Geodesics on Regular Constant Distance Surfaces

J. J. P. Veerman*†

September 4, 2023

Abstract

Suppose that the surfaces K_0 and K_r are the boundaries of two convex, complete, connected C^2 bodies in $\mathbb{R}^3$. Assume further that the (Euclidean) distance between any point x in K_r and K_0 is always r ($r > 0$). For x in K_r, let $\Pi(x)$ denote the nearest point to x in K_0. We show that the projection Π preserves geodesics in these surfaces if and only if both surfaces are concentric spheres or co-axial round cylinders. This is optimal in the sense that the main step to establish this result is false for $C^{1,1}$ surfaces. Finally, we give a non-trivial example of a geodesic preserving projection of two C^2 *non*-constant distance surfaces. The question whether for any C^2 convex surface S_0, there is a surface S whose projection to S_0 preserves geodesics is open.

2020 Mathematics Subject Classification: 52A15, 53A05.

1 Introduction

Suppose $\gamma(t)$ is a trajectory of an object in $\mathbb{R}^3$ outside a convex body. In this paper, $\Pi(\gamma(t))$ is called the *projection* of $\gamma(t)$. In many applications it is important to track the point $\Pi(\gamma(t))$ on the surface of the body nearest to the moving object[1]. In [10], a method to compute and track the projection was considered. Instead, here we consider the question whether this projection can take geodesics to (reparametrized) geodesics.

Before describing the main result, we give some general background about this problem. A diffeomorphism $\phi : S_1 \to S_2$ between (sub) manifolds is called a *geodesic mapping* if it carries geodesics to geodesics. We restrict our discussion to surfaces in $\mathbb{R}^3$. It is well-known that if S_1 has constant Gaussian curvature, then there is a geodesic mapping from S_1 to the plane. Vice versa, Beltrami's theorem says that if S_1 admits a (local) geodesic mapping to the plane near every point in S_1, then S_1 has constant Gaussian curvature ([4], Section 4.6, exercises 12 and 13). There is a fairly

*Weizmann Institute, Rehovot, Israel.

†Maseeh Dept. of Math. and Stat., Portland State Univ., Portland, OR, USA; e-mail: veerman@pdx.edu.

[1]A case in point is the event on September 26, 2022, when an unmanned spacecraft hit the asteroid Didymos on purpose [9], thereby changing the orbit of the asteroid. Clearly, the change in orbit of the asteroid is related to the locus, angle, and speed of the missile at the time of impact. The asteroid itself is in good approximation a convex set, but far from round [9].

large body of literature on geodesic mappings, [7, 8] and the references therein. Our own interest here is to find out whether *projections* from one surface to another can be *geodesic mappings*.

Our main result concerns projections to the convex set from a surface whose distance to the convex set is exactly r (a constant). We call such a surface a surface of constant distance (the word 'equidistant' is already in use for a slightly different concept [11]). Very little has been written about sets of constant distance (but see [3, 2]). What we aim to show here is essentially a rigidity result in $\mathbb{R}^3$: a constant distance surface whose projection takes geodesics to geodesics must be a sphere or a cylinder. We proceed with the details.

We imagine a C^2 convex body in $\mathbb{R}^3$ whose boundary we denote by K_0. Let p be any point in the surface. By applying an isometry, we may assume that p is located at the origin of $\mathbb{R}^3$ and that the tangent plane to K_0 at p is given by $z = 0$. Thus the coordinate patch near the origin can be written as

$$K_0(x_1, x_2) = \left(x_1, x_2, -\frac{1}{2}(a_1 x_1^2 + a_2 x_2^2) - h(x_1, x_2) \right) , \tag{1.1}$$

where the a_i are the *principal curvatures* and h is twice continuously differentiable with $h(0, 0)$ is zero and the same holds for all first and second derivatives. By convexity, the principal curvatures a_i are non-negative.

Because of the smoothness and the convexity, we can smoothly coordinatize the space Ω surrounding the convex body by using these coordinate patches as follows [10]:

$$S(x_1, x_2, r) = K_0(x_1, x_2) + r\hat{n}(x_1, x_2) . \tag{1.2}$$

where $\hat{n}$ is the unit normal to K_0. These 3-dimensional coordinate patches form a differentiable atlas of Ω. Denote by $\Pi : \Omega \to K_0$ the orthogonal 'projection' from Ω onto K_0, defined as follows [10]: $\gamma := \Pi(z)$ is the unique point on K_0 nearest to $z \in \Omega$. Clearly, the inverse of Π at a point γ of K_0 consists of a ray normal to K_0 at γ.

$$\Pi^{-1}(\gamma) = \cup_{r>0} \{\gamma + r\hat{n}(\gamma)\} ,$$

where $\hat{n}$ is the unit normal at γ pointing outwards.

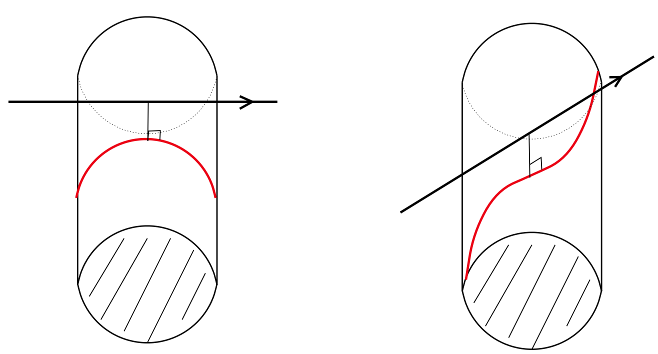

Figure 1.1: *Left, the projection (red) of straight line orthogonal to the axis of symmetry of a solid cylinder. Right, the projection of a line at an arbitrary angle with the axis of symmetry of the cylinder. The former is a geodesic, the latter clearly not.*

The simple example of Figure 1.1 shows that the projection of a straight line in $\mathbb{R}^3$ does not usually result in a geodesic in K_0. The question arises when is it that geodesics *do* project to geodesics? In this paper, any non-singular reparametrization (i.e. with non-zero, possibly variable, speed) of a unit speed geodesic will also be called a geodesic.

Definition 1.1 *i) Given two closed C^2 surfaces S_1 and S_2 in $\mathbb{R}^3$. The* projection $\Pi : S_1 \to S_2$ *is defined as follows. For $x \in S_1$,*

$$\Pi(x) := \{y \in S_2 \,:\, y \textit{ minimizes the Euclidean distance } d(x,y)\}\,.$$

ii) Two surfaces are called regular constant distance surfaces *if the Euclidean distance from any point x in S_1 to S_2 equals r (fixed), and the nearest point on S_2 is always unique.*

It is a curious fact that in general $\Pi : S_1 \to S_2$ and $\Pi' : S_2 \to S_1$ are *not* inverses of one another. However, if the S_i are at least C^1 and regular constant distance, then Π and Π' *are* inverses. This is the content of Proposition 2.1. In the remainder of this paper, we deal with this case (except where mentioned otherwise).

Let K_r denote the surface that has distance r to K_0, or

$$K_r := \{S(x_1, x_2, r) \,:\, r > 0 \text{ fixed}\}\,.$$

We are interested in determining when the projection $\Pi : K_r \to K_0$ between these surfaces have the property that they send geodesics to (reparametrizations of) geodesics. We call this property *preservation of geodesics* and Π a *geodesic mapping.* The proof of the following result takes up most of this paper.

Theorem 1.2 *Let K_0 be C^2 surface patch given by* (1.1) *with $a_1 \geq 0$ and $a_2 \geq 0$ and fix $r > 0$. Then the projection $\Pi : K_r \to K_0$ does* not *preserve geodesics, unless (in that patch) (i) the Gaussian curvature is zero (i.e. $a_1 a_2 = 0$) or (ii) the patch consists of umbilic points (i.e. $a_1 = a_2$)*

A moment's reflection, will tell us that in $\mathbb{R}^3$, projections between concentric spheres or between co-axial round cylinders *do* preserve geodesics. The interesting question is, are there any others? Here is a (to the author) surprising corollary of Theorem 1.2.

Corollary 1.3 *Let K_0 and K_r be regular constant distance, complete, convex, connected, C^2 surfaces in $\mathbb{R}^3$ at a distance $r > 0$. The projection from K_r to K_0 preserves geodesics if and only if both are either spheres or (infinite) round cylinders.*

Remark. In this context, a (generalized) cylinder C is a set of points such that for every point $p \in C$ there is a unique line $\ell(p)$ in C and any two such lines are either the same or parallel. A 'perfect' or 'round' cylinder is a cylinder that rotationally symmetric around its axis. In particular, its principal curvatures are constant.

Remark. In view of Proposition 2.1, $\Pi : K_r \to K_0$ and $\Pi' : K_0 \to K_r$ are inverses. So Π preserves geodesics if and only if Π' preserves geodesics.

Proof of Corollary 1.3. It is clear that if K_0 and K_r both are either spheres or (infinite) round cylinders, then the geodesics are preserved.

Vice versa, if the projection preserves geodesics, then by Theorem 1.2, every C^2 surface patch is either a piece of a sphere or piece of a cylinder. The two cannot occur in the same C^2 patch, because at any 'intermediate' point, (i) or (ii) in that theorem will be violated, and then geodesics will not be preserved. Thus all of K_0 must satisfy either (i) or (ii).

It is well-known that a C^2 complete surface whose principal curvatures are the identical (or *umbilic* surface) must be a part of a sphere ([4], Section 3.2). Similarly ([4], section 5.8), a complete surface with Gaussian curvature zero, must be a generalized cylinder. Finally, Proposition 4.1 implies that if K_0 and K_r are cylinders and the projection preserves geodesics, then they must be round cylinders. ■

Remark. Interestingly, this corollary is clearly false in $\mathbb{R}^2$. For instance, if K_0 is an ellipse in $\mathbb{R}^2$ and K_r a circle that contains it, the projection $K_r \to K_0$ is surjective. On the other hand, in dimension 4 or higher, nothing appears to be known.

We furthermore prove that Corollary 1.3 is optimal in the sense that if we drop C^2 in favor of $C^{1,1}$, that is: once continuously differentiable with a Lipschitz derivatives, then the result does not hold.

Theorem 1.4 *There exist regular constant distance, complete, convex, $C^{1,1}$ surfaces K_0 and K_r in $\mathbb{R}^3$ with the property that (wherever the surfaces are C^2) either (i) $a_1 a_2 = 0$ or (ii) $a_1 = a_2$ holds, but the projection from K_0 to K_r does not preserve geodesics.*

Proof. The result follows directly from Proposition 5.1. ■

Remark. In [1] (see also [10]), a related, but more complicated, counter-example was constructed which carries over to cylinders in $\mathbb{R}^3$. It says that here is a convex $C^{1,1}$ cylinder such that that the projection Π onto this cylinder does not have a derivative.

Finally, we are interested in the question whether, given the boundary S_0 of a convex body, there is *any* surface S outside it, whose projection onto S_0 preserves geodesics. For cylinders in $\mathbb{R}^3$, the answer is affirmative, as we show in Section 6. In fact, in that case, the space outside S_0 can be foliated by surfaces S_k, $k \geq 0$ so that each projection $\Pi_k : S_k \to S_0$ preserves geodesics. However, as we will show, these surfaces S_k generally are not convex.

Remark. For general C^2 convex bodies, even in $\mathbb{R}^3$, it is unknown at the time of this writing whether the space outside them can be foliated by surfaces S_k so that each projection $\Pi_k : S_k \to S_0$ preserves geodesics.

2 Preliminaries

We first prove that the projections between two regular constant distant surfaces (see Definition 1.1) are inverses of one another. Then we discuss the strategy to prove Theorem 1.2.

Proposition 2.1 *Let S_1 and S_2 be C^1 surfaces in $\mathbb{R}^3$ such that the Euclidean distance from any point x in S_1 to S_2 equals r (fixed), and the nearest point on S_2 is always unique. Then the projections*

$\Pi : S_1 \to S_2$ *and* $\Pi' : S_2 \to S_1$ *are inverses of one another.*

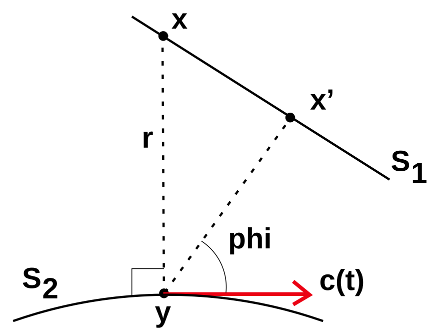

Figure 2.1: *This figure illustrates that* $\Pi : S_1 \to S_2$ *and* $\Pi' : S_2 \to S_1$ *are not generally inverses of one another. Traveling from* y *along* S_2 *in the direction of* $c(t)$ *will (initially) decrease the distance to* x'.

Proof. Consider $\Pi : S_1 \to S_2$ and $\Pi' : S_2 \to S_1$ and suppose $\Pi(x) = y$ (see Figure 2.1). Suppose there is x' in S_2 not equal to x such that $x' \in \Pi'(y)$. Denote the Euclidean distance by $d(x, y)$. Now

$$\begin{aligned} x' \in \Pi'(y) \quad &\Longrightarrow \quad d(y, x') \leq d(y, x) = r \\ d(x', S_2) = r \quad &\Longrightarrow \quad d(x', y) \geq r \end{aligned}$$

So $d(y, x') = r$.

Consider the plane P through x, x', and y, and parametrize S_2 by the arclength t and let the geodesic $c(t)$ be the tangent to $S_2(t)$ as drawn in Figure 2.1. Then, by differentiability of S_2,

$$\lim_{t \searrow 0^+} \frac{d(S_2(t), x') - d(S_2(0), x')}{t} = \lim_{t \searrow 0^+} \frac{d(c(t), x') - d(c(0), x')}{t} = -\cos\phi\,.$$

The last equality is a special case[2] of Theorem 4.3 in [5]. Thus for some positive t, $d(S_2(t), x') < r$, contradicting the assumption that $d(x', S_2) = r$. ∎

To prove Theorem 1.2, we pick a family Γ of geodesics in the patch given by (1.1) as follows. A geodesic $\gamma(t)$ in Γ is determined by initial condition $\gamma(0) = (0, x_2(0), x_3(0))$, where $x_2(0)$ is not zero but small and $\dot{x}_1(0) > 0$ is of order unity, while $x_3(0)$ is determined by the fact that γ is a curve in the surface K_0 (see Figure 2.2).

Since we are interested not in geodesics per se, but in geodesics modulo (non-singular) reparametrization, we establish a simple characterization of geodesics in Γ that does not depend on the parametrization (Lemma 3.1). We then consider the projection $\Pi : K_0 \to K_r$, with $r > 0$, which maps γ to a curve γ_r in K_r. And finally, we prove that γ_r is not a (reparametrization of a) geodesic by showing that it fails the criterion just mentioned. To do this, we will need to determine the terms of the leading order of magnitude in a fairly involved expression. We will employ the standard 'big-oh'

[2] In this simple case, it can also be derived easily from an explicit computation.

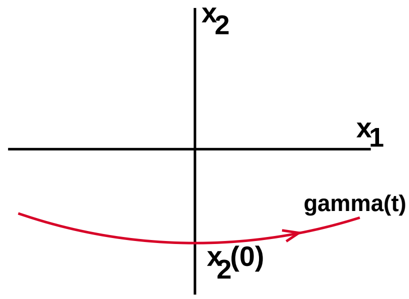

Figure 2.2: *A view of a geodesic γ in K_0 from 'above' (i.e. $x_3 > 0$). At $t = 0$, γ passes through the point $(0, x_2(0))$ with velocity $(\dot{x}_1(0), 0)$.*

and 'small-oh' notation as follows. We consider curves such as the ones in Figure 2.2, and evaluate certain quantities as these curves cross the $x_1 = 0$ axis. Thus[3] using x as shorthand for (x_1, x_2):

$$\begin{aligned} f(x_1, x_2) &= Ok \quad \text{means} \quad \limsup_{|x|\to 0} \frac{|f(x)|}{|x|^k} < \infty\,, \\ \text{and} \qquad f(x_1, x_2) &= ok \quad \text{means} \quad \lim_{|x|\to 0} \frac{|f(x)|}{|x|^k} = 0\,. \end{aligned}$$

It will be convenient to have a more compact notation. Hence the following definition.

Definition 2.2 *We define $z_i := a_i x_i + \partial_i h(x)$, where $x(t) = (x_1(t), x_2(t))$ is the projection to the x_1-x_2 plane of the geodesic $\gamma(t)$ in Figure 2.2.*

We compute the leading orders at $t = 0$ of z_i, $\dot{z}_i$, and $\ddot{z}_i$.

$$\dot{z}_i = a_i \dot{x}_i + d_t \partial_i h \qquad \text{and} \qquad \ddot{z}_i = a_i \ddot{x}_i + d_t^2 \partial_i h\,.$$

We know that $h = o2$ and so $\partial_i h = o1$. Furthermore, at $t = 0$, $x_1 = 0$, and $x_2 = O1$. Thus

$$z_1 = \partial_1 h \qquad \text{and} \qquad z_2 = a_2 x_2 + o1\,. \tag{2.1}$$

Each of these is $O1$ or less. Now,

$$d_t \partial_i h = \partial_1 \partial_i h\ \dot{x}_1 + \partial_2 \partial_i h\ \dot{x}_2\,.$$

Along the geodesic in the patch, $\dot{x}_1$ is order unity (or $O1$), and even though $\dot{x}_2$ may be small, we see that $d_t \partial_2 h = o0$. In fact, we are only interested in evaluating these quantities at $t = 0$ at which point we have $\dot{x}_2 = \ddot{x}_2 = 0$. Putting this together results at $t = 0$ in

$$\dot{z}_1 = a_1 \dot{x}_i + o0 \qquad \text{and} \qquad \dot{z}_2 = \partial_1 \partial_2 h\ \dot{x}_1\,, \tag{2.2}$$

and so $\dot{z}_2 = o0$. The next derivative, $\ddot{z}_i$, is a little trickier. The reason is that $d_t^2 \partial_2 h$ cannot be bounded by some order. It may be large, or, depending on h, it may be small. To ensure we have the leading terms of $\ddot{z}_i$, we have to include both terms and the expression does not simplify. Setting $t = 0$, we will see that $\ddot{x}_1 = O1$ and we know that $\ddot{x}_1 = 0$. So at $t = 0$,

$$\ddot{z}_1 = a_1 \ddot{x}_i + d_t^2 \partial_2 h \qquad \text{and} \qquad \ddot{z}_2 = \partial_1 \partial_2 h\ \dot{x}_1\,. \tag{2.3}$$

[3]Care should be exercised with the "=" sign. It is not reflexive in this context.

3 Proof of Theorem 1.2

To distinguish the standard inner product in $\mathbb{R}^3$ from a 2-tuple, we indicate the former by a dot: $x \cdot y$. Also, to avoid cluttering the formulas with the repetitive occurrence of the argument "(0)", we will not write it, except when its omission might lead to misunderstandings.

Lemma 3.1 *Suppose the family of curves $\gamma(t) = (x_1(t), x_2(t), -\frac{1}{2}[a_1x_1(t)^2 + a_2x_2(t)^2] - h)$ in K_0 are (a reparametrization of) geodesics with $x_1 = 0$, $\dot{x}_1 > 0$, $x_2 \neq 0$, and $\dot{x}_2 = 0$. Then at $t = 0$*

$$\lim_{x_2 \to 0} \frac{\ddot{x}_2}{\dot{x}_1^2 x_2} = -a_1a_2 \,.$$

Furthermore, this characterization is independent of the (smooth) parametrization of γ.

Proof. Set $e_i := \partial_i K_0$, where K_0 is given by (1.1). The metric tensor $g_{ij} = e_i \cdot e_j$ and its inverse are given by (see Definition 2.2)

$$g = \begin{pmatrix} 1+z_1^2 & z_1z_2 \\ z_1z_2 & 1+z_2^2 \end{pmatrix} \qquad \text{and} \qquad g^{-1} = \Delta^{-1} \begin{pmatrix} 1+z_2^2 & -z_1z_2 \\ -z_1z_2 & 1+z_1^2 \end{pmatrix} \,,$$

where Δ is the determinant of g. The coefficients of g^{-1} are denoted by g^{ij}. The Christoffel symbols of the second kind are now given by

$$\Gamma_{ij}^k := \partial_i e_j \cdot \sum_n g^{kn} e_n \,.$$

We have that

$$\partial_i e_j = (0, 0, -\partial_i z_j) \,.$$

So we only need the 3rd component of $\sum_n g^{kn} e_n$. A straightforward computation gives that these are $-\Delta^{-1} z_k$. This yields

$$\Gamma_{ij}^k = \Delta^{-1} z_k \;\, \partial_i z_j \,.$$

Employing the rules for order calculation, one checks that this gives an $O1$ term only if $i = j$, namely $a_i x_i$. Everything else gives at best $o1$ terms. So $\Gamma_{ii}^k = a_k a_i x_k + o1$, and $\Gamma_{ij}^k = o1$ if $i \neq j$. The geodesic equations are

$$\ddot{x}_k + \sum_{i,j} \Gamma_{ij}^k \dot{x}_i \, \dot{x}_j = 0 \,.$$

So in our case, the equation for $\ddot{x}_2$ is

$$\ddot{x}_2 + (a_1a_2x_2 + o1)\, \dot{x}_1^2 + (o1)\dot{x}_1\dot{x}_2 + (a_2^2x_2 + o1)\, \dot{x}_2^2 = 0 \,.$$

Setting $\dot{x}_2 = 0$, proves the first part of the lemma.

To prove that this is invariant under the parametrization $t \to s(t)$, define $c(t) = \gamma \circ s$. Set $s(0) = 0$. Using $\dot{x}_2(0) = 0$ again, it is trivial to show that at $t = 0$

$$\frac{d_t^2(x_2(s))}{(d_t x(s))^2 \, x_2(s)} = \frac{\ddot{x}_2}{\dot{x}_1^2 x_2} \,,$$

where we use d_t for $\frac{d}{dt}$. ∎

Lemma 3.2 *Given the surface K_0 of* (1.1)*, then the constant distance surface K_r can be parametrized as follows*

$$K_r(u_1, u_2) = \left(u_1, u_2, r - \frac{1}{2}(a_{1r}u_1^2 + a_{2r}u_2^2) + o2 \right) ,$$

where

$$a_{ir} = \frac{a_i}{1 + ra_i} .$$

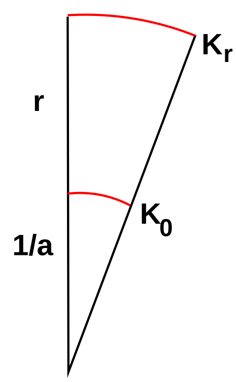

Figure 3.1: *The radius of curvature in the x-direction of K_0 at the origin equals $1/a$. The orthogonal projection to K_r then gives a radius of curvature of $r + 1/a$. The principal curvature is the reciprocal of this.*

Proof. Fix $r > 0$. The inverse projection $\Pi_r^{-1} : K_0 \to K_r$ is well-defined, and given by

$$K_r(x_1, x_2) := \Pi_r^{-1}(K_0(x_1, x_2)) = K_0(x_1, x_2) + r\hat{n}(x_1, x_2) . \tag{3.1}$$

We'll call K_0, somewhat informally, the 'downstairs' surface and K_r is 'upstairs'. We compute, using Definition 2.2

$$\hat{n}(x_1, x_2) = \frac{(z_1, z_2, 1)}{\sqrt{1 + z_1^2 + z_2^2}} . \tag{3.2}$$

So

$$\begin{aligned} K_r(x_1, x_2) &= \left(x_1 + \frac{rz_1}{V}, x_2 + \frac{rz_2}{V}, -\frac{1}{2}(a_1x^2 + a_1x_2^2) - h(x_1, x_2) + \frac{r}{V} \right) , \\ \text{where } V &= \sqrt{1 + z_1^2 + z_2^2} \end{aligned}$$

There are no mixed quadratic terms of the form x_1x_2 in the expansion of $K_r(x_1, x_2)$. So if we rewrite this as $K_r(u_1, u_2) = (u_1, u_2, r + u_3(u_1, u_2))$, then the u_1- and u_2-axes of K_r are the axes of principal curvature at $(u_1, u_2) = (0, 0)$. All we need to do to complete the proof, is a computation of the curvature in the x_1-r plane to get a_{1r}. This is done in Figure 3.1 by employing osculating circles. The computation is the same in the x_2-r plane. ■

Part of the difficulty here is that it is pretty clear that if K_0 does not have constant curvature along a geodesic $\gamma(t)$, then the curve traced in K_r by projecting γ will certainly not be a constant speed curve, let alone a constant speed geodesic. It is thus a priori clear that the projected curve will not satisfy the geodesic equations. What we wish to establish, however, is whether it can be *reparametrized* as a geodesic. We use Lemma 3.1 that the images γ_r $(r > 0)$ under the projection are not geodesics.

Lemma 3.3 *The geodesic γ depicted in Figure 2.2 with $\gamma(0) = (0, x_2)$ and $\dot\gamma(0) = (\dot x_1, 0)$ projects to a curve*

$$\gamma_r(t) = (u_1(t), u_2(t), u_3(t))$$

in K_r $(r > 0)$, where at $t = 0$, we have

$$\begin{aligned} \dot u_1 &= (1 + ra_1)\dot x_1 + o0 \\ u_2 &= (1 + ra_2)x_2 + o1 \\ \ddot u_2 &= -(1 + ra_1 + ra_2)a_1a_2\dot x_1^2 x_2 + rd_t^2\partial_2 h + o1\,. \end{aligned}$$

Proof. We trace a possibly reparametrized geodesic $\gamma(t)$ in K_0 satisfying Lemma 3.1, and determine the curvature of its projection γ_r 'upstairs' in K_r. Note that $\gamma_r(t)$ is given by $\gamma(t) + r\hat n(x_1(t), x_2(t))$. The unit-normal $\hat n$ is given in (3.2). We use the rules of evaluating the orders given in Section 2.

The x_1 and x_2 coordinates of γ_r will be called u_1 and u_2 and, noting that $z_i = O1$ (Section 2), we get

$$u_1 = x_1 + rz_1(1 - \tfrac{1}{2}z_1^2 - \tfrac{1}{2}z_2^2 + O4)$$
$$u_2 = x_2 + rz_2(1 - \tfrac{1}{2}z_1^2 - \tfrac{1}{2}z_2^2 + O4)\,.$$

Referring to (2.1), this gives for u_2 the following:

$$u_2 = (1 + ra_2)x_2 + o1\,. \tag{3.3}$$

Now, differentiate the u_i with respect to time.

$$\dot u_1 = \dot x_1 + r\dot z_1 - r\dot z_1(\tfrac{1}{2}z_1^2 + \tfrac{1}{2}z_2^2 + O4) - r\dot z_1(z_1\dot z_1 + z_2\dot z_2 + O3)$$
$$\dot u_2 = \dot x_2 + r\dot z_2 - r\dot z_2(\tfrac{1}{2}z_1^2 + \tfrac{1}{2}z_2^2 + O4) - r\dot z_2(z_1\dot z_1 + z_2\dot z_2 + O3)\,.$$

Use (2.2), to see that the leading term appearing in $\dot u_1$ is $\dot x_1$ (which is $O0$), and thus

$$\dot u_1 = (1 + ra_1)\dot x_1 + o0\,. \tag{3.4}$$

We need to differentiate $\dot u_2$ one more time with respect to time.

$$\ddot u_2 = \underbrace{\ddot x_2 + \ddot z_2}_{A} - \underbrace{r\ddot z_2\, O2}_{B} - \underbrace{2r\dot z_2(z_1\dot z_1 + z_2\dot z_2 + O3)}_{C} - \underbrace{rz_2(\dot z_1^2 + z_1\ddot z_1 + \dot z_2^2 + z_2\ddot z_2 + O2)}_{D}\,.$$

To analyze this, we denote the four terms A through D, and look at each individually. In A, $\ddot z_2$ can be evaluated via (2.3) and $\ddot x_2$ can be eliminated via Lemma 3.1. This gives $-(1+ra_2)a_1a_2\dot x_1^2 x_2 + rd_t^2\partial_2 h$ for the term marked A. Clearly, B is negligible compared to A. In C, $\dot z_2 = o0$ as noted before, and the term in parentheses is $O1$. So all together this term is $o1$ and therefore negligible compared to A.

Finally, in C, we use (2.1) to establish that $-r\dot{z}_1^2 z_2$ is $O1$ and, by (2.1), all other terms are smaller. This last expression can be simplified using (2.1) and (2.2) to $-ra_1^2 a_2 \dot{x}_1^2 x_2 + o1$. Collecting terms and adding the relations (3.3) and (3.4) yields the lemma. ■

Proof of Theorem 1.2. On the one hand, if the projected curve γ_r is also a geodesic, then it itself must satisfy Lemma 3.1 with the curvatures given by Lemma 3.2. So

$$\ddot{u}_2 = -\frac{a_1 a_2}{(1+ra_1)(1+ra_2)} \dot{u}_1^2 u_2 \,.$$

Eliminating $\dot{u}_1$ and u_2 in favor of $\dot{x}_1$ and x_2 via Lemma 3.3 gives

$$\ddot{u}_2 = -(1+ra_1)a_1 a_2 \dot{x}_1^2 x_2 + o1 \,.$$

On the other hand, another equation for $\ddot{u}_2$ is given by Lemma 3.3. If we equate the two expressions, we obtain

$$-(1+ra_1)a_1 a_2 \dot{x}_1^2 x_2 + o1 = -(1+ra_1+ra_2)a_1 a_2 \dot{x}_1^2 x_2 + r d_t^2 \partial_2 h + o1 \,.$$

Upon simplification, this gives

$$-ra_1 a_2^2 \dot{x}_1^2 x_2 + r d_t^2 \partial_2 h = o1 \,. \tag{3.5}$$

This equation has two possible solutions. The first is if the left hand is $o1$ and so $a_1 a_2 = 0$ and $d_t^2 \partial_2 h$ is $o1$. From the rules about manipulating the order symbols in Section 2, it follows that then $h = o4$. The other possibility is if $a_1 a_2 > 0$ and so $d_t^2 \partial_2 h = a_1 a_2^2 \dot{x}_1^2 x_2 + o1$. This happens if and only if $h = \frac{1}{4} a_1 a_2^2 x_1^2 x_2^2 + g$ and $d_t^2 \partial_2 g = o1$. So $h = \frac{1}{4} a_1 a_2^2 x_1^2 x_2^2 + o4$.

Now consider the geodesic η which is just γ rotated by $\pi/2$. Then, by the same reasoning, if the projection η_r in K_r is a geodesic, we must have that $h = \frac{1}{4} a_1^2 a_2 x_1^2 x_2^2 + o4$. Since both[4] must hold, we get $\frac{1}{4} a_1 a_2^2 x_1^2 x_2^2 = \frac{1}{4} a_1^2 a_2 x_1^2 x_2^2$ or $a_1 = a_2$. ■

Remark. The two types of solutions of (3.5) in this proof do indeed occur. For if K_0 is a plane or a cylinder with radius $1/a$, we get

$$K(x_1, x_2) = \sqrt{a^{-2} - x_1^2} - a^{-1} = -\tfrac{1}{2} a x_1^2 - \tfrac{1}{8} a^3 x_1^4 + O6 \,.$$

In this case, the Gaussian curvature is zero and $h = o4$. On the other hand, for a sphere of radius $1/a$, we have

$$K(x_1, x_2) = \sqrt{a^{-2} - x_1^2 - x_2^2} - a^{-1} = -\frac{1}{2}(a x_1^2 + a x_2^2) - \frac{1}{8}(a^3 x_1^4 + 2a^3 x_1^2 x_2^2 + a^3 x_2^4) + O6 \,.$$

Here, the principal curvatures are equal and $d_t^2 \partial_2 h = a^3 \dot{x}_1^2 x_2 + o1$.

4 Cylinders Must Be Round

Now let K_0 be a convex, but not necessarily round, cylinder, invariant under translations along the x_3-axis. Consider "polar" coordinates (ρ, x_3, r) in $\mathbb{R}^3$ where ρ is the arclength along the simple, closed curve in the x_1-x_2 plane that defines K_0 as illustrated in Figure 4.1.

Proposition 4.1 *With the above assumptions, the projection $\Pi : K_0 \to K_r$ preserves geodesics if and only if the non-zero principal curvature of K_0 is constant.*

[4] Note that the powers of a_i are distinct.

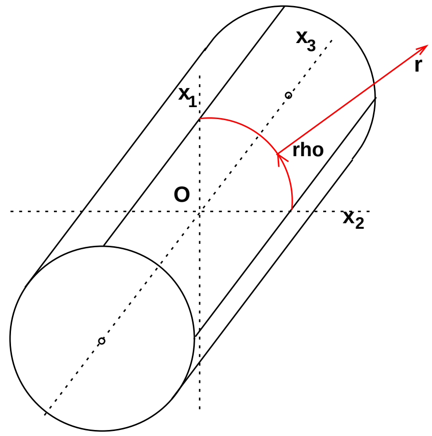

Figure 4.1: *"Polar" coordinates in* $\mathbb{R}^3$*.*

Proof. Since the Gaussian curvature is zero, the map from the cylinder to the plane, given by $K_0(\rho, x_3) \to (\rho, x_3)$ is a bijective isometry and so maps geodesics to geodesics. A geodesic γ in K_0 is (a) parallel to the x_3-axis, or (b) a circle in the x_1-x_2 plane, or (c) a curve $\gamma(x_3) = (\rho(x_3), x_3)$. Since γ is a geodesic, $\rho(x_3)$ is affine and has a constant derivative $\frac{d\rho}{dx_3}$. Assume γ is a geodesic of type (c).

Now consider the projection γ_r of γ onto K_r. As with K_0, we parametrize K_r by the arclength ρ_r of the defining curve and x_3. It is clear that γ_r is a curve $x_3 \to \rho_r(x_3)$. Again, if γ_r is a geodesic, then $\frac{d\rho_r}{dx_3}$ is constant. Denote the non-zero principal curvature of K_0 by $a(\rho)$. A reasoning similar to that of Lemma 3.2 gives that arclengths ρ_r and ρ travelled along each geodesic relate as

$$d\rho_r = \frac{1/a(\rho) + r}{1/a(\rho)}\; d\rho = (1 + a(\rho) r)\; d\rho\,.$$

Since the x_3 coordinates of $\gamma(t)$ and and its projection $\gamma_r(t)$ are the same, we get

$$\frac{d\rho_r}{dx_3} = (1 + a(\rho) r)\; \frac{d\rho}{dx_3}\,. \tag{4.1}$$

Thus $\frac{d\rho_r}{dx_3}$ is constant if and only if $a(\rho)$ is constant. ∎

5 A $C^{1,1}$ Counter-example

We consider the round cylinder 'topped off' by a hemisphere both of radius r, which gives a $C^{1,1}$ surface (see Figure 5.1). Denote this surface by S_r. It is easy to convince oneself that S_1 and S_r $(r > 1)$ are regular, constant distance surfaces. Clearly, at every point (except where C^2 does not hold) either (i) $a_1 a_2 = 0$ or (ii) $a_1 = a_2$.

Proposition 5.1 *Let S_1, S_r, and $\Pi : S_1 \to S_r$ be given as above. Let γ_1 be the shortest geodesic connecting $P_1 = (0, 1, \pi/2)$ and $Q_1 = (0, -1/\sqrt{2}, -1/\sqrt{2})$. The projection of γ_1 by Π to γ_r (connecting P_r to Q_r in S_r $(r > 1)$) is* not *a local geodesic near the point M_r where γ_r intersects the boundary of the cylinder (see Figure5.1).*

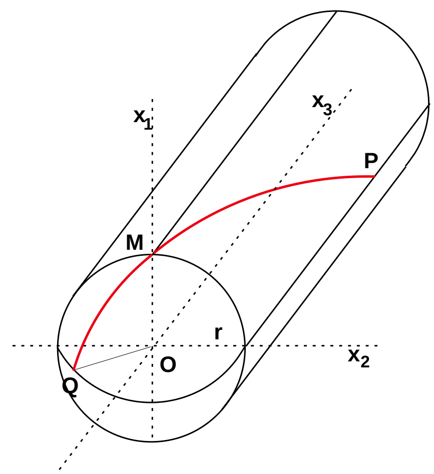

Figure 5.1: *A straight cylinder of radius r topped off by a hemisphere of radius r. This surface is C^2, except on the circle where the cylinder and the hemisphere meet. Here $\partial^2/\partial z^2$ has a discontinuity. Since the first derivative changes gradually, this surface is $C^{1,1}$.*

Proof. The geodesic γ connects P_1 to Q_1, but we do not know where it crosses over from the cylinder to sphere. So let us call that point $M_1(\theta)$. We have

$$P_1 = (0, 1, \pi/2) \ , \quad M_1(\theta) = (\sin\theta, \cos\theta, 0) \ , \quad Q_1 = \left(0, \tfrac{-1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}\right) \ .$$

It is easy to see that then the projection γ_r of γ connects P_r to Q_r via $M_r(\theta)$ (the same θ), where

$$P_r = (0, r, \pi/2) \ , \quad M_r(\theta) = (r\sin\theta, r\cos\theta, 0) \ , \quad Q_r = \left(0, \tfrac{-r}{\sqrt{2}}, \tfrac{-r}{\sqrt{2}}\right) \ . \tag{5.1}$$

The projection of γ_r consists of two pieces that live on C^2 surfaces with either curvature zero (the cylinder) or the sphere, and so each of these two pieces is a geodesic in S_r.

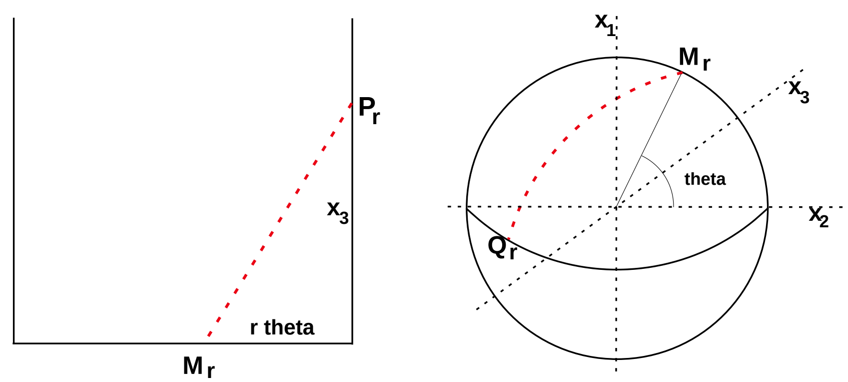

Figure 5.2: *The two geodesic pieces of γ_r in S_r. To the left, the piece in the flattened out cylinder. To the right the piece that lies in the hemisphere.*

The first piece connects P_r to M_r, see the left of Figure 5.2. In the flattened out cylinder, it is the hypotenuse of the triangle with sides $\pi/2$ and $r\theta$ and thus has length $\sqrt{\pi^2/4 + r^2\theta^2}$. The second piece lives in the sphere. Its length is r times the angle α between M_r and Q_r. The cosine of α is given by the dot product of the unit vectors parallel to M_r and Q_r, which gives $-\cos\theta/\sqrt{2}$. Thus the length of the second piece equals $r \arccos\left(\frac{-\cos\theta}{\sqrt{2}}\right)$. Therefore, the length of the projected curve γ_r is given by

$$\ell_\theta(\gamma_r) = \sqrt{\frac{\pi^2}{4} + r^2\theta^2} + r \arccos\left(\frac{-\cos\theta}{\sqrt{2}}\right) .$$

We need to minimize this over $\theta \in [0.\pi]$. It is an elementary calculus exercise[5] to see that this is minimized at θ satisfying $\frac{\theta}{\sin\theta} = \frac{\pi/2}{r}$. We know that γ is minimizing in S_1. Therefore if we substitute $r = 1$, we get $\theta = \pi/2$. That same calculation for $r > 1$ implies then that γ_r (where $r > 1$) is not globally minimizing in S_r.

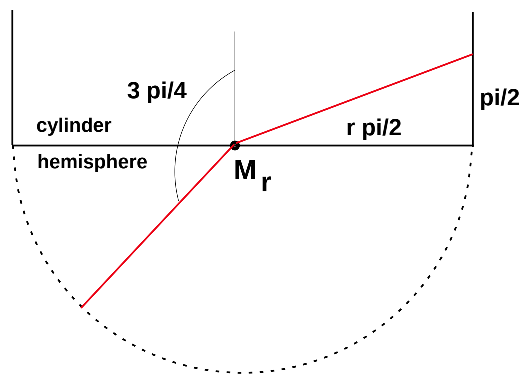

Figure 5.3: *The tangent space TS_r at M_r. The red curve corresponds to the lift of γ_r. The branch to the left of the base point M_r travels to Q_r and the branch to the right travels P_r (see* (5.1)*).*

Figure 5.3 is a slightly impressionistic image of the tangent space TS_r at M_r. The slope of γ_r restricted to the lower half plane that projects to the hemisphere equals 1. However the slope restricted the upper half plane which can be identified with the rolled out cylinder, the slope equals $1/r$. Thus γ_r is not locally minimizing at M_r. ■

6 There are Other Projections that Preserve Geodesics

In this Section, we find a beautiful example of a family of projections $\Pi_k : S_k \to S_0$ such that the surfaces $\{S_k\}_{k\geq 0}$ foliate the space surrounding the boundary S_0 of a convex body in $\mathbb{R}^3$. It is not known whether this is possible for all such surfaces S_0. Our construction is based on Section 4 and works for (convex) cylinders.

Consider a general, not necessarily round, convex, cylinder S_0. It consists of parametrized closed curve $c(t)$ and lines though that curve, orthogonal to it, as sketched in Figure 4.1. We can define a S outside S_0 by first defining a new curve in $\mathbb{R}^2$:

$$C(t) = c(t) + r(t)\hat{n}(t) .$$

[5] Use that the derivative of $\arccos q$ equals $-1/\sqrt{1-q^2}$.

Here $\hat{n}(t)$ is the unit normal to $c(t)$ and $r(t)$ is a non-negative distance. Let us denote the curvature of $c(t)$ by $a(t)$. According to (4.1), the projection $\Pi : S \to S_0$ between the corresponding cylinders preserves geodesics if

$$r(t) = \frac{k}{a(t)}\,,$$

where k is a positive constant. The cylinder S_k, $k \geq 0$ is given by the lines through C_k orthogonal to plane of C_k. Thus the projection $\Pi_k : S_k \to S_0$ preserves geodesics. Notice that we have to be careful here, because now the back and forth projections are not inverses of one another anymore (see Proposition 2.1).

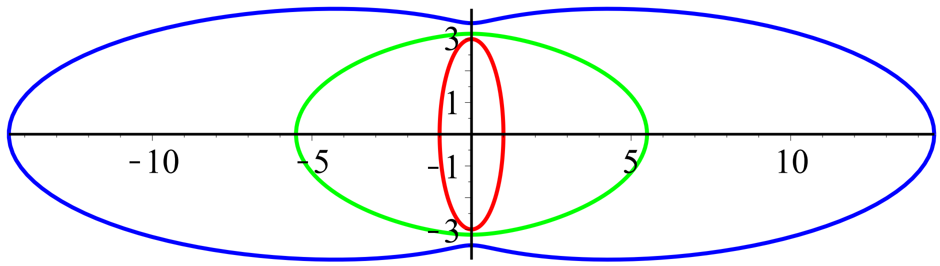

Figure 6.1: *the cylinders orthogonal to the plane of the figure through the blue and the green curves, have projections to the cylinder orthogonal to the red ellipse that preserve geodesics.*

We take as an example the ellipse given by

$$c(t) := (\alpha\cos(t), \beta\sin(t)) \qquad \text{and} \qquad C_k(t) = c(t) + \frac{k}{a(t)}\,\hat{n}(t)\,,$$

where k is a non-negative constant. Standard calculations give $C(t)$ explicitly as

$$C_k(t) = \left(\left(\alpha + \frac{k}{\alpha}(\alpha^2\sin(t)^2 + \beta^2\cos(t)^2)\right)\cos(t), \left(\beta + \frac{k}{\beta}(\alpha^2\sin(t)^2 + \beta^2\cos(t)^2)\right)\sin(t)\right)\,.$$

We used MAPLE in Figure 6.1, to draw the ellipse $c(t) = (\cos(t), 3\sin(t))$ in red, $C_k(t)$ for $k = 0.5$ in green, and for $k = 1.5$ in blue. Note that these remarkable curves lose convexity for large enough k. We leave it to the reader to establish that for large k, the projection $\Pi'_k : S_0 \to S_k$ is not single-valued and therefore does not preserve geodesics.

References

[1] S. S. Akmal, M. N. Nam, J. J. P. Veerman, *On a Convex Set with Nondifferentiable Metric Projection*, **Optimization Letters**, Vol 9, Issue 6, 1039-1052, 2015.

[2] A. Blokh, M. Misiurewicz, L. Oversteegen, *Sets of Constant Distance from a Compact Set in 2-Manifolds with a Geodesic Metric*, **Proceedings of the American Mathematical Society**, 137, 733-743, 2009.

[3] M. Brown, *Sets of Constant Distance from a Planar Set*, **Michigan Math. J.** 19, 321-323, 1972.

[4] M. P. do Carmo (1976), *Differentiable Geometry of Curves and Surfaces*, Prentice-Hall, London.

[5] L. S. Fox, P. Oberly, J. J. P. Veerman, *One-Sided Derivative of Distance to a Compact Set*, **Rocky Mountain Journal of Mathematics** 51 (2), 491-508, 2021.

[6] P. Michel, *The ESA Hera Mission: Detailed Characterization of the DART Impact Outcome and of the Binary Asteroid (65803) Didymos*, **The Planetary Science Journal**, 3: 160, 2022 (July).

[7] J. Mikes, *Geodesic Mappings of Affine-Connected and Riemannian Spaces*, J. Math. Sciences, 78 (3), 1996, 311-333.

[8] J. Mikes, V. E. Berezovski, E. Stepanova, and H. Chudá, *Geodesic Mappings and their Generalizations*, J. Math. Sciences, 217 (5), 2016, 608-623.

[9] S.P. Naidu e. a., *Radar observations and a physical model of binary near-Earth asteroid 65803 Didymos, target of the DART mission*, **Icarus**, 348, 133777, 2020.

[10] J. J. P. Veerman (2020), *Navigating Around Convex Sets*, **The American Mathematical Monthly**, 127:6, 504-517,

[11] J. B. Wilker, *Equidistant Sets and their Properties*, **Proceedings of the American Mathematical Society**, 47 (2): 446-452, 1975.