ON A CONVEX SET WITH NONDIFFERENTIABLE METRIC PROJECTION

Shyan S. Akmal[1], Nguyen Mau Nam[2], and J. J. P. Veerman [3]

Abstract. A remarkable example of a nonempty closed convex set in the Euclidean plane for which the directional derivative of the metric projection mapping fails to exist was constructed by A. Shapiro. In this paper, we revisit and modify that construction to obtain a convex set with $C^{1,1}$ *boundary* which possesses the same property.

Key words. metric projection; directional derivative

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1 A Convex Set with Smooth Boundary

Define a strictly decreasing sequence of real numbers $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0, \pi/2]$ with

$$\lim_{n\to\infty} \alpha_n = 0 \text{ and } \alpha_{n+1} \le \frac{\alpha_n + \alpha_{n+2}}{2} \text{ for all } n \in \mathbb{N}. \tag{1.1}$$

Now we identify $\mathbb{R}^2$ equipped with the Euclidean norm $\|\cdot\|$ with $\mathbb{C}$ and let $A_n = e^{i\alpha_n}$. A beautiful and surprisingly simple example of a nonempty closed convex set for which the directional derivative of the metric projection mapping fails to exist was constructed by A. Shapiro in [13]. This set is essentially the convex hull J of the collection of points 0, 1, and $\{A_n\}_{n\in\mathbb{N}}$. Note that this set does not have smooth boundary. More positive and negative results on the existence of directional derivatives to the metric projection mapping as well as applications to optimization can be found in [1, 3, 6, 9, 10, 12, 13, 14] and the references therein.

To define a convex set with smooth boundary, we start by choosing $\alpha_1 = \pi/2$ and proceeding as before to obtain the set J. The strategy to obtain a convex set K with smooth boundary is to replace the pointy parts of this figure by circular arcs; see Figure 1. Let T_n be the midpoint of the line segment A_nA_{n+1} and let S_n the point in the line segment $A_{n-1}A_n$ so that

$$\|A_n - S_n\| = \|A_n - T_n\| = \sin\left(\frac{\alpha_n - \alpha_{n+1}}{2}\right). \tag{1.2}$$

Replace the two line segments T_nA_n and A_nS_n by a circular arc C_n tangent to both segments. Let O_n be the center of the circle that contains C_n as an arc and let r_n denote the radius of the circle. Let J_1 be the convex hull of the points 0, 1, the circular arcs $\{C_n\}_{n\in\mathbb{N}}$, and the

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line segments connecting them. Let J_2 be the image of J_1 under reflection in the real axis and let J_3 be the reflection of $J_1 \cup J_2$ in the imaginary axis. Then we define $K := J_1 \cup J_2 \cup J_3$. The set obtained has *smooth boundary* in the sense we will define shortly.

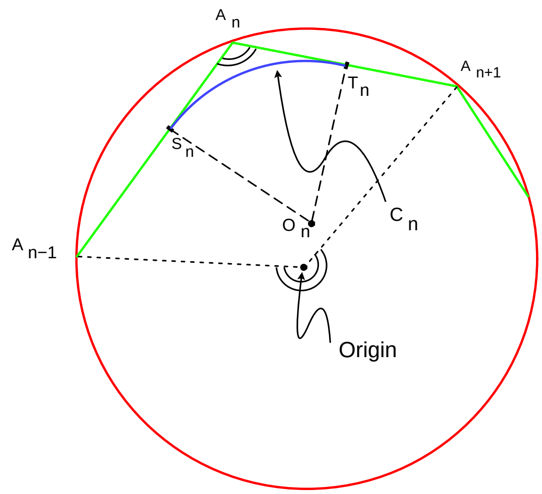

Figure 1: The construction of a convex set with smooth boundary.

Lemma 1.1. *For any sequence $\{\alpha_n\}$ that defines the set K, one has*

$$\lim_{n\to\infty} \left| r_n - 2\,\frac{\alpha_n - \alpha_{n+1}}{\alpha_{n-1} - \alpha_{n+1}} \right| = 0.$$

Proof: Consider the angle ψ_n at A_n and the angle ϕ_n and the origin as indicated by the double arcs in Figure 1. From our definition of α_n, we see that $\phi_n = 2\pi - (\alpha_{n-1} - \alpha_{n+1})$. By the Inscribed Angle Theorem, we have $\psi_n = \frac{1}{2}\phi_n$. Thus,

$$\psi_n = \pi - \frac{1}{2}(\alpha_{n-1} - \alpha_{n+1}). \tag{1.3}$$

The figure $A_n S_n O_n T_n$ is a *right kite* with right angles at S_n and at T_n. Therefore,

$$\frac{\|O_n - T_n\|}{\|A_n - T_n\|} = \tan\left(\frac{\psi_n}{2}\right) = \left[\tan\left(\frac{\pi - \psi_n}{2}\right)\right]^{-1}. \tag{1.4}$$

Using (1.2), (1.3), and (1.4) in the relation

$$r_n = \frac{\|O_n - T_n\|}{\|A_n - T_n\|}\,\|A_n - T_n\|,$$

we see that

$$r_n = \frac{\sin(\frac{1}{2}(\alpha_n - \alpha_{n+1}))}{\tan(\frac{1}{4}(\alpha_{n-1} - \alpha_{n+1}))}. \tag{1.5}$$

The result then follows easily. $\square$

Recall that a subset Ω of $\mathbb{R}^m$ is called convex if

$$\alpha x + (1-\alpha)y \in \Omega \text{ whenever } x, y \in \Omega \text{ and } \alpha \in (0,1).$$

A function $f : \Omega \to \mathbb{R}$ is called convex if

$$f(\alpha x + (1-\alpha)y)) \leq \alpha f(x) + (1-\alpha) f(y) \text{ for all } x, y \in \Omega \text{ and } \alpha \in (0,1).$$

If $-f$ is convex, then f is called concave.

The lemma below will be important in what follows.

Corollary 1.2. *Consider the sequence $\{\alpha_n\}$ given by $\alpha_n = C\lambda^n$, where $C > 0$ and $\lambda \in (0,1)$. Then $\{\alpha_n\}$ is strictly decreasing and satisfies the conditions in* (1.1)*. Moreover,*

$$\lim_{n\to\infty} r_n = \frac{2\lambda}{1+\lambda}.$$

Proof: It is obvious that $\{\alpha_n\}$ is strictly decreasing. Since the function $g(x) := C\lambda^x$ is convex on $\mathbb{R}$,

$$g(n+1) \leq \frac{g(n) + g(n+2)}{2},$$

which implies that $\alpha_{n+1} \leq \dfrac{\alpha_n + \alpha_{n+2}}{2}$.

By Lemma 1.1, we also see that

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} 2\,\frac{\alpha_n - \alpha_{n+1}}{\alpha_{n-1} - \alpha_{n+1}} = 2\lim_{n\to\infty} \frac{C\lambda^n - C\lambda^{n+1}}{C\lambda^{n-1} - C\lambda^{n+1}} = \frac{2\lambda}{1+\lambda},$$

which completes the proof. □

In the theorem below, we show that the set K obtained using a strictly decreasing sequence of real numbers $\{\alpha_n\} \subset (0, \pi/2]$ that satisfies condition (1.1) has smooth boundary. Moreover, we can choose $\{\alpha_n\}$ such that the boundary of K is $C^{1,1}$.

Theorem 1.3. *Let $x : [-1,1] \to \mathbb{R}$ be the function whose graph is the intersection of the boundary ∂K of K with the half plane $\{(x,y) \in \mathbb{R}^2 \mid x \geq 0\}$. Then $x(\cdot)$ has continuous derivatives. In particular, if the set K is obtained using the sequence $\{\alpha_n\}$ defined in Corollary 1.2, then the derivative x' is Lipschitz on $[-1,1]$.*

Proof: We first prove that x' exists and is continuous at $y = 0$. We use standard (x,y) coordinates (for real and imaginary parts). Observe that $x(0) = 1$. The concavity of $x(\cdot)$ implies that for $y > 0$ the slopes

$$s(y) := \frac{x(y) - x(0)}{y}$$

have the property: $s(y_2) \geq s(y_1)$ if $y_2 \leq y_1$. To calculate the limit of $s(y)$ as $y \to 0^+$, it is sufficient to choose a sequence $y_n \searrow 0$ and consider the limit

$$s(0+) := \lim_{n\to\infty} \frac{x(y_n) - x(0)}{y_n}.$$

The same calculation for negative y will result in the limit $s(0-)$. To conclude that x is differentiable at 0, we show that $s(0+)$ and $s(0-)$ both exist and equal 0. Note that $s(0-) = -s(0+)$.

Here is the calculation that establishes that $s(0+) = 0$. Recall that

$$T_n = \frac{e^{i\alpha_n} + e^{i\alpha_{n+1}}}{2} = \cos\left(\frac{\alpha_n - \alpha_{n+1}}{2}\right) e^{i\frac{\alpha_n + \alpha_{n+1}}{2}}$$

We now set

$$y_n := \mathrm{Im}\,(T_n) \quad \text{and} \quad x(y_n) := \mathrm{Re}\,(T_n)$$

and evaluate

$$\lim_{n\to\infty} \frac{x(y_n) - x(0)}{y_n} = \lim_{n\to\infty} \frac{\cos\left(\frac{\alpha_n - \alpha_{n+1}}{2}\right)\cos\left(\frac{\alpha_n + \alpha_{n+1}}{2}\right) - 1}{\cos\left(\frac{\alpha_n - \alpha_{n+1}}{2}\right)\sin\left(\frac{\alpha_n + \alpha_{n+1}}{2}\right)} = 0.$$

Thus, $x(\cdot)$ is differentiable at $y = 0$ and $x'(0) = 0$. It follows that $x(\cdot)$ is differentiable on $[-1, 1]$, and x' is continuous away from the point $y = 0$.

By the monotonicity of x' on $[-1, 1]$, the continuity of the derivative can be established by a similar argument. It is sufficient to show that $x'(y_n)$ tends to zero as n tends to infinity. We have

$$x'(y_n) = \frac{-\sin\left(\frac{\alpha_n + \alpha_{n+1}}{2}\right)}{\cos\left(\frac{\alpha_n + \alpha_{n+1}}{2}\right)}.$$

Again the limit is zero which proves the continuity of the derivative.

Now consider the set K obtained using the sequence $\{\alpha_n\}$ defined in Corollary 1.2. We see that in this case the sequence $\{r_n\}$ is bounded. The curve ∂K is given by a linear function in the flat pieces which gives $x''(y) = 0$, or by $x = x(y)$ where the second derivative of $x(\cdot)$ (except at the joints of the construction) is related to the curvature $1/r_n$ by

$$\frac{1}{r_n} = \frac{x''}{(1 + x'^2)^{3/2}}.$$

We need to prove that $x'(\cdot)$ is Lipschitz on $[-1, 1]$. As noted above, in these cases $x''(\cdot)$ exists (except at the joints) and is uniformly bounded on $[-1, 1]$ by the facts that $\{r_n\}$ is bounded and $x'(\cdot)$ is continuous on $[-1, 1]$. Thus, it is well-known that x' is absolutely continuous on $[-1, 1]$; see, e.g., [7, Exercise 3.23, p.p.82]. By Lebesgue's Theorem ([5, Theorem 6, Section 33]), we have

$$x'(y_2) - x'(y_1) = \int_{y_1}^{y_2} x''(s)\, ds,$$

where $y_1, y_2 \in [-1, 1]$. By the bounded property of $x''(\cdot)$, the function $x'(\cdot)$ is Lipschitz on $[-1, 1]$. □

Remark 1.4. Since $x'(\cdot)$ is decreasing, $x(\cdot)$ is a concave function on $[-1,1]$. Equivalently, $-x(\cdot)$ is a convex on $[-1,1]$, and hence $x(\cdot)$ is locally Lipschitz on $(-1,1)$. Thus, we can apply [2, Corollary 2.2.4, p.p.33] to obtain the continuity of $x'(\cdot)$ on $(-1,1)$, and hence on $[-1,1]$, from its differentiability on this interval. However, we give a direct proof as above for the convenience of the reader.

Definition 1.5. *For the set K obtained using the sequence $\{\alpha_n\}$ defined in Corollary 1.2 with the properties specified in Theorem 1.3, we say that the ∂K is $C^{1,1}$ around $(1,0)$.*

2 The Metric Projection

Given a nonempty closed convex set $\Omega \subset \mathbb{R}^m$, the metric projection from a given point $x_0 \in \mathbb{R}^m$ to Ω is defined by

$$\Pi(x_0;\Omega) := \{w \in \Omega \mid d(x_0;\Omega) = \|x_0 - w\|\},$$

where $d(x_0;\Omega) := \inf\{\|x_0 - w\| \mid w \in \Omega\}$. It is well-known that $\Pi(x_0;\Omega) \in \Omega$ is always a singleton. Moreover, the mapping $\Pi(\cdot;\Omega)$ is nonexpansive in the sense that

$$\|\Pi(x;\Omega) - \Pi(y;\Omega)\| \le \|x - y\| \text{ for all } x, y \in \mathbb{R}^m.$$

The readers are referred to [4, 8, 11] for more details on the metric projection mapping.

In what follows, we consider the metric projection mapping $\Pi(\cdot;K)$, where the set K is defined in the previous section. We omit K in $\Pi(\cdot;K)$ if no confusion occurs.

The *directional derivative* of the metric projection mapping at $x_0 \notin \Omega$ in the direction v is given by

$$D_v\Pi(x_0) := \lim_{t\to 0+} \frac{\Pi(x_0 + tv) - \Pi(x_0)}{t}.$$

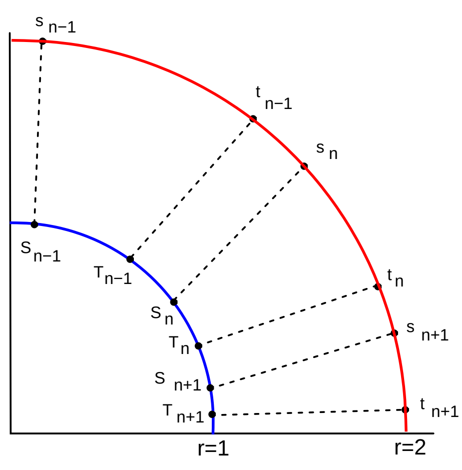

Figure 2: *The construction of the projection of the convex set K.*

Now consider the parametrization of the circle $\mathcal{C}$ centered at the origin with radius 2: $x(\theta) = 2e^{i\theta/2}$.

Lemma 2.1. *The directional derivative of* Π *at* $x(0)$ *in the direction* $v := x'(0)$ *exists if and only if the limit*

$$\lim_{\theta \to 0^+} \frac{\Pi(x(\theta)) - \Pi(x(0))}{\theta - 0}$$

exists.

Proof: By the nonexpansive property of the metric projection mapping, the following holds for any $\theta > 0$:

$$\begin{aligned} \left\| \frac{\Pi(x(\theta)) - \Pi(x(0))}{\theta - 0} - \frac{\Pi(x(0) + \theta v) - \Pi(x(0))}{\theta - 0} \right\| &= \left\| \frac{\Pi(x(\theta)) - \Pi(x(0) + \theta v)}{\theta - 0} \right\| \\ &\leq \left\| \frac{x(\theta) - x(0) - \theta v}{\theta - 0} \right\| \\ &= \left\| \frac{x(\theta) - x(0)}{\theta - 0} - v \right\|. \end{aligned}$$

Since $\lim_{\theta \to 0^+} \left\| \frac{x(\theta) - x(0)}{\theta - 0} - v \right\| = 0$, the conclusion follows easily. □

By Lemma 2.1, the directional derivative of the metric projection mapping at $(2, 0)$ in the direction of the unit vector i exists if and only if $\frac{d}{d\theta}\Pi(x(\theta))|_{\theta=0}$ exists.

To better understand the metric projection mapping from the circle $\mathcal{C}$ onto K, we define two points $2e^{it_n/2}$ and $2e^{is_n/2}$ such that

$$\Pi(2e^{it_n/2}) = T_n \quad \text{and} \quad \Pi(2e^{is_n/2}) = S_n,$$

where T_n and S_n are defined as before. The situation is depicted in Figure 2.

Lemma 2.2. *For any sequence* $\{\alpha_n\}$ *that defines the convex set* K*, we have*

$$\lim_{n \to \infty} \frac{\Pi(2e^{it_{n-1}/2}) - \Pi(2e^{is_n/2})}{t_{n-1} - s_n} = i. \tag{2.1}$$

Proof: Let

$$z_n := \frac{\Pi(2e^{it_{n-1}/2}) - \Pi(2e^{is_n/2})}{t_{n-1} - s_n}.$$

It suffices to show that

$$\|z_n\| \to 1 \text{ and } \arg z_n \to \pi/2 \text{ as } n \to \infty.$$

For the magnitude, let P_n denote point $2e^{is_n/2}$ and consider the orthogonal projection H_n of P_n onto the radii connecting the origin and $2e^{it_{n-1}/2}$ as seen in Figure 3. Obviously, $P_n H_n \Pi(2e^{it_{n-1}/2})\Pi(2e^{is_n/2})$ forms a rectangle. Opposite side lengths are equal, so

$$\|\Pi(2e^{it_{n-1}/2}) - \Pi(2e^{is_n/2})\| = \|P_n - H_n\|.$$

Considering the radii connecting the origin to the points $2e^{it_{n-1}/2}$ and $2e^{is_n/2}$ which mark

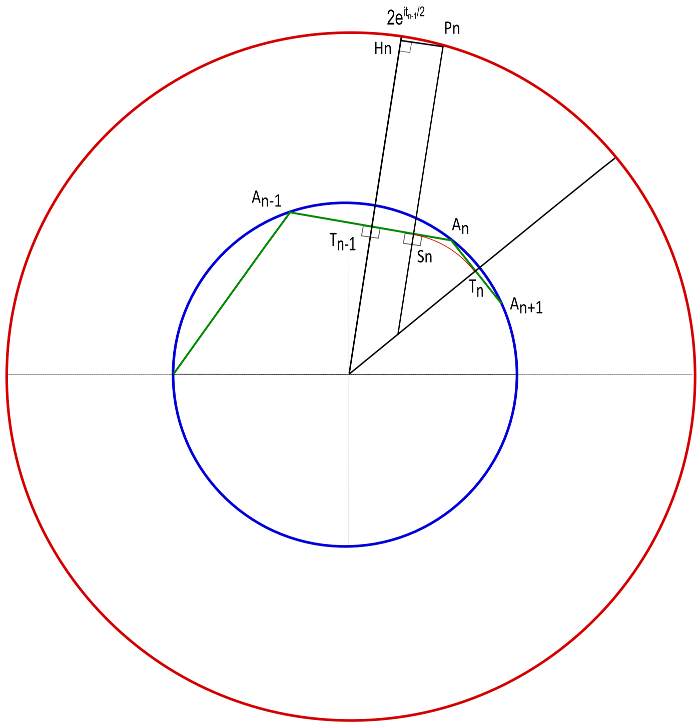

Figure 3: An illustration for the proof of Lemma 2.2.

off the angle $(t_{n-1} - s_n)/2$, we see that

$$\|\Pi(2e^{it_{n-1}/2}) - \Pi(2e^{is_n/2})\| = \|P_n - H_n\| = 2\sin\left(\frac{t_{n-1} - s_n}{2}\right).$$

By the fundamental sine identity,

$$\lim_{n\to\infty} \left\| \frac{\Pi(2e^{it_{n-1}/2}) - \Pi(2e^{is_n/2})}{t_{n-1} - s_n} \right\| = 1.$$

To show that the argument tends to $\frac{\pi}{2}$, observe that since T_{n-1} is the midpoint of $A_{n-1}A_n$, the line segment $A_{n-1}A_n$ is perpendicular to the line through T_{n-1} and the origin. Since $\Pi(2e^{it_{n-1}/2})$ and $\Pi(2e^{is_n/2})$ are on the line segment $A_{n-1}A_n$ by definition, we get that

$$\arg\left(\frac{\Pi(2e^{it_{n-1}/2}) - \Pi(2e^{is_n/2})}{t_{n-1} - s_n}\right) = \frac{\pi}{2} + \arg\left(2e^{it_{n-1}/2}\right) = \frac{\pi + t_{n-1}}{2}.$$

Observe that $2e^{it_{n-1}/2}, T_{n-1}$, and the origin are collinear (as in Figure 1), we have $t_{n-1} = \alpha_{n-1} + \alpha_n$. Thus,

$$\lim_{n\to\infty} \arg\left(\frac{\Pi(2e^{it_{n-1}/2}) - \Pi(2e^{is_n/2})}{t_{n-1} - s_n}\right) = \lim_{n\to\infty} \frac{\pi + \alpha_{n-1} + \alpha_n}{2} = \frac{\pi}{2}.$$

We have shown that the limit in (2.1) is i as desired. $\square$

Throughout the next few lemmas, we use $f(n) \sim g(n)$ to denote $\lim_{n\to\infty} f(n)/g(n) = 1$.

Lemma 2.3. *If positive functions $f(n), g(n), h(n)$ satisfy $g(n) \sim h(n)$ and there exists a constant $c > 0$ such that $\left|\frac{f(n)}{h(n)} - 1\right| \geq c$ for all sufficiently large n, then $f(n) - g(n) \sim f(n) - h(n)$.*

Proof: For all sufficiently large n, one has

$$\left|\frac{f(n) - g(n)}{f(n) - h(n)} - 1\right| = \left|\frac{g(n) - h(n)}{f(n) - h(n)}\right| = \left|\frac{\frac{g(n)}{h(n)} - 1}{\frac{f(n)}{h(n)} - 1}\right| \leq \frac{1}{c}\left|\frac{g(n)}{h(n)} - 1\right|.$$

Then the conclusion follows easily. □

Lemma 2.4. *For any sequence $\{\alpha_n\}$ that defines the set K, consider*

$$f(n) := \alpha_{n-1} - \alpha_{n+1},\ h(n) := \frac{\alpha_{n-1} - 2\alpha_n + \alpha_{n+1}}{2}.$$

Then $f(n)$ and $h(n)$ satisfy the condition in Lemma 2.3, i.e., exists a constant $c > 0$ such that $\left|\frac{f(n)}{h(n)} - 1\right| \geq c$ for all sufficiently large n.

Proof: Define $b_n = \alpha_{n-1} - \alpha_n$. By condition (1.1), $\{b_n\}$ is a positive decreasing sequencing that tends to 0. Then $f(n) = b_n + b_{n+1}$ and $h(n) = \frac{b_n - b_{n+1}}{2}$. It suffices to show that there exists a constant $c > 0$ such that $\left|\frac{2(b_n + b_{n+1})}{b_n - b_{n+1}} - 1\right| \geq c$ for all sufficiently large n. Indeed,

$$\left|\frac{2(b_n + b_{n+1})}{b_n - b_{n+1}} - 1\right| = \left|\frac{b_n + 3b_{n+1}}{b_n - b_{n+1}}\right| \geq \frac{b_n + 3b_{n+1}}{b_n} \geq 1 \text{ for all } n \in \mathbb{N}.$$

The proof is now complete. □

Lemma 2.5. *For any sequence $\{\alpha_n\}$ that defines the convex set K, we have*

$$\lim_{n\to\infty}\left(\frac{\Pi(2e^{is_n/2}) - \Pi(2e^{it_n/2})}{s_n - t_n} - \frac{2(\alpha_n - \alpha_{n+1})}{\alpha_{n-1} + 2\alpha_n - 3\alpha_{n+1}} \cdot i\right) = 0.$$

Proof: Following the proof of Lemma 2.2, we compute the argument and magnitude separately.

Observe from the proof of Lemma 1.1 that $A_nS_nO_nT_n$ is a right kite, and thus has perpendicular diagonals. In particular, this implies

$$\arg\left(\frac{\Pi(2e^{is_n/2}) - \Pi(2e^{it_n/2})}{s_n - t_n}\right) = \frac{\pi}{2} + \arg\left(A_n - O_n\right) = \frac{\pi}{2} + \arg(T_n) + \frac{\pi - \psi_n}{2},$$

where ψ_n refers to the double-marked angle in Figure 1.

As noted from the proofs of Lemma 1.1 and Theorem 1.3,

$$\pi - \psi_n = \frac{\alpha_{n-1} - \alpha_{n+1}}{2} \text{ and } \arg\left(T_n\right) = t_n/2 = \frac{\alpha_n + \alpha_{n+1}}{2}.$$

Then

$$\lim_{n\to\infty} \arg\left(\frac{\Pi(2e^{is_n/2}) - \Pi(2e^{it_n/2})}{s_n - t_n}\right) = \lim_{n\to\infty}\left(\frac{\pi}{2} + \frac{\alpha_n + \alpha_{n+1}}{2} + \frac{\alpha_{n-1} - \alpha_{n+1}}{4}\right) = \frac{\pi}{2}.$$

Now we compute the magnitude of the expression in question. By formula (1.5), as S_n and T_n are on the circle of radius r_n centered at O_n, we see that

$$\|\Pi(2e^{is_n/2}) - \Pi(2e^{it_n/2})\| = 2r_n \sin\left(\frac{\pi - \psi_n}{2}\right) \sim r_n \cdot \frac{\alpha_{n-1} - \alpha_{n+1}}{2} \sim \alpha_n - \alpha_{n+1}.$$

We also have

$$s_n - t_n = (t_{n-1} - t_n) - (t_{n-1} - s_n) = (\alpha_{n-1} - \alpha_{n+1}) - (t_{n-1} - s_n).$$

By Lemma 2.2,

$$t_{n-1} - s_n \sim \|\Pi(2e^{it_{n-1}/2}) - \Pi(2e^{is_n/2})\|.$$

By the definition of $T_{n-1} = \Pi(2e^{it_{n-1}/2})$ and $S_n = \Pi(2e^{is_n/2})$, we see that

$$\|\Pi(2e^{it_{n-1}/2}) - \Pi(2e^{is_n/2})\| = \frac{\|A_n - A_{n-1}\| - \|A_{n+1} - A_n\|}{2},$$

so that

$$\|\Pi(2e^{it_{n-1}/2}) - \Pi(2e^{is_n/2})\| = \sin\left(\frac{\alpha_{n-1} - \alpha_n}{2}\right) - \sin\left(\frac{\alpha_n - \alpha_{n+1}}{2}\right) \sim \frac{\alpha_{n-1} - 2\alpha_n + \alpha_{n+1}}{2}.$$

Applying Lemma 2.3 and Lemma 2.4 with

$$f(n) = \alpha_{n-1} - \alpha_{n+1}, g(n) = t_{n-1} - s_n, h(n) = \frac{\alpha_{n-1} - 2\alpha_n + \alpha_{n+1}}{2}$$

yields

$$s_n - t_n \sim (\alpha_{n-1} - \alpha_{n+1}) - \frac{\alpha_{n-1} - 2\alpha_n + \alpha_{n+1}}{2} = \frac{\alpha_{n-1} + 2\alpha_n - 3\alpha_{n+1}}{2}.$$

Then using the above three equations together, we get that

$$\lim_{n\to\infty}\left\|\frac{\Pi(2e^{is_n/2}) - \Pi(2e^{it_n/2})}{s_n - t_n}\right\| = \lim_{n\to\infty}\frac{2(\alpha_n - \alpha_{n+1})}{\alpha_{n-1} + 2\alpha_n - 3\alpha_{n+1}}$$

as desired. □

It is well-known the differentiability and the directional differentiability of the metric projection mapping are related to the second-order behavior of the boundary of the set involved; see [1, 3, 9, 12] and the references therein. Note that the differentiability implies the directional differentiability. In the theorem below, we provide an example of a set with $C^{1,1}$ boundary but the metric projection mapping fails to be directionally differentiable.

Theorem 2.6. *Consider the set K obtained using the sequence $\{\alpha_n\}$ defined in Corollary*

1.2. Then ∂K is $C^{1,1}$ around $(1,0)$ and $D_v\Pi$ does not exist at $x(0)=(2,0)$, where $v=x'(0)=(0,1)$.

Proof. By Lemma 2.1, it suffices to study the limit:

$$\lim_{\theta\to 0+}\frac{\Pi(2e^{i\theta/2})-\Pi(2e^{0})}{\theta} \tag{2.2}$$

Applying Lemma 2.5, we see that

$$\lim_{n\to\infty}\frac{\Pi(2e^{is_n/2})-\Pi(2e^{it_n/2})}{s_n-t_n}=\frac{2\lambda i}{3\lambda+1}. \tag{2.3}$$

By definition $T_n=\ell_n e^{\frac{i}{2}(\alpha_n+\alpha_{n+1})}$ where $\ell_n=\cos\left(\dfrac{\alpha_n-\alpha_{n+1}}{2}\right)$ tends to 1. Note that in Figure 1 T_n, O_n, and the origin are collinear. It follows that $t_n=\alpha_n+\alpha_{n+1}$. Since $2e^{it_n/2}$ projects to T_n, we must have

$$\lim_{n\to\infty}\frac{\Pi(2e^{it_n/2})-\Pi(2e^{0})}{t_n}=\frac{i}{2}$$

We write $\dfrac{\Pi(2e^{it_{n-1}/2})-\Pi(2e^{0})}{t_{n-1}}$ as a weighted mean of three fractions:

$$\frac{\Pi(2e^{it_{n-1}/2})-\Pi(2e^{is_n/2})}{t_{n-1}-s_n}\cdot\frac{t_{n-1}-s_n}{t_{n-1}}+\frac{\Pi(2e^{is_n/2})-\Pi(2e^{it_n/2})}{s_n-t_n}\cdot\frac{s_n-t_n}{t_{n-1}}+\frac{\Pi(2e^{it_n/2})-\Pi(2e^{0})}{t_n}\cdot\frac{t_n}{t_{n-1}}. \tag{2.4}$$

Similarly, we write

$$\frac{\Pi(2e^{is_n/2})-\Pi(2e^{0})}{s_n}=\frac{\Pi(2e^{is_n/2})-\Pi(2e^{it_n/2})}{s_n-t_n}\cdot\frac{s_n-t_n}{s_n}+\frac{\Pi(2e^{it_n/2})-\Pi(2e^{0})}{t_n}\cdot\frac{t_n}{s_n}. \tag{2.5}$$

Now we will show that the limit in (2.2), and hence the directional derivative of the metric projection mapping at $x(0)=(2,0)$ in the direction $v=(0,1)$, does not exist. Suppose to the contrary that that this limit does exist. Then

$$\lim_{n\to\infty}\frac{\Pi(2e^{is_n/2})-\Pi(2e^{0})}{s_n}=\lim_{n\to\infty}\frac{\Pi(2e^{it_n/2})-\Pi(2e^{0})}{t_n}=i/2.$$

Let $\lambda_n=\dfrac{s_n-t_n}{s_n}$ and $\beta_n=\dfrac{t_n}{s_n}$. Obviously, $\{\lambda_n\}$ and $\{\beta_n\}$ are nonnegative bounded sequences with

$$\lambda_n+\beta_n=1 \text{ for all } n\in\mathbb{N}.$$

We will show that $\{\lambda_n\}$ converges to 0. By a contradiction, suppose that this is not the case. Then there exist $\epsilon_0>0$ and a subsequence of $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_k}\geq\epsilon_0$ for all $k\in\mathbb{N}$. By extracting a further convergent subsequence, we can assume without loss of generality that $\lim_{k\to\infty} \lambda_{n_k} = c > 0$. From (2.3) and (2.5), one has

$$\frac{i}{2} = c\frac{2\lambda i}{3\lambda+1} + (1-c)\frac{i}{2},$$

which implies

$$\frac{1}{2} = c\frac{2\lambda}{3\lambda+1} + (1-c)\frac{1}{2}.$$

Since $\frac{2\lambda}{3\lambda+1} < 1/2$, one has

$$\frac{1}{2} = c\frac{2\lambda}{3\lambda+1} + (1-c)\frac{1}{2} < c/2 + (1-c)/2 = 1/2,$$

a contradiction. We have shown that $\lim_{n\to\infty} \dfrac{s_n - t_n}{s_n} = 0$, and hence $\lim_{n\to\infty} \dfrac{t_n}{s_n} = 1$.

Now, taking the limit as n approaches infinity in (2.4), we get that

$$\frac{i}{2} = \lim_{n\to\infty} \left(i \cdot \frac{t_{n-1} - s_n}{t_{n-1}} + \frac{2\lambda i}{3\lambda+1} \cdot \frac{s_n - t_n}{t_{n-1}} + \frac{i}{2} \cdot \frac{t_n}{t_{n-1}} \right). \tag{2.6}$$

Of course, from $t_n = \alpha_n + \alpha_{n+1}$, we must have

$$\lim_{n\to\infty} \frac{t_n}{t_{n-1}} = \lambda.$$

Since $\lim_{n\to\infty} \dfrac{t_n}{s_n} = 1$, we get

$$\lim_{n\to\infty} \frac{s_n}{t_{n-1}} = \lambda.$$

Plugging these limits into (2.6) yields

$$\frac{i}{2} = i(1-\lambda) + \frac{i}{2}\lambda,$$

which is absurd. Therefore, the limit from (2.2) does not exist, and hence $D_v\Pi$ does not exist at $x(0) = (2,0)$ in the direction $v = (0,1)$.

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References

[1] Abatzoglou, T.J: The minimum norm projection on C^2 manifolds in $\mathbb{R}^n$, Trans. Amer. Math. Soc. **243**, 115–122 (1978)

[2] Clarke, F.H.: *Optimization and Nonsmooth Analysis*, John Wiley & Sons, Inc, New York (1983)

[3] Fitzpatrick, S., Phelps, R.R.: Differentiability of the metric projection in Hilbert space, Trans. Amer. Math. Soc. **270**, 483–501 (1982)

[4] Hiriart-Urruty, J.-B., Lemaréchal, C.: Fundamentals of Convex Analysis, Springer (2001)

[5] Kolmogorov, A.N., Fomin, S.V.: Introductory Real Analysis, Dover Publications, New York (1975)

[6] Kruskal, J. B.: Two convex counterexamples: a discontinuous envelope function and a nondifferentiable nearest-point mapping, Proc. Amer. Math. Soc. **23**, 697–703 (1969)

[7] Leoni, G.: A First Course in Sobolev Spaces, Graduate Studies in Mathematics, 105. American Mathematical Society, Providence, RI (2009)

[8] Mordukhovich, B. S., Nam, M.N.: An Easy Path to Convex Analysis and Applications, Morgan & Claypool Publishers (2014)

[9] Mordukhovich,B.S., Outrata, J. V., Ramirez, H.: Second-order variational analysis in conic programming with applications to optimality conditions and stability, to appear in SIAM J. Optim.

[10] Outrata, J. V., Sun, D.: On the coderivative of the projection operator onto the second- order cone, Set-Valued Anal. **16**, 999–1014 (2008)

[11] Rockafellar, R. T.: Convex Analysis, Princeton University Press, Princeton, NJ (1970)

[12] Shapiro, A.: Existance and differentiability of metric projections in Hilbert spaces, SIAM J. Optim. **4**, 130–141 (1994)

[13] Shapiro, A.: Directionally nondifferentiable metric projection, J. Optim. Theory Appl. **81**, 203–204 (1994)

[14] Shapiro, A.: Sensitivity analysis of nonlinear programs and differentiability properties of metric projections, SIAM J. Control Optim. **26**, 628-645 (1998)