xi = \{x_i | y_i = y_{i+1} \text{ or } y_i = n_2\}$ and $\xi' = \xi \backslash \{(x_i, y_i) | x_i \in A_\xi\}$. See Figure 1.

This is a bijective correspondence, and we now exhibit the inverse map which associates a chain in $L(n_1, n_2)$ with each pair (A, ξ') consisting of a subset $A \subseteq \{0, 1, \ldots, n_1\}$ and a y-strict chain ξ' in $L(n_1, n_2-1)$. Let (A, ξ') be such a pair. For each i $(0 \le i \le n_1)$ let $\text{top}_{\xi'}(i)$ denote the maximum y such that $\xi' \cup \{(i, y)\}$ is a chain in $L(n_1, n_2)$. Then the chain associated with the pair (A, ξ') is the chain $\xi = \xi' \cup \{(i, \text{top}_{\xi'}(i)) | i \in A\}$.

Given this correspondence, it remains only to count the number of pairs (A, ξ'). Let ξ' be an i-element y-strict chain in $L(n_1, n_2 - 1)$. Then there are $\binom{n_2}{i}$ choices for the y-coordinates that appear in ξ'. Once the y-coordinates have been chosen, we may choose the x-coordinates freely, as long as the resulting choice maintains the chain condition for ξ'. That is, we must choose i x-coordinates such that $0 \le x_1 \le x_2 \le \cdots \le x_i \le n_1$. The number of such choices for the x-coordinates is given by $\binom{n_1+i}{i}$. Thus, we have

$$\sum_{i=0}^{n_2} \binom{n_1+i}{i} \binom{n_2}{i}$$

y-strict chains in $L(n_1, n_2 - 1)$. For each such chain ξ', we have 2^{n_1+1} choices for a subset A. It follows that the number of pairs (A, ξ') is given by the right side of (4). $\square$

Remark. A recursive proof of (4), albeit somewhat less illuminating, can also be given as follows. A simple inclusion/exclusion argument shows that for positive integers n_1 and n_2

$$|C(n_1, n_2)| = 2|C(n_1, n_2 - 1)| + 2|C(n_1 - 1, n_2)| - 2|C(n_1 - 1, n_2 - 1)|. \tag{5}$$

It is readily verified that the expression on the right side of (4) also satisfies this recurrence, along with appropriate initial conditions.

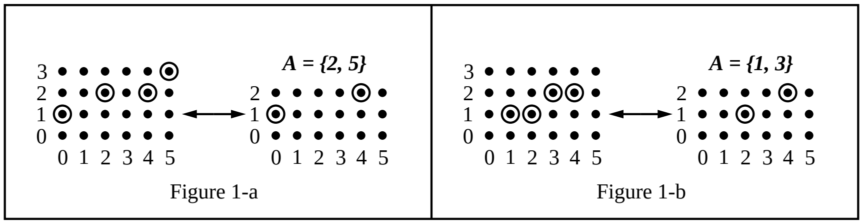

Figure 1: Mapping Chains

We close this section with a few comments on symmetry. Observe that although the quantity $|C(n_1, n_2)|$ is, by its definition, symmetric in n_1 and n_2, the expression in (4) is less obviously so. It is perhaps also worth noting, then, that our expression for $|C(n_1, n_2)|$ is quite conveniently stated using the notation of hypergeometric series. Recall that for any complex number a and any natural number n, we define $(a)_n := (a)(a+1)\cdots(a+n-1)$. Using this notation, the ${}_2F_1$ hypergeometric series is defined as follows:

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} $$

for complex a, b, c, z with $c \neq 0$. There are many identities involving such series. For example, one of the so-called Euler transformations (see [6, p.33]) gives that

$$ {}_2F_1(a, b; c; 1/2) = 2^a\, {}_2F_1(a, c-b; c; -1). \tag{6} $$

By Theorem 2, we see that

$$ C(n_1, n_2) = 2^{n_2+1}\, {}_2F_1(-n_1, n_2+1; 1; -1). $$

Applying (6), we obtain the more symmetric expression

$$ C(n_1, n_2) = 2^{n_1+n_2+1}\, {}_2F_1(-n_1, -n_2; 1; 1/2). $$

3 Delannoy numbers

The Delannoy numbers count the number of lattice paths in $L(n_1, n_2)$ from $(0,0)$ to (n_1, n_2) in which only vertical $\mathfrak{v} = (0,1)$, horizontal $\mathfrak{h} = (1,0)$, and diagonal $\mathfrak{d} = (1,1)$ steps are allowed. Such a path is sometimes referred to as a (restricted) king's walk. When n_1 and n_2 share a common value n, we refer to $d_n = D(n, n)$ as the *central Delannoy numbers*. For arbitrary dimension d, we set $D(n_1, ..., n_d)$ equal to the number of lattice paths in $L(n_1, ..., n_d)$ that begin from the origin and terminate at $(n_1, ..., n_d)$, in which only positive steps from the d-dimensional unit hypercube are allowed. This follows [4]. Indeed, for more about generalizations of the Delannoy numbers, we refer the reader to [4, 1].

Although the Delannoy numbers are typically derived recursively or with generating functions, we can count the number of restricted king's walks as follows. Observe that $D(n_1, n_2) = D(n_2, n_1)$ so we may assume without loss of generality that $n_1 \leq n_2$.

Theorem 3 *Let n_1, n_2 be nonnegative integers such that $n_1 \leq n_2$. Then*

$$ D(n_1, n_2) = \sum_{i=0}^{n_1} \binom{n_2+i}{i} \binom{n_2}{n_1-i}. \tag{7} $$

Proof. A walk is a sequence of $\mathfrak{h}, \mathfrak{v}$, and $\mathfrak{d}$ steps. To reach (n_1, n_2), the number of $\mathfrak{h}, \mathfrak{d}$ steps must sum to n_1 and the number of $\mathfrak{v}, \mathfrak{d}$ steps must sum to n_2. Let i denote the number of diagonal $\mathfrak{d}$ steps in a given walk. Clearly, $0 \leq i \leq n_1$. Also, the total number of steps of such a walk is $n_1 + n_2 - i$. The number of $\mathfrak{h}$

steps is given by $n_1 - i$. The number of $\mathfrak{v}$ steps is $n_2 - i$. So the total number of such walks is given by the trinomial coefficient $\binom{n_1+n_2-i}{i,n_1-i,n_2-i} = \binom{n_1+n_2-i}{n_1-i}\binom{n_2}{i}$. Thus, the total number of walks is

$$\sum_{i=0}^{n_1} \binom{n_1 + n_2 - i}{n_1 - i}\binom{n_2}{i}.$$

Reindexing the sum (replacing i by $n_1 - i$) we obtain the desired result. □

Comparing lines (4) and (7), we have the following.

Corollary 4 *[7, 8] For any nonnegative integer n, $C(n,n) = 2^{n+1}D(n,n)$.* □

Remark. We note that the expression in (7) can also be established recursively. Indeed, a simple inclusion/exclusion argument shows that

$$D(n_1, n_2) = D(n_1, n_2 - 1) + D(n_1 - 1, n_2) + D(n_1 - 1, n_2 - 1).$$

It is then readily verified that the expression on the right side of (7) also satisfies this recurrence, along with appropriate initial conditions.

We conclude this paper with a conjecture, analogous to Corollary 4, that appears to be supported by numerical evidence.

Conjecture 1 *If $n_1 = n_2 = \cdots = n_d$, then*

$$C(n_1, n_2, \ldots, n_d) = 2^{n_1+1}D(n_1, n_2, \ldots, n_d).$$

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