Lecture Notes on Operator Algebras

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It is not essential for the value of an education that every idea be understood at the time of its accession. Any person with a genuine intellectual interest and a wealth of intellectual content acquires much that he only gradually comes to understand fully in the light of its correlation with other related ideas. ... Scholarship is a progressive process, and it is the art of so connecting and recombining individual items of learning by the force of one's whole character and experience that nothing is left in isolation, and each idea becomes a commentary on many others.

- NORBERT WIENER

CHAPTER 1

LINEAR ALGEBRA AND THE SPECTRAL THEOREM

1.1. Vector Spaces and the Decomposition of Diagonalizable Operators

1.1.1. Convention. In this course, unless the contrary is explicitly stated, all vector spaces will be assumed to be vector spaces over $\mathbb{C}$. That is, *scalar* will be taken to mean *complex number*.

1.1.2. Definition. The triple $(V, +, M)$ is a (COMPLEX) VECTOR SPACE if $(V, +)$ is an Abelian group and $M : \mathbb{C} \to \mathrm{Hom}(V)$ is a unital ring homomorphism (where $\mathrm{Hom}(V)$ is the ring of group homomorphisms on V).

A function $T : V \to W$ between vector spaces is LINEAR if $T(u + v) = Tu + Tv$ for all u, $v \in V$ and $T(\alpha v) = \alpha Tv$ for all $\alpha \in \mathbb{C}$ and $v \in V$. Linear functions are frequently called *linear transformations* or *linear maps*. When $V = W$ we say that T is an OPERATOR on V. The collection of all linear maps from V to W is denoted by $\mathfrak{L}(V, W)$ and the set of operators on V is denoted by $\mathfrak{L}(V)$. Depending on context we denote the identity operator $x \mapsto x$ on V by id_V or I_V or just I. Recall that if $T : V \to W$ is a linear map, then the KERNEL of T, denoted by $\ker T$, is $T^{\leftarrow}(\{0\}) = \{x \in V : Tx = \mathbf{0}\}$. Also, the RANGE of T, denoted by $\mathrm{ran}\, T$, is $T^{\rightarrow}(V) = \{Tx : x \in V\}$.

1.1.3. Definition. A linear map $T : V \to W$ between vector spaces is INVERTIBLE (or is an ISOMORPHISM) if there exists a linear map $T^{-1} : W \to V$ such that $T^{-1}T = \mathrm{id}_V$ and $TT^{-1} = \mathrm{id}_W$.

Recall that if a linear map is invertible its inverse is unique. Recall also that for a linear operator T on a finite dimensional vector space the following are equivalent:

(a) T is an isomorphism;
(b) T is injective;
(c) the kernel of T is $\{0\}$; and
(d) T is surjective.

1.1.4. Definition. Two operators R and T on a vector space V are SIMILAR if there exists an invertible operator S on V such that $R = S^{-1}TS$.

1.1.5. Proposition. *If V is a vector space, then similarity is an equivalence relation on $\mathfrak{L}(V)$.*

1.1.6. Definition. Let V be a finite dimensional vector space and $B = \{e^1, \dots, e^n\}$ be a basis for V. An operator T on V is DIAGONAL if there exist scalars $\alpha_1, \dots, \alpha_n$ such that $Te^k = \alpha_k e^k$ for each $k \in \mathbb{N}_n$. Equivalently, T is diagonal if its matrix representation $[T] = [T_{ij}]$ has the property that $T_{ij} = 0$ whenever $i \neq j$.

Asking whether a particular operator on some finite dimensional vector space is diagonal is, strictly speaking, nonsense. As defined the operator property of being diagonal is definitely *not* a vector space concept. It makes sense only for a vector space *for which a basis has been specified.* This important, if obvious, fact seems to go unnoticed in beginning linear algebra courses, due, I suppose, to a rather obsessive fixation on $\mathbb{R}^n$ in such courses. Here is the relevant *vector space* property.

1.1.7. Definition. An operator T on a finite dimensional vector space V is DIAGONALIZABLE if there exists a basis for V with respect to which T is diagonal. Equivalently, an operator on a finite dimensional vector space *with basis* is diagonalizable if it is similar to a diagonal operator.

1.1.8. Definition. Let M and N be subspaces of a vector space V. If $M \cap N = \{\mathbf{0}\}$ and $M + N = V$, then V is the (INTERNAL) DIRECT SUM of M and N. In this case we write

$$V = M \oplus N .$$

We say that M and N are COMPLEMENTARY SUBSPACES and that each is a (vector space) COMPLEMENT of the other. The CODIMENSION of the subspace M is the dimension of its complement N.

1.1.9. Example. Let $\mathcal{C} = \mathcal{C}[-1,1]$ be the vector space of all continuous real valued functions on the interval $[-1,1]$. A function f in $\mathcal{C}$ is EVEN if $f(-x) = f(x)$ for all $x \in [-1,1]$; it is ODD if $f(-x) = -f(x)$ for all $x \in [-1,1]$. Let $\mathcal{C}_o = \{f \in \mathcal{C} \colon f \text{ is odd }\}$ and $\mathcal{C}_e = \{f \in \mathcal{C} \colon f \text{ is even }\}$. Then $\mathcal{C} = \mathcal{C}_o \oplus \mathcal{C}_e$.

1.1.10. Proposition. *If M is a subspace of a vector space V, then there exists a subspace N of V such that $V = M \oplus N$.*

1.1.11. Proposition. *Let V be a vector space and suppose that $V = M \oplus N$. Then for every $v \in V$ there exist unique vectors $m \in M$ and $n \in N$ such that $v = m + n$.*

1.1.12. Definition. Let V be a vector space and suppose that $V = M \oplus N$. We know from 1.1.11 that for each $\mathbf{v} \in V$ there exist unique vectors $\mathbf{m} \in M$ and $\mathbf{n} \in N$ such that $\mathbf{v} = \mathbf{m} + \mathbf{n}$. Define a function $E_{MN} \colon V \to V$ by $E_{MN} v = n$. The function E_{MN} is the PROJECTION OF V ALONG M ONTO N. (Frequently we write E for E_{MN}. But keep in mind that E depends on both M and N.)

1.1.13. Proposition. *Let V be a vector space and suppose that $V = M \oplus N$. If E is the projection of V along M onto N, then*

(i) *E is linear;*
(ii) *$E^2 = E$ (that is, E is* IDEMPOTENT*);*
(iii) *$\operatorname{ran} E = N$; and*
(iv) *$\ker E = M$.*

1.1.14. Proposition. *Let V be a vector space and suppose that $E \colon V \to V$ is a function which satisfies*

(i) *E is linear, and*
(ii) *$E^2 = E$.*

Then

$$V = \ker E \oplus \operatorname{ran} E$$

and E is the projection of V along $\ker E$ onto $\operatorname{ran} E$.

It is important to note that an obvious consequence of the last two propositions is that a function $T \colon V \to V$ from a finite dimensional vector space into itself is a projection if and only if it is linear and idempotent.

1.1.15. Proposition. *Let V be a vector space and suppose that $V = M \oplus N$. If E is the projection of V along M onto N, then $I - E$ is the projection of V along N onto M.*

As we have just seen, if E is a projection on a vector space V, then the identity operator on V can be written as the sum of two projections E and $I - E$ whose corresponding ranges form a direct sum decomposition of the space $V = \operatorname{ran} E \oplus \operatorname{ran}(I - E)$. We can generalize this to more than two projections.

1.1.16. Definition. Suppose that on a vector space V there exist projection operators $E_1, \dots, E_n$ such that

(i) $I_V = E_1 + E_2 + \dots + E_n$ and
(ii) $E_i E_j = 0$ whenever $i \neq j$.

Then we say that $I_V = E_1 + E_2 + \dots + E_n$ is a RESOLUTION OF THE IDENTITY.

1.1.17. Proposition. *If $I_V = E_1 + E_2 + \cdots + E_n$ is a resolution of the identity on a vector space V, then $V = \bigoplus_{k=1}^{n} \operatorname{ran} E_k$.*

1.1.18. Example. Let P be the plane in $\mathbb{R}^3$ whose equation is $x - z = 0$ and L be the line whose equations are $y = 0$ and $x = -z$. Let E be the projection of $\mathbb{R}^3$ along L onto P and F be the projection of $\mathbb{R}^3$ along P onto L. Then

$$[E] = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \qquad \text{and} \qquad [F] = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

1.1.19. Definition. A complex number λ is an EIGENVALUE of an operator T on a vector space V if $\ker(T - \lambda I_V)$ contains a nonzero vector. Any such vector is an EIGENVECTOR of T associated with λ and $\ker(T - \lambda I_V)$ is the EIGENSPACE of T associated with λ. The set of all eigenvalues of the operator T is its POINT SPECTRUM and is denoted by $\sigma_p(T)$.

If M is an $n \times n$ matrix, then $\det(M - \lambda I_n)$ (where I_n is the $n \times n$ identity matrix) is a polynomial in λ of degree n. This is the CHARACTERISTIC POLYNOMIAL of M. A standard way of computing the eigenvalues of an operator T on a finite dimensional vector space is to find the zeros of the characteristic polynomial of its matrix representation. It is an easy consequence of the multiplicative property of the determinant function that the characteristic polynomial of an operator T on a vector space V is independent of the basis chosen for V and hence of the particular matrix representation of T that is used.

1.1.20. Example. The eigenvalues of the operator on (the real vector space) $\mathbb{R}^3$ whose matrix representation is $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ are -2 and $+2$, the latter having (both algebraic and geometric) multiplicity 2. The eigenspace associated with the negative eigenvalue is $\operatorname{span}\{(1, 0, -1)\}$ and the eigenspace associated with the positive eigenvalue is $\operatorname{span}\{(1, 0, 1), (0, 1, 0)\}$.

The central fact asserted by the finite dimensional vector space version of the *spectral theorem* is that every diagonalizable operator on such a space can be written as a linear combination of projection operators where the coefficients of the linear combination are the eigenvalues of the operator and the ranges of the projections are the corresponding eigenspaces. Thus if T is a diagonalizable operator on a finite dimensional vector space V, then V has a basis consisting of eigenvectors of T.

Here is a formal statement of the theorem.

1.1.21. Theorem (Spectral Theorem: vector space version)**.** *Suppose that T is a diagonalizable operator on a finite dimensional vector space V. Let $\lambda_1, \dots, \lambda_n$ be the (distinct) eigenvalues of T. Then there exists a resolution of the identity $I_V = E_1 + \cdots + E_n$, where for each k the range of the projection E_k is the eigenspace associated with λ_k, and furthermore*

$$T = \lambda_1 E_1 + \cdots + \lambda_n E_n\,.$$

PROOF. A good proof of this theorem can be found in [17] on page 212. □

1.1.22. Example. Let T be the operator on (the real vector space) $\mathbb{R}^2$ whose matrix representation is $\begin{bmatrix} -7 & 8 \\ -16 & 17 \end{bmatrix}$.

(a) The characteristic polynomial for T is $c_T(\lambda) = \lambda^2 - 10\lambda + 9$.

(b) The eigenspace M_1 associated with the eigenvalue 1 is $\operatorname{span}\{(1, 1)\}$.

(c) The eigenspace M_2 associated with the eigenvalue 9 is $\operatorname{span}\{(1, 2)\}$.

(d) We can write T as a linear combination of projection operators. In particular,

$$T = 1 \cdot E_1 + 9 \cdot E_2 \text{ where } [E_1] = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \text{ and } [E_2] = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}.$$

(e) Notice that the sum of $[E_1]$ and $[E_2]$ is the identity matrix and that their product is the zero matrix.

(f) The matrix $S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ diagonalizes $[T]$. That is, $S^{-1}\,[T]\,S = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$.

(g) A matrix representing $\sqrt{T}$ is $\begin{bmatrix} -1 & 2 \\ -4 & 5 \end{bmatrix}$.

1.2. Normal Operators on an Inner Product Space

1.2.1. Definition. Let V be a vector space. A function which associates to each pair of vectors x and y in V a complex number $\langle x, y\rangle$ is an INNER PRODUCT (or a DOT PRODUCT) on V provided that the following four conditions are satisfied:

(a) If x, y, $z \in V$, then
$$\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle.$$
(b) If x, $y \in V$, then
$$\langle \alpha x, y\rangle = \alpha\langle x, y\rangle.$$
(c) If x, $y \in V$, then
$$\langle x, y\rangle = \overline{\langle y, x\rangle}.$$
(d) For every nonzero x in V we have $\langle x, x\rangle > 0$.

Conditions (a) and (b) show that an inner product is linear in its first variable. Conditions (a) and (b) of proposition 1.2.3 say that an inner product is CONJUGATE LINEAR in its second variable. When a mapping is linear in one variable and conjugate linear in the other, it is often called SESQUILINEAR (the prefix “sesqui-” means “one and a half”). Taken together conditions (a)–(d) say that the inner product is a *positive definite conjugate symmetric sesquilinear form*.

1.2.2. Notation. If V is a vector space which has been equipped with an inner product and $x \in V$ we introduce the abbreviation
$$\|x\| := \sqrt{\langle x, x\rangle}$$
which is read *the norm of* x or *the length of* x. (This somewhat optimistic terminology is justified in proposition 1.2.11 below.)

1.2.3. Proposition. *If x, y, and z are vectors in an inner product space and $\alpha \in \mathbb{C}$, then*

(a) $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$,
(b) $\langle x, \alpha y\rangle = \overline{\alpha}\langle x, y\rangle$, *and*
(c) $\langle x, x\rangle = 0$ *if and only if* $x = 0$.

1.2.4. Example. For vectors $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ belonging to $\mathbb{C}^n$ define
$$\langle x, y\rangle = \sum_{k=1}^{n} x_k \overline{y_k}\,.$$
Then $\mathbb{C}^n$ is an inner product space.

1.2.5. Example. Let l_2 be the set of all square summable sequences of complex numbers. (A sequence $x = (x_k)_{k=1}^{\infty}$ is SQUARE SUMMABLE if $\sum_{k=1}^{\infty}|x_k|^2 < \infty$.) For vectors $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ belonging to l_2 define
$$\langle x, y\rangle = \sum_{k=1}^{\infty} x_k \overline{y_k}\,.$$

Then l_2 is an inner product space. (It must be shown, among other things, that the series in the preceding definition actually converges.)

1.2.6. Example. For $a < b$ let $\mathcal{C}([a, b], \mathbb{C})$ be the family of all continuous complex valued functions on the interval $[a, b]$. For every $f, g \in \mathcal{C}([a, b])$ define

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx.$$

Then $\mathcal{C}([a, b])$ is an inner product space.

1.2.7. Theorem. *In every inner product space the* Schwarz inequality

$$|\langle x, y\rangle| \le \|x\|\,\|y\|.$$

holds for all vectors x and y.

1.2.8. Proposition. *If (x_n) is a sequence in an inner product space V which converges to a vector $a \in V$, then $\langle x_n, y\rangle \to \langle a, y\rangle$ for every $y \in V$.*

1.2.9. Definition. Let V be a vector space. A function $\|\ \|\colon V \to \mathbb{R}\colon x \mapsto \|x\|$ is a NORM on V if

(i) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in V$;
(ii) $\|\alpha x\| = |\alpha|\,\|x\|$ for all $x \in V$ and $\alpha \in \mathbb{R}$; and
(iii) if $\|x\| = 0$, then $x = \mathbf{0}$.

The expression $\|x\|$ may be read as "the *norm* of x" or "the *length* of x". If the function $\|\ \|$ satisfies (i) and (ii) above (but perhaps not (iii)) it is a SEMINORM on V.

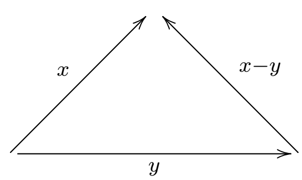
A vector space on which a norm has been defined is a NORMED LINEAR SPACE (or NORMED VECTOR SPACE). A vector in a normed linear space which has norm 1 is a UNIT VECTOR.

1.2.10. Proposition. *If $\|\ \|$ is norm (or a seminorm) on a vector space V, then $\|x\| \ge 0$ for every $x \in V$ and $\|\mathbf{0}\| = 0$.*

Every inner product space is a normed linear space.

1.2.11. Proposition. *Let V be an inner product space. The map $x \mapsto \|x\|$ defined on V in 1.2.2 is a norm on V.*

Every normed linear space is a metric space. More precisely, a norm on a vector space induces a metric d, which is defined by $d(x, y) = \|x - y\|$. That is, the distance between two vectors is the length of their difference.



If no other metric is specified we always regard a normed linear space as a metric space under this induced metric. Thus every metric (and hence every topological) concept makes sense in a (semi)normed linear space.

1.2.12. Proposition. *Let V be a normed linear space. Define $d\colon V \times V \to \mathbb{R}$ by $d(x, y) = \|x - y\|$. Then d is a metric on V. If V is only a seminormed space, then d is a pseudometric.*

When there is a topology on a vector space, in particular in normed linear spaces, we reserve the word "operator" for those linear mappings from the space into itself which are continuous. We are usually not made aware of this conflicting terminology in elementary linear algebra because that subject focuses primarily on finite dimensional vector and inner product spaces where the question is moot: on finite dimensional normed linear spaces all linear maps are automatically continuous (see proposition 1.2.14 below).

1.2.13. Definition. An OPERATOR on a normed linear space V is a *continuous* linear map from V into itself.

1.2.14. Proposition. *If V and W are normed linear spaces and V is finite dimensional, then every linear map $T\colon V \to W$ is continuous.*

PROOF. See [5], proposition III.3.4.

1.2.15. Definition. Vectors x and y in an inner product space V are ORTHOGONAL (or PERPENDICULAR) if $\langle x, y\rangle = 0$. In this case we write $x \perp y$. Subsets A and B of V are ORTHOGONAL if $a \perp b$ for every $a \in A$ and $b \in B$. In this case we write $A \perp B$.

1.2.16. Definition. If M and N are subspaces of an inner product space V we use the notation $V = M \oplus N$ to indicate not only that V is the (vector space) direct sum of M and N but also that M and N are orthogonal. Thus we say that V is the (INTERNAL) ORTHOGONAL DIRECT SUM of M and N.

1.2.17. Proposition. *Let a be a vector in an inner product space V. Then $a \perp x$ for every $x \in V$ if and only if $a = 0$.*

1.2.18. Proposition (The Pythagorean theorem)**.** *If $x \perp y$ in an inner product space, then*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 .$$

1.2.19. Definition. Let V and W be inner product spaces. For (v, w) and (v', w') in $V \times W$ and $\alpha \in \mathbb{C}$ define

$$(v, w) + (v', w') = (v + v', w + w')$$

and

$$\alpha(v, w) = (\alpha v, \alpha w) .$$

This results in a vector space, which is the *(external) direct sum* of V and W. To make it into an inner product space define

$$\langle (v, w), (v', w')\rangle = \langle v, v'\rangle + \langle w, w'\rangle.$$

This makes the direct sum of V and W into an inner product space. It is the (EXTERNAL ORTHOGONAL) DIRECT SUM of V and W and is denoted by $V \oplus W$.

Notice that the same notation $\oplus$ is used for both internal and external direct sums and for both vector space direct sums (see definition 1.1.8) and orthogonal direct sums. So when we see the symbol $V \oplus W$ it is important to know which category we are in: vector spaces or inner product spaces, especially as it is common practice to omit the word "orthogonal" as a modifier to "direct sum" even in cases when it is intended.

1.2.20. Example. In $\mathbb{R}^2$ let M be the x-axis and L be the line whose equation is $y = x$. If we think of $\mathbb{R}^2$ as a (real) vector space, then it is correct to write $\mathbb{R}^2 = M \oplus L$. If, on the other hand, we regard $\mathbb{R}^2$ as a (real) inner product space, then $\mathbb{R}^2 \neq M \oplus L$ (because M and L are not perpendicular).

1.2.21. Proposition. *Let V be an inner product space. The inner product on V, regarded as a map from $V \oplus V$ into $\mathbb{C}$, is continuous. So is the norm, regarded as a map from V into $\mathbb{R}$.*

Concerning the proof of the preceding proposition, notice that the maps $(v, v') \mapsto \|v\| + \|v'\|$, $(v, v') \mapsto \sqrt{\|v\|^2 + \|v'\|^2}$, and $(v, v') \mapsto \max\{\|v\|, \|v'\|\}$ are all norms on $V \oplus V$. Which one is induced by the inner product on $V \oplus V$? Why does it not matter which one we use in proving that the inner product is continuous?

1.2.22. Proposition (The parallelogram law)**.** *If x and y are vectors in an inner product space, then*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 .$$

1.2.23. Example. Consider the space $\mathcal{C}([0,1])$ of continuous complex valued functions defined on $[0,1]$. Under the UNIFORM NORM

$$\|f\|_u := \sup\{|f(x)| : 0 \le x \le 1\}$$

$\mathcal{C}([0,1])$ is a normed linear space. There is no inner product on $\mathcal{C}([0,1])$ which induces this norm.

Hint for proof. Use the preceding proposition.

1.2.24. Proposition (The polarization identity)**.** *If x and y are vectors in an inner product space, then*

$$\langle x, y\rangle = \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2)\,.$$

1.2.25. Notation. Let V be an inner product space, $x \in V$, and A, $B \subseteq V$. If $x \perp a$ for every $a \in A$, we write $x \perp A$; and if $a \perp b$ for every $a \in A$ and $b \in B$, we write $A \perp B$. We define $A^\perp$, the ORTHOGONAL COMPLEMENT of A, to be $\{x \in V : x \perp A\}$. We write $A^{\perp\perp}$ for $\left(A^\perp\right)^\perp$.

1.2.26. Proposition. *If A is a subset of an inner product space V, then $A^\perp$ is a closed linear subspace of V.*

1.2.27. Theorem (Gram-Schmidt Orthogonalization)**.** *If $\{v^1, \dots, v^n\}$ is a linearly independent subset of an inner product space V, then there exists an orthogonal set $\{e^1, \dots, e^n\}$ of vectors such that* $\operatorname{span}\{v^1, \dots, v^n\} = \operatorname{span}\{e^1, \dots, e^n\}$.

1.2.28. Corollary. *If M is a subspace of a finite dimensional inner product space V then $V = M \oplus M^\perp$.*

For a counterexample showing that the preceding result need not hold in an infinite dimensional space, see example 2.1.6.

1.2.29. Definition. A LINEAR FUNCTIONAL on a vector space V is a linear map from V into its scalar field. The set of all linear functionals on V is the (ALGEBRAIC) DUAL SPACE of V. We will use the notation $V^\#$ (rather than $\mathfrak{L}(V, \mathbb{C})$) for the algebraic dual space.

1.2.30. Theorem (Riesz-Fréchet Theorem)**.** *If $f \in V^\#$ where V is a finite dimensional inner product space, then there exists a unique vector a in V such that*

$$f(x) = \langle x, a\rangle$$

for all x in V.

We will prove shortly that every continuous linear functional on an arbitrary inner product space has the above representation. The finite dimensional version stated here is a special case, since every linear map on a finite dimensional inner product space is continuous.

1.2.31. Definition. Let $T : V \to W$ be a linear transformation between complex inner product spaces. If there exists a function $T^* : W \to V$ which satisfies

$$\langle T\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{v}, T^*\mathbf{w}\rangle$$

for all $\mathbf{v} \in V$ and $\mathbf{w} \in W$, then T^* is the ADJOINT (or CONJUGATE TRANSPOSE, or HERMITIAN CONJUGATE) of T.

1.2.32. Proposition. *If $T : V \to W$ is a linear map between finite dimensional inner product spaces, then T^* exists.*

Hint for proof. The functional $\phi : V \times W \to \mathbb{C} : (v, w) \mapsto \langle Tv, w\rangle$ is sesquilinear. Fix $w \in W$ and define $\phi_w : V \to \mathbb{C} : v \mapsto \phi(v, w)$. Then $\phi_w \in V^\#$. Use the *Riesz-Fréchet theorem* (1.2.30).

1.2.33. Proposition. *If $T : V \to W$ is a linear map between finite dimensional inner product spaces, then the function T^* defined above is linear and $T^{**} = T$.*

1.2.34. Theorem (The fundamental theorem of linear algebra)**.** *If $T\colon V \to W$ is a linear map between finite dimensional inner product spaces, then*

$$\ker T^* = (\operatorname{ran} T)^{\perp} \quad \textit{and} \quad \operatorname{ran} T^* = (\ker T)^{\perp}.$$

1.2.35. Definition. An operator U on an inner product space is UNITARY if $UU^* = U^*U = I$, that is if $U^* = U^{-1}$.

1.2.36. Definition. Two operators R and T on an inner product space V are UNITARILY EQUIVALENT if there exists a unitary operator U on V such that $R = U^*TU$.

1.2.37. Proposition. *If V is an inner product space, then unitary equivalence is in fact an equivalence relation on $\mathfrak{L}(V)$.*

1.2.38. Definition. An operator T on a finite dimensional inner product space V is UNITARILY DIAGONALIZABLE if there exists an orthonormal basis for V with respect to which T is diagonal. Equivalently, an operator on a finite dimensional inner product space *with basis* is diagonalizable if it is unitarily equivalent to a diagonal operator.

1.2.39. Definition. An operator T on an inner product space is SELF-ADJOINT (or HERMITIAN) if $T^* = T$.

1.2.40. Definition. A projection P in an inner product space is an ORTHOGONAL PROJECTION if it is self-adjoint. If M is the range of an orthogonal projection we will adopt the notation P_M for the projection rather than the more cumbersome $E_{M^{\perp}M}$.

CAUTION. A projection on a vector space or a normed linear space is linear and idempotent, while an orthogonal projection on an inner product space is linear, idempotent, and self-adjoint. This otherwise straightforward situation is somewhat complicated by a common tendency to refer to orthogonal projections simply as "projections". In fact, later in these notes we will adopt this very convention. In inner product spaces $\oplus$ usually indicates orthogonal direct sum and "projection" usually means "orthogonal projection". In many elementary linear algebra texts, where everything happens in $\mathbb{R}^n$, it can be quite exasperating trying to divine whether on any particular page the author is treating $\mathbb{R}^n$ as a vector space or as an inner product space.

1.2.41. Proposition. *If P is an orthogonal projection on an inner product space V, then we have the orthogonal direct sum decomposition $V = \ker P \oplus \operatorname{ran} P$.*

1.2.42. Definition. If $I_V = P_1 + P_2 + \cdots + P_n$ is a resolution of the identity in an inner product space V and each P_k is an orthogonal projection, then we say that $I = P_1 + P_2 + \cdots + P_n$ is an ORTHOGONAL RESOLUTION OF THE IDENTITY.

1.2.43. Proposition. *If $I_V = P_1 + P_2 + \cdots + P_n$ is an orthogonal resolution of the identity on an inner product space V, then $V = \bigoplus_{k=1}^{n} \operatorname{ran} P_k$.*

1.2.44. Definition. An operator N on an inner product space is NORMAL if $NN^* = N^*N$.

Two great triumphs of linear algebra are the *spectral theorem* for operators on a (complex) finite dimensional inner product space (see 1.2.45), which gives a simply stated necessary and sufficient condition for an operator to be unitarily diagonalizable, and theorem 1.2.46, which gives a complete classification of those operators.

1.2.45. Theorem (Spectral Theorem for Complex Inner Product Spaces)**.** *Let T be an operator on a finite dimensional inner product space V with (distinct) eigenvalues $\lambda_1, \dots, \lambda_n$. Then T is unitarily diagonalizable if and only if it is normal. If T is normal, then there exists an orthogonal resolution of the identity $I_V = P_1 + \cdots + P_n$, where for each k the range of the orthogonal projection P_k is the eigenspace associated with λ_k, and furthermore*

$$T = \lambda_1 P_1 + \cdots + \lambda_n P_n.$$

Proof. See [26], page 227. □

1.2.46. Theorem. *Two normal operators on a finite dimensional inner product space are unitarily equivalent if and only if they have the same eigenvalues each with the same multiplicity; that is, if and only if they have the same characteristic polynomial.*

Proof. See [17], page 357. □

Much of the remainder of this course is about the long and difficult adventure of finding appropriate generalizations of the preceding two results to the infinite dimensional setting and the astonishing landscapes which came into view along the way.

CHAPTER 2

THE ALGEBRA OF HILBERT SPACE OPERATORS

2.1. Hilbert Space Geometry

Inner products induce norms (see proposition 1.2.11) and norms induce metrics (see 1.2.12), so the notion of completeness makes sense in both cases. The next definition tells us that a *Banach space* is a complete normed linear space and a *Hilbert space* is a complete inner product space.

2.1.1. Definition. If a normed linear space is complete with respect to the metric induced by its norm, it is a BANACH SPACE. If an inner product space is complete with respect to the metric induced by the norm induced by the inner product on V, it is a HILBERT SPACE.

2.1.2. Example. The space $\mathbb{C}^n$ of n-tuples of complex numbers is a Hilbert space under the inner product defined in example 1.2.4.

2.1.3. Example. The space l_2 of all square summable sequences of complex numbers is a Hilbert space under the inner product defined in example 1.2.5.

2.1.4. Definition. Let (x_n) be a sequence of elements of a set S and P be some property that members of S may possess. We say that the sequence (x_n) EVENTUALLY has property P if there exists $n_0 \in \mathbb{N}$ such that x_n has property P for every $n \geq n_0$. (Another way to say the same thing: x_n has property P for all but finitely many n.)

2.1.5. Example. We denote by l_c, the vector space of all sequences (a_n) of complex numbers which are eventually zero. (The vector space operations are defined pointwise.) We make the space l_c into an inner product space by defining $\langle a, b \rangle = \langle\, (a_n), (b_n)\, \rangle := \sum_{k=1}^{\infty} a_n \overline{b_n}$. The resulting space is not a Hilbert space.

2.1.6. Example. The vector subspace l_c of the Hilbert space l_2 shows that corollary 1.2.28 need not hold for infinite dimensional spaces.

2.1.7. Example. Let l_1 be the family of all sequences $(a_1, a_2, a_3, \dots)$ which are ABSOLUTELY SUMMABLE; that is, such that $\sum_{k=1}^{\infty} |a_k| < \infty$. This is a Banach space under pointwise operations of addition and scalar multiplication and norm given by

$$\|a\| = \sum_{k=1}^{\infty} |a_k| \,.$$

The norm does not arise from an inner product, so it is not a Hilbert space.

2.1.8. Example. Let μ be a measure on a σ-algebra $\mathfrak{A}$ of subsets of a set S. A complex valued function f on S is MEASURABLE if the inverse image under f of every Borel set (equivalently, every open set) in $\mathbb{C}$ belongs to $\mathfrak{A}$. We define an equivalence relation $\sim$ on the family of measurable complex valued functions by setting $f \sim g$ whenever f and g differ on a set of measure zero, that is, whenever $\mu(\{x \in S \colon f(x) \neq g(x)\}) = 0$. We adopt conventional notation and denote the equivalence class containing f by f itself (rather than something more logical such as $[f]$). We denote the family of (equivalence classes) of measurable complex valued functions on S by $\mathcal{M}(S)$. A function $f \in \mathcal{M}(S)$ is SQUARE INTEGRABLE if $\int_S |f(x)|^2 \, d\mu(x) < \infty$. We denote the family of (equivalence classes of) square integrable functions on S by $L_2(S)$. For every $f, g \in L_2(S)$ define

$$\langle f, g \rangle = \int_S f(x)\overline{g(x)} \, d\mu(x) \,.$$

With this inner product (and the obvious pointwise vector space operations) $L_2(S)$ is a Hilbert space.

Here is a standard example of a space which is a Banach space but not a Hilbert space.

2.1.9. Example. As above let μ be a measure on a σ-algebra $\mathfrak{A}$ of subsets of a set S. A function $f \in \mathcal{M}(S)$ is INTEGRABLE if $\int_S |f(x)|\,d\mu(x) < \infty$. We denote the family of (equivalence classes of) integrable functions on S by $L_1(S)$. An attempt to define an inner product on $L_1(S)$ as we did on $L_2(S)$ by setting $\langle f, g\rangle = \int_S f(x)\overline{g(x)}\,d\mu(x)$ fails. (Why?) Nevertheless, the function $f \mapsto \int_S |f(x)|\,d\mu(x)$ is a norm on $L_1(S)$. It is denoted by $\|\cdot\|_1$. With respect to this norm (and the obvious pointwise vector space operations), $L_1(S)$ is a Banach space.

2.1.10. Example. If X is a compact Hausdorff space the family $\mathcal{C}(X)$ of continuous complex valued functions on X is a Banach space under the UNIFORM NORM, which is defined by

$$\|f\|_u := \sup\{|f(x)| \colon x \in X\}.$$

We have seen in example 1.2.23 that this norm does not arise from an inner product.

2.1.11. Example. If X is a locally compact Hausdorff space the uniform norm may not be defined on the family $\mathcal{C}(X)$ of continuous complex valued functions on X. (Why not?) However, it is defined on the family $\mathcal{C}_b(X)$ of bounded continuous complex valued functions on X and on $\mathcal{C}_0(X)$ the family of continuous complex valued functions on X that vanish at infinity (A complex valued function on X is said to VANISH AT INFINITY if for every $\epsilon > 0$ there exists a compact subset K of X such that $|f(x)| < \epsilon$ whenever $x \notin K$.) Both $\mathcal{C}_b(X)$ and $\mathcal{C}_0(X)$ are Banach spaces under the uniform norm (and the obvious pointwise vector space operations).

2.1.12. Example. Let H be the set of all absolutely continuous functions on $[0,1]$ such that f' belongs to $L^2([0,1])$ and $f(0) = 0$. For f and g in H define

$$\langle f, g\rangle = \int_0^1 f'(t)\overline{g'(t)}\,dt.$$

This is an inner product on H under which H becomes a Hilbert space.

2.1.13. Convention. In the context of Hilbert (and, more generally, Banach) spaces the word "subspace" will always mean *closed vector subspace*. To indicate that M is a subspace of H we write $M \preccurlyeq H$. A (not necessarily closed) vector subspace of a Hilbert space is often called by other names such as *linear subspace* or *linear manifold*.

2.1.14. Definition. Let A be a nonempty subset of a Banach space B. We define the CLOSED LINEAR SPAN of A (denoted by $\bigvee A$) to be the intersection of the family of all subspaces of B which contain A. This is frequently referred to as *the smallest subspace of B containing A*.

2.1.15. Proposition. *The preceding definition makes sense. It is equivalent to defining $\bigvee A$ to be the closure of the (linear) span of A.*

2.1.16. Definition. Let V be a vector space and a, $b \in V$. Then the SEGMENT between a and b, denoted by $\operatorname{Seg}[a, b]$, is defined to be

$$\{(1-t)a + tb \colon 0 \le t \le 1\}.$$

A subset C of V is CONVEX if $\operatorname{Seg}[a, b] \subseteq C$ whenever a, $b \in C$.

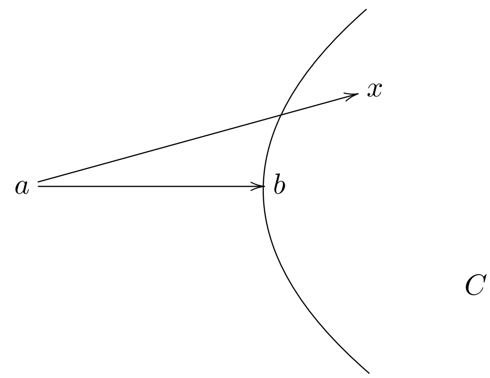
2.1.17. Proposition. *The intersection of a family of convex subsets of a vector space is convex.*

2.1.18. Proposition. *If $T \colon V \to W$ is a linear map between vector spaces and C is a convex subset of V, then $T^{\to}(C)$ is a convex subset of W.*

2.1.19. Definition. Let a be a element of a normed linear space V and $r > 0$. The OPEN BALL of radius r about a is $\{x \in V \colon \|x - a\| < r\}$. The CLOSED BALL of radius r about a is $\{x \in V \colon \|x - a\| \le r\}$. And the SPHERE of radius r about a is $\{x \in V \colon \|x - a\| = r\}$.

2.1.20. Proposition. *Every open ball (and every closed ball) in a normed linear space is convex.*

2.1.21. Theorem (Minimizing Vector Theorem)**.** *If C is a nonempty closed convex subset of a Hilbert space H and $a \in C^c$, then there exists a unique $b \in C$ such that $\|b - a\| \le \|x - a\|$ for every $x \in C$.*



2.1.22. Example. The vector space $\mathbb{R}^2$ under the uniform metric is a Banach space. To see that in this space the *minimizing vector theorem* does not hold take C to be the closed unit ball about the origin and a to be the point $(2, 0)$.

2.1.23. Example. The sets $C_1 = \{f \in \mathcal{C}([0,1],\mathbb{R})\colon \int_0^{1/2} f - \int_{1/2}^1 f = 1\}$ and $C_2 = \{f \in L_1([0,1],\mathbb{R})\colon \int_0^1 f = 1\}$ are examples that show that neither the existence nor the uniqueness claims of the *minimizing vector theorem* necessarily holds in a Banach space.

2.1.24. Theorem (Vector decomposition theorem)**.** *Let H be a Hilbert space, M be a subspace of H, and $x \in H$. Then there exist unique vectors $y \in M$ and $z \in M^\perp$ such that $x = y + z$.*

2.1.25. Proposition. *Let H be a Hilbert space. Then the following hold:*

(a) *if $M \subseteq H$, then $M \subseteq M^{\perp\perp}$;*
(b) *if $M \subseteq N \subseteq H$, then $N^\perp \subseteq M^\perp$;*
(c) *if M is a subspace of H, then $M = M^{\perp\perp}$; and*
(d) *if $M \subseteq H$, then $\bigvee M = M^{\perp\perp}$.*

2.1.26. Proposition. *If M is a subspace of a Hilbert space H, then $H = M \oplus M^\perp$. (This conclusion need not follow if M is assumed only to be a linear subspace of H.)*

2.1.27. Example. The preceding result says that every subspace of a Hilbert space is a direct summand. This is not true if M is assumed to be just a linear subspace of the space. For example, notice that $M = l$ (see example 2.1.5) is a linear subspace of the Hilbert space l_2 (see example 2.1.3) but $M^\perp = \{\mathbf{0}\}$.

2.1.28. Proposition. *Let M and N be (closed linear) subspaces of a Hilbert space. Then*

(a) *$(M + N)^\perp = (M \cup N)^\perp = M^\perp \cap N^\perp$, and*
(b) *$(M \cap N)^\perp = M^\perp + N^\perp$.*

2.1.29. Definition. Let V and W be (complex) vector spaces. A function $\phi\colon V \times W \to \mathbb{C}$ is a SESQUILINEAR FUNCTIONAL if it satisfies

(i) $\phi(u + v, w) = \phi(u, w) + \phi(v, w)$,
(ii) $\phi(\alpha v, w) = \alpha\phi(v, w)$,
(iii) $\phi(v, w + x) = \phi(v, w) + \phi(v, x)$, and
(iv) $\phi(v, \alpha w) = \overline{\alpha}\,\phi(v, w)$

for all u,$v \in V$, all w, $x \in W$ and all $\alpha \in \mathbb{C}$. When $W = V$ there are two more terms we will need. A sesquilinear functional $\phi\colon V \times V \to \mathbb{C}$ is (CONJUGATE) SYMMETRIC if

(v) $\phi(u,v) = \overline{\phi(v,u)}$ for all $u,\, v \in V$.

A sesquilinear functional $\phi\colon V \times V \to \mathbb{C}$ is POSITIVE SEMIDEFINITE if

(vi) $\phi(v,v) \geq 0$ for all $v \in V$.

If $\phi\colon V \times V \to \mathbb{C}$ is sesquilinear, then the function $\hat{\phi}$ defined on V by $\hat{\phi}(v) = \phi(v,v)$ is the QUADRATIC FORM ASSOCIATED WITH ϕ.

2.1.30. Proposition. *Let ϕ be a sesquilinear functional on a vector space V and $\hat{\phi}$ be its associated quadratic form. Then*

(a) $4\phi(u,v) = \hat{\phi}(u+v) - \hat{\phi}(u-v) + i\hat{\phi}(u+iv) - i\hat{\phi}(u-iv)$ *for all $u,\, v \in V$;*

(b) *ϕ is uniquely determined by $\hat{\phi}$; and*

(c) *ϕ is symmetric if and only if $\hat{\phi}$ is real valued.*

Notice that part (a) of the preceding proposition is just a slight generalization of the *polarization identity* (proposition 1.2.24).

2.1.31. Definition. A positive symmetric sesquilinear form is a SEMI-INNER PRODUCT.

2.1.32. Proposition. *The* Schwarz inequality *(see 1.2.7) holds for semi-inner products. That is, if x and y are elements of a semi-inner product space, then*

$$|\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle\,.$$

2.1.33. Proposition. *Let V be a semi-inner product space and $z \in V$. Then $\langle z,z\rangle = 0$ if and only if $\langle z,y\rangle = 0$ for all $y \in V$.*

2.1.34. Proposition. *If V is a semi-inner product space, then the set $L := \{z \in V\colon \langle z,z\rangle = 0\}$ is a vector subspace of V and the quotient vector space V/L can be made into an inner product space by defining*

$$\langle\, [x], [y]\,\rangle := \langle x,y\rangle$$

for all $[x],\, [y] \in V/L$.

2.2. Operators on Hilbert Spaces

2.2.1. Definition. A linear transformation $T\colon V \to W$ between normed linear spaces is BOUNDED if $T(B)$ is a bounded subset of W whenever B is a bounded subset of V. In other words, a bounded linear map takes bounded sets to bounded sets. We denote by $\mathfrak{B}(V,W)$ the family of all bounded linear transformations from V into W. A bounded linear map from a space V into itself is an OPERATOR and we denote the family of all operators on V by $\mathfrak{B}(V)$.

2.2.2. Proposition. *If $T\colon V \to W$ is a linear transformation between normed linear spaces, then the following are equivalent:*

(i) *T is continuous at 0.*
(ii) *T is continuous on V.*
(iii) *T is uniformly continuous on V.*
(iv) *The image of the closed unit ball under T is bounded.*
(v) *T is bounded.*
(vi) *There exists a number $M > 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in V$.*

2.2.3. Proposition. *Let V and W be normed linear spaces. If $S,\, T \in \mathfrak{B}(V,W)$ and $\alpha \in \mathbb{C}$, then $S+T$ and αT belong to $\mathfrak{B}(V,W)$.*

2.2.4. Proposition. *Let $T\colon V \to W$ be a bounded linear transformation between normed linear spaces. Then the following four numbers (exist and) are equal.*

(i) $\sup\{\|Tx\|\colon \|x\| \leq 1\}$
(ii) $\sup\{\|Tx\|\colon \|x\| = 1\}$

(iii) $\sup\{\|Tx\|\,\|x\|^{-1} \colon x \neq \mathbf{0}\}$
(iv) $\inf\{M > 0 \colon \|Tx\| \leq M\|x\| \ \ \textit{for all } x \in V\}$

2.2.5. Definition. If T is a bounded linear map between normed linear spaces, then $\|T\|$, called the NORM of T, is defined to be any one of the four expressions in the previous proposition.

2.2.6. Proposition. *Let V and W be normed linear spaces. The function*

$$\| \quad \| \colon \mathfrak{B}(V,W) \to \mathbb{R} \colon T \mapsto \|T\|$$

is a norm. Under this norm $\mathfrak{B}(V,W)$ is a Banach space whenever W is.

2.2.7. Proposition. *Let U, V, and W be normed linear spaces. If $S \in \mathfrak{B}(U,V)$ and $T \in \mathfrak{B}(V,W)$, then $TS \in \mathfrak{B}(U,W)$ and $\|TS\| \leq \|T\|\|S\|$.*

2.2.8. Example. On any normed linear space V the *identity operator*

$$\mathrm{id}_V = I_V = I \colon V \to V \colon v \mapsto v$$

is bounded and $\|I_V\| = 1$. The *zero operator*

$$\mathbf{0}_V = \mathbf{0} \colon V \to V \colon v \mapsto \mathbf{0}$$

is also bounded and $\|0_V\| = 0$.

2.2.9. Example. Let $T \colon \mathbb{R}^2 \to \mathbb{R}^3 \colon (x,y) \mapsto (3x, x + 2y, x - 2y)$. Then T is bounded and $\|T\| = \sqrt{11}$.

2.2.10. Example. Let X be a compact Hausdorff space and ϕ be a continuous real valued function on X. Define

$$M_\phi \colon \mathcal{C}(X) \to \mathcal{C}(X) \colon f \mapsto \phi f\,.$$

Then M_ϕ is an operator on the Banach space $\mathcal{C}(X)$ (see example 2.1.10) and $\|M_\phi\| = \|\phi\|_u$. The operator M_ϕ is called a MULTIPLICATION OPERATOR.)

2.2.11. Example. Under the uniform norm the family $\mathcal{D}([0,1],\mathbb{R})$ of all continuously differentiable real valued functions on $[0,1]$ is a real normed linear space. The differentiation map

$$D \colon \mathcal{D}([0,1],\mathbb{R}) \to \mathcal{C}([0,1],\mathbb{R}) \colon f \mapsto f'$$

(although linear) is *not* bounded.

2.2.12. Notation. If V is a normed linear space we denote by V^* the family of all continuous linear functionals on V. We call this the DUAL SPACE of V.

If V is finite dimensional, then $V^* = V^\#$ (see proposition 1.2.14). By proposition 2.2.6 V^* is always a Banach space (whether or not V itself is complete).

2.2.13. Convention. When we say *dual space* we always mean V^*; when referring to $V^\#$ we will say *algebraic dual space.*

2.2.14. Example. Let X be a compact topological space. For $a \in X$ define

$$E_a \colon \mathcal{C}(X) \to \mathbb{C} \colon f \mapsto f(a)\,.$$

Then $E_a \in \mathcal{C}^*(X)$ (where $\mathcal{C}^*(X)$ is a shortened notation for $\big(\mathcal{C}(X)\big)^*$) and $\|E_a\| = 1$.

2.2.15. Example. Recall from example 2.1.3 that the family l_2 of all square summable sequences of complex numbers is a Hilbert space. Let

$$S \colon l_2 \to l_2 \colon (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots)\,.$$

Then S is an operator on l_2,called the UNILATERAL SHIFT OPERATOR, and $\|S\| = 1$.

2.2.16. Example. Let $a = (a_1, a_2, a_3, \dots)$ be a bounded sequence of complex numbers. Define a mapping

$$D_a : l_2 \to l_2 \colon (x_1, x_2, x_3, \dots) \mapsto (a_1x_1, a_2x_2, a_3x_3, \dots) .$$

Then D_a is an operator on l_2 whose norm is $\|a\|_u$. This is a DIAGONAL OPERATOR.

2.2.17. Example. Let $(S, \mathfrak{A}, \mu)$ be a sigma-finite measure space and $L_2(S)$ be the Hilbert space of all (equivalence classes of) complex valued functions on S which are square integrable with respect to μ (see example 2.1.8). Let $k\colon S \times S \to \mathbb{C}$ be square integrable with respect to the product measure $\mu \times \mu$ on $S \times S$. Define $\operatorname{int} k$ on $L_2(S)$ by

$$\operatorname{int} k(f)(x) := \int_S k(x, y)f(y)\, d\mu(y)$$

for every $x \in S$. Then $\operatorname{int} k$ is an operator on $L_2(S)$. If $K = \operatorname{int} k$ for some $k \in L_2(\mu \times \mu)$, then K is an INTEGRAL OPERATOR and k is its KERNEL. (This is another use of the word "kernel"; it has nothing whatever to do with the more common use of the word: $\ker K = K^{\leftarrow}(\{\mathbf{0}\})$—see definition 1.1.2).

2.2.18. Example. Let $H = L_2([0, 1])$ be the real Hilbert space of all (equivalence classes of) real valued functions on $[0, 1]$ which are square integrable with respect to Lebesgue measure. Define V on H by

$$Vf(x) = \int_0^x f(t)\, dt\,. \tag{2.1}$$

Then V is an operator on H. This is a VOLTERRA OPERATOR and is an example of an integral operator. (What is its kernel k?)

There is nothing important about the choice of the space $L_2([0, 1])$ in this example. Many spaces admit *Volterra operators*. For instance, we could just as well have used equation (2.1) to define an operator V on the Banach space $\mathcal{C}([0, 1])$.

2.2.19. Proposition. *The kernel of a bounded linear map $A\colon H \to K$ between Hilbert spaces is a closed linear subspace of H.*

2.2.20. Proposition. *Let A, $B \in \mathfrak{B}(H, K)$ where H and K are Hilbert spaces. Then $A = B$ if and only if $\langle Ax, y\rangle = \langle Bx, y\rangle$ for all $x \in H$ and $y \in K$.*

2.2.21. Proposition. *If H is a complex Hilbert space and $T \in \mathfrak{B}(H)$ satisfies $\langle Tz, z\rangle = 0$ for all $z \in H$, then $T = 0$. The corresponding result fails in real Hilbert spaces.*

Hint for proof. In the hypothesis replace z first by $x + y$ and then by $x + iy$.

Despite our convention that in these notes all vector spaces are complex, the word "complex" was added to the hypotheses of the preceding proposition to draw attention to the fact that it is one of the few facts which holds *only* for complex spaces. While proposition 2.2.20 holds for both real and complex Hilbert spaces 2.2.21 does not. (Consider the operator which rotates the plane $\mathbb{R}^2$ by 90 degrees.)

2.2.22. Example. Let H be a Hilbert space and $a \in H$. Define $\psi_a\colon H \to \mathbb{C}\colon x \mapsto \langle x, a\rangle$. Then $\psi_a \in H^*$.

Now we generalize theorem 1.2.30 to the infinite dimensional setting. This result is sometimes called the *Riesz representation theorem* (which invites confusion with the more substantial results about representing certain linear functionals as measures) or the *little Riesz representation theorem.*

2.2.23. Theorem (Riesz-Fréchet Theorem)**.** *If $f \in H^*$ where H is a Hilbert space, then there exists a unique vector a in H such that $f = \psi_a$. Furthermore, $\|a\| = \|\psi_a\|$.*

PROOF. See [5], I.3.4.

2.3. Algebras

2.3.1. Definition. A (complex) ALGEBRA is a (complex) vector space A together with a binary operation $(x, y) \mapsto xy$, called *multiplication*, which satisfy

(i) $(ab)c = a(bc)$,
(ii) $(a + b)c = ac + bc$,
(iii) $a(b + c) = ab + ac$, and
(iv) $\alpha(ab) = (\alpha a)b = a(\alpha b)$

for all a, b, $c \in A$ and $\alpha \in \mathbb{C}$. In other words, an algebra is a vector space which is also a ring and satisfies (iv). If an algebra A has a *multiplicative identity* (or *unit*), that is, a nonzero element $\mathbf{1}$ (or $\mathbf{1}_A$) which satisfies

(v) $\mathbf{1}\, a = a\, \mathbf{1} = a$

for every $a \in A$, then the algebra is UNITAL. An algebra A for which $ab = ba$ for all a, $b \in A$ is a COMMUTATIVE ALGEBRA.

A map $\phi\colon A \to B$ between algebras is an ALGEBRA HOMOMORPHISM if it is linear and *multiplicative* (meaning that $\phi(aa') = \phi(a)\phi(a')$ for all a, $a' \in A$). If the algebras A and B are unital a homomorphism $\phi\colon A \to B$ is UNITAL if $\phi(\mathbf{1}_A) = \mathbf{1}_B$.

A subset B of an algebra A is a SUBALGEBRA of A if it is an algebra under the operations it inherits from A. A subalgebra B of a unital algebra A is a UNITAL SUBALGEBRA if it contains the multiplicative identity of A. **CAUTION.** To be a unital subalgebra it is *not* enough for B to have a multiplicative identity of its own; it must contain the identity of A. Thus, an algebra can be both unital and a subalgebra of A without being a unital subalgebra of A. (An example is given later in 3.1.4.)

2.3.2. Convention. As with vector spaces, all algebras in the sequel will be assumed to be complex algebras unless the contrary is explicitly stated.

2.3.3. Proposition. *An algebra can have at most one multiplicative identity.*

2.3.4. Definition. An element a of a unital algebra A is LEFT INVERTIBLE if there exists an element a_l in A (called a LEFT INVERSE of a) such that $a_l a = \mathbf{1}$ and is RIGHT INVERTIBLE if there exists an element a_r in A (called a RIGHT INVERSE of a)such that $aa_r = \mathbf{1}$. The element is INVERTIBLE if it is both left invertible and right invertible. The set of all invertible elements of A is denoted by $\operatorname{inv} A$.

An element of a unital algebra can have at most one multiplicative inverse. In fact, more is true.

2.3.5. Proposition. *If an element of a unital algebra has both a left inverse and a right inverse, then these two inverses are equal (and so the element is invertible).*

When an element a of a unital algebra is invertible its (unique) inverse is denoted by a^{-1}.

2.3.6. Proposition. *If a is an invertible element of a unital algebra, then a^{-1} is also invertible and*

$$\left(a^{-1}\right)^{-1} = a\,.$$

2.3.7. Proposition. *If a and b are invertible elements of a unital algebra, then their product ab is also invertible and*

$$(ab)^{-1} = b^{-1}a^{-1}\,.$$

2.3.8. Proposition. *If a and b are invertible elements of a unital algebra, then*

$$a^{-1} - b^{-1} = a^{-1}(b - a)b^{-1}\,.$$

2.3.9. Proposition. *Let a and b be elements of a unital algebra. If both ab and ba are invertible, then so are a and b.*

Hint for proof. Use proposition 2.3.5.

2.3.10. Proposition. *Let a and b be elements of a unital algebra. Then $\mathbf{1} - ab$ is invertible if and only if $\mathbf{1} - ba$ is.*

PROOF. If $\mathbf{1} - ab$ is invertible, then $\mathbf{1} + b(\mathbf{1} - ab)^{-1}a$ is the inverse of $\mathbf{1} - ba$. □

2.3.11. Example. The simplest example of a (complex) algebra is the family $\mathbb{C}$ of complex numbers. It is both unital and commutative.

2.3.12. Example. If X is a nonempty topological space, then the family $\mathcal{C}(X)$ of all continuous complex valued functions on X is an algebra under the usual pointwise operations of addition, scalar multiplication, and multiplication. It is both unital and commutative.

2.3.13. Example. If X is a locally compact Hausdorff space which is not compact, then (under pointwise operations) the family $\mathcal{C}_0(X)$ of all continuous complex valued functions on X which vanish at infinity (see example 2.1.11) is a commutative algebra. However, it is *not* a unital algebra.

2.3.14. Example. The family $\mathbf{M}_n$ of $n \times n$ matrices of complex numbers is a unital algebra under the usual matrix operations of addition, scalar multiplication, and multiplication. It is not commutative when $n > 1$.

We will be making considerable use of a generalization of the preceding example.

2.3.15. Example. Let A be an algebra. Make the family $\mathbf{M}_n(A)$ of $n \times n$ matrices of elements of A into an algebra by using the same rules for matrix operations that are used for $\mathbf{M}_n$. Thus $\mathbf{M}_n$ is just $\mathbf{M}_n(\mathbb{C})$. The algebra $\mathbf{M}_n(A)$ is unital if and only if A is.

2.3.16. Example. If V is a normed linear space, then $\mathfrak{B}(V)$ is a unital algebra. If $\dim V > 1$, the algebra is not commutative. (See propositions 2.2.3 and 2.2.7.)

2.3.17. Definition. An LEFT IDEAL in an algebra A is a vector subspace J of A such that $AJ \subseteq J$. (For RIGHT IDEALS, of course, we require $JA \subseteq J$.) We say that J is an IDEAL if it is a two-sided ideal, that is, both a left and a right ideal. A PROPER ideal is an ideal which is a proper subset of A.

The ideals $\{0\}$ and A are often referred to as the TRIVIAL IDEALS of A. The algebra A is SIMPLE if it has no nontrivial ideals.

A MAXIMAL ideal is a proper ideal that is properly contained in no other proper ideal. We denote the family of all maximal ideals in an algebra A by $\operatorname{Max} A$. A MINIMAL ideal is a nonzero ideal that properly contains no other nonzero ideal.

2.3.18. Convention. Whenever we refer to an *ideal* in an algebra we understand it to be a two-sided ideal (unless the contrary is stated).

2.3.19. Proposition. *No invertible element in a unital algebra can belong to a proper ideal.*

2.3.20. Proposition. *Every proper ideal in a unital algebra A is contained in a maximal ideal. Thus, in particular, $\operatorname{Max} A$ is nonempty whenever A is a unital algebra.*

Hint for proof. Zorn's lemma.

2.3.21. Proposition. *Let a be an element of a commutative algebra A. If A is unital and a is not invertible, then aA is a proper ideal in A.*

2.3.22. Definition. The ideal aA in the preceding proposition is the PRINCIPAL IDEAL generated by a.

2.3.23. Definition. Let J be a proper ideal in an algebra A. Define an equivalence relation $\sim$ on A by

$$a \sim b \qquad \text{if and only if} \qquad b - a \in J.$$

For each $a \in A$ let $[a]$ be the equivalence class containing a. Let A/J be the set of all equivalence classes of elements of A. For $[a]$ and $[b]$ in A/J define

$$[a] + [b] := [a + b] \qquad \text{and} \qquad [a][b] := [ab]$$

and for $\alpha \in \mathbb{C}$ and $[a] \in A/J$ define

$$\alpha[a] := [\alpha a]\,.$$

Under these operations A/J becomes an algebra. It is the QUOTIENT ALGEBRA of A by J. The notation A/J is usually read "A mod J". The surjective algebra homomorphism

$$\pi \colon A \to A/J \colon a \mapsto [a]$$

is called the QUOTIENT MAP.

2.3.24. Proposition. *The preceding definition makes sense and the claims made therein are correct. Furthermore, the quotient algebra A/J is unital if A is.*

Hint for proof. You will need to show that:

(a) $\sim$ is an equivalence relation.
(b) Addition and multiplication of equivalence classes is well defined.
(c) Multiplication of an equivalence class by a scalar is well defined.
(d) A/J is an algebra.
(e) The "quotient map" π really is a surjective algebra homomorphism.

(At what point is it necessary that we factor out an ideal and not just a subalgebra?)

2.4. Spectrum

2.4.1. Definition. Let a be an element of a unital algebra A. The SPECTRUM of a, denoted by $\sigma_A(a)$ or just $\sigma(a)$, is the set of all complex numbers λ such that $a - \lambda\mathbf{1}$ is not invertible.

2.4.2. Example. If z is an element of the algebra $\mathbb{C}$ of complex numbers, then $\sigma(z) = \{z\}$.

2.4.3. Example. Let X be a compact Hausdorff space. If f is an element of the algebra $\mathcal{C}(X)$ of continuous complex valued functions on X, then the spectrum of f is its range.

2.4.4. Example. Let X be an arbitrary topological space. If f is an element of the algebra $\mathcal{C}_b(X)$ of bounded continuous complex valued functions on X, then the spectrum of f is the closure of its range.

2.4.5. Example. Let S be a positive measure space and $[f] \in \mathrm{L}_\infty(S)$ be an (equivalence class of) essentially bounded function(s) on S. Then the spectrum of $[f]$ is its essential range.

2.4.6. Example. The family $\mathfrak{M}_3$ of 3×3 matrices of complex numbers is a unital algebra under the usual matrix operations. The spectrum of the matrix $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ is $\{1, 2\}$.

2.4.7. Proposition. *Let a be an element of a unital algebra such that $a^2 = \mathbf{1}$. Then either*

(i) *$a = \mathbf{1}$, in which case $\sigma(a) = \{1\}$, or*
(ii) *$a = -\mathbf{1}$, in which case $\sigma(a) = \{-1\}$, or*
(iii) *$\sigma(a) = \{-1, 1\}$.*

Hint for proof. In (iii) to prove $\sigma(a) \subseteq \{-1, 1\}$, consider $\dfrac{1}{1 - \lambda^2}(a + \lambda\mathbf{1})$.

2.4.8. Proposition. *An element a of an algebra is* IDEMPOTENT *if $a^2 = a$. Let a be an idempotent element of a unital algebra. Then either*

(i) $a = \mathbf{1}$, *in which case $\sigma(a) = \{1\}$, or*
(ii) $a = \mathbf{0}$, *in which case $\sigma(a) = \{0\}$, or*
(iii) $\sigma(a) = \{0, 1\}$.

Hint for proof. In (iii) to prove $\sigma(a) \subseteq \{0,1\}$, consider $\dfrac{1}{\lambda - \lambda^2}\big(a + (\lambda - 1)\mathbf{1}\big)$.

2.4.9. Proposition. *Let a be an invertible element of a unital algebra. Then $\lambda \in \sigma(a)$ if and only if $\frac{1}{\lambda} \in \sigma(a^{-1})$.*

The next proposition is a simple corollary of proposition 2.3.10.

2.4.10. Proposition. *If a and b are elements of a unital algebra, then, except possibly for 0, the spectra of ab and ba are the same.*

CHAPTER 3

BANACH ALGEBRAS

3.1. Definition and Elementary Properties

3.1.1. Definition. If an algebra A is equipped with a norm $\|\cdot\|$ satisfying

$$\|xy\| \leq \|x\| \, \|y\|$$

for all x, $y \in A$, then we say that it is a NORMED ALGEBRA and that $\|\cdot\|$ is an ALGEBRA NORM. We also require that if A is unital, then to be a normed algebra it must also satisfy

$$\|\mathbf{1}_A\| = 1$$

where $\mathbf{1}_A$ is the multiplicative identity of A. A complete normed algebra is a BANACH ALGEBRA.

A map $f\colon A \to B$ between Banach algebras is a BANACH ALGEBRA HOMOMORPHISM if it is both an algebraic homomorphism and a bounded linear map between A and B (regarded as Banach spaces).

3.1.2. Convention. In the context of Banach algebras, when we speak of a *homomorphism*, or just a *morphism*, we mean a *Banach algebra homomorphism*.

3.1.3. Proposition. *Let A and B be Banach algebras. The* DIRECT SUM *of A and B, denoted by $A \oplus B$, is defined to be the vector space direct sum of A and B on which a multiplication has been defined by*

$$(a, b)\,(a', b') = (aa', bb')$$

(for a, $a' \in A$ and b, $b' \in B$) and a norm defined by

$$\|(a, b)\| = \max\{\|a\|, \|b\|\}\,.$$

This makes $A \oplus B$ into a Banach algebra.

3.1.4. Example. The algebra $\mathbb{C} \cong \mathbb{C} \oplus \{\mathbf{0}\}$ is unital and is a subalgebra of $\mathbb{C} \oplus \mathbb{C}$, but it is not a unital subalgebra of $\mathbb{C} \oplus \mathbb{C}$.

3.1.5. Proposition. *In a Banach algebra the operations of addition and multiplication (regarded as maps from $A \oplus A$ to A) are continuous and scalar multiplication (regarded as a map from $\mathbb{C} \oplus A$ to A) is also continuous.*

3.1.6. Example. Let X be a compact Hausdorff space. The family $\mathcal{C}(X)$ of all continuous complex valued functions on X under pointwise operations of addition, multiplication, and scalar multiplication is a unital commutative Banach algebra under the uniform norm

$$\|f\|_u := \sup\{|f(x)| : x \in X\}\,.$$

(See examples 2.1.10 and 2.3.12.) Similarly if X is locally compact and Hausdorff, then $\mathcal{C}_0(X)$ and $\mathcal{C}_b(X)$ are commutative Banach algebras (see example 2.1.11).

3.1.7. Example. If V is a Banach space, then with composition as multiplication $\mathfrak{B}(V)$ is a unital Banach algebra (see example 2.3.16 and proposition 2.2.6).

3.1.8. Example. Let $l_1(\mathbb{Z})$ be the family of all bilateral sequences

$$(\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots)$$

which are ABSOLUTELY SUMMABLE; that is, such that $\sum_{k=-\infty}^{\infty}|a_k| < \infty$. This is a Banach space under pointwise operations of addition and scalar multiplication and norm given by

$$\|a\| = \sum_{k=-\infty}^{\infty} |a_k| \, .$$

For a, $b \in l_1(\mathbb{Z})$ define $a * b$ to be the sequence whose n^{th} entry is given by

$$(a * b)_n = \sum_{k=-\infty}^{\infty} a_{n-k} \, b_k \, .$$

The operation $*$ is called CONVOLUTION. (To see where the definition comes from try multiplying the power series $\sum_{-\infty}^{\infty} a_k z^k$ and $\sum_{-\infty}^{\infty} b_k z^k$.) With this additional operation $l_1(\mathbb{Z})$ becomes a unital commutative Banach algebra.

3.1.9. Example. The Banach space $L_1(\mathbb{R})$ (see example 2.1.9) can be made into a commutative Banach algebra. For f, $g \in L_1(\mathbb{R})$ define

$$h(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\, dy \tag{1}$$

whenever the function $y \mapsto f(x-y)g(y)$ belongs to $L_1(\mathbb{R})$. Then $h(x)$ is defined and finite for almost all $x \in \mathbb{R}$. Set $h(x) = 0$ whenever (1) is undefined. Furthermore h belongs to $L_1(\mathbb{R})$ and $\|h\|_1 \le \|f\|_1 \|g\|_1$. The function h is usually denoted by $f * g$; this is the CONVOLUTION of f and g. (In this definition does any problem arise from the fact that members of L_1 are in fact equivalence classes of functions?)

3.1.10. Definition. Let (a_k) be a sequence of vectors in a normed linear space V. We say that the sequence (a_k) is SUMMABLE, or that the series $\sum_{k=1}^{\infty} a_k$ converges if there exists an element $b \in V$ such that $\|b - \sum_{k=1}^{n} a_k\| \to 0$ as $n \to \infty$. In this case we write $\sum_{k=1}^{\infty} a_k = b$.

3.1.11. Proposition (The Neumann series)**.** *Let a be an element of a unital Banach algebra A. If $\|a\| < 1$, then $\mathbf{1} - a \in \operatorname{inv} A$ and $(\mathbf{1} - a)^{-1} = \sum_{k=0}^{\infty} a^k$.*

Hint for proof. In a unital algebra we take a^0 to mean $\mathbf{1}_A$. Start by proving that the sequence $(\mathbf{1}, a, a^2, \dots)$ is summable by showing that the sequence of partial sums $\sum_{k=0}^{n} a^k$ is Cauchy.

3.1.12. Proposition. *If A is a unital Banach algebra, then $\operatorname{inv} A \overset{\circ}{\subseteq} A$.*

Hint for proof. Let $a \in \operatorname{inv} A$. Show, for sufficiently small h, that $\mathbf{1} - a^{-1}h$ is invertible.

3.1.13. Proposition. *If a belongs to a unital Banach algebra and $\|a\| < 1$, then*

$$\big\|(\mathbf{1} - a)^{-1} - \mathbf{1}\big\| \le \frac{\|a\|}{1 - \|a\|}.$$

3.1.14. Proposition. *Let A be a unital Banach algebra. The map $a \mapsto a^{-1}$ from $\operatorname{inv} A$ into itself is a homeomorphism.*

3.1.15. Notation. Let f be a complex valued function on some set S. Denote by Z_f the set of all points x in S such that $f(x) = 0$. This is the ZERO SET of f.

3.1.16. Proposition. *The invertible elements in the Banach algebra $\mathcal{C}(X)$ of all continuous complex valued functions on a compact Hausdorff space X are the functions which vanish nowhere. That is,*

$$\operatorname{inv} \mathcal{C}(X) = \{ f \in \mathcal{C}(X) \colon Z_f = \emptyset \} \, .$$

3.1.17. Proposition. *Let A be a unital Banach algebra. The map $r \colon a \mapsto a^{-1}$ from $\operatorname{inv} A$ into itself is differentiable and at each invertible element a, we have $dr_a(h) = -a^{-1}ha^{-1}$ for all $h \in A$.*

3.1.18. Proposition. *Let a be an element of a unital Banach algebra A. Then the spectrum of a is compact and $|\lambda| \le \|a\|$ for every $\lambda \in \sigma(a)$.*

Hint for proof. Use the *Heine-Borel theorem.* To prove that the spectrum is closed notice that $(\sigma(a))^c = f^{\leftarrow}(\operatorname{inv} A)$ where $f(\lambda) = a - \lambda\mathbf{1}$ for every complex number λ. Also show that if $|\lambda| > \|a\|$, then $\mathbf{1} - \lambda^{-1}a$ is invertible.

3.1.19. Definition. Let a be an element of a unital Banach algebra. The RESOLVENT MAPPING for a is defined by

$$R_a \colon \mathbb{C} \setminus \sigma(a) \to A \colon \lambda \mapsto (a - \lambda\mathbf{1})^{-1}\,.$$

3.1.20. Definition. Let $U \overset{\circ}{\subseteq} \mathbb{C}$ and A be a unital Banach algebra. A function $f \colon U \to A$ is ANALYTIC on U if

$$f'(a) := \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists for every $a \in U$. A complex valued function which is analytic on all of $\mathbb{C}$ is an ENTIRE function.

3.1.21. Proposition. *For a an element of a unital Banach algebra A and ϕ a bounded linear functional on A let $f := \phi \circ R_a \colon \mathbb{C} \setminus \sigma(a) \to \mathbb{C}$. Then*

(i) *f is analytic on its domain, and*
(ii) *$f(\lambda) \to 0$ as $|\lambda| \to \infty$.*

Hint for proof. For (i) use proposition 2.3.8.

In order to prove our next major result, that the spectrum of an element is never empty (see theorem 3.1.25), we need two theorems: *Liouville's theorem* from complex variables and the *Hahn-Banach theorem* from functional analysis.

3.1.22. Theorem (Liouville's theorem)**.** *Every bounded entire function on $\mathbb{C}$ is constant.*

A proof of this theorem can be found in nearly any text on complex variables.

What is known as the *Hahn-Banach theorem* is really a family of related theorems that guarantee the existence of a generous supply of linear functionals. Some authors refer to the version given below, which says that linear functionals on subspaces can be extended without increasing their norm, as the *Bohnenblust-Sobczyk-Suhomlinov theorem.*

3.1.23. Theorem (Hahn-Banach theorem)**.** *If M is a linear subspace of a normed linear space V and $f \in M^*$, then there exists an extension $\hat{f}$ of f to all of V such that $\|\hat{f}\| = \|f\|$.*

PROOF. See [15], theorem 14.12.

3.1.24. Corollary. *Let M be a linear subspace of a normed linear space V. If z is a vector in M^c such that the distance $d(z, M)$ between z and M is strictly greater than zero, then there exists a linear functional $g \in V^*$ such that $g^{\to}(M) = \{\mathbf{0}\}$, $g(z) = d(z, M)$, and $\|g\| = 1$.*

PROOF. See [15], corollary 14.13.

3.1.25. Theorem. *The spectrum of every element of a unital Banach algebra is nonempty.*

Hint for proof. Argue by contradiction. Use *Liouville's theorem* to show that $\phi \circ R_a$ is constant for every bounded linear functional ϕ on A. Then use the (corollary to the) *Hahn-Banach theorem* to prove that R_a is constant. Why must this constant be 0?

It is important to keep in mind that we are working only with *complex* algebras. This result is *false* for real Banach algebras. An easy counterexample is the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ regarded as an element of the (real) Banach algebra of all 2×2 matrices of real numbers.

The next result says that essentially the only (complex) Banach division algebra is the field of complex numbers.

3.1.26. Theorem (Gelfand-Mazur theorem)**.** *If A is a unital Banach algebra in which every nonzero element is invertible, then A is isometrically isomorphic to $\mathbb{C}$.*

Hint for proof. Let $B = \{\lambda\mathbf{1} \colon \lambda \in \mathbb{C}\}$. Use the preceding result (theorem 3.1.25) to show that $B = A$.

3.1.27. Proposition. *Let a be an element of a unital algebra. Then $\sigma(a^n) = [\sigma(a)]^n$ for every $n \in \mathbb{N}$.* (The notation $[\sigma(a)]^n$ means $\{\lambda^n \colon \lambda \in \sigma(a)\}$.)

3.1.28. Definition. Let a be an element of a unital algebra. The SPECTRAL RADIUS of a, denoted by $\rho(a)$, is defined to be $\sup\{|\lambda| \colon \lambda \in \sigma(a)\}$.

3.1.29. Proposition. *If a is an element of a unital Banach algebra, then $\rho(a) \leq \|a\|$ and $\rho(a^n) = \big(\rho(a)\big)^n$.*

3.1.30. Theorem (Spectral radius formula)**.** *If a is an element of a unital Banach algebra, then*

$$\rho(a) = \inf\big\{\|a^n\|^{1/n} \colon n \in \mathbb{N}\big\} = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

PROOF. The proof while not terribly deep is a bit complicated. A good version can be found in [1], pages 19–20.

3.1.31. Exercise. The Volterra operator V (defined in example 2.2.18) is an integral operator on the Banach space $\mathcal{C}([0,1])$.

(a) Compute the spectrum of the operator V. (*Hint.* Show that $\|V^n\| \leq (n!)^{-1}$ for every $n \in \mathbb{N}$ and use the *spectral radius formula*.)

(b) What does (a) say about the possibility of finding a continuous function f which satisfies the integral equation

$$f(x) - \mu \int_0^x f(t)\,dt = h(x),$$

where μ is a scalar and h is a given continuous function on $[0,1]$?

(c) Use the idea in (b) and the Neumann series expansion for $\mathbf{1} - \mu V$ (see proposition 3.1.11) to calculate explicitly (and in terms of elementary functions) a solution to the integral equation

$$f(x) - \tfrac{1}{2}\int_0^x f(t)\,dt = e^x.$$

3.1.32. Proposition. *Let A be a unital Banach algebra and B a closed subalgebra containing $\mathbf{1}_A$. Then*

(a) $\operatorname{inv} B$ *is both open and closed in* $B \cap \operatorname{inv} A$*;*

(b) $\sigma_A(b) \subseteq \sigma_B(b)$ *for every* $b \in B$*; and*

(c) *if* $b \in B$ *and* $\sigma_A(b)$ *has no holes (that is, if its complement in* $\mathbb{C}$ *is connected), then* $\sigma_A(b) = \sigma_B(b)$*.*

Hint for proof. For part (c) consider (for $b \in B$) the function

$$f_b \colon (\sigma_A(b))^c \to B \cap \operatorname{inv} A \colon \lambda \mapsto b - \lambda\mathbf{1}\,.$$

3.2. Maximal Ideal Space

3.2.1. Proposition. *If J is a proper ideal in a unital Banach algebra, then so is its closure.*

3.2.2. Corollary. *Every maximal ideal in a unital Banach algebra is closed.*

3.2.3. Proposition. *Let J be a proper closed ideal in a Banach algebra A. On the quotient algebra A/J (see definition 2.3.23) define a function*

$$\|\ \ \| \colon A/J \to \mathbb{R} \colon \alpha \mapsto \inf\{\|u\| \colon u \in \alpha\}\,.$$

This function is a norm on A/J, under this norm A/J is a Banach algebra, and the quotient map is continuous with $\|\pi\| \leq 1$. The Banach algebra A/J is the QUOTIENT ALGEBRA *of A by J.*

3.2.4. Proposition. *Let I be a proper closed ideal in a unital commutative Banach algebra A. Then I is maximal if and only if A/I is a field.*

The following is an immediate consequence of the preceding proposition and the Gelfand-Mazur theorem (3.1.26).

3.2.5. Corollary. *If I is a maximal ideal in a commutative unital Banach algebra, then A/I is isometrically isomorphic to $\mathbb{C}$.*

3.2.6. Example. For every subset C of a topological space X the set

$$J_C := \big\{ f \in \mathcal{C}(X) \colon f^{\rightarrow}(C) = \{0\} \big\}$$

is an ideal in $\mathcal{C}(X)$. Furthermore, $J_C \supseteq J_D$ whenever $C \subseteq D \subseteq X$. (In the following we will write J_x for the ideal $J_{\{x\}}$.)

3.2.7. Proposition. *Let X be a compact topological space and I be a proper ideal in $\mathcal{C}(X)$. Then there exists $x \in X$ such that $I \subseteq J_x$.*

3.2.8. Proposition. *Let x and y be points in a compact Hausdorff space. If $J_x \subseteq J_y$, then $x = y$.*

3.2.9. Proposition. *Let X be a compact Hausdorff space. A subset I of $\mathcal{C}(X)$ is a maximal ideal in $\mathcal{C}(X)$ if and only if $I = J_x$ for some $x \in X$.*

3.2.10. Corollary. *If X is a compact Hausdorff space, then the map $x \mapsto J_x$ from X to* $\operatorname{Max} \mathcal{C}(X)$ *is bijective.*

Compactness is an important ingredient in proposition 3.2.9.

3.2.11. Example. In the Banach algebra $\mathcal{C}_b\big((0,1)\big)$ of bounded continuous functions on the interval $(0,1)$ there exists a maximal ideal I such that for *no* point $x \in (0,1)$ is $I = J_x$. Let I be a maximal ideal containing the ideal S of all functions f in $\mathcal{C}_b\big((0,1)\big)$ for which there exists a neighborhood U_f of 0 in $\mathbb{R}$ such that $f(x) = 0$ for all $x \in U_f \cap (0,1)$.

3.3. Characters

3.3.1. Definition. A CHARACTER (or NONZERO MULTIPLICATIVE LINEAR FUNCTIONAL) on an algebra A is a nonzero homomorphism from A into $\mathbb{C}$. The set of all characters on A is denoted by ΔA.

3.3.2. Proposition. *Let A be a unital algebra and $\phi \in \Delta A$. Then*

(a) $\phi(\mathbf{1}) = 1$;
(b) *if $a \in \operatorname{inv} A$, then $\phi(a) \neq 0$;*
(c) *if a is* NILPOTENT *(that is, if $a^n = 0$ for some $n \in \mathbb{N}$), then $\phi(a) = 0$;*
(d) *if a is* IDEMPOTENT *(that is, if $a^2 = a$), then $\phi(a)$ is 0 or 1; and*
(e) *$\phi(a) \in \sigma(a)$ for every $a \in A$.*

We note in passing that part (e) of the preceding proposition does not give us an easy way of showing that the spectrum $\sigma(a)$ of an algebra element is nonempty. This would depend on knowing that $\Delta(A)$ is nonempty.

3.3.3. Example. The identity map is the only character on the algebra $\mathbb{C}$.

3.3.4. Example. Let A be the algebra of 2×2 matrices $a = \big[a_{ij}\big]$ such that $a_{12} = 0$. This algebra has exactly two characters $\phi(a) = a_{11}$ and $\psi(a) = a_{22}$. *Hint.* Use proposition 3.3.2.

3.3.5. Example. The algebra of all 2×2 matrices of complex numbers has no characters.

3.3.6. Proposition. *Let A be a unital algebra and ϕ be a linear functional on A. Then $\phi \in \Delta A$ if and only if* $\ker \phi$ *is closed under multiplication and $\phi(\mathbf{1}) = 1$.*

Hint for proof. For the converse apply ϕ to the product of $a - \phi(a)\mathbf{1}$ and $b - \phi(b)\mathbf{1}$ for $a, b \in A$.

3.3.7. Proposition. *Every multiplicative linear functional on a unital Banach algebra A is continuous. In fact, if $\phi \in \Delta(A)$, then ϕ is contractive and $\|\phi\| = 1$.*

3.3.8. Example. Let X be a topological space and $x \in X$. We define the EVALUATION FUNCTIONAL AT x, denoted by $E_X x$, by

$$E_X x \colon \mathcal{C}(X) \to \mathbb{C} \colon f \mapsto f(x)\,.$$

This functional is a character on $\mathcal{C}(X)$ and its kernel is J_x. When there is only one topological space under discussion we simplify the notation from $E_X x$ to E_x. Thus, in particular, for $f \in \mathcal{C}(X)$ we often write $E_x(f)$ for the more cumbersome $E_X x(f)$.

Proposition 3.3.7 turns out to be very important: it says that characters on a unital Banach algebra A all live on the unit sphere of the dual space A^*. The trouble with the unit sphere in the dual space is that, while it is closed and bounded, it is not compact in the usual (norm) topology on A^*. We need a new topology on A^*, one that is weak enough to make the closed unit ball (and hence the unit sphere) compact and yet strong enough to be Hausdorff.

3.4. The Gelfand Topology

3.4.1. Proposition. *Let B be a Banach space. For each $x \in B$ let*

$$x^{**} \colon B^* \to \mathbb{C} \colon f \mapsto f(x)\,.$$

Then $x^{**} \in B^{**}$ for every $x \in B$ and the map

$$J_B \colon B \to B^{**} \colon x \mapsto x^{**}$$

is a linear isometry.

3.4.2. Definition. The map J_B in the preceding proposition is the NATURAL INJECTION of a Banach space B into its second dual B^{**}. (The rationale for the use of the word "natural" will appear shortly—see example 4.3.2.) In situations where there is a single Banach space under discussion we usually write J for J_B. A Banach space B is REFLEXIVE if the map J_B is surjective.

CAUTION. To show that a Banach space is reflexive it is not enough to show that *there exists* an isometric isomorphism from the space to its second dual; what must be proved is that the natural injection J is an isometric isomorphism.

3.4.3. Definition. Let B be a Banach space and $J \colon B \to B^{**}$ be the natural injection of B into its second dual. The w^*-topology (pronounced *weak star topology*) is the weak topology on B^* determined by $\operatorname{ran} J$. That is, it is the weakest topology on B^* which makes every functional on B^* of the form x^{**} continuous. Thus we take as a subbase for the w^*-topology on B^* all sets of the form $(x^{**})^{\leftarrow}(U) = \{f \in B^* \colon f(x) \in U\}$ where $x \in B$ and $U \overset{\circ}{\subseteq} \mathbb{C}$.

3.4.4. Proposition. *The w^*-topology on the dual of a Banach space is Hausdorff and is weaker than the norm topology.*

3.4.5. Notation. In the following proposition we use the notation $\operatorname{Fin}(S)$ to denote the family of finite subsets of a set S.

3.4.6. Proposition. *Let B be a Banach space. The family of all subsets of B^* of the form*

$$V(f; A; \epsilon) := \{g \in B^* \colon |f(x) - g(x)| < \epsilon \text{ for all } x \in A\}\,,$$

where $f \in B^*$, $A \in \operatorname{Fin}(B)$, and $\epsilon > 0$, is a base for the w^*-topology on B^*.

3.4.7. Notation. Let (f_λ) be a net in the dual B^* of a Banach space and let $g \in B^*$. To indicate that the net (f_λ) converges to g in the w^*-topology we write $f_\lambda \xrightarrow{w^*} g$.

The following result is a great joy. It tells us that the w^*-topology on the dual of a Banach space is nothing more than the topology of pointwise convergence.

3.4.8. Proposition. *Let (f_λ) be a net in the dual B^* of a Banach space B and $g \in B^*$. Then $f_\lambda \xrightarrow{w^*} g$ if and only if $f_\lambda(x) \to g(x)$ in $\mathbb{C}$ for every $x \in B$.*

3.4.9. Definition. In proposition 3.3.7 we discovered that every character on a unital Banach algebra A lives on the unit sphere of the dual A^*. Thus we may give the set $\Delta(A)$ of characters on A the relative w^*-topology it inherits from A^*. This is the GELFAND TOPOLOGY on $\Delta(A)$ and the resulting topological space we call the CHARACTER SPACE (or the STRUCTURE SPACE) of A.

In order to show that the character space is compact we need an important theorem from functional analysis.

3.4.10. Theorem (Alaoglu's theorem)**.** *If V is a normed linear space, then the closed unit ball of its dual V^* is compact in the w^*-topology.*

PROOF. See [5], theorem V.3.1.

3.4.11. Proposition. *The character space of a unital Banach algebra is a compact Hausdorff space.*

3.4.12. Example. The maximal ideal space of the unital Banach algebra $l_1(\mathbb{Z})$ (see example 3.1.8) is (homeomorphic to) the unit circle $\mathbb{T}$.

Hint for proof. For each $z \in \mathbb{T}$ define

$$\psi_z \colon l_1(\mathbb{Z}) \to \mathbb{C} \colon a \mapsto \sum_{k=-\infty}^{\infty} a_k z^k .$$

Show that $\psi_z \in \Delta l_1(\mathbb{Z})$. Then show that the map

$$\psi \colon \mathbb{T} \to \Delta l_1(\mathbb{Z}) \colon z \mapsto \psi_z$$

is a homeomorphism.

3.4.13. Proposition. *If $\phi \in \Delta A$ where A is a unital algebra, then $\ker \phi$ is a maximal ideal in A.*

Hint for proof. To show maximality, suppose I is an ideal in A which properly contains $\ker \phi$. Choose $z \in I \setminus \ker \phi$. Consider the element $x - \big(\phi(x)/\phi(z)\big)\, z$ where x is an arbitrary element of A.

3.4.14. Proposition. *A character on a unital algebra is completely determined by its kernel.*

Hint for proof. Let a be an element of the algebra and ϕ be a character. For how many complex numbers λ can $a^2 - \lambda a$ belong to the kernel of ϕ?

3.4.15. Corollary. *If A is a unital algebra, then the map $\phi \mapsto \ker \phi$ from ΔA to $\operatorname{Max} A$ is injective.*

3.4.16. Proposition. *Let I be a maximal ideal in a unital commutative Banach algebra A. Then there exists a character on A whose kernel is I.*

Hint for proof. Use corollary 3.2.5 Why can we think of the quotient map as a character?

3.4.17. Corollary. *If A is a unital commutative Banach algebra, then the map $\phi \mapsto \ker \phi$ is a bijection from ΔA onto $\operatorname{Max} A$.*

3.4.18. Definition. Let A be a unital commutative Banach algebra. In light of the preceding corollary we can give $\operatorname{Max} A$ a topology under which it is homeomorphic to the character space ΔA. This is the MAXIMAL IDEAL SPACE of A. Since ΔA and $\operatorname{Max} A$ are homeomorphic it is common practice to identify them and so ΔA is often called the *maximal ideal space* of A.

3.4.19. Definition. Let X be a compact Hausdorff space and $x \in X$. Recall that in example 3.3.8 we defined $E_X x$, the *evaluation functional* at x by

$$E_X x(f) := f(x)$$

for every $f \in \mathcal{C}(X)$. The map

$$E_X : X \to \Delta\mathcal{C}(X) : x \mapsto E_X x$$

is the EVALUATION MAP on X. As was mentioned earlier when only one topological space is being considered we usually shorten E_X to E and $E_X x$ to E_x.

3.4.20. Notation. To indicate that two topological spaces X and Y are homeomorphic we write $X \approx Y$.

3.4.21. Proposition. *Let X be a compact Hausdorff space. Then the evaluation map on X*

$$E_X : X \to \Delta\mathcal{C}(X) : x \mapsto E_X x$$

is a homeomorphism. Thus we have

$$X \approx \Delta\mathcal{C}(X) \approx \operatorname{Max}\mathcal{C}(X) .$$

More is true: not only is each E_X a homeomorphism between compact Hausdorff spaces, but E itself is a *natural equivalence* between functors—the identity functor and the $\Delta\mathcal{C}$ functor. Discussion of this will have to wait until we discuss the language of categories. (See example 4.3.3.) It suffices for the moment to say that the identification between a compact Hausdorff space X and its character space and its maximal ideal space is so strong that many people speak of them as if they were actually equal. It is very common to hear, for example, that "the maximal ideals in $\mathcal{C}(X)$ are just the points of X". Although not literally true, it does sound a bit less intimidating than "the maximal ideals of $\mathcal{C}(X)$ are the kernels of the evaluation functionals at points of X".

3.4.22. Proposition. *Let X and Y be compact Hausdorff spaces and $F : X \to Y$ be continuous. Define $\mathcal{C}(F)$ on $\mathcal{C}(Y)$ by*

$$\mathcal{C}(F)(g) = g \circ F$$

for all $g \in \mathcal{C}(Y)$. Then

(a) *$\mathcal{C}(F)$ maps $\mathcal{C}(Y)$ into $\mathcal{C}(X)$.*
(b) *The map $\mathcal{C}(F)$ is a contractive unital Banach algebra homomorphism.*
(c) *$\mathcal{C}(F)$ is injective if and only if F is surjective.*
(d) *$\mathcal{C}(F)$ is surjective if and only if F is injective.*
(e) *If X is homeomorphic to Y, then $\mathcal{C}(X)$ is isometrically isomorphic to $\mathcal{C}(Y)$.*

3.4.23. Proposition. *Let A and B be unital commutative Banach algebras and $T : A \to B$ be a unital algebra homomorphism. Define ΔT on ΔB by*

$$\Delta T(\psi) = \psi \circ T$$

for all $\psi \in \Delta B$. Then

(a) *ΔT maps ΔB into ΔA.*
(b) *The map $\Delta T : \Delta B \to \Delta A$ is continuous.*
(c) *If T is surjective, then ΔT is injective.*
(d) *If T is an (algebra) isomorphism, then ΔT is a homeomorphism.*
(e) *If A and B are (algebraically) isomorphic, then ΔA and ΔB are homeomorphic.*

3.4.24. Corollary. *Let X and Y be compact Hausdorff spaces. If $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are algebraically isomorphic, then X and Y are homeomorphic.*

3.4.25. Corollary. *Two compact Hausdorff spaces are homeomorphic if and only if their algebras of continuous complex valued functions are (algebraically) isomorphic.*

3.4.26. Corollary. *Let X and Y be compact Hausdorff spaces. If $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are algebraically isomorphic, then they are isometrically isomorphic.*

3.5. The Gelfand Transform

3.5.1. Definition. Let A be a commutative Banach algebra and $a \in A$. Define

$$\Gamma_A a \colon \Delta A \to \mathbb{C} \colon \phi \mapsto \phi(a)$$

for every $\phi \in \Delta(A)$. (Alternative notations: when no confusion seems likely we frequently write Γa or $\widehat{a}$ for $\Gamma_A a$.) The map Γ_A is the GELFAND TRANSFORM ON A.

Since $\Delta A \subseteq A^*$ it is clear that $\Gamma_A a$ is just the restriction of a^{**} to the character space of A. Furthermore the Gelfand topology on ΔA is the relative w^*-topology, the weakest topology under which a^{**} is continuous on ΔA for each $a \in A$, so $\Gamma_A a$ is a continuous function on ΔA. Thus $\Gamma_A \colon A \to \mathcal{C}(\Delta A)$.

As a matter of brevity and convenience the element $\Gamma_A a = \Gamma a = \widehat{a}$ is usually called just the *Gelfand transform of* a—because the phrase *the Gelfand transform on* A *evaluated at* a is awkward.

3.5.2. Definition. We say that a family $\mathcal{F}$ of functions on a set S SEPARATES POINTS of S (or is a SEPARATING FAMILY of functions on S) if for every pair of distinct elements x and y of S there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.

3.5.3. Proposition. *Let X be a compact topological space. Then $\mathcal{C}(X)$ is separating if and only if X is Hausdorff.*

3.5.4. Proposition. *If A is a unital commutative Banach algebra, then $\Gamma_A \colon A \to \mathcal{C}(\Delta A)$ is a unital contractive algebra homomorphism having norm one. Furthermore the range of Γ_A is a separating subalgebra of $\mathcal{C}(\Delta A)$.*

3.5.5. Proposition. *Let a be an element of a unital commutative Banach algebra A. Then a is invertible in A if and only if $\hat{a}$ is invertible in $\mathcal{C}(\Delta A)$.*

3.5.6. Proposition. *Let A be a unital commutative Banach algebra and a be an element of A. Then* $\operatorname{ran} \hat{a} = \sigma(a)$ *and* $\|\hat{a}\|_u = \rho(a)$.

3.5.7. Definition. An element a of a Banach algebra is QUASINILPOTENT if $\lim_{n\to\infty} \|a^n\|^{1/n} = 0$.

3.5.8. Proposition. *Let a be an element of a unital commutative Banach algebra A. Then the following are equivalent:*

(a) *a is quasinilpotent;*
(b) $\rho(a) = 0$*;*
(c) $\sigma(a) = \{0\}$*;*
(d) $\Gamma a = 0$*;*
(e) $\phi(a) = 0$ *for every* $\phi \in \Delta A$*;*
(f) $a \in \bigcap \operatorname{Max} A$.

3.5.9. Definition. A Banach algebra is SEMISIMPLE if it has no nonzero quasinilpotent elements.

3.5.10. Proposition. *Let A be a unital commutative Banach algebra. Then the following are equivalent:*

(a) *A is semisimple;*
(b) *if* $\rho(a) = 0$*, then* $a = 0$*;*
(c) *if* $\sigma(a) = \{0\}$*, then* $a = 0$*;*
(d) *the Gelfand transform* Γ_A *is a monomorphism (that is, an injective homomorphism);*
(e) *if* $\phi(a) = 0$ *for every* $\phi \in \Delta A$*, then* $a = 0$*;*
(f) $\bigcap \operatorname{Max} A = \{0\}$.

3.5.11. Proposition. *Let A be a unital commutative Banach algebra. Then the following are equivalent:*

(a) $\|a^2\| = \|a\|^2$ *for all* $a \in A$*;*

(b) $\rho(a) = \|a\|$ *for all* $a \in A$*; and*
(c) *the Gelfand transform is an isometry; that is,* $\|\widehat{a}\|_u = \|a\|$ *for all* $a \in A$.

3.5.12. Example. The Gelfand transform on $l_1(\mathbb{Z})$ is not an isometry.

Recall from proposition 3.4.21 that when X is a compact Hausdorff space the evaluation mapping E_X identifies the space X with the maximal ideal space of the Banach algebra $\mathcal{C}(X)$. Thus according to proposition 3.4.22 the mapping $\mathcal{C}E_X$ identifies the algebra $\mathcal{C}(\Delta(\mathcal{C}(X)))$ of continuous functions on this maximal ideal space with the algebra $\mathcal{C}(X)$ itself. It turns out that the Gelfand transform on the algebra $\mathcal{C}(X)$ is just the inverse of this identification map.

3.5.13. Example. Let X be a compact Hausdorff space. Then the Gelfand transform on the Banach algebra $\mathcal{C}(X)$ is an isometric isomorphism. In fact, on $\mathcal{C}(X)$ the Gelfand transform Γ_X is $(\mathcal{C}E_X)^{-1}$.

3.6. The Fourier Transform

3.6.1. Definition. For each f in the commutative Banach algebra $L_1(\mathbb{R})$ (see example 3.1.9) define a function $\widetilde{f}$ by

$$\widetilde{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-itx}\,dt\,.$$

The function $\widetilde{f}$ is the Fourier transform of f.

An important fact from real analysis, the so-called *Riemann-Lebesgue lemma*, is very useful in studying the Fourier transform.

3.6.2. Proposition (Riemann-Lebesgue Lemma)**.** *If* $f \in L_1(\mathbb{R})$*, then* $\widetilde{f} \in \mathcal{C}_0(\mathbb{R})$.

Proof. See any good real analysis text, for example [15](21.39).

3.6.3. Proposition. *The Fourier transform*

$$\mathfrak{F}\colon L_1(\mathbb{R}) \to \mathcal{C}_0(\mathbb{R})\colon f \mapsto \widetilde{f}$$

is a homomorphism of Banach algebras. (See example 3.1.9.)

What particular property of the homomorphism in the preceding proposition would you guess is particularly useful in applications?

3.6.4. Proposition. *The Banach algebra* $L_1(\mathbb{R})$ *is not unital.*

3.6.5. Example. The maximal ideal space of the Banach algebra $L_1(\mathbb{R})$ is $\mathbb{R}$ itself and the Gelfand transform on $L_1(\mathbb{R})$ is the Fourier transform.

Proof. See [2], pages 169–171.

The function $t \mapsto e^{it}$ is a bijection from the interval $[-\pi, \pi)$ to the unit circle $\mathbb{T}$ in the complex plane. One consequence of this is that we need not distinguish between

(i) 2π-periodic functions on $\mathbb{R}$,
(ii) all functions on $[-\pi, \pi)$,
(iii) functions f on $[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$, and
(iv) functions on $\mathbb{T}$.

In the sequel we will frequently without further explanation identify these classes of functions.

Another convenient identification is the one between the unit circle $\mathbb{T}$ in $\mathbb{C}$ and the maximal ideal space of the algebra $l_1(\mathbb{Z})$. The homeomorphism ψ between these two compact Hausdorff space was defined in example 3.4.12. It is often technically more convenient in working with the

Gelfand transform Γ_a of an element $a \in l_1(\mathbb{Z})$ to treat it as a function, let's call it G_a, whose domain is $\mathbb{T}$ as the following diagram suggests.

$$\begin{tikzcd}
\mathbb{T} \arrow[d, "\psi"'] \arrow[dr, "G_a"] & \\
\Delta l_1(\mathbb{Z}) \arrow[r, "\Gamma_a"'] & \mathbb{C}
\end{tikzcd}$$

Thus for $a \in l_1(\mathbb{Z})$ and $z \in \mathbb{T}$ we have

$$G_a(z) = \Gamma_a(\psi_z) = \psi_z(a) = \sum_{k=-\infty}^{\infty} a_n z^n \,.$$

3.6.6. Definition. If $f \in \mathcal{L}_1([-\pi, \pi))$, the FOURIER SERIES for f is the series

$$\sum_{n=-\infty}^{\infty} \widetilde{f}(n) \exp(int) \qquad -\pi \leq t \leq \pi$$

where the sequence $\widetilde{f}$ is defined by

$$\widetilde{f}(n) = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \exp(-int)\, dt$$

for all $n \in \mathbb{Z}$. The doubly infinite sequence $\widetilde{f}$ is the FOURIER TRANSFORM of f, and the number $\widetilde{f}(n)$ is the n^{th} FOURIER COEFFICIENT of f. If $\widetilde{f} \in l_1(\mathbb{Z})$ we say that f has an ABSOLUTELY CONVERGENT FOURIER SERIES. The set of all continuous functions on $\mathbb{T}$ with absolutely convergent Fourier series is denoted by $\mathcal{AC}(\mathbb{T})$.

3.6.7. Proposition. *If f is a continuous 2π-periodic function on $\mathbb{R}$ whose Fourier transform is zero, then $f = 0$.*

3.6.8. Corollary. *The Fourier transform on $\mathcal{C}(\mathbb{T})$ is injective.*

3.6.9. Proposition. *The Fourier transform on $\mathcal{C}(\mathbb{T})$ is a left inverse of the Gelfand transform on $l_1(\mathbb{Z})$.*

3.6.10. Proposition. *The range of the Gelfand transform on $l_1(\mathbb{Z})$ is the unital commutative Banach algebra $\mathcal{AC}(\mathbb{T})$.*

3.6.11. Proposition. *There are continuous functions whose Fourier series diverge at 0.*

PROOF. See, for example, [15], exercise 18.45.)

What does the preceding result say about the Gelfand transform $\Gamma\colon l_1(\mathbb{Z}) \to \mathcal{C}(\mathbb{T})$?

Suppose a function f belongs to $\mathcal{AC}(\mathbb{T})$ and is never zero. Then $1/f$ is certainly continuous on $\mathbb{T}$, but does it have an absolutely convergent Fourier series? One of the first triumphs of the abstract study of Banach algebras was a very simple proof of the answer to this question given originally by Norbert Wiener. Wiener's original proof by comparison was quite difficult.

3.6.12. Theorem (Wiener's theorem)**.** *Let f be a continuous function on $\mathbb{T}$ which is never zero. If f has an absolutely convergent Fourier series, then so does its reciprocal $1/f$.*

3.6.13. Example. The Laplace transform can also be viewed as a special case of the Gelfand transform. For details see [2], pages 173–175.

CHAPTER 4

INTERLUDE: THE LANGUAGE OF CATEGORIES

4.1. Objects and Morphisms

4.1.1. Definition. Let $\mathfrak{A}$ be a class, whose members we call OBJECTS. For every pair (S, T) of objects we associate a set $\mathfrak{Mor}(S, T)$, whose members we call MORPHISMS from S to T. We assume that $\mathfrak{Mor}(S, T)$ and $\mathfrak{Mor}(U, V)$ are disjoint unless $S = U$ and $T = V$.

We suppose further that there is an operation $\circ$ (called COMPOSITION) that associates with every $\alpha \in \mathfrak{Mor}(S, T)$ and every $\beta \in \mathfrak{Mor}(T, U)$ a morphism $\beta \circ \alpha \in \mathfrak{Mor}(S, U)$ in such a way that:

(1) $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$ whenever $\alpha \in \mathfrak{Mor}(S, T)$, $\beta \in \mathfrak{Mor}(T, U)$, and $\gamma \in \mathfrak{Mor}(U, V)$;

(2) for every object S there is a morphism $I_S \in \mathfrak{Mor}(S, S)$ satisfying $\alpha \circ I_S = \alpha$ whenever $\alpha \in \mathfrak{Mor}(S, T)$ and $I_S \circ \beta = \beta$ whenever $\beta \in \mathfrak{Mor}(R, S)$.

Under these circumstances the class $\mathfrak{A}$, together with the associated families of morphisms, is a CATEGORY.

4.1.2. Example. The category **SET** comprises the class of all sets (as objects) and all functions between these sets (as morphisms).

4.1.3. Example. The category **CpH** comprises the class of all compact Hausdorff spaces (as objects) and all continuous functions between these spaces (as morphisms).

4.1.4. Example. We will denote by $\mathbf{BAN}_\infty$ the category of Banach spaces and continuous linear maps between these spaces.

4.1.5. Example. The category $\mathbf{BA}_\infty$ comprises the class of all Banach algebras (as objects) and all continuous algebra homomorphisms between these algebras (as morphisms). We will further denote by $\mathbf{CBA}_\infty$ the category of commutative Banach algebras and continuous algebra homomorphisms, and by $\mathbf{UCBA}_\infty$ the category of unital commutative Banach algebras and unital continuous algebra homomorphisms. Following Palmer [21] we will refer to these categories as TOPOLOGICAL categories of Banach algebras.

4.1.6. Example. We will denote by $\mathbf{BAN_1}$ the category of Banach spaces and contractive linear maps between these spaces.

4.1.7. Example. The category $\mathbf{BA_1}$ comprises the class of all Banach algebras (as objects) and all contractive algebra homomorphisms between these algebras (as morphisms). We will further denote by $\mathbf{CBA_1}$ the category of commutative Banach algebras and contractive algebra homomorphisms, and by $\mathbf{UCBA_1}$ the category of unital commutative Banach algebras and unital contractive algebra homomorphisms. Again following Palmer [21] we will refer to these categories as GEOMETRIC categories of Banach algebras.

4.1.8. Example. The category **HIL** comprises the class of all Hilbert spaces (as objects) and all bounded linear maps between these spaces (as morphisms).

4.1.9. Example. We may regard the set $\mathbb{N}$ of natural numbers as a category where for m, $n \in \mathbb{N}$ the morphisms in $\mathfrak{Mor}(m, n)$ are the $m \times n$ matrices and composition is taken to be ordinary matrix multiplication.

4.1.10. Example. A MONOID is a semigroup with identity. We may regard a monoid $(M, *)$ as a category with a single object M (or any other object you choose). Then there is a single class of morphisms $\mathfrak{Mor}(M,M)$ and the morphisms in this class are the elements of the monoid M. The composite of two morphisms a and b is their product $a * b$ in M.

4.1.11. Notation. If $\mathbf{C}$ is a category we denote by $\mathbf{C^0}$ the class of objects in $\mathbf{C}$ and by $\mathbf{C^1}$ the class of morphism in $\mathbf{C}$. Thus, for example, the notation $X \in \mathbf{CpH^0}$ would indicate that X is a compact Hausdorff space and $f \in \mathbf{CpH^1}$ would mean that f is a continuous function between compact Hausdorff spaces.

4.1.12. Notation. In these notes we restrict the notation $A \xrightarrow{\phi} B$ to morphisms. When this notation appears it should be clear from context what category is being discussed. We then infer that A and B are objects in that category and $\phi \colon A \to B$ is a morphism.

4.1.13. Definition. The terminology for inverses of morphisms in categories is essentially the same as for functions. Let $S \xrightarrow{\alpha} T$ and $T \xrightarrow{\beta} S$ be morphisms in a category. If $\beta \circ \alpha = I_S$, then β is a LEFT INVERSE of α and, equivalently, α is a RIGHT INVERSE of β. We say that the morphism α is an ISOMORPHISM (or is INVERTIBLE) if there exists a morphism $T \xrightarrow{\beta} S$ which is both a left and a right inverse for α. Such a function is denoted by α^{-1} and is called the INVERSE of α. To indicate that objects A and B in some category are isomorphic we will, in general, use the notation $A \cong B$. There is one exception however: in various categories of topological spaces $X \approx Y$ means that the spaces X and Y are homeomorphic (topologically isomorphic).

4.2. Functors

4.2.1. Definition. If $\mathbf{A}$ and $\mathbf{B}$ are categories a (COVARIANT) FUNCTOR F from $\mathbf{A}$ to $\mathbf{B}$ (written $\mathbf{A} \xrightarrow{F} \mathbf{B}$) is a pair of maps: an OBJECT MAP F which associates with each object S in $\mathbf{A}$ an object $F(S)$ in $\mathbf{B}$ and a MORPHISM MAP (also denoted by F) which associates with each morphism $f \in \mathfrak{Mor}(S,T)$ in $\mathbf{A}$ a morphism $F(f) \in \mathfrak{Mor}(F(S),F(T))$ in $\mathbf{B}$, in such a way that

(1) $F(g \circ f) = F(g) \circ F(f)$ whenever $g \circ f$ is defined in $\mathbf{A}$; and
(2) $F(\mathrm{id}_S) = \mathrm{id}_{F(S)}$ for every object S in $\mathbf{A}$.

The definition of a CONTRAVARIANT FUNCTOR (or COFUNCTOR) $\mathbf{A} \xrightarrow{F} \mathbf{B}$ differs from the preceding definition only in that, first, the morphism map associates with each morphism $f \in \mathfrak{Mor}(S,T)$ in $\mathbf{A}$ a morphism $F(f) \in \mathfrak{Mor}(F(T),F(S))$ in $\mathbf{B}$ and, second, condition (1) above is replaced by

(1′) $F(g \circ f) = F(f) \circ F(g)$ whenever $g \circ f$ is defined in $\mathbf{A}$.

4.2.2. Definition. A LATTICE is a partially ordered set in which every pair of elements has a supremum and an infimum. A lattice L is ORDER COMPLETE if the $\sup A$ and $\inf A$ exist (in L) for every nonempty subset A of L.

4.2.3. Example. Let S be a nonempty set.

(a) The power set $\mathfrak{P}(S)$ of S partially ordered by $\subseteq$ is an order complete lattice.
(b) The class of order complete lattices and order preserving maps is a category.
(c) For each function f between sets let $\mathfrak{P}(f) = f^{\rightarrow}$. Then $\mathfrak{P}$ is a covariant functor from the category of sets and functions to the category of order complete lattices and order preserving maps.
(d) For each function f between sets let $\mathfrak{P}(f) = f^{\leftarrow}$. Then $\mathfrak{P}$ is a contravariant functor from the category of sets and functions to the category of order complete lattices and order preserving maps.

4.2.4. Example. Let $T \colon B \to C$ be a bounded linear map between Banach spaces. Define

$$T^* \colon C^* \to B^* \colon g \mapsto gT\,.$$

The map T^* is the ADJOINT of T.

(a) The composite gT is in B^* and the pair of maps $V \mapsto V^*$ and $T \mapsto T^*$ is a contravariant functor from the category $\mathbf{BAN}_\infty$ of Banach spaces and bounded linear maps into itself. The pair of maps $B \mapsto B^{**}$ and $T \mapsto T^{**}$ is a covariant functor. We will call this the *second dual functor*.

(b) If T is contractive, then $g \circ T \in B^*$ and the pair of maps $B \mapsto B^*$ and $T \mapsto T^*$ is a contravariant functor from the category $\mathbf{BAN_1}$ of Banach spaces and contractive linear maps into itself.

(c) We may regard the functor $\mathcal{C}$ (see example 4.2.5) as a contravariant functor from the category **CpH** of compact Hausdorff spaces and continuous maps into the category $\mathbf{BAN_1}$ of Banach spaces and contractive linear maps (because every Banach algebra is a Banach space). The composite of this functor with the one in part (b) is a covariant functor from **CpH** to $\mathbf{BAN_1}$. We will denote this composite by $\mathcal{C}^*$ so that $C^*(X)$ means $\big(\mathcal{C}(X)\big)^*$.

4.2.5. Example. Let X and Y be compact Hausdorff spaces and $F\colon X \to Y$ be continuous. As in proposition 3.4.22 define $\mathcal{C}(F)$ on $\mathcal{C}(Y)$ by

$$\mathcal{C}(F)(g) = g \circ F$$

for all $g \in \mathcal{C}(Y)$. Then the pair of maps $X \mapsto \mathcal{C}(X)$ and $F \mapsto \mathcal{C}(F)$ is a contravariant functor from the category **CpH** of compact Hausdorff spaces and continuous maps to the category $\mathbf{UCBA_1}$ of unital commutative Banach algebras and contractive unital algebra homomorphisms. Since every contractive homomorphism is continuous we may, if we choose, regard $\mathcal{C}$ as a functor from the category **CpH** to the category $\mathbf{UCBA}_\infty$ of unital commutative Banach algebras and unital continuous algebra homomorphisms.

4.2.6. Example. Let A and B be unital commutative Banach algebras and $T\colon A \to B$ be a unital algebra homomorphism. As in proposition 3.4.23 define ΔT on ΔB by

$$\Delta T(\psi) = \psi \circ T$$

for all $\psi \in \Delta B$. Then the pair of maps $A \mapsto \Delta A$ and $T \mapsto \Delta T$ is a contravariant functor from the category **UCBA** of unital commutative Banach algebras and unital algebra homomorphisms to the category **CpH** of compact Hausdorff spaces and continuous maps.

4.2.7. Example. All the categories that are of interest in this course are concrete categories. A CONCRETE CATEGORY is, roughly speaking, one in which the objects are sets usually with additional structure (algebraic operations, inner products, norms, topologies, and the like) and the morphisms are maps (functions) which preserve, in some sense, the additional structure. If A is an object in some concrete category **C**, we denote by $\Box A$ its underlying set. And if $A \xrightarrow{f} B$ is a morphism in **C** we denote by $\Box f$ the map from $\Box A$ to $\Box B$ regarded simply as a function between sets. It is easy to see that $\Box$, which takes objects in **C** to objects in **SET** (the category of sets and maps) and morphisms in **C** to morphisms in **SET**, is a functor. It is referred to as the FORGETFUL FUNCTOR. (The definite article here is inaccurate but nearly universal.) In the category **VEC** of vector spaces and linear maps, for example, $\Box$ causes a vector space V to "forget" about its addition and scalar multiplication; $\Box V$ is just a set. And if $T\colon V \to W$ is a linear transformation, then $\Box T\colon \Box V \to \Box W$ is just a map between sets—it has "forgotten" about preserving operations (which are no longer there to preserve). For more precise definitions of *concrete categories* and *forgetful functors* consult any text on category theory.

4.2.8. Convention. In these notes all categories are concrete.

4.2.9. Definition. In light of the preceding convention it makes sense to define in any (concrete) category a MONOMORPHISM to be an injective morphism and an EPIMORPHISM to be a surjective morphism. This terminology, which makes no sense in arbitrary categories, deviates from the usage in category theory proper where *monomorphism* (or *monic*) means left-cancellable and *epimorphism* (or *epic*) means right-cancellable. In most of the concrete categories which we consider morphisms are injective if and only if they are left-cancellable. The agreement between surjective and right-cancellable is less general. In the category of topological spaces and continuous maps, for example, morphisms are right-cancellable if and only if they have dense range.

4.2.10. Example. If $\mathbf{C}$ is a category let $\mathbf{C^2}$ be the category whose objects are ordered pairs of objects in $\mathbf{C}$ and whose morphisms are ordered pairs of morphisms in $\mathbf{C}$. Thus if $A \xrightarrow{f} C$ and $B \xrightarrow{g} D$ are morphisms in $\mathbf{C}$, then $(A,B) \xrightarrow{(f,g)} (C,D)$ (where $(f,g)(a,b) = \big(f(a), g(b)\big)$ for all $a \in A$ and $b \in B$) is a morphism in $\mathbf{C^2}$. Composition of morphism is defined in the obvious way: $(f,g) \circ (h,j) = (f \circ h, g \circ j)$. We define the DIAGONAL FUNCTOR $\mathbf{C} \xrightarrow{\mathsf{D}} \mathbf{C^2}$ by $\mathsf{D}(A) := (A,A)$. This is a covariant functor.

4.3. Natural Transformations

4.3.1. Definition. Let $\mathbf{A}$ and $\mathbf{B}$ be categories and $\mathbf{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathbf{B}$ be covariant functors. A NATURAL TRANSFORMATION from F to G is a map τ which assigns to each object A in category $\mathbf{A}$ a morphism $F(A) \xrightarrow{\tau_A} G(A)$ in category $\mathbf{B}$ in such a way that for every morphism $A \xrightarrow{f} A'$ in $\mathbf{A}$ the following diagram commutes.

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\tau_A"'] & F(A') \arrow[d, "\tau_{A'}"] \\
G(A) \arrow[r, "G(f)"'] & G(A')
\end{tikzcd}$$

We denote such a transformation by $F \xrightarrow{\tau} G$. (The definition of a natural transformation between two contravariant functors should be obvious: just reverse the horizontal arrows in the preceding diagram.)

A natural transformation $F \xrightarrow{\tau} G$ is a NATURAL EQUIVALENCE (or a NATURAL ISOMORPHISM) if each morphism τ_A is an isomorphism in $\mathbf{B}$. Two functors are NATURALLY EQUIVALENT if there exists a natural equivalence between them.

4.3.2. Example. On the category $\mathbf{BAN}_\infty$ of Banach spaces and bounded linear transformations let I be the identity functor and $(\,\cdot\,)^{**}$ be the second dual functor (see example 4.2.4(a)). Show that the map J which takes each Banach space B to its natural injection J_B (see definition 3.4.2) is a natural transformation from I to $(\,\cdot\,)^{**}$. In the category of *reflexive* Banach spaces and bounded linear maps this natural transformation is a natural equivalence.

4.3.3. Example. The mapping $E\colon X \mapsto E_X$, which takes compact Hausdorff spaces to their corresponding evaluation maps is a natural equivalence between the identity functor and the functor $\Delta\mathcal{C}$ in the category of compact Hausdorff spaces and continuous maps. (See example 3.3.8 and proposition 3.4.21.)

Thus X and $\Delta\mathcal{C}(X)$ are not only homeomorphic, they are *naturally homeomorphic*. This is the justification for the very common informal assertion that the maximal ideals of $\mathcal{C}(X)$ “are” just the points of X.

4.3.4. Example. The Gelfand transform Γ is a natural transformation from the identity functor to the $\mathcal{C}\Delta$ functor on the category $\mathbf{UCBA}$ of unital commutative Banach algebras and unital algebra homomorphisms.

4.4. Universal Morphisms

Much of mathematics involves the construction of new objects from old ones—things such as products, coproducts, quotients, completions, compactifications, and unitizations. More often than not it is possible to characterize such a construction by the existence of a unique morphism having some particular property. Because this morphism and its corresponding property characterize the construction in question, they are referred to as a *universal morphism* and a *universal property*, respectively. Here is one very common way in which such morphisms arise.

4.4.1. Definition. Let $\mathbf{A} \xrightarrow{\mathsf{F}} \mathbf{B}$ be a functor between categories $\mathbf{A}$ and $\mathbf{B}$ and B be an object in $\mathbf{B}$. A pair (A, u) with A an object in $\mathbf{A}$ and u a $\mathbf{B}$-morphism from B to $\mathsf{F}(A)$ is a UNIVERSAL MORPHISM for B (with respect to the functor F) if for every object A' in $\mathbf{A}$ and every $\mathbf{B}$-morphism $B \xrightarrow{f} \mathsf{F}(A')$ there exists a unique $\mathbf{A}$-morphism $A \xrightarrow{\tilde{f}} A'$ such that the following diagram commutes.

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[dr, "f"'] & \mathsf{F}(A) \arrow[d, dashed, "\mathsf{F}(\tilde{f})"] & & A \arrow[d, dashed, "\tilde{f}"] \\ & \mathsf{F}(A') & & A' \end{tikzcd} \tag{4.1}$$

In this context the object A is often referred to as a UNIVERSAL OBJECT in $\mathbf{A}$.

4.4.2. Example. Let S be a set and $\mathbf{VEC} \xrightarrow{\square} \mathbf{SET}$ be the forgetful functor from the category $\mathbf{VEC}$ to the category $\mathbf{SET}$. If there exists a vector space V and an injection $S \xrightarrow{\iota} \square V$ which constitute a universal morphism for S (with respect to $\square$), then V is the FREE VECTOR SPACE over S. Of course merely *defining* an object does not guarantee its existence. In fact, free vector spaces exist over arbitrary sets. Given the set S let V be the set of all complex valued functions on S which have finite support. Define addition and scalar multiplication pointwise. The map $\iota\colon s \mapsto \chi_{\{s\}}$ of each element $s \in S$ to the characteristic function of $\{s\}$ is the desired injection. To verify that V is free over S it must be shown that for every vector space W and every function $S \xrightarrow{f} \square W$ there exists a unique linear map $V \xrightarrow{\tilde{f}} W$ which makes the following diagram commute.

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[dr, "f"'] & \square V \arrow[d, dashed, "\square\tilde{f}"] & & V \arrow[d, dashed, "\tilde{f}"] \\ & \square W & & W \end{tikzcd}$$

4.4.3. Example. Let S be a nonempty set and let $S' = S \cup \{\mathbf{1}\}$ where $\mathbf{1}$ is any element not belonging to S. A WORD in the language S is a sequence s of elements of S' which is eventually $\mathbf{1}$ and satisfies: if $s_k = \mathbf{1}$, then $s_{k+1} = \mathbf{1}$. The constant sequence $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \dots)$ is called the EMPTY WORD. Let F be the set of all words of members in the language S. Suppose that $s = (s_1, \dots, s_m, \mathbf{1}, \mathbf{1}, \dots)$ and $t = (t_1, \dots, t_n, \mathbf{1}, \mathbf{1}, \dots)$ are words (where $s_1, \dots, s_m, t_1, \dots, t_n \in S$). Define

$$x * y := (s_1, \dots, s_m, t_1, \dots, t_n, \mathbf{1}, \mathbf{1}, \dots).$$

This operation is called CONCATENATION. It is not difficult to see that the set F under concatenation is a monoid (a semigroup with identity) where the empty word is the identity element. This is the FREE MONOID generated by S. If we exclude the empty word we have the FREE SEMIGROUP generated by S. The associated diagram is

$$\begin{tikzcd}
S \arrow[r, "\iota"] \arrow[dr, "f"'] & \Box F \arrow[d, dashed, "\Box\tilde{f}"] & F \arrow[d, dashed, "\tilde{f}"] \\
& \Box G & G
\end{tikzcd}$$

where ι is the obvious injection $s \mapsto (s, \mathbf{1}, \mathbf{1}, \dots)$ (usually treated as an inclusion mapping), G is an arbitrary semigroup, $f \colon S \to G$ is an arbitrary function, and $\Box$ is the forgetful functor from the category of monoids and homomorphisms (or the category of semigroups and homomorphisms) to the category **SET**.

4.4.4. Example. Here is the usual presentation of the coproduct of two objects in a category. Let A_1 and A_2 be two objects in a category $\mathbf{C}$. A COPRODUCT of A_1 and A_2 is a triple (Q, ι_1, ι_2) with Q an object in $\mathbf{C}$ and $A_k \xrightarrow{\iota_k} Q$ $(k = 1, 2)$ morphisms in $\mathbf{C}$ which satisfies the following condition: if B is an arbitrary object in $\mathbf{C}$ and $A_k \xrightarrow{f_k} B$ $(k = 1, 2)$ are arbitrary morphisms in $\mathbf{C}$, then there exists a unique $\mathbf{C}$-morphism $Q \xrightarrow{f} B$ which makes the following diagram commute.

$$\begin{tikzcd}
& B & \\
A_1 \arrow[ur, "f_1"] \arrow[r, "\iota_1"'] & Q \arrow[u, "f"] & A_2 \arrow[ul, "f_2"'] \arrow[l, "\iota_2"]
\end{tikzcd} \tag{4.2}$$

It may not be obvious at first glance that this construction is *universal* in the sense of definition 4.4.1. To see that it in fact is, let D be the diagonal functor from a category $\mathbf{C}$ to the category of pairs $\mathbf{C^2}$ (see example 4.2.10). Suppose that (Q, ι_1, ι_2) is a coproduct of the $\mathbf{C}$-objects A_1 and A_2 in the sense defined above. Then $A = (A_1, A_2)$ is an object in $\mathbf{C^2}$, $A \xrightarrow{\iota} \mathsf{D}(Q)$ is a $\mathbf{C^2}$-morphism, and the pair (A, ι) is universal in the sense of 4.4.1. The diagram corresponding to diagram (4.1) is

$$\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathsf{D}(Q) \arrow[d, dashed, "\mathsf{D}(\tilde{f})"] & Q \arrow[d, dashed, "\tilde{f}"] \\
& \mathsf{D}(B) & B
\end{tikzcd} \tag{4.3}$$

where B is an arbitrary object in $\mathbf{C}$ and (for $k = 1, 2$) $A_k \xrightarrow{f_k} B$ are arbitrary $\mathbf{C}$-morphisms so that $f = (f_1, f_2)$ is a $\mathbf{C^2}$-morphism.

4.4.5. Example. The coproduct of two objects H and K in the category **HIL** of Hilbert spaces and bounded linear maps (and more generally in the category of inner product spaces and linear maps) is their (external orthogonal) direct sum $H \oplus K$ (see 1.2.19).

4.4.6. Example. The coproduct of two objects S and T in the category **SET** is their disjoint union $S \uplus T$.

4.4.7. Example. Let A and B be Banach spaces. On the Cartesian product $A \times B$ define addition and scalar multiplication pointwise. For every $(a, b) \in A \times B$ let $\|(a, b)\| = \max\{\|a\|, \|b\|\}$. This makes $A \times B$ into a Banach space, which is denoted by $A \oplus B$ and is called the DIRECT SUM of A and B. The direct sum is a coproduct in the topological category $\mathbf{BAN}_\infty$ of Banach spaces but not in the corresponding geometrical category $\mathbf{BAN_1}$.

4.4.8. Example. To construct a coproduct on the geometrical category $\mathbf{BAN_1}$ of Banach spaces define the vector space operations pointwise on $A \times B$ but as a norm use $\|(a,b)\|_1 = \|a\| + \|b\|$ for all $(a,b) \in A \times B$.

Virtually everything in category theory has a dual concept—one that is obtained by reversing all the arrows. We can, for example, reverse all the arrows in diagram (4.1).

4.4.9. Definition. Let $\mathbf{A} \xrightarrow{\mathsf{F}} \mathbf{B}$ be a functor between categories $\mathbf{A}$ and $\mathbf{B}$ and B be an object in $\mathbf{B}$. A pair (A, u) with A an object in $\mathbf{A}$ and u a $\mathbf{B}$-morphism from $\mathsf{F}(A)$ to B is a CO-UNIVERSAL MORPHISM FOR B (WITH RESPECT TO F) if for every object A' in $\mathbf{A}$ and every $\mathbf{B}$-morphism $\mathsf{F}(A') \xrightarrow{f} B$ there exists a unique $\mathbf{A}$-morphism $A' \xrightarrow{\tilde{f}} A$ such that the following diagram commutes.

$$
\begin{tikzcd}
B & \mathsf{F}(A) \arrow[l, "u"'] & A \\
 & \mathsf{F}(A') \arrow[ul, "f"] \arrow[u, dashed, "\mathsf{F}(\tilde{f})"'] & A' \arrow[u, dashed, "\tilde{f}"']
\end{tikzcd}
\tag{4.4}
$$

Some authors reverse the convention and call the morphism in 4.4.1 *co-universal* and the one here *universal*. Other authors, this one included, call both *universal morphisms*.

4.4.10. Convention. Morphisms of both the types defined in 4.4.1 and 4.4.9 will be referred to as *universal morphisms*.

4.4.11. Example. The usual categorical approach to products is as follows. Let A_1 and A_2 be two objects in a category $\mathbf{C}$. A PRODUCT of A_1 and A_2 is a triple (P, π_1, π_2) with P an object in $\mathbf{C}$ and $Q \xrightarrow{\iota_k} A_k$ $(k = 1, 2)$ morphisms in $\mathbf{C}$ which satisfies the following condition: if B is an arbitrary object in $\mathbf{C}$ and $B \xrightarrow{f_k} A_k$ $(k = 1, 2)$ are arbitrary morphisms in $\mathbf{C}$, then there exists a unique $\mathbf{C}$-morphism $B \xrightarrow{f} P$ which makes the following diagram commute.

$$
\begin{tikzcd}
 & B \arrow[dl, "f_1"'] \arrow[d, "f"] \arrow[dr, "f_2"] & \\
A_1 & P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & A_2
\end{tikzcd}
\tag{4.5}
$$

This is (co)-universal in the sense of definition 4.4.9.

4.4.12. Example. The product of two objects H and K in the category $\mathbf{HIL}$ of Hilbert spaces and bounded linear maps (and more generally in the category of inner product spaces and linear maps) is their (external orthogonal) direct sum $H \oplus K$ (see 1.2.19).

4.4.13. Example. The product of two objects S and T in the category $\mathbf{SET}$ is their Cartesian product $S \times T$.

4.4.14. Example. If A and B are Banach algebras their direct sum $A \oplus B$ is a product in both the topological and geometric categories, $\mathbf{BA_\infty}$ and $\mathbf{BA_1}$, of Banach algebras. Compare this to the situation discussed in examples 4.4.7 and 4.4.8.

4.4.15. Proposition. *Universal objects in a category are essentially unique.*

4.4.16. Definition. Let M and N be metric spaces. We say that N is a COMPLETION of M if N is complete and M is isometric to a dense subset of N.

4.4.17. Proposition. *Every metric space has a completion.*

Hint for proof. Let (M, d) be a metric space and fix $a \in M$. For each $x \in M$ define $\phi_x : M \to \mathbb{R}$ by $\phi_x(u) = d(x, u) - d(u, a)$. Show first that for every $x \in M$ the function ϕ_x belongs to the space $\mathcal{C}_b(M, \mathbb{R})$ of bounded real valued continuous functions on M. Then show that $\phi : M \to \mathcal{C}_b(M, \mathbb{R})$ is an isometry. (To verify that $d_u(\phi_x, \phi_y) \geq d(x, y)$ notice that $\|\phi_x - \phi_y\|_u \geq |\phi_x(y) - \phi_y(y)|$.) Explain why the closure of $\operatorname{ran} \phi$ in $\mathcal{C}_b(M)$ is a completion of M.

4.4.18. Example. Let $\mathbf{MS_1}$ be the category of metric spaces and contractive maps and $\mathbf{CMS_1}$ be the category of complete metric spaces and contractive maps. The map $M \mapsto \widehat{M}$ from $\mathbf{MS_1}$ to $\mathbf{CMS_1}$ which takes a metric space to its completion is universal in the sense of definition 4.4.1.

The following consequence of proposition 4.4.15 allows us to speak of *the* completion of a metric space.

4.4.19. Corollary. *Metric space completions are unique (up to isometry).*

Every inner product space is a metric space by propositions 1.2.11 and 1.2.12. It is an important fact that completing an inner product space as a metric space produces a Hilbert space.

4.4.20. Proposition. *Let V be an inner product space and H be the metric space completion of V. Then the inner product on V can be extended to an inner product on H and the metric on H induced by this inner product is the same as the original metric on H.*

4.4.21. Proposition. *Let V be an inner product space and H be its completion (see proposition 4.4.20). Then every operator on V extends to an operator on H.*

Next is a closely related result which we will need in the sequel.

4.4.22. Proposition. *Let D be a dense subset of a Hilbert space H and suppose that S and T are operators on H such that $\langle Su, v\rangle = \langle u, Tv\rangle$ for all u, $v \in D$. Then $\langle Sx, y\rangle = \langle x, Ty\rangle$ for all x, $y \in H$.*

4.4.23. Definition. A category $\mathbf{A}$ is a SUBCATEGORY of category $\mathbf{B}$ if $\mathbf{A}^0 \subseteq \mathbf{B}^0$ and $\mathbf{A}^1 \subseteq \mathbf{B}^1$. It is a FULL subcategory of $\mathbf{B}$ if, additionally, for all objects A_1 and A_2 in $\mathbf{A}$ every $\mathbf{B}$-morphism $A_1 \xrightarrow{\tau} A_2$ is also an $\mathbf{A}$-morphism.

4.4.24. Example. The category of unital algebras and unital algebra homomorphism is a subcategory of the category of algebras and algebra homomorphisms, but not a full subcategory.

4.4.25. Example. The category of complete metric spaces and continuous maps is a full subcategory of the category of metric spaces and continuous maps.

CHAPTER 5

C^*-ALGEBRAS

5.1. Adjoints of Hilbert Space Operators

5.1.1. Definition. Let H and K be Hilbert spaces. A sesquilinear functional $\phi\colon H \times K \to \mathbb{C}$ on $H \times K$ is BOUNDED if there exists a constant $M > 0$ such that

$$|\phi(x, y)| \le M\|x\|\|y\|$$

for all $x \in H$ and $y \in K$.

5.1.2. Proposition. *If $\phi\colon H \times K \to \mathbb{C}$ is a bounded sesquilinear functional on the product of two Hilbert spaces, then the following numbers (exist and) are equal:*

- $\sup\{|\phi(x, y)|\colon \|x\| \le 1, \|y\| \le 1\}$
- $\sup\{|\phi(x, y)|\colon \|x\| = \|y\| = 1\}$
- $\sup\left\{\dfrac{|\phi(x, y)|}{\|x\|\,\|y\|}\colon x, y \ne 0\right\}$
- $\inf\{M > 0\colon |\phi(x, y)| \le M\|x\|\,\|y\|$ *for all* $x \in H,\ y \in K\}$.

The proof of the preceding proposition is virtually identical to the one for linear maps (see proposition 2.2.4).

5.1.3. Definition. Let $\phi\colon H \times K \to \mathbb{C}$ be a bounded sesquilinear functional on the product of two Hilbert spaces. We define $\|\phi\|$, the *norm* of ϕ, to be any of the (equal) expressions in the preceding result.

5.1.4. Proposition. *Let $A\colon H \to K$ be a bounded linear map between Hilbert spaces. Then $\phi\colon H \times K \to \mathbb{C}\colon (x, y) \mapsto \langle Ax, y\rangle$ is a bounded sesquilinear functional on $H \times K$ and $\|\phi\| = \|A\|$.*

5.1.5. Proposition. *Let $\phi\colon H \times K \to \mathbb{C}$ be a bounded sesquilinear functional on the product of two Hilbert spaces. Then there exist unique bounded linear maps $B \in \mathfrak{B}(H, K)$ and $C \in \mathfrak{B}(K, H)$ such that*

$$\phi(x, y) = \langle Bx, y\rangle = \langle x, Cy\rangle$$

for all $x \in H$ and $y \in K$. Also, $\|B\| = \|C\| = \|\phi\|$.

Hint for proof. Show that for every $x \in H$ the map $y \mapsto \overline{\phi(x, y)}$ is a bounded linear functional on K. Use the *Riesz-Fréchet theorem* (2.2.23).

The next proposition provides an entirely satisfactory extension of proposition 1.2.32 to the infinite dimensional setting: adjoints of Hilbert space operators always exist.

5.1.6. Proposition. *Let $A\colon H \to K$ be a bounded linear map between Hilbert spaces. The mapping $(x, y) \mapsto \langle Ax, y\rangle$ from $H \times K$ into $\mathbb{C}$ is a bounded sesquilinear functional. Then there exists an unique bounded linear map $A^*\colon K \to H$ such that*

$$\langle Ax, y\rangle = \langle x, A^*y\rangle$$

for all $x \in H$ and $y \in K$. This is the ADJOINT *of A (see definition 1.2.31). Also $\|A^*\| = \|A\|$.*

5.1.7. Example. If S is the unilateral shift operator on l_2 (see example 2.2.15)

$$S\colon l_2 \to l_2\colon (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots),$$

then its adjoint is given by

$$S^*\colon l_2 \to l_2\colon (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots).$$

Example 2.2.10 dealt with multiplication operators on $\mathcal{C}(X)$ where X is a compact Hausdorff space. Multiplication operators can be defined on many spaces; there is nothing special in this respect about $\mathcal{C}(X)$. For example, we can just as well consider multiplication operators on spaces of square integrable functions.

5.1.8. Example. Let $(S, \mathcal{A}, \mu)$ be a sigma-finite measure space and $L_2(S)$ be the Hilbert space of all (equivalence classes of) complex valued functions on S which are square integrable with respect to μ. Let ϕ be an essentially bounded complex valued μ-measurable function on S. Define M_ϕ on $L_2(S)$ by $M_\phi(f) := \phi f$. Then M_ϕ is an operator on $L_2(S)$; it is called a MULTIPLICATION OPERATOR. Its norm is given by $\|M_\phi\| = \|\phi\|_\infty$ and its adjoint by ${M_\phi}^* = M_{\overline{\phi}}$.

5.1.9. Example. Let $(S, \mathcal{A}, \mu)$ be a sigma-finite measure space and $L_2(X)$ be the Hilbert space of all (equivalence classes of) complex valued functions on S which are square integrable with respect to μ. Let K be the integral operator whose kernel is the function $k\colon S \times S \to \mathbb{C}$ which is square integrable with respect to the product measure $\mu \times \mu$ on $S \times S$ (see example 2.2.17). The adjoint K^* of K is also an integral operator and its kernel is the function k^* defined by $k^*(x, y) = \overline{k(y, x)}$.

5.1.10. Proposition. *Let A and B be operators on a Hilbert space H and $\alpha \in \mathbb{C}$. Then*

(i) $(A + B)^* = A^* + B^*$;
(ii) $(\alpha A)^* = \overline{\alpha} A^*$;
(iii) $A^{**} = A$; *and*
(iv) $(AB)^* = B^* A^*$.

5.1.11. Proposition. *Let A be an operator on a Hilbert space. Then*

$$\|A^* A\| = \|A\|^2 .$$

We now give an infinite dimensional generalization of the *fundamental theorem of linear algebra* 1.2.34. We cannot expect 1.2.34 to hold exactly in the infinite dimensional case because the range of an operator is not necessarily closed, while orthogonal complements of subspaces always are. (An example of an operator with non-closed range is given in 10.2.5.)

5.1.12. Theorem. *If A is an operator on a Hilbert space, then*

(i) $\ker A^* = (\operatorname{ran} A)^\perp$, *and*
(ii) $\overline{\operatorname{ran} A^*} = (\ker A)^\perp$.

5.1.13. Proposition. *An operator A on a Hilbert space H is self-adjoint if and only if $\langle Ax, x\rangle \in \mathbb{R}$ for every $x \in H$.*

5.1.14. Proposition. *The pair of maps $H \mapsto H$ and $A \mapsto A^*$ taking every Hilbert space to itself and every bounded linear map between Hilbert spaces to its adjoint is a contravariant functor from the category* **HIL** *to itself.*

5.2. Algebras with Involution

5.2.1. Definition. An INVOLUTION on an algebra A is a map $x \mapsto x^*$ from A into A which satisfies

(i) $(x + y)^* = x^* + y^*$,
(ii) $(\alpha x)^* = \overline{\alpha}\, x^*$,
(iii) $x^{**} = x$, and
(iv) $(xy)^* = y^* x^*$

for all x, $y \in A$ and $\alpha \in \mathbb{C}$. An algebra on which an involution has been defined is a $*$-ALGEBRA (pronounced "star algebra"). An algebra homomorphism ϕ between $*$-algebras which preserves involution (that is, such that $\phi(a^*) = (\phi(a))^*$) is a $*$-HOMOMORPHISM (pronounced "star homomorphism". A $*$-homomorphism $\phi\colon A \to B$ between unital algebras is said to be UNITAL if $\phi(\mathbf{1}_A) = \mathbf{1}_B$. In the category of $*$-algebras and $*$-homomorphisms, the isomorphisms (called for emphasis $*$-ISOMORPHISMS) are the bijective $*$-homomorphisms.

5.2.2. Example. In the algebra $\mathbb{C}$ of complex numbers the map $z \mapsto \overline{z}$ of a number to its complex conjugate is an involution.

5.2.3. Example. The map of an $n \times n$ matrix to its conjugate transpose is an involution on the unital algebra M_n (see example 2.3.14).

5.2.4. Example. Let X be a compact Hausdorff space. The map $f \mapsto \overline{f}$ of a function to its complex conjugate is an involution in the algebra $\mathcal{C}(X)$.

5.2.5. Example. The map $T \mapsto T^*$ of a Hilbert space operator to its adjoint is an involution in the algebra $\mathfrak{B}(H)$ (see proposition 5.1.10).

5.2.6. Proposition. *Let a and b be elements of a $*$-algebra. Then a commutes with b if and only if a^* commutes with b^*.*

5.2.7. Proposition. *In a unital $*$-algebra $\mathbf{1}^* = \mathbf{1}$.*

5.2.8. Proposition. *If a $*$-algebra A has a left multiplicative identity e, then A is unital and $e = \mathbf{1}_A$.*

5.2.9. Proposition. *Let a be an element of a unital $*$-algebra. Then a^* is invertible if and only if a is. And when a is invertible we have*

$$(a^*)^{-1} = \left(a^{-1}\right)^*.$$

5.2.10. Proposition. *Let a be an element of a unital $*$-algebra. Then $\lambda \in \sigma(a)$ if and only if $\overline{\lambda} \in \sigma(a^*)$.*

5.2.11. Corollary. *For every element a of a *-algebra $\rho(a^*) = \rho(a)$.*

5.2.12. Definition. An element a of a $*$-algebra A is SELF-ADJOINT (or HERMITIAN) if $a^* = a$. It is NORMAL if $a^*a = aa^*$. And it is UNITARY if $a^*a = aa^* = \mathbf{1}$. The set of all self-adjoint elements of A is denoted by $\mathfrak{H}(A)$, the set of all normal elements by $\mathfrak{N}(A)$, and the set of all unitary elements by $\mathfrak{U}(A)$.

5.2.13. Proposition. *Let a be an element of a $*$-algebra. Then there exist unique self-adjoint elements u and v such that $a = u + iv$.*

Hint for proof. Think of the special case of writing a complex number in terms of its real and imaginary parts.

5.2.14. Definition. Let S be a subset of a $*$-algebra A. Then $S^* = \{s^*\colon s \in S\}$. The subset S is SELF-ADJOINT if $S^* = S$.

A nonempty self-adjoint subalgebra of A is a $*$-SUBALGEBRA (or a SUB-$*$-ALGEBRA).

CAUTION. The preceding definition does *not* say that the elements of a self-adjoint subset of a $*$-algebra are themselves self-adjoint.

5.2.15. Definition. In an algebra A with involution a $*$-IDEAL is a self-adjoint ideal in A.

5.2.16. Proposition. *Let $\phi\colon A \to B$ be a $*$-homomorphism between $*$-algebras. Then the kernel of ϕ is a $*$-ideal in A and the range of ϕ is a $*$-subalgebra of B.*

5.2.17. Proposition. *If J is a $*$-ideal in a $*$-algebra A, then defining $[a]^* = [a^*]$ for each $a \in A$ makes the quotient A/J into a $*$-algebra and the quotient map $a \mapsto [a]$ is a $*$-homomorphism. The $*$-algebra A/J is,of course, the QUOTIENT of A by J.*

5.3. C^*-Algebras

5.3.1. Definition. A C^*-ALGEBRA is a Banach algebra A with involution which satisfies

$$\|a^*a\| = \|a\|^2$$

for every $a \in A$. This property of the norm is usually referred to as *the C^*-condition*. An algebra norm satisfying this condition is a C^*-NORM. A C^*-SUBALGEBRA of a C^*-algebra A is a closed $*$-subalgebra of A.

5.3.2. Example. The vector space $\mathbb{C}$ of complex numbers with the usual multiplication of complex numbers and complex conjugation $z \mapsto \overline{z}$ as involution is a unital commutative C^*-algebra.

5.3.3. Example. If X is a compact Hausdorff space, the algebra $\mathcal{C}(X)$ of continuous complex valued functions on X is a unital commutative C^*-algebra when involution is taken to be complex conjugation.

5.3.4. Example. If X is a locally compact Hausdorff space, the Banach algebra $\mathcal{C}_0(X) = \mathcal{C}_0(X, \mathbb{C})$ of continuous complex valued functions on X which vanish at infinity is a (not necessarily unital) commutative C^*-algebra when involution is taken to be complex conjugation.

5.3.5. Example. If (X, μ) is a measure space, the algebra $\mathcal{L}^\infty(X, \mu)$ of essentially bounded measurable complex valued functions on X (again with complex conjugation as involution) is a C^*-algebra. (Technically, of course, the members of $L^\infty(X, \mu)$ are equivalence classes of functions which differ on sets of measure zero.)

5.3.6. Example. The algebra $\mathfrak{B}(H)$ of bounded linear operators on a Hilbert space H is a unital C^*-algebra when addition and scalar multiplication of operators are defined pointwise, composition is taken as multiplication, the map $T \mapsto T^*$, which takes an operator to its adjoint, is the involution, and the norm is the usual operator norm. (See proposition 5.1.11 for the crucial C^*-property $\|T^*T\| = \|T\|^2$.)

5.3.7. Example. A special case of the preceding example is the set $\mathbf{M}_n$ of $n \times n$ matrices of complex numbers. We saw in example 5.2.3 that $\mathbf{M}_n$ is a unital algebra with involution. To make it into a C^*-algebra simply identify each matrix with the (necessarily bounded) linear operator in $\mathfrak{B}(\mathbb{C}^n)$ which it represents.

5.3.8. Proposition. *In every C^*-algebra involution is an isometry. That is, $\|a^*\| = \|a\|$ for every element a in the algebra.*

In definition 3.1.1 of *normed algebra* we made the special requirement that the identity element of a unital normed algebra have norm one. In C^*-algebras this requirement is redundant.

5.3.9. Corollary. *In a unital C^*-algebra $\|\mathbf{1}\| = 1$.*

5.3.10. Corollary. *Every unitary element in a unital C^*-algebra has norm one.*

5.3.11. Corollary. *If a is an element of a C^*-algebra A such that $ab = \mathbf{0}$ for every $b \in A$, then $a = \mathbf{0}$.*

5.3.12. Proposition. *Let a be a normal element of a unital C^*-algebra. Then $\|a^2\| = \|a\|^2$ and therefore $\rho(a) = \|a\|$.*

5.3.13. Corollary. *Let A be a unital commutative C^*-algebra. Then $\|a^2\| = \|a\|^2$ and $\rho(a) = \|a\|$ for every $a \in A$.*

5.3.14. Corollary. *On a unital commutative C^*-algebra A the Gelfand transform Γ is an isometry; that is, $\|\Gamma_a\|_u = \|\widehat{a}\|_u = \|a\|$ for every $a \in A$.*

5.3.15. Corollary. *The norm of a unital C^*-algebra is unique in the sense that given a unital algebra A with involution there is at most one norm which makes A into a C^*-algebra.*

5.3.16. Proposition. *If h is a self-adjoint element of a unital C^*-algebra, then $\sigma(h) \subseteq \mathbb{R}$.*

5.3.17. Proposition. *If u is a unitary element of a unital C^*-algebra, then $\sigma(u) \subseteq \mathbb{T}$.*

5.3.18. Proposition. *If a is an element in a C^*-algebra, then*

$$\begin{aligned} \|a\| &= \sup\{\|xa\| : \|x\| \le 1\} \\ &= \sup\{\|ax\| : \|x\| \le 1\}. \end{aligned}$$

5.3.19. Corollary. *If a is an element of a C^*-algebra A, the operator L_a, called* LEFT MULTIPLICATION BY *a and defined by*

$$L_a \colon A \to A \colon x \mapsto ax$$

is a (bounded linear) operator on A. Furthermore, the map

$$L \colon A \to \mathfrak{B}(A) \colon a \mapsto L_a$$

is both an isometry and an algebra homomorphism.

5.3.20. Proposition. *The closure of a $*$-subalgebra of a C^*-algebra A is a C^*-subalgebra of A.*

5.3.21. Definition. Let S be a nonempty subset of a C^*-algebra A. The intersection of the family of all C^*-subalgebras of A which contain S is the C^*-SUBALGEBRA GENERATED BY S. We denote it by $C^*(S)$. (It is easy to see that the intersection of a family of C^*-subalgebras really is a C^*-algebra.) In some cases we shorten the notation slightly: for example, if $a \in A$ we write $C^*(a)$ for $C^*(\{a\})$.

5.3.22. Proposition. *Let S be a nonempty subset of a C^*-algebra A. For each natural number n define the set W_n to be the set of all elements a of A for which there exist $x_1, x_2, \dots, x_n$ in $S \cup S^*$ such that $a = x_1 x_2 \cdots x_n$. Let $W = \bigcup_{n=1}^{\infty} W_n$. Then*

$$C^*(S) = \overline{\operatorname{span} W}\,.$$

5.4. The Gelfand-Naimark Theorem—Version I

5.4.1. Notation. The category **CSA** comprises C^*-algebras as objects and $*$-homomorphisms as morphisms. Closely related, of course, is the category **UCSA** of unital C^*-algebras and unital $*$-homomorphisms.

In the categories **CSA** and **UCSA** every bijective morphism is an isomorphism. We will write $A \overset{*}{\cong} B$ to indicate that C^*-algebras A and B are $*$-isomorphic.

5.4.2. Proposition. *A $*$-homomorphism $\phi \colon A \to B$ between C^*-algebras is a $*$-isomorphism if and only if it is bijective.*

5.4.3. Proposition. *Let A be a unital C^*-algebra. If $a \in A$ is self-adjoint, then its Gelfand transform $\widehat{a}$ is real valued.*

5.4.4. Proposition. *On a unital C^*-algebra A every character preserves involution, thus the Gelfand transform Γ_A is a $*$-homomorphism.*

5.4.5. Theorem (Stone-Weierstrass Theorem)**.** *Let X be a compact Hausdorff space. Every unital separating $*$-subalgebra of $\mathcal{C}(X)$ is dense.*

PROOF. See [9], theorem 2.40.

The first version of the *Gelfand-Naimark theorem* says that any unital commutative C^*-algebra is the algebra of all continuous functions on some compact Hausdorff space. Of course the word *is* in the preceding sentence means *is isometrically $*$-isomorphic to.* The compact Hausdorff space referred to is the character space of the algebra.

5.4.6. Theorem (Gelfand-Naimark Theorem I)**.** *Let A be a unital commutative C^*-algebra. Then the Gelfand transform $\Gamma_A \colon a \mapsto \widehat{a}$ is an isometric $*$-isomorphism of A onto $\mathcal{C}(\Delta A)$.*

5.4.7. Theorem (Abstract Spectral Theorem)**.** *If a is a normal element of a unital C^*-algebra A, then the C^*-algebra $\mathcal{C}(\sigma(a))$ is isometrically $*$-isomorphic to $C^*(\mathbf{1}_A, a)$.*

Hint for proof. Use the Gelfand transform of a to identify the maximal ideal space of $C^*(\mathbf{1}_A, a)$ with the spectrum of a. Apply the functor $\mathcal{C}$. Compose the resulting map with Γ^{-1} where Γ is the Gelfand transform on the C^*-algebra $C^*(\mathbf{1}_A, a)$.

5.4.8. Example. Suppose that in the preceding theorem $\psi\colon \mathcal{C}(\sigma(a)) \to C^*(\mathbf{1}, a)$ implements the isometric $*$-isomorphism. Then the image under ψ of the constant function $\mathbf{1}$ on the spectrum of a is $\mathbf{1}_A$ and the image under ψ of the identity function $\lambda \mapsto \lambda$ on the spectrum of a is a.

5.4.9. Example. Let T be a normal operator on a Hilbert space H whose spectrum is contained in $[0, \infty)$. Suppose that $\psi\colon \mathcal{C}(\sigma(T)) \to C^*(I, T)$ implements the isometric *-isomorphism between these two C^*-algebras. Then there is at least one operator $\sqrt{T}$ whose square is T. Indeed, whenever f is a continuous function on the spectrum of a normal operator T, we may meaningfully speak of the operator $f(T)$.

5.4.10. Proposition (Spectral mapping theorem)**.** *Let a be a self-adjoint element in a unital C^*-algebra A and f be a continuous complex valued function on the spectrum of a. Then*

$$\sigma(f(a)) = f^{\rightarrow}(\sigma(a))\,.$$

CHAPTER 6

SURVIVAL WITHOUT IDENTITY

6.1. Unitization of Banach Algebras

6.1.1. Definition. Let A be an algebra. Define $A \bowtie \mathbb{C}$ to be the set $A \times \mathbb{C}$ on which addition and scalar multiplication are defined pointwise and multiplication is defined by

$$(a, \lambda) \cdot (b, \mu) = (ab + \mu a + \lambda b, \lambda\mu).$$

If the algebra A is equipped with an involution, define an involution on $A \bowtie \mathbb{C}$ pointwise; that is, $(a, \lambda)^* := (a^*, \overline{\lambda})$. (The notation $A \bowtie \mathbb{C}$ is not standard.)

6.1.2. Proposition. *If A is an algebra, then $A \bowtie \mathbb{C}$ is a unital algebra in which A is embedded as (that is, isomorphic to) an ideal such that $(A \bowtie \mathbb{C})/A \cong \mathbb{C}$. The identity of $A \bowtie \mathbb{C}$ is $(0, 1)$. If A is a $*$-algebra, then $A \bowtie \mathbb{C}$ is a unital $*$-algebra in which A is a $*$-ideal such that $(A \bowtie \mathbb{C})/A \cong \mathbb{C}$.*

6.1.3. Definition. The algebra $A \bowtie \mathbb{C}$ is the UNITIZATION of the algebra (or $*$-algebra) A.

6.1.4. Notation. In the preceding construction elements of the algebra $A \bowtie \mathbb{C}$ are technically ordered pairs (a, λ). They are usually written differently. Let $\iota\colon A \to A \bowtie \mathbb{C}\colon a \mapsto (a, 0)$ and $\pi\colon A \bowtie \mathbb{C} \to \mathbb{C}\colon (a, \lambda) \mapsto \lambda$. It follows, since $(0, 1)$ is the identity in $A \bowtie \mathbb{C}$, that

$$\begin{aligned}(a, \lambda) &= (a, 0) + (\mathbf{0}, \lambda)\\ &= \iota(a) + \lambda \mathbf{1}_{A \bowtie \mathbb{C}}\end{aligned}$$

It is conventional to treat ι as an inclusion mapping. Thus it is reasonable to write (a, λ) as $a + \lambda \mathbf{1}_{A \bowtie \mathbb{C}}$ or simply as $a + \lambda \mathbf{1}$. No ambiguity seems to follow from omitting reference to the multiplicative identity, so a standard notation for the pair (a, λ) is $a + \lambda$.

6.1.5. Definition. Let A be a *nonunital* algebra. Define the SPECTRUM of an element $a \in A$ to be the spectrum of a regarded as an element of $A \bowtie \mathbb{C}$; that is, $\sigma_A(a) := \sigma_{A \bowtie \mathbb{C}}(a)$.

6.1.6. Definition. Let A be a normed algebra (with or without involution). On the unitization $A \bowtie \mathbb{C}$ of A define $\|(a, \lambda)\| := \|a\| + |\lambda|$.

6.1.7. Proposition. *Let A be a normed algebra. The mapping $(a, \lambda) \mapsto \|a\| + |\lambda|$ defined above is a norm under which $A \bowtie \mathbb{C}$ is a normed algebra. If A is a Banach algebra (respectively, a Banach $*$-algebra), then $A \bowtie \mathbb{C}$ is a Banach algebra (respectively, a Banach $*$-algebra). The resulting Banach algebra (or Banach $*$-algebra) is the* UNITIZATION *of A and will be denoted by A_e.*

With this expanded definition of *spectrum* many of the earlier facts for unital Banach algebras remain true in the more general setting. In particular, for future reference we restate items 3.1.18, 3.1.25, 3.1.29, and 3.1.30.

6.1.8. Proposition. *Let a be an element of a Banach algebra A. Then the spectrum of a is compact and $|\lambda| \le \|a\|$ for every $\lambda \in \sigma(a)$.*

6.1.9. Proposition. *The spectrum of every element of a Banach algebra is nonempty.*

6.1.10. Proposition. *For every element a of a Banach algebra $\rho(a) \le \|a\|$ and $\rho(a^n) = \big(\rho(a)\big)^n$.*

6.1.11. Theorem (Spectral radius formula)**.** *If a is an element of a Banach algebra, then*

$$\rho(a) = \inf\big\{\|a^n\|^{1/n} \colon n \in \mathbb{N}\big\} = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

6.1.12. Definition. Let A be an algebra. A left ideal J in A is a MODULAR LEFT IDEAL if there exists an element u in A such that $au - a \in J$ for every $a \in A$. Such an element u is called a RIGHT IDENTITY WITH RESPECT TO J. Similarly, a right ideal J in A is a MODULAR RIGHT IDEAL if there exists an element v in A such that $va - a \in J$ for every $a \in A$. Such an element v is called a LEFT IDENTITY WITH RESPECT TO J. A two-sided ideal J is a MODULAR IDEAL if there exists an element e which is both a left and a right identity with respect to J.

6.1.13. Proposition. *An ideal J in an algebra is modular if and only if it is both left modular and right modular.*

Hint for proof. Show that if u is a right identity with respect to J and v is a left identity with respect to J, then vu is both a right and left identity with respect to J.

6.1.14. Proposition. *An ideal J in an algebra A is modular if and only if the quotient algebra A/J is unital.*

6.1.15. Example. Let X be a locally compact Hausdorff space. For every $x \in X$ the ideal J_x is a maximal modular ideal in the C^*-algebra $\mathcal{C}_0(X)$ of continuous complex valued functions on X.

PROOF. By the locally compact Hausdorff space version of *Urysohn's lemma* (see, for example, [11], theorem 17.2.10) there exists a function $g \in \mathcal{C}_0(X)$ such that $g(x) = 1$. Thus J_x is modular because g is an identity with respect to J_x. Since $\mathcal{C}_0(X) = J_x \oplus \operatorname{span}\{g\}$ the ideal J_x has codimension 1 and is therefore maximal. □

6.1.16. Proposition. *If J is a proper modular ideal in a Banach algebra, then so is its closure.*

6.1.17. Corollary. *Every maximal modular ideal in a Banach algebra is closed.*

6.1.18. Proposition. *Every proper modular ideal in a Banach algebra is contained in a maximal modular ideal.*

6.1.19. Proposition. *Let A be a commutative Banach algebra and $\phi \in \Delta A$. Then $\ker\phi$ is a maximal modular ideal in A and $A/\ker\phi$ is a field. Furthermore, every maximal modular ideal is the kernel of exactly one character in ΔA.*

6.1.20. Proposition. *Every multiplicative linear functional on a commutative Banach algebra A is continuous. In fact, every character is contractive.*

6.1.21. Example. Let A_e be the unitization of a commutative Banach algebra A (see proposition 6.1.7). Define

$$\phi_\infty \colon A_e \to \mathbb{C} \colon (a, \lambda) \mapsto \lambda\,.$$

Then ϕ_∞ is a character on A_e.

6.1.22. Proposition. *Every character ϕ on a commutative Banach algebra A has a unique extension to a character ϕ_e on the unitization A_e of A. And the restriction to A of any character on A_e, with the obvious exception of ϕ_∞, is a character on A.*

6.1.23. Proposition. *If A is a commutative Banach algebra, then*

(a) *ΔA is a locally compact Hausdorff space,*
(b) *$\Delta A_e = \Delta A \cup \{\phi_\infty\}$,*
(c) *ΔA_e is the one-point compactification of ΔA, and*
(d) *the map $\phi \mapsto \phi_e$ is a homeomorphism from ΔA onto $\Delta A_e \setminus \{\phi_\infty\}$.*

If A is unital (so that ΔA is compact), then ϕ_∞ is an isolated point of ΔA_e.

6.1.24. Theorem. *If A is a commutative Banach algebra with a nonempty character space, then the Gelfand transform*

$$\Gamma = \Gamma_A \colon \mathcal{C}_0(\Delta A) \colon a \mapsto \widehat{a} = \Gamma_a$$

is a contractive algebra homomorphism and $\rho(a) = \|\widehat{a}\|_u$. Furthermore, if A is not *unital, then $\sigma(a) = \operatorname{ran}\widehat{a} \cup \{0\}$ for every $a \in A$.*

6.2. Exact Sequences and Extensions

The rather simple procedure for the unitization of Banach $*$-algebras (see 6.1.1 and 6.1.6) does not carry over to C^*-algebras. The norm defined in 6.1.6 does not satisfy the C^*-condition (definition 5.3.1). It turns out that the unitization of C^*-algebras is a bit more complicated. Before examining the details we look at some preparatory material on *exact sequences* and *extensions* of C^*-algebras.

6.2.1. Definition. A sequence of C^*-algebras and $*$-homomorphisms

$$\cdots \longrightarrow A_{n-1} \xrightarrow{\phi_n} A_n \xrightarrow{\phi_{n+1}} A_{n+1} \longrightarrow \cdots$$

is said to be EXACT AT A_n if $\operatorname{ran}\phi_n = \ker\phi_{n+1}$. A sequence is EXACT if it is exact at each of its constituent C^*-algebras. A sequence of C^*-algebras and $*$-homomorphisms of the form

$$0 \longrightarrow A \xrightarrow{\phi} E \xrightarrow{\psi} B \longrightarrow 0 \tag{6.1}$$

is a SHORT EXACT SEQUENCE. (Here $\mathbf{0}$ denotes the trivial 0-dimensional C^*-algebra, and the unlabeled arrows are the obvious $*$-homomorphisms.) The short exact sequence of C^*-algebras (6.1) is SPLIT EXACT if there exists a $*$-homomorphism $\xi\colon B \to E$ such that $\psi\circ\xi = \mathrm{id}_B$.

The preceding definitions were for the category **CSA** of C^*-algebras and $*$-homomorphisms. Of course there is nothing special about this particular category. Exact sequences make sense in many situations, in, for example, various categories of Banach spaces, Banach algebras, Hilbert spaces, vector spaces, Abelian groups, modules, and so on.

Often in the context of C^*-algebras the exact sequence (6.1) is referred to as an EXTENSION. Some authors refer to it as an *extension of A by B* (for example, [29] and [7]) while others say it is an *extension of B by A* ([20], [12], and [3]). In [12] and [3] the *extension* is defined to be the sequence 6.1; in [29] it is defined to be the ordered triple (ϕ, E, ψ); and in [20] and [7] it is defined to be the C^*-algebra E itself. Regardless of the formal definitions it is common to say that E *is an extension of A by B (or of B by A).*

6.2.2. Convention. In a C^*-algebra the word *ideal* will always mean a closed two-sided $*$-ideal (unless, of course, the contrary is explicitly stated). We will show shortly (in proposition 6.6.6) that requiring an ideal in a C^*-algebra to be self-adjoint is redundant. A two-sided algebra ideal of a C^*-algebra which is not necessarily closed will be called an ALGEBRAIC IDEAL. A self-adjoint algebraic ideal of a C^*-algebra will be called an ALGEBRAIC $*$-IDEAL.

6.2.3. Proposition. *The kernel of a $*$-homomorphism $\phi\colon A \to B$ between C^*-algebras is an ideal in A and its range is a $*$-subalgebra of B.*

6.2.4. Proposition. *Consider the following diagram in the category of C^*-algebras and *-homomorphisms*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "j"] \arrow[d, "f"] & B \arrow[r, "k"] \arrow[d, "g"] & C \arrow[r] \arrow[d, dashed, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "j'"'] & B' \arrow[r, "k'"'] & C' \arrow[r] & 0
\end{tikzcd}$$

If the rows are exact and the left square commutes, then there exists a unique *-homomorphism $h\colon C \to C'$ which makes the right square commute.

6.2.5. Proposition (The Five Lemma)**.** *Suppose that in the following diagram of C^*-algebras and *-homomorphisms*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "j"] \arrow[d, "f"] & B \arrow[r, "k"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "j'"'] & B' \arrow[r, "k'"'] & C' \arrow[r] & 0
\end{tikzcd}$$

the rows are exact and the squares commute. Prove the following.

(1) *If g is surjective, so is h.*
(2) *If f is surjective and g is injective, then h is injective.*
(3) *If f and h are surjective, so is g.*
(4) *If f and h are injective, so is g.*

6.2.6. Definition. Let A and B be C^*-algebras. We define the (EXTERNAL) DIRECT SUM of A and B, denoted by $A \oplus B$, to be the Cartesian product $A \times B$ with pointwise defined algebraic operations and norm given by

$$\|(a,b)\| = \max\{\|a\|, \|b\|\}$$

for all $a \in A$ and $b \in B$. An alternative notation for the element (a,b) in $A \oplus B$ is $a \oplus b$.

6.2.7. Example. Let A and B be C^*-algebras. Then the direct sum of A and B is a C^*-algebra and the following sequence is split short exact:

$$\mathbf{0} \longrightarrow A \xrightarrow{\iota_1} A \oplus B \xrightarrow{\pi_2} B \longrightarrow \mathbf{0}$$

The indicated maps in the preceding are the obvious ones:

$$\iota_1 \colon A \to A \oplus B \colon a \mapsto (a, \mathbf{0}) \qquad \text{and} \qquad \pi_2 \colon A \oplus B \to B : (a,b) \mapsto b\,.$$

This is the DIRECT SUM EXTENSION.

6.2.8. Proposition. *If A and B are nonzero C^*-algebras, then their direct sum $A \oplus B$ is a product in the category* **CSA** *of C^*-algebras and $*$-homomorphisms (see example 4.4.11). The direct sum is unital if and only if both A and B are.*

6.2.9. Definition. Let A and B be C^*-algebras and E and E' be extensions of A by B. These extensions are STRONGLY EQUIVALENT if there exists a *-isomorphism $\theta \colon E \to E'$ that makes the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\theta"] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & E' \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

commute.

6.2.10. Proposition. *In the preceding definition it is enough to require θ to be a *-homomorphism.*

6.2.11. Proposition. *Let A and B be C^*-algebras. An extension*

$$\mathbf{0} \longrightarrow A \xrightarrow{\phi} E \xrightarrow{\psi} B \longrightarrow \mathbf{0}$$

is strongly equivalent to the direct sum extension $A \oplus B$ if and only if there exists a *-homomorphism $\nu \colon E \to A$ such that $\nu \circ \phi = \mathrm{id}_A$.

6.2.12. Proposition. *If the sequences of C^*-algebras*

$$0 \longrightarrow A \xrightarrow{\phi} E \xrightarrow{\psi} B \longrightarrow 0 \tag{6.2}$$

and

$$0 \longrightarrow A \xrightarrow{\phi'} E' \xrightarrow{\psi'} B \longrightarrow 0 \tag{6.3}$$

are strongly equivalent and (6.2) splits, then so does (6.3).

6.3. Unitization of C^*-algebras

6.3.1. Proposition. *Let A be a C^*-algebra. Then there exists a unital C^*-algebra $\widetilde{A}$ in which A is embedded as an ideal such that the sequence*

$$\mathbf{0} \longrightarrow A \longrightarrow \widetilde{A} \longrightarrow \mathbb{C} \longrightarrow \mathbf{0} \tag{6.4}$$

is split exact. If A is unital then the sequence (6.4) is strongly equivalent to the direct sum extension, so that $\widetilde{A} \cong A \oplus \mathbb{C}$. If A is not *unital, then $\widetilde{A}$ is not isomorphic to $A \oplus \mathbb{C}$.*

Hint for proof. The proof this result is a little complicated. Everyone should go through all the details at least once in his/her life. What follows is an outline of a proof.

Notice that we speak of the unitization of C^*-algebra A *whether or not A already has a unit (multiplicative identity).* We divide the argument into two cases.

Case 1: the algebra A is unital.

(1) On the algebra $A \bowtie \mathbb{C}$ define
$$\|(a, \lambda)\| := \max\{\|a + \lambda \mathbf{1}_A\|, |\lambda|\}$$
and let $\widetilde{A} := A \bowtie \mathbb{C}$ together with this norm.
(2) Prove that the map $(a, \lambda) \mapsto \|(a, \lambda)\|$ is a norm on $A \bowtie \mathbb{C}$.
(3) Prove that this norm is an algebra norm.
(4) Show that it is, in fact, a $C*$-norm on $A \bowtie \mathbb{C}$.
(5) Observe that it is an extension of the norm on A.
(6) Prove that $A \bowtie \mathbb{C}$ is a C^*-algebra by verifying completeness of the metric space induced by the preceding norm.
(7) Prove that the sequence
$$\begin{tikzcd} \mathbf{0} \arrow[r] & A \arrow[r, "\iota"] & \widetilde{A} \arrow[r, shift left, "Q"] & \mathbb{C} \arrow[l, shift left, "\psi"] \arrow[r] & \mathbf{0} \end{tikzcd}$$
is split exact (where $\iota\colon a \mapsto (a, 0)$, $Q\colon (a, \lambda) \mapsto \lambda$, and $\psi\colon \lambda \mapsto (0, \lambda)$).
(8) Prove that $\widetilde{A} \overset{*}{\cong} A \oplus \mathbb{C}$.

Case 2: the algebra A is *not* unital.

(9) Prove the following simple fact.

6.3.2. Lemma. *Let A be an algebra and B be a normed algebra. If $\phi\colon A \to B$ is an algebra homomorphism, the function $a \mapsto \|a\| := \|\phi(a)\|$ is a seminorm on A. The function is a norm if ϕ is injective.*

(10) Recall that we defined the operator L_a, *left multiplication by a*, in 5.3.19. Now let
$$A^\sharp := \{L_a + \lambda I_A \in \mathfrak{B}(A) \colon a \in A \text{ and } \lambda \in \mathbb{C}\}$$
and show that $A^\sharp$ is a normed algebra.
(11) Make $A^\sharp$ into a $*$-algebra by defining
$$(L_a + \lambda I_A)^* := L_{a^*} + \overline{\lambda} I_A$$
for all $a \in A$ and $\lambda \in \mathbb{C}$.
(12) Define
$$\phi\colon A \bowtie \mathbb{C} \to A^\sharp \colon (a, \lambda) \mapsto L_a + \lambda I_A$$
and verify that ϕ is a $*$-homomorphism.
(13) Prove that ϕ is injective.
(14) Use (9) to endow $A \bowtie \mathbb{C}$ with a norm which makes it into a unital normed algebra. Let $\widetilde{A} := A \bowtie \mathbb{C}$ with the norm pulled back by ϕ from $A^\sharp$

(15) Verify the following facts.
   (a) The map $\phi\colon \widetilde{A} \to A^\sharp$ is an isometric isomorphism.
   (b) $\operatorname{ran} L$ is a closed subalgebra of $\operatorname{ran}\phi = A^\sharp \subseteq \mathfrak{B}(A)$.
   (c) $I_A \notin \operatorname{ran} L$.

(16) Prove that the norm on $\widetilde{A}$ satisfies the C^*-condition.

(17) Prove that $\widetilde{A}$ is a unital C^*-algebra. (To show that $\widetilde{A}$ is complete we need only show that $A^\sharp$ is complete. To this end let $\big(\phi(a_n, \lambda_n)\big)_{n=1}^{\infty}$ be a Cauchy sequence in $A^\sharp$. To show that this sequence converges it suffices to show that it has a convergent subsequence. Showing that the sequence (λ_n) is bounded allows us to extract from it a convergent subsequence $\big(\lambda_{n_k}\big)$. Prove that $(L_{a_{n_k}})$ converges.)

(18) Prove that the sequence

$$0 \longrightarrow A \longrightarrow \widetilde{A} \longrightarrow \mathbb{C} \longrightarrow 0 \tag{6.5}$$

is split exact.

(19) The C^*-algebra $\widetilde{A}$ is *not* equivalent to $A \oplus \mathbb{C}$.

6.3.3. Definition. The C^*-algebra $\widetilde{A}$ constructed in the preceding proposition is the UNITIZATION of A.

Note that the expanded definition of *spectrum* given in 6.1.5 applies to C^*-algebras since the added identity is purely an algebraic matter and is the same for C^*-algebras as it is for general Banach algebras. Thus many of the earlier facts stated for unital C^*-algebras remain true. In particular, for future reference we restate items 5.3.12, 5.3.13, 5.3.14, 5.3.15, 5.3.16, 5.4.3, and 5.4.4.

6.3.4. Proposition. *Let a be a normal element of a C^*-algebra. Then $\|a^2\| = \|a\|^2$ and therefore $\rho(a) = \|a\|$.*

6.3.5. Corollary. *If A is a commutative C^*-algebra, then $\|a^2\| = \|a\|^2$ and $\rho(a) = \|a\|$ for every $a \in A$.*

6.3.6. Corollary. *On a commutative C^*-algebra A the Gelfand transform Γ is an isometry; that is, $\|\Gamma_a\|_u = \|\widehat{a}\|_u = \|a\|$ for every $a \in A$.*

6.3.7. Corollary. *The norm of a C^*-algebra is unique in the sense that given a algebra A with involution there is at most one norm which makes A into a C^*-algebra.*

6.3.8. Proposition. *If h is a self-adjoint element of a C^*-algebra, then $\sigma(h) \subseteq \mathbb{R}$.*

6.3.9. Proposition. *If a is a self-adjoint element in a C^*-algebra, then its Gelfand transform $\widehat{a}$ is real valued.*

6.3.10. Proposition. *Every character on a C^*-algebra A preserves involution, thus the Gelfand transform Γ_A is a $*$-homomorphism.*

An immediate result of the preceding results is the second version of the *Gelfand-Naimark theorem*, which says that any commutative C^*-algebra is (isometrically unitally $*$-isomorphic to) the algebra of all those continuous functions on some locally compact Hausdorff space which vanish at infinity. As was the case with the first version of this theorem (see 5.4.6) the locally compact Hausdorff space referred to is the character space of the algebra.

6.3.11. Theorem (Gelfand-Naimark Theorem II)**.** *Let A be a commutative C^*-algebra. Then the Gelfand transform $\Gamma_A\colon a \mapsto \widehat{a}$ is an isometric unital $*$-isomorphism of A onto $\mathcal{C}_0(\Delta A)$.*

It follows from the next proposition that the unitization process is functorial.

6.3.12. Proposition. *Every $*$-homomorphism $\phi\colon A \to B$ between C^*-algebras has a unique extension to a unital $*$-homomorphism $\widetilde{\phi}\colon \widetilde{A} \to \widetilde{B}$ between their unitizations.*

In contrast to the situation in general Banach algebras there is no distinction between topological and geometric categories of C^*-algebras. One of the most remarkable aspects of C^*-algebra theory is that $*$-homomorphisms between such algebras are automatically continuous—in fact, contractive. It follows that if two C^*-algebras are algebraically $*$-isomorphic, then they are isometrically isomorphic.

6.3.13. Proposition. *Every $*$-homomorphism between C^*-algebras is contractive.*

6.3.14. Proposition. *Every injective $*$-homomorphism between C^*-algebras is an isometry.*

6.3.15. Proposition. *Let X be a locally compact Hausdorff space and $\widetilde{X} = X \cup \{\infty\}$ be its one-point compactification. Define*

$$\iota\colon \mathcal{C}_0(X) \to \mathcal{C}(\widetilde{X})\colon f \mapsto \widetilde{f}$$

where

$$\widetilde{f}(x) = \begin{cases} f(x), & \text{if } x \in X; \\ 0, & \text{if } x = \infty. \end{cases}$$

Also let E_∞ be defined on $\mathcal{C}(\widetilde{X})$ by $E_\infty(g) = g(\infty)$. Then the sequence

$$\mathbf{0} \longrightarrow \mathcal{C}_0(X) \xrightarrow{\ \iota\ } \mathcal{C}(\widetilde{X}) \xrightarrow{\ E_\infty\ } \mathbb{C} \longrightarrow \mathbf{0}$$

is exact.

In the preceding proposition we refer to $\widetilde{X}$ as the one-point compactification of X even in the case that X is compact to begin with. Most definitions of *compactification* require a space to be dense in any compactification. (See my remarks in the beginning of section 17.3 of [11].) We have previously adopted the convention that the unitization of a unital algebra gets a new multiplicative identity. In the spirit of consistency with this choice we will in the sequel subscribe to the convention that the one-point compactification of a compact space gets an additional (isolated) point. (See also the terminology introduced in 9.3.1.)

From the point of view of the Gelfand-Naimark theorem (6.3.11) the fundamental insight prompted by the next proposition is that the unitization of a commutative C^*-algebra is, in some sense, the "same thing" as the one-point compactification of a locally compact Hausdorff space.

6.3.16. Proposition. *If X is a locally compact Hausdorff space, then the unital C^*-algebras $\big(\mathcal{C}_0(X)\big)^\sim$ and $\mathcal{C}(\widetilde{X})$ are isometrically $*$-isomorphic.*

Proof. Define

$$\theta\colon \big(\mathcal{C}_0(X)\big)^\sim \to \mathcal{C}(\widetilde{X})\colon (f,\lambda) \mapsto \widetilde{f} + \lambda \mathbf{1}_{\widetilde{X}}$$

(where $\mathbf{1}_{\widetilde{X}}$ is the constant function 1 on $\widetilde{X}$). Then consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}_0(X) \arrow[r] \arrow[d, equal] & \big(\mathcal{C}_0(X)\big)^\sim \arrow[r] \arrow[d, "\theta"] & \mathbb{C} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{C}_0(X) \arrow[r, "\iota"] & \mathcal{C}(\widetilde{X}) \arrow[r, "E_\infty"] & \mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

The top row is exact by proposition 6.3.1, the bottom row is exact by proposition 6.3.15, and the diagram obviously commutes. It is routine to check that θ is a $*$-homomorphism. Therefore θ is an isometric $*$-isomorphism by proposition 6.2.10 and corollary 6.3.14. $\square$

6.4. Quasi-inverses

6.4.1. Definition. An element b of an algebra A is a LEFT QUASI-INVERSE for $a \in A$ if $ba = a + b$. It is a RIGHT QUASI-INVERSE for a if $ab = a + b$. If b is both a left and a right quasi-inverse for a it is a QUASI-INVERSE for a. When a has a quasi-inverse denote it by a'.

6.4.2. Proposition. *If b is a left quasi-inverse for a in an algebra A and c is a right quasi-inverse for a in A, then $b = c$.*

6.4.3. Proposition. *Let A be a unital algebra and let a, $b \in A$. Then*

(a) *a' exists if and only if $(\mathbf{1} - a)^{-1}$ exists; and*
(b) *b^{-1} exists if and only if $(\mathbf{1} - b)'$ exists.*

Hint for proof. For the reverse direction in (a) consider $a + c - ac$ where $c = \mathbf{1} - (\mathbf{1} - a)^{-1}$.

6.4.4. Proposition. *An element a of a Banach algebra A is quasi-invertible if and only if it is not an identity with respect to any maximal modular ideal in A.*

6.4.5. Proposition. *Let A be a Banach algebra and $a \in A$. If $\rho(a) < 1$, then a' exists and $a' = -\sum_{k=1}^{\infty} a^k$.*

6.4.6. Proposition. *Let A be a Banach algebra and $a \in A$. If $\|a\| < 1$, then a' exists and*

$$\frac{\|a\|}{1 + \|a\|} \le \|a'\| \le \frac{\|a\|}{1 - \|a\|}.$$

6.4.7. Proposition. *Let A be a unital Banach algebra and $a \in A$. If $\rho(\mathbf{1} - a) < 1$, then a is invertible in A.*

6.4.8. Proposition. *Let A be a Banach algebra and a, $b \in A$. If b' exists and $\|a\| < (1 + \|b'\|)^{-1}$, then $(a + b)'$ exists and*

$$\|(a + b)' - b'\| \le \frac{\|a\| \, (1 + \|b'\|)^2}{1 - \|a\| \, (1 + \|b'\|)}.$$

Hint for proof. Show first that $u = (a - b'a)'$ exists and that $u + b' - ub'$ is a left quasi-inverse for $a + b$.

Compare the next result with propositions 3.1.12 and 3.1.14

6.4.9. Proposition. *The set Q_A of quasi-invertible elements of a Banach algebra A is open in A, and the map $a \mapsto a'$ is a homeomorphism of Q_A onto itself.*

6.4.10. Notation. If a and b are elements in an algebra we define $a \circ b := a + b - ab$.

6.4.11. Proposition. *If A is an algebra, then Q_A is a group under $\circ$.*

6.4.12. Proposition. *If A is a unital Banach algebra and* inv A *is the set of its invertible elements, then*

$$\Psi \colon Q_A \to \operatorname{inv} A \colon a \mapsto \mathbf{1} - a$$

is an isomorphism.

6.4.13. Definition. If A is a algebra and $a \in A$, we define the Q-SPECTRUM of a in A by

$$\breve{\sigma}_A(a) := \{\lambda \in \mathbb{C} \colon \lambda \neq 0 \text{ and } \frac{a}{\lambda} \notin Q_A\} \cup \{0\}.$$

In a unital algebra, the preceding definition is "almost" the usual one.

6.4.14. Proposition. *Let A be a unital algebra and $a \in A$. Then for all $\lambda \neq 0$, we have $\lambda \in \sigma(a)$ if and only if $\lambda \in \breve{\sigma}(a)$.*

6.4.15. Proposition. *If an algebra A is not unital and $a \in A$, then $\breve{\sigma}_A(a) = \breve{\sigma}_{\widetilde{A}}(a)$, where $\widetilde{A}$ is the unitization of A.*

6.5. Positive Elements in C^*-algebras

6.5.1. Definition. A self-adjoint element a of a C^*-algebra A is POSITIVE if $\sigma(a) \subseteq [0, \infty)$. In this case we write $a \geq \mathbf{0}$. We denote the set of all positive elements of A by A^+. This is the POSITIVE CONE of A. For any subset B of A let $B^+ = B \cap A^+$. We will use the positive cone to induce a partial ordering on A: we write $a \leq b$ when $b - a \in A^+$.

6.5.2. Definition. Let $\leq$ be a relation on a nonempty set S. If the relation $\leq$ is reflexive and transitive, it is a PREORDERING. If $\leq$ is a preordering and is also antisymmetric, it is a PARTIAL ORDERING.

A partial ordering $\leq$ on a real vector space V is COMPATIBLE with (or RESPECTS) the operations (addition and scalar multiplication) on V if for all x, y, $z \in V$

(a) $x \leq y$ implies $x + z \leq y + z$, and
(b) $x \leq y$, $\alpha \geq 0$ imply $\alpha x \leq \alpha y$.

A real vector space equipped with a partial ordering which is compatible with the vector space operations is an ORDERED VECTOR SPACE.

6.5.3. Definition. Let V be a vector space. A subset C of V is a CONE in V if $\alpha C \subseteq C$ for every $\alpha \geq 0$. A cone C in V is PROPER if $C \cap (-C) = \{\mathbf{0}\}$.

6.5.4. Proposition. *A cone C in a vector space is convex if and only if $C + C \subseteq C$.*

6.5.5. Example. If V is an ordered vector space, then the set

$$V^+ := \{x \in V : x \geq \mathbf{0}\}$$

is a proper convex cone in V. This is the POSITIVE CONE of V and its members are the POSITIVE ELEMENTS of V.

6.5.6. Proposition. *Let V be a real vector space and C be a proper convex cone in V. Define $x \leq y$ if $y - x \in C$. Then the relation $\leq$ is a partial ordering on V and is compatible with the vector space operations on V. This relation is the* PARTIAL ORDERING INDUCED BY *the cone C. The positive cone V^+ of the resulting ordered vector space is just C itself.*

6.5.7. Proposition. *If a is a self-adjoint element of a unital C^*-algebra and $t \in \mathbb{R}$, then*

(i) *$a \geq \mathbf{0}$ whenever $\|a - t\mathbf{1}\| \leq t$; and*
(ii) *$\|a - t\mathbf{1}\| \leq t$ whenever $\|a\| \leq t$ and $a \geq \mathbf{0}$.*

6.5.8. Example. The positive cone of a C^*-algebra A is a closed proper convex cone in the real vector space $\mathfrak{H}(A)$.

6.5.9. Proposition. *If a and b are positive elements of a C^*-algebra and $ab = ba$, then ab is positive.*

6.5.10. Proposition. *Every positive element of a C^*-algebra A has a unique positive n^{th} root ($n \in \mathbb{N}$). That is, if $a \in A^+$, then there exists a unique $b \in A^+$ such that $b^n = a$.*

PROOF. *Hint.* The existence part is a simple application of the C^*-functional calculus (that is, the *abstract spectral theorem* 5.4.7). The element b given by the functional calculus is positive in the algebra $C^*(\mathbf{1}, a)$. Explain why it is also positive in A. The uniqueness argument deserves considerable care.

6.5.11. Theorem (Jordan Decomposition)**.** *If c is a self-adjoint element of a C^*-algebra A, then there exist positive elements c^+ and c^- of A such that $c = c^+ - c^-$ and $c^+c^- = \mathbf{0}$.*

6.5.12. Lemma. *If c is an element of a C^*-algebra such that $-c^*c \geq \mathbf{0}$, then $c = \mathbf{0}$.*

6.5.13. Proposition. *If a is an element of a C^*-algebra, then $a^*a \geq \mathbf{0}$.*

6.5.14. Proposition. *If c is an element of a C^*-algebra, then the following are equivalent:*

(i) $c \geq \mathbf{0}$;
(ii) *there exists* $b \geq \mathbf{0}$ *such that* $c = b^2$; *and*
(iii) *there exists* $a \in A$ *such that* $c = a^*a$,

6.5.15. Example. If T is an operator on a Hilbert space H, then T is a positive member of the C^*-algebra $\mathfrak{B}(H)$ if and only if $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

Hint for proof. Showing that if $T \geq \mathbf{0}$ in $\mathfrak{B}(H)$, then $\langle Tx, x\rangle \geq 0$ for all $x \in H$ is easy: use proposition 6.5.14 to write T as S^*S for some operator S.

For the converse suppose that $\langle Tx, x\rangle \geq 0$ for all $x \in H$. It is easy to see that this implies that T is self-adjoint. Use the *Jordan decomposition theorem*6.5.11 to write T as $T^+ - T^-$. For arbitrary $u \in H$ let $x = T^-u$ and verify that $0 \leq \langle Tx, x\rangle = -\langle (T^-)^3u, u\rangle$. Now $(T^-)^3$ is a positive element of $\mathfrak{B}(H)$. (Why?) Conclude that $(T^-)^3 = \mathbf{0}$ and therefore $T^- = \mathbf{0}$. (For additional detail see [8], page 37.)

6.5.16. Definition. For an arbitrary element a of a C^*-algebra we define $|a|$ to be $\sqrt{a^*a}$.

6.5.17. Proposition. *If a is a self-adjoint element of a C^*-algebra, then*

$$|a| = a^+ + a^- .$$

6.5.18. Example. The absolute value in a C^*-algebra need not be subadditive; that is, $|a+b|$ need not be less than $|a| + |b|$. For example, in $\mathbf{M}_2(\mathbb{C})$ take $a = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$. Then $|a| = a$, $|b| = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, and $|a+b| = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$. If $|a+b| - |a| - |b|$ were positive, then, according to example 6.5.15, $\langle\, (|a+b| - |a| - |b|)\, x \,,\, x \,\rangle$ would be positive for every vector $x \in \mathbb{C}^2$. But this is not true for $x = (1,0)$.

6.5.19. Proposition. *If $\phi\colon A \to B$ is a $*$-homomorphism between C^*-algebras, then $\phi(a) \in B^+$ whenever $a \in A^+$. If ϕ is a $*$-isomorphism, then $\phi(a) \in B^+$ if and only if $a \in A^+$.*

6.5.20. Proposition. *Let a be a self-adjoint element of a C^*-algebra A and f a continuous complex valued function on the spectrum of a. Then $f \geq \mathbf{0}$ in $\mathcal{C}(\sigma(a))$ if and only if $f(a) \geq \mathbf{0}$ in A.*

6.5.21. Proposition. *If a is a self-adjoint element of a C^*-algebra A, then $\|a\|\mathbf{1}_A \pm a \geq \mathbf{0}$.*

6.5.22. Proposition. *If a and b are self-adjoint elements of a C^*-algebra A and $a \leq b$, then $x^*ax \leq x^*bx$ for every $x \in A$.*

6.5.23. Proposition. *If a and b are elements of a C^*-algebra with $0 \leq a \leq b$, then $\|a\| \leq \|b\|$.*

6.5.24. Proposition. *Let A be a unital C^*-algebra and $c \in A^+$. Then c is invertible if and only if $c \geq \epsilon\mathbf{1}$ for some $\epsilon > 0$.*

6.5.25. Proposition. *Let A be a unital C^*-algebra and $c \in A$. If $c \geq \mathbf{1}$, then c is invertible and $\mathbf{0} \leq c^{-1} \leq \mathbf{1}$.*

6.5.26. Proposition. *If a is a positive invertible element in a unital C^*-algebra, then a^{-1} is positive.*

Next we show that the notation $a^{-\frac{1}{2}}$ is unambiguous.

6.5.27. Proposition. *Let $a \in A^+$ where A is a unital C^*-algebra. If a is invertible, so is $a^{\frac{1}{2}}$ and $\big(a^{\frac{1}{2}}\big)^{-1} = \big(a^{-1}\big)^{\frac{1}{2}}$.*

6.5.28. Proposition. *Let a and b be elements of a C^*-algebra. If $\mathbf{0} \leq a \leq b$ and a is invertible, then b is invertible and $b^{-1} \leq a^{-1}$.*

6.5.29. Proposition. *If a and b are elements of a C^*-algebra and $\mathbf{0} \leq a \leq b$, then $\sqrt{a} \leq \sqrt{b}$.*

6.5.30. Example. Let a and b be elements of a C^*-algebra with $\mathbf{0} \le a \le b$. It is *not* necessarily the case that $a^2 \le b^2$.

Hint for proof. In the C^*-algebra $\mathbf{M}_2$ let $a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $b = a + \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

6.6. Approximate Identities

6.6.1. Definition. An APPROXIMATE IDENTITY (or APPROXIMATE UNIT) in a C^*-algebra A is an increasing net $(e_\lambda)_{\lambda\in\Lambda}$ of positive elements of A such that $\|e_\lambda\| \le 1$ and $ae_\lambda \to a$ (equivalently, $e_\lambda\, a \to a$) for every $a \in A$. If such a net is in fact a sequence, we have a SEQUENTIAL APPROXIMATE IDENTITY. In the literature be careful of varying definitions: many authors omit the requirements that the net be increasing and/or that it be bounded.

6.6.2. Example. Let $A = \mathcal{C}_0(\mathbb{R})$. For each $n \in \mathbb{N}$ let $U_n = (-n, n)$ and let $e_n \colon \mathbb{R} \to [0, 1]$ be a function in A whose support is contained in U_{n+1} and such that $e_n(x) = 1$ for every $x \in U_n$. Then (e_n) is a (sequential) approximate identity for A.

6.6.3. Proposition. *If A is a C^*-algebra, then the set*

$$\Lambda := \{a \in A^+ \colon \|a\| < 1\}$$

is a directed set (under the ordering it inherits from $\mathfrak{H}(A)$).

Hint for proof. Verify that the function

$$f \colon [0, 1) \to \mathbb{R}^+ \colon t \mapsto (1 - t)^{-1} - 1 = \frac{t}{1 - t}$$

induces an order isomorphism $f \colon \Lambda \to A^+$. (An ORDER ISOMORPHISM between partially ordered sets is an order preserving bijection whose inverse also preserves order.) A careful proof of this result involves checking a rather large number of details.

6.6.4. Proposition. *If A is a C^*-algebra, then the set*

$$\Lambda := \{a \in A^+ \colon \|a\| < 1\}$$

is an approximate identity for A.

6.6.5. Corollary. *Every C^*-algebra A has an approximate identity. If A is separable then it has a sequential approximate identity.*

6.6.6. Proposition. *Every closed (two-sided) algebraic ideal in a C^*-algebra is self-adjoint.*

6.6.7. Proposition. *If J is an ideal in a C^*-algebra A, then A/J is a C^*-algebra.*

PROOF. See [16], theorem 1.7.4, or [7], pages 13–14, or [12], theorem 2.5.4.

6.6.8. Example. If J is an ideal in a C^*-algebra A, then the sequence

$$\mathbf{0} \longrightarrow J \longrightarrow A \xrightarrow{\ \pi\ } A/J \longrightarrow \mathbf{0}$$

is short exact.

6.6.9. Proposition. *The range of a $*$-homomorphism between C^*-algebras is closed (and therefore itself a C^*-algebra).*

6.6.10. Definition. A C^*-subalgebra B of a C^*-algebra A is HEREDITARY if $a \in B$ whenever $a \in A$, $b \in B$, and $\mathbf{0} \le a \le b$.

6.6.11. Proposition. *Suppose $x^*x \le a$ in a C^*-algebra A. Then there exists $b \in A$ such that $x = ba^{\frac{1}{4}}$ and $\|b\| \le \|a\|^{\frac{1}{4}}$.*

PROOF. See [7], page 13.

6.6.12. Proposition. *Suppose J is an ideal in a C^*-algebra A, $j \in J^+$, and $a^*a \le j$. Then $a \in J$. Thus ideals in C^*-algebras are hereditary.*

6.6.13. Theorem. *Let A and B be C^*-algebras and J be an ideal in A. If ϕ is a $*$-homomorphism from A to B and $\ker\phi \supseteq J$, then there exists a unique $*$-homomorphism $\widetilde{\phi}\colon A/J \to B$ which makes the following diagram commute.*

$$\begin{tikzcd} A \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \\ A/J \arrow[r, "\widetilde{\phi}"'] & B \end{tikzcd}$$

Furthermore, $\widetilde{\phi}$ is injective if and only if $\ker\phi = J$; and $\widetilde{\phi}$ is surjective if and only if ϕ is.

6.6.14. Corollary. *If $\mathbf{0} \longrightarrow A \xrightarrow{\ \phi\ } E \longrightarrow B \longrightarrow \mathbf{0}$ is a short exact sequence of C^*-algebras, then $E/\operatorname{ran}\phi$ and B are isometrically $*$-isomorphic.*

6.6.15. Corollary. *Every C^*-algebra A has codimension one in its unitization $\widetilde{A}$; that is, $\dim \widetilde{A}/A = 1$.*

6.6.16. Proposition. *Let A be a C^*-algebra, B be a C^*-subalgebra of A, and J be an ideal in A. Then*

$$B/(B \cap J) \cong (B + J)/J\,.$$

Let B be a unital subalgebra of an arbitrary algebra A. It is clear that if an element $b \in B$ is invertible in B, then it is also invertible in A. The converse turns out to be true in C^*-algebras: if b is invertible in A, then its inverse lies in B. This is usually expressed by saying that *every unital C^*-subalgebra of a C^*-algebra is* INVERSE CLOSED.

6.6.17. Proposition. *Let B be a unital C^*-subalgebra of a C^*-algebra A. If $b \in \operatorname{inv}(A)$, then $b^{-1} \in B$.*

6.6.18. Corollary. *Let B be a unital C^*-subalgebra of a C^*-algebra A and $b \in B$. Then*

$$\sigma_B(b) = \sigma_A(b)\,.$$

6.6.19. Corollary. *Let $\phi\colon A \to B$ be a unital C^*-monomorphism of a C^*-algebra A and $a \in A$. Then*

$$\sigma(a) = \sigma(\phi(a))\,.$$

CHAPTER 7

SOME IMPORTANT CLASSES OF HILBERT SPACE OPERATORS

7.1. Orthonormal Bases in Hilbert Spaces

7.1.1. Definition. A BASIS for a Hilbert space H is a maximal orthonormal set in H.

CAUTION. This notion of a basis for a Hilbert space (sometimes, for emphasis, called an *orthonormal basis*) should not be confused with the usual vector space (Hamel) basis for the underlying vector space. For a Hilbert space the basis vectors are required to be of length one and to be mutually perpendicular; but it is *not* necessary that every vector in the space be a linear combination of basis vectors (but see proposition 7.1.10(d) below).

7.1.2. Proposition. *Every orthonormal set in a Hilbert space H can be extended to a basis for H.*

7.1.3. Corollary. *Every nonzero Hilbert space has a basis.*

7.1.4. Example. For each $n \in \mathbb{N}$ let $e^n = \big(\delta_{nk}\big)_{k=1}^{\infty}$ be the constant zero sequence except for the n^{th} entry whose value is one. Then $\{e^n : n \in \mathbb{N}\}$ is a basis for the Hilbert space l_2 (see example 2.1.3).

7.1.5. Example. For each integer n define the function $\mathbf{e}_n$ on $[0, 2\pi]$ by

$$\mathbf{e}_n(t) = \tfrac{1}{\sqrt{2\pi}}\, e^{int}$$

for $0 \le t \le 2\pi$. Then $\{\mathbf{e}_n : n \in \mathbb{Z}\}$ is a basis for the Hilbert space $L_2([0, 2\pi])$ (see example 2.1.8).

7.1.6. Definition. Let V be a normed linear space and $A \subseteq V$. For every $F \in \operatorname{Fin} A$ define

$$s_F = \sum F\,.$$

Then $s = \big(s_F\big)_{F \in \operatorname{Fin} A}$ is a net in V. If this net converges, the set A is said to be SUMMABLE; the limit of the net is the SUM of A and is denoted by $\sum A$. Indexed sets require a slightly different notation. Suppose, for example, that $A = \{x_\lambda : \lambda \in \Lambda\}$ where Λ is an arbitrary index set. Then for each $F \in \operatorname{Fin} \Lambda$

$$s_F = \sum_{\lambda \in F} x_\lambda \quad \big(= \sum \{x_\lambda : \lambda \in F\}\big).$$

As above s is a net in H. If it converges $\{x_\lambda : \lambda \in \Lambda\}$ is summable and its sum is denoted by $\sum_{\lambda\in\Lambda} x_\lambda$ (or by $\sum\{x_\lambda : \lambda \in \Lambda\}$). An alternative way of saying that $\{x_\lambda : \lambda \in \Lambda\}$ is summable is to say that the *series* $\sum_{\lambda\in\Lambda} x_\lambda$ CONVERGES or that the *sum* $\sum_{\lambda\in\Lambda} x_\lambda$ EXISTS.

7.1.7. Proposition (Bessel's inequality)**.** *If E is an orthonormal subset of a Hilbert space H and $x \in H$, then*

$$\sum_{e \in E} |\langle x, e\rangle|^2 \le \|x\|^2\,.$$

7.1.8. Corollary. *If E is an orthonormal subset of a Hilbert space H and $x \in H$, then $\{e \in E : \langle x, e\rangle \ne 0\}$ is countable.*

7.1.9. Proposition. *If E is an orthonormal subset of a Hilbert space H and $x \in H$, then $\{\langle x, e\rangle e : e \in E\}$ is summable.*

7.1.10. Proposition. *Let E be an orthonormal set in a Hilbert space H. Then the following are equivalent.*

(a) *E is maximal (that is, E is a basis).*
(b) *E is* total *(that is, if $x \perp E$, then $x = \mathbf{0}$).*
(c) *E is* complete *(that is, $\bigvee E = H$).*
(d) $x = \displaystyle\sum_{e \in E} \langle x, e\rangle e$ *for all $x \in H$.* (Fourier expansion)
(e) $\langle x, y\rangle = \displaystyle\sum_{e \in E} \langle x, e\rangle \langle e, y\rangle$ *for all x, $y \in H$.* (Parseval's identity)
(f) $\|x\|^2 = \displaystyle\sum_{e \in E} |\langle x, e\rangle|^2$ *for all $x \in H$.* (Parseval's identity)

The coefficients $\langle x, e\rangle$ in the Fourier expansion of the vector x (in part (d) of proposition 7.1.10) are the FOURIER COEFFICIENTS of x with respect to the basis E. It is easy to see that these coefficients are unique.

7.1.11. Proposition. *Let E be a basis for a Hilbert space H and $x \in H$. If $x = \sum_{e \in E} \alpha_e e$, then $\alpha_e = \langle x, e\rangle$ for each $e \in E$.*

7.1.12. Example. By writing out the Fourier expansion of the identity function $f : x \mapsto x$ in the Hilbert space $L_2([0, 2\pi])$ with respect to the basis given in example 7.1.5, we demonstrate that the sum of the infinite series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is $\frac{1}{6}\pi^2$.

7.1.13. Definition. A mapping $T : V \to W$ between vector spaces is CONJUGATE LINEAR if $T(u + v) = Tu + Tv$ and $T(\alpha v) = \overline{\alpha}\, Tv$ for all u, $v \in V$ and all $\alpha \in \mathbb{C}$. A bijective conjugate linear map between vector spaces is an ANTI-ISOMORPHISM.

7.1.14. Example. Every inner product is conjugate linear in its second variable.

7.1.15. Example. For a Hilbert space H, the mapping $\psi : H \to H^* : a \mapsto \psi_a$ defined in example 2.2.22 is an isometric anti-isomorphism between H and its dual space.

7.1.16. Definition. A conjugate linear mapping $C : V \to V$ from a vector space into itself which satisfies $C^2 = \mathrm{id}_V$ is called a CONJUGATION on V.

7.1.17. Example. Complex conjugation $z \mapsto \overline{z}$ is an example of a conjugation in the vector space $\mathbb{C}$.

7.1.18. Example. Let H be a Hilbert space and B be a basis for H. Then the map $x \mapsto \sum_{e \in B} \langle e, x\rangle e$ is a conjugation and an isometry on H.

The term "anti-isomorphism" is a bit misleading. It suggests something entirely different from an isomorphism. In fact, an anti-isomorphism is nearly as good as an isomorphism. The next proposition says in essence that a Hilbert space and its dual are isomorphic *because* they are anti-isomorphic.

7.1.19. Proposition. *Let H be a Hilbert space, $\psi : H \to H^*$ be the anti-isomorphism defined in 2.2.22 (see 7.1.15), and C be the conjugation defined in 7.1.18. Then the composite $C\psi^{-1}$ is an isometric isomorphism from H^* onto H.*

7.1.20. Corollary. *Every Hilbert space is isometrically isomorphic to its dual.*

7.1.21. Proposition. *Let H be a Hilbert space and $\psi : H \to H^*$ be the anti-isomorphism defined in 2.2.22 (see 7.1.15). If we define $\langle f, g\rangle := \langle \psi^{-1} g, \psi^{-1} f\rangle$ for all f, $g \in H^*$, then H^* becomes a Hilbert space isometrically isomorphic to H. The resulting norm on H^* is its usual norm.*

7.1.22. Corollary. *Every Hilbert space is reflexive.*

In example 4.2.4 we defined the adjoint of a linear operator between Banach spaces and in proposition 5.1.6 we defined the adjoint of a Hilbert space operator. In the case of an operator on a Hilbert space (which is also a Banach space) what is the relationship between these two "adjoints"? They certainly are not equal since the former acts between the dual spaces and the latter between the original spaces. In the next proposition we make use of the anti-isomorphism ψ defined in 2.2.22 (see 7.1.15) to demonstrate that the two adjoints are "essentially" the same.

7.1.23. Proposition. *Let T be an operator on a Hilbert space H and ψ be the anti-isomorphism defined in 2.2.22. If we denote (temporarily) the Banach space dual of T by $T'\colon H^* \to H^*$, then $T' = \psi T^* \psi^{-1}$. That is, the following diagram commutes.*

$$\begin{tikzcd} H^* \arrow[r, "T'"] & H^* \\ H \arrow[u, "\psi"] \arrow[r, "T^*"'] & H \arrow[u, "\psi"'] \end{tikzcd}$$

7.1.24. Definition. Let A be a subset of a Banach space B. Then the ANNIHILATOR of A, denoted by $A^\perp$, is $\{f \in B^* \colon f(a) = 0 \text{ for every } a \in A\}$.

It is possible for the conflict in notations between 1.2.25 and 7.1.24 to cause confusion. To see that the annihilator of a subspace and its orthogonal complement are "essentially" the same thing, use the isometric anti-isomorphism ψ between H and H^* discussed in 7.1.15 to identify them.

7.1.25. Proposition. *Let M be a subspace of a Hilbert space H. If we (temporarily) denote the annihilator of M by M^a, then $M^a = \psi^{\to}\big(M^\perp\big)$.*

7.2. Projections and Partial Isometries

7.2.1. Convention. In this and subsequent sections our attention is focused primarily on operators on Hilbert spaces. In this context the term *projection* is always taken to mean *orthogonal projection* (see definition 1.2.40). Thus a Hilbert space operator is called a projection if $P^2 = P$ and $P^* = P$.

We generalize the definition of "(orthogonal) projection" from Hilbert spaces to $*$-algebras.

7.2.2. Definition. A PROJECTION in a $*$-algebra A is an element p of the algebra which is idempotent ($p^2 = p$) and self-adjoint ($p^* = p$). The set of all projections in A is denoted by $\mathcal{P}(A)$. In the case of $\mathfrak{B}(H)$, the bounded operators on a Hilbert space, we write $\mathcal{P}(H)$, or, if H is understood, just $\mathcal{P}$, for $\mathcal{P}(\mathfrak{B}(H))$.

7.2.3. Proposition. *Every operator on a Hilbert space that is an isometry on the orthogonal complement of its kernel has closed range.*

7.2.4. Proposition. *Let P be a projection on a Hilbert space H. Then*

(i) *$Px = x$ if and only if $x \in \operatorname{ran} P$;*
(ii) *$\ker P = (\operatorname{ran} P)^\perp$; and*
(iii) *$H = \ker P \oplus \operatorname{ran} P$.*

In part (iii) of the preceding proposition the symbol $\oplus$ stands of course for *orthogonal* direct sum (see the paragraph following 1.2.40).

7.2.5. Proposition. *Let M be a subspace of a Hilbert space H. If $P \in \mathfrak{B}(H)$, if $Px = x$ for every $x \in M$, and if $Px = \mathbf{0}$ for every $x \in M^\perp$, then P is the projection of H onto M.*

7.2.6. Proposition. *Let p and q be projections in a $*$-algebra. Then the following are equivalent:*

(i) *$pq = \mathbf{0}$;*
(ii) *$qp = \mathbf{0}$;*

(iii) *$qp = -pq$;*
(iv) *$p + q$ is a projection.*

7.2.7. Definition. Let p and q be projections in a $*$-algebra. If any of the conditions in the preceding result holds, then we say that p and q are ORTHOGONAL and write $p \perp q$. (Thus for operators on a Hilbert space we would correctly speak of orthogonal orthogonal projections!)

7.2.8. Proposition. *Let P and Q be projections on a Hilbert space H. Then $P \perp Q$ if and only if $\operatorname{ran} P \perp \operatorname{ran} Q$. In this case $P + Q$ is an orthogonal projection whose kernel is $\ker P \cap \ker Q$ and whose range is $\operatorname{ran} P + \operatorname{ran} Q$.*

7.2.9. Example. On a Hilbert space (orthogonal) projections need not commute. For example let P be the projection of the (real) Hilbert space R^2 onto the line $y = x$ and Q be the projection of R^2 onto the x-axis. Then $PQ \neq QP$.

7.2.10. Proposition. *Let p and q be projections in a $*$-algebra. Then pq is a projection if and only if $pq = qp$.*

7.2.11. Proposition. *Let P and Q be projections on a Hilbert space H. If $PQ = QP$, then PQ is a projection whose kernel is $\ker P + \ker Q$ and whose range is $\operatorname{ran} P \cap \operatorname{ran} Q$.*

7.2.12. Proposition. *Let p and q be projections in a $*$-algebra. Then the following are equivalent:*
(i) *$pq = p$;*
(ii) *$qp = p$;*
(iii) *$q - p$ is a projection.*

7.2.13. Definition. Let p and q be projections in a $*$-algebra. If any of the conditions in the preceding result holds, then we write $p \preceq q$.

7.2.14. Proposition. *Let P and Q be projections on a Hilbert space H. Then the following are equivalent:*
(i) *$P \preceq Q$;*
(ii) *$\|Px\| \leq \|Qx\|$ for all $x \in H$; and*
(iii) *$\operatorname{ran} P \subseteq \operatorname{ran} Q$.*

In this case $Q - P$ is a projection whose kernel is $\operatorname{ran} P + \ker Q$ and whose range is $\operatorname{ran} Q \ominus \operatorname{ran} P$.

Notation: Let H, M, and N be subspaces of a Hilbert space. The assertion $H = M \oplus N$, may be rewritten as $M = H \ominus N$ (or $N = H \ominus M$).

7.2.15. Proposition. *The operation $\preceq$ defined in 7.2.13 for projections on a $*$-algebra A is a partial ordering on $\mathcal{P}(A)$. If p is a projection in A, then $\mathbf{0} \preceq p \preceq \mathbf{1}$.*

7.2.16. Proposition. *Suppose p and q are projections on a $*$-algebra A. If $pq = qp$, then the infimum of p and q, which we denote by $p \curlywedge q$, exists with respect to the partial ordering $\preceq$ and $p \curlywedge q = pq$. The infimum $p \curlywedge q$ may exist even when p and q do not commute. A necessary and sufficient condition that $p \perp q$ hold is that both $p \curlywedge q = \mathbf{0}$ and $pq = qp$ hold.*

7.2.17. Proposition. *Suppose p and q are projections on a $*$-algebra A. If $p \perp q$, then the supremum of p and q, which we denote by $p \curlyvee q$, exists with respect to the partial ordering $\preceq$ and $p \curlyvee q = p + q$. The supremum $p \curlyvee q$ may exist even when p and q are not orthogonal.*

7.2.18. Proposition. *Let A be C^*-algebra, $a \in A^+$, and $0 < \epsilon \leq \frac{1}{2}$. If $\|a^2 - a\| < \epsilon/2$, then there exists a projection p in A such that $\|p - a\| < \epsilon$.*

7.2.19. Definition. An element v of a C^*-algebra A is a PARTIAL ISOMETRY if v^*v is a projection in A. (Since v^*v is always self-adjoint, it is enough to require that v^*v be idempotent.) The element v^*v is the INITIAL (or SUPPORT) PROJECTION of v and vv^* is the FINAL (or RANGE) PROJECTION of v. (It is an obvious consequence of the next proposition that if v is a partial isometry, then vv^* is in fact a projection.)

7.2.20. Proposition. *If v is a partial isometry in a C^*-algebra, then*

(i) $vv^*v = v$ *; and*

(ii) v^* *is a partial isometry.*

Hint for proof. Let $z = v - vv^*v$ and consider z^*z.

7.2.21. Proposition. *Let v be a partial isometry in a C^*-algebra A. Then its initial projection p is the smallest projection (with respect to the partial ordering $\preceq$ on $\mathcal{P}(A)$) such that $vp = v$ and its final projection q is the smallest projection such that $qv = v$.*

7.2.22. Proposition. *If V is a partial isometry on a Hilbert space H (that is, if V is a partial isometry in the C^*-algebra $\mathfrak{B}(H)$), then the initial projection V^*V is the projection of H onto $(\ker V)^{\perp}$ and the final projection VV^* is the projection of H onto $\operatorname{ran} V$.*

Because of the preceding result $(\ker V)^{\perp}$ is called the INITIAL (or SUPPORT) SPACE of V and $\operatorname{ran} V$ is sometimes called the FINAL SPACE of V.)

7.2.23. Proposition. *An operator on a Hilbert space is a partial isometry if and only if it is an isometry on the orthogonal complement of its kernel.*

7.2.24. Theorem (Polar Decomposition)**.** *If T is an operator on a Hilbert space H, then there exists a partial isometry V on H such that*

(i) *the initial space of V is $(\ker T)^{\perp}$,*

(ii) *the final space of V is $\overline{\operatorname{ran} T}$, and*

(iii) $T = V|T|$.

This decomposition of T is unique in the sense that if $T = V_0P$ where V_0 is a partial isometry and P is a positive operator such that $\ker V_0 = \ker P$, then $P = |T|$ and $V_0 = V$.

PROOF. See [5], VIII.3.11; [7], theorem I.8.1; [18], theorem 6.1.2; [20], theorem 2.3.4; or [23], theorem 3.2.17.

7.2.25. Corollary. *If T is an invertible operator on a Hilbert space, then the partial isometry in the polar decomposition (see 7.2.24) is unitary.*

7.2.26. Proposition. *If T is a normal operator on a Hilbert space H, then there exists a unitary operator U on H which commutes with T and satisfies $T = U|T|$.*

PROOF. See [23], proposition 3.2.20.

7.2.27. Exercise. Let $(S, \mathcal{A}, \mu)$ be a σ-finite measure space and $\phi \in L_\infty(\mu)$. Find the polar decomposition of the multiplication operator M_ϕ (see example 5.1.8).

7.3. Finite Rank Operators

7.3.1. Definition. The RANK of an operator is the dimension of its range. Thus an operator of FINITE RANK is one which has finite dimensional range. We denote by $\mathfrak{FR}(V)$ the collection of finite rank operators on a vector space V.

7.3.2. Proposition. *Let H be a Hilbert space. The family of all finite rank operators on H is a minimal (algebraic) $*$-ideal in the $*$-algebra $\mathfrak{B}(H)$.*

PROOF. See [9], proposition 5.5.

The next definition is the usual one for a *positive* operator on a Hilbert space. But recall example 6.5.15.

7.3.3. Definition. A self-adjoint operator T on a Hilbert space H is POSITIVE if $\langle Tx, x\rangle \geq 0$ for every $x \in H$. In this case we may write $T \geq \mathbf{0}$. For self-adjoint operators S and T on H (as for elements of any C^*-algebra), we write $S \leq T$ (or $T \geq S$) if $T - S \geq \mathbf{0}$.

7.3.4. Proposition. *For a self-adjoint operator T on a Hilbert space the following are equivalent:*

(i) $\sigma(T) \subseteq [0, \infty)$;
(ii) *there exists an operator S on H such that $T = S^*S$; and*
(iii) *T is positive.*

7.3.5. Notation. For vectors x and y in a Hilbert space H let

$$x \otimes y \colon H \to H \colon z \mapsto \langle z, y\rangle x\,.$$

7.3.6. Proposition. *If x and y are nonzero vectors in a Hilbert space H, then $x \otimes y$ is a rank 1 operator in $\mathfrak{B}(H)$.*

Let us extend the terminology of definition 2.1.29 from complex valued functions to functions whose codomain is a *-algebra.

7.3.7. Definition. Suppose V is a vector space and A is a *-algebra. A function $\phi \colon V \times V \to A$ is SESQUILINEAR if it is linear in its first variable and conjugate linear in its second. It is CONJUGATE SYMMETRIC if $\big(\phi(x,y)\big)^* = \phi(y,x)$ for all x, $y \in V$.

If A is a C^*-algebra we say that the mapping ϕ is POSITIVE SEMIDEFINITE if $\phi(x,x) \geq \mathbf{0}$ for every $x \in V$.

7.3.8. Proposition. *If H is a Hilbert space, then*

$$\phi \colon H \times H \to \mathfrak{B}(H) \colon (x,y) \mapsto x \otimes y$$

is a positive semidefinite, conjugate symmetric, sesquilinear mapping.

7.3.9. Proposition. *If H is a Hilbert space, x and y are elements of H, and $T \in \mathfrak{B}(H)$, then*

$$T(x \otimes y) = (Tx) \otimes y$$

and therefore

$$(x \otimes y)T = x \otimes (T^*y)\,.$$

7.3.10. Lemma. *If x is a vector in a Hilbert space H, then $x \otimes x$ is a rank 1 projection if and only if x is a unit vector.*

7.3.11. Proposition. *Let $M = \operatorname{span}\{e^1, \dots, e^n\}$ where $\{e^1, \dots, e^n\}$ is an orthonormal subset of a Hilbert space H. Then*

$$P_M = \sum_{k=1}^{n} e^k \otimes e^k\,.$$

7.3.12. Proposition. *Every finite rank Hilbert space operator is a linear combination of rank one projections.*

PROOF. See [20], theorem 2.4.6.

7.4. Compact Operators

7.4.1. Definition. An operator K on a Hilbert space H is COMPACT if the image of the closed unit ball in H under K is a compact (equivalently, totally bounded) subset of H. We denote by $\mathfrak{K}(H)$ the family of all compact operators on H. (More generally, a bounded linear map K between normed linear spaces is COMPACT if the image under K of the closed unit ball is relatively compact.)

7.4.2. Example. The integral operator $K = \operatorname{int} k$ defined in example 2.2.17 on the Hilbert space $L_2(S)$ (where $(S, \mathfrak{A}, \mu)$ is a sigma-finite measure space) is a compact Hilbert space operator.

7.4.3. Example. The Volterra operator defined in example 2.2.18 on $L_2([0,1])$ is a compact Hilbert space operator.

7.4.4. Example. The identity operator on a Banach space is compact if and only if the space is finite dimensional.

PROOF. See [4], theorem 4.4.6.

7.4.5. Proposition. *Let H be an infinite dimensional Hilbert space. Then the family $\mathfrak{K}(H)$ of compact operators on H is the closure of the set of all finite rank operators on H.*

PROOF. See [5] theorem II.4.4 or [9], theorem 5.9.

7.4.6. Proposition. *Let H be an infinite dimensional Hilbert space. Then the family $\mathfrak{K}(H)$ of compact operators on H is a minimal ideal in the C^*-algebra $\mathfrak{B}(H)$.*

PROOF. See [5], theorem VI.3.4 or [9], corollary 5.11.

CHAPTER 8

THE GELFAND-NAIMARK-SEGAL CONSTRUCTION

8.1. Positive Linear Functionals

8.1.1. Definition. Let A be an algebra with involution. For each linear functional τ on A and each $a \in A$ define $\tau^\star(a) = \overline{\tau(a^*)}$. We say that τ is HERMITIAN if $\tau^\star = \tau$. Notice that a linear functional $\tau\colon A \to \mathbb{C}$ is Hermitian if and only if it preserves involution; that is, $\tau^\star = \tau$ if and only if $\tau(a^*) = \overline{\tau(a)}$ for all $a \in A$.

CAUTION. The $\tau^\star$ defined above should not be confused with the usual adjoint mapping $\tau^*\colon \mathbb{C}^* \to A^*$. Context (or use of a magnifying glass) should make it clear which is intended.

8.1.2. Proposition. *A linear functional τ on a C^*-algebra A is Hermitian if and only if $\tau(a) \in \mathbb{R}$ whenever a is self-adjoint.*

As is always the case with maps between ordered vector spaces, *positive* maps are the ones that take positive elements to positive elements.

8.1.3. Definition. A linear functional τ on a C^*-algebra A is POSITIVE if $\tau(a) \geq 0$ whenever $a \geq \mathbf{0}$ in A for all $a \in A$.

8.1.4. Proposition. *Every positive linear functional on a C^*-algebra is Hermitian.*

8.1.5. Proposition. *The family of positive linear functionals is a proper convex cone in the real vector space of all Hermitian linear functionals on a C^*-algebra. The cone induces a partial ordering on the vector space: $\tau_1 \leq \tau_2$ whenever $\tau_2 - \tau_1$ is positive.*

8.1.6. Definition. A STATE of a C^*-algebra A is a positive linear functional τ on A such that $\tau(\mathbf{1}) = 1$.

8.1.7. Example. Let x be a vector in a Hilbert space H. Define

$$\omega_x : \mathfrak{B}(H) \to \mathbb{C}\colon T \mapsto \langle Tx, x\rangle\,.$$

Then ω_x is a positive linear functional on $\mathfrak{B}(H)$. If x is a unit vector, then ω_x is a state of $\mathfrak{B}(H)$. A state τ is a VECTOR STATE if $\tau = \omega_x$ for some unit vector x.

8.1.8. Proposition (Schwarz inequality)**.** *If τ is a positive linear functional on a C^*-algebra A, then*

$$|\tau(b^*a)|^2 \leq \tau(a^*a)\tau(b^*b)$$

for all a, $b \in A$.

8.1.9. Proposition. *A linear functional τ on a C^*-algebra A is positive if and only if it is bounded and $\|\tau\| = \tau(\mathbf{1}_A)$.*

PROOF. See [18], pages 256–257.

8.2. Representations

8.2.1. Definition. Let A be a C^*-algebra. A REPRESENTATION of A is a pair (π, H) where H is a Hilbert space and $\pi\colon A \to \mathfrak{B}(H)$ is a $*$-homomorphism. Usually one says simply that π is a representation of A. When we wish to emphasize the role of the particular Hilbert space we say that π *is a representation of A on H.* Depending on context we may write either π_a or $\pi(a)$ for the Hilbert space operator which is the image of the algebra element a under π. A representation π of A on H is NONDEGENERATE if $\pi(A)H$ is dense in H.

8.2.2. Convention. We add to the preceding definition the following requirement: if the C^*-algebra A is unital, then a representation of A must be a *unital* $*$-homomorphism.

8.2.3. Definition. A representation π of a C^*-algebra A on a Hilbert space H is FAITHFUL if it is injective. If there exists a vector $x \in H$ such that $\pi^{\rightarrow}(A)x = \{\pi_a(x) \colon a \in A\}$ is dense in H, then we say that the representation π is CYCLIC and that x is a CYCLIC VECTOR for π.

8.2.4. Example. Let $(S, \mathfrak{A}, \mu)$ be a σ-finite measure space and $L_\infty = L_\infty(S, \mathfrak{A}, \mu)$ be the C^*-algebra of essentially bounded μ-measurable functions on S. As we saw in example 5.1.8 for each $\phi \in L_\infty$ the corresponding multiplication operator M_ϕ is an operator on the Hilbert space $L_2 = L_2(S, \mathfrak{A}, \mu)$. The mapping $M \colon L_\infty \to \mathfrak{B}(L_2) \colon \phi \mapsto M_\phi$ is a faithful representation of the C^*-algebra L_∞ on the Hilbert space L_2.

8.2.5. Example. Let $\mathcal{C}([0,1])$ be the C^*-algebra of continuous functions on the interval $[0,1]$. For each $\phi \in \mathcal{C}([0,1])$ the corresponding multiplication operator M_ϕ is an operator on the Hilbert space $L_2 = L_2([0,1])$ of functions on $[0,1]$ which are square-integrable with respect to Lebesgue measure. The mapping $M \colon \mathcal{C}([0,1]) \to \mathfrak{B}(L_2) \colon \phi \mapsto M_\phi$ is a faithful representation of the C^*-algebra $\mathcal{C}([0,1])$ on the Hilbert space L_2.

8.2.6. Example. Suppose that π is a representation of a unital C^*-algebra A on a Hilbert space H and x is a unit vector in H. If ω_x is the corresponding vector state of $\mathfrak{B}(H)$, then $\omega_x \circ \pi$ is a state of A.

8.2.7. Exercise. Let X be a locally compact Hausdorff space. Find an isometric (therefore faithful) representation (π, H) of the C^*-algebra $C_0(X)$ on some Hilbert space H..

8.2.8. Definition. Let ρ be a state of a C^*-algebra A. Then

$$L_\rho := \{a \in A \colon \rho(a^*a) = 0\}$$

is called the LEFT KERNEL of ρ.

Recall that as part of the proof of *Schwarz inequality* 8.1.8 for positive linear functionals we verified the following result.

8.2.9. Proposition. *If ρ is a state of a C^*-algebra A, then $\langle a, b\rangle_0 := \rho(b^*a)$ defines a semi-inner product on A.*

8.2.10. Corollary. *If ρ is a state of a C^*-algebra A, then its left kernel L_ρ is a vector subspace of A and $\langle\, [a], [b]\,\rangle := \langle a, b\rangle_0$ defines an inner product on the quotient algebra A/L_ρ.*

8.2.11. Proposition. *Let ρ be a state of a C^*-algebra A and $a \in L_\rho$. Then $\rho(b^*a) = 0$ for every $b \in A$.*

8.2.12. Proposition. *If ρ is a state of a C^*-algebra A, then its left kernel L_ρ is a closed left ideal in A.*

8.3. The GNS-Construction and the Third Gelfand-Naimark Theorem

The following theorem is known as the *Gelfand-Naimark-Segal construction* (the *GNS-construction*).

8.3.1. Theorem (GNS-construction)**.** *Let ρ be a state of a C^*-algebra A. Then there exists a cyclic representation π_ρ of A on a Hilbert space H_ρ and a unit cyclic vector x_ρ for π_ρ such that $\rho = \omega_{x_\rho} \circ \pi_\rho$.*

8.3.2. Notation. In the following material π_ρ, H_ρ, and x_ρ are the cyclic representation, the Hilbert space, and the unit cyclic vector guaranteed by the GNS-construction starting with a given state ρ of a C^*-algebra

8.3.3. Proposition. *Let ρ be a state of a C^*-algebra A and π be a cyclic representation of A on a Hilbert space H such that $\rho = \omega_x \circ \pi$ for some unit cyclic vector x for π. Then there exists a unitary map U from H_ρ to H such that $x = Ux_\rho$ and $\pi(a) = U\pi_\rho(a)U^*$ for all $a \in A$.*

8.3.4. Definition. Let $\{H_\lambda \colon \lambda \in \Lambda\}$ be a family of Hilbert spaces. Denote by $\bigoplus\limits_{\lambda\in\Lambda} H_\lambda$ the set of all functions $x \colon \Lambda \to \bigcup\limits_{\lambda\in\Lambda} H_\lambda \colon \lambda \mapsto x_\lambda$ such that $x_\lambda \in H_\lambda$ for each $\lambda \in \Lambda$ and $\sum\limits_{\lambda\in\Lambda} \|x_\lambda\|^2 < \infty$. On $\bigoplus\limits_{\lambda\in\Lambda} H_\lambda$ define addition and scalar multiplication pointwise; that is. $(x+y)_\lambda = x_\lambda + y_\lambda$ and $(\alpha x)_\lambda = \alpha x_\lambda$ for all $\lambda \in \Lambda$, and define an inner product by $\langle x, y\rangle = \sum_{\lambda\in\Lambda}\langle x_\lambda, y_\lambda\rangle$. These operations (are well defined and) make $\bigoplus\limits_{\lambda\in\Lambda} H_\lambda$ into a Hilbert space. It is the DIRECT SUM of the Hilbert spaces H_λ. Various notations for elements of this direct sum occur in the literature: x, (x_λ), $(x_\lambda)_{\lambda\in\Lambda}$, and $\oplus_\lambda x_\lambda$ are common.

Now suppose that $\{T_\lambda \colon \lambda \in \Lambda\}$ is a family of Hilbert space operators where $T_\lambda \in \mathfrak{B}(H_\lambda)$ for each $\lambda \in \Lambda$. Suppose further that $\sup\{\|T_\lambda\| \colon \lambda \in \Lambda\} < \infty$. Then $T(x_\lambda)_{\lambda\in\Lambda} = (T_\lambda x_\lambda)_{\lambda\in\Lambda}$ defines an operator on the Hilbert space $\bigoplus_\lambda H_\lambda$. The operator T is usually denoted by $\bigoplus_\lambda T_\lambda$ and is called the DIRECT SUM of the operators T_λ.

8.3.5. Proposition. *The claims made in the preceding definition that $\bigoplus\limits_{\lambda\in\Lambda} H_\lambda$ is a Hilbert space and $\bigoplus_\lambda T_\lambda$ is an operator on $\bigoplus\limits_{\lambda\in\Lambda} H_\lambda$ are correct.*

8.3.6. Example. Let A be a C^*-algebra and $\{\pi_\lambda \colon \lambda \in \Lambda\}$ be a family of representations of A on Hilbert spaces H_λ so that $\pi_\lambda(a) \in \mathfrak{B}(H_\lambda)$ for each $\lambda \in \Lambda$ and each $a \in A$. Then

$$\pi = \bigoplus\nolimits_\lambda \pi_\lambda \colon A \to \mathfrak{B}\big(\bigoplus\nolimits_\lambda H_\lambda\big) \colon a \mapsto \bigoplus\nolimits_\lambda \pi_\lambda(a)$$

is a representation of A on the Hilbert space $\bigoplus_\lambda H_\lambda$. It is the DIRECT SUM of the representations π_λ.

8.3.7. Theorem. *Every C^*-algebra has a faithful representation.*

PROOF. See [18], page 281 or [5], page 253.

An obvious restatement of the preceding theorem is a third version of the *Gelfand-Naimark theorem*, which says that every C^*-algebra is (essentially) an algebra of Hilbert space operators.

8.3.8. Corollary (Gelfand-Naimark Theorem III)**.** *Every C^*-algebra is isometrically $*$-isomorphic to a C^*-subalgebra of $\mathfrak{B}(H)$ for some Hilbert space H.*

CHAPTER 9

MULTIPLIER ALGEBRAS

9.1. Hilbert Modules

9.1.1. Notation. The inner products that occur previously in these notes and that one encounters in standard textbooks and monographs on Hilbert spaces, functional analysis, and so on, are linear in the first variable and conjugate linear in the second. Most contemporary operator algebraists have chosen to work with objects called *right* Hilbert A-modules (A being a C^*-algebra). For such modules it turns out to be more convenient to have "inner products" that are linear in the second variable and conjugate linear in the first. While this switch in conventions may provoke some slight irritation, it is, mathematically speaking, of little consequence. Of course, we would like Hilbert spaces to be examples of Hilbert $\mathbb{C}$-modules. To make this possible we equip a Hilbert space, whose inner product is denoted by $\langle\ \ ,\ \ \rangle$ with a new "inner product" defined by $\langle x \,|\, y\rangle := \langle y, x\rangle$. This "inner product" is linear in the second variable and conjugate linear in the first. I will try to be consistent in using $\langle\ \ ,\ \ \rangle$ for the standard inner product and $\langle\ \ |\ \ \rangle$ for the one which is linear in its second variable.

Another (very common) solution to this problem is to insist that an inner product is always linear in its second variable and "correct" standard texts, monographs, and papers accordingly.

9.1.2. Convention. In light of the preceding remarks we will in the sequel use the word "sesquilinear" to mean either *linear in the first variable and conjugate linear in the second* or *linear in the second variable and conjugate linear in the first*.

9.1.3. Definition. Let A be a nonzero C^*-algebra. A vector space V is an A-MODULE if there is a bilinear map

$$B\colon V \times A \to V\colon (x, a) \mapsto xa$$

such that $x(ab) = (xa)b$ holds for all $x \in V$ and $a, b \in A$. We also require that $x\mathbf{1}_A = x$ for every $x \in V$ if A is unital. (A function of two variables is BILINEAR if it is linear in both of its variables.)

9.1.4. Definition. For clarity in some of the succeeding material it will be convenient to have the following formal definition. A (COMPLEX) VECTOR SPACE is a triple $(V, +, M)$ where $(V, +)$ is an Abelian group and $M\colon \mathbb{C} \to \operatorname{Hom}(V, +)$ is a unital ring homomorphism. (As you would guess, $\operatorname{Hom}(V, +)$ denotes the unital ring of endomorphisms of the group $(V, +)$.) Thus an ALGEBRA is an ordered quadruple $(A, +, M, \cdot\,)$ where $(A, +, M)$ is a vector space and $\cdot : A \times A \to A\colon (a, b) \mapsto a \cdot b = ab$ is a binary operation on A satisfying the four conditions of 2.3.1.

9.1.5. Exercise. Make sure the preceding definition of *vector space* is equivalent to the one you are accustomed to.

9.1.6. Definition. Let $(A, +, M, \cdot\,)$ be a (complex) algebra. Then A^{op} is the algebra $(A, +, M, *)$ where $a * b = b \cdot a$ for all $a, b \in A$. This is the OPPOSITE ALGEBRA of A. It is just A with the order of multiplication reversed. If A and B are algebras, then any function $f\colon A \to B$ induces in an obvious fashion a function from A into B^{op} (or from A^{op} into B, or from A^{op} into B^{op}). We will denote all these functions simply by f.

9.1.7. Definition. Let A and B be algebras. A function $\phi\colon A \to B$ is an ANTIHOMOMORPHISM if the function $f\colon A \to B^{\mathrm{op}}$ is a homomorphism. A bijective antihomomorphism is an ANTI-ISOMORPHISM. In example 7.1.15 we encountered *anti-isomorphisms* of Hilbert spaces. These were a bit different since instead of reversing multiplication (which Hilbert spaces don't have) they conjugate scalars. In either case an anti-isomorphism is not something terribly different from an isomorphism but actually something quite similar.

The notion of an A-module (where A is an algebra) was defined in 9.1.3. You may prefer the following alternative definition, which is more in line with the definition of *vector space* given in 9.1.4.

9.1.8. Definition. Let A be an algebra. An A-MODULE is an ordered quadruple $(V, +, M, \Phi)$ where $(V, +, M)$ is a vector space and $\Phi\colon A \to \mathfrak{L}(V)$ is an algebra homomorphism. If A is unital we require also that Φ be unital.

9.1.9. Exercise. Check that the definitions of A-*module* given in 9.1.3 and 9.1.8 are equivalent.

We now say precisely what it means,when A is a C^*-algebra, to give an A-module an A-valued inner product.

9.1.10. Definition. Let A be a C^*-algebra. A SEMI-INNER PRODUCT A-MODULE is an A-module V together with a mapping

$$\beta\colon V \times V \to A\colon (x, y) \mapsto \langle x|\, y\rangle$$

which is linear in its second variable and satisfies

(i) $\langle x|\, ya\rangle = \langle x|\, y\rangle a$,
(ii) $\langle x|\, y\rangle = \langle y|\, x\rangle^*$, and
(iii) $\langle x|\, x\rangle \geq \mathbf{0}$

for all x, $y \in V$ and $a \in A$. It is an INNER PRODUCT A-MODULE (or a PRE-HILBERT A-MODULE) if additionally

(iv) $\langle x|\, x\rangle = \mathbf{0}$ implies that $x = \mathbf{0}$

when $x \in V$. We will refer to the mapping β as an A-VALUED (SEMI-)INNER PRODUCT on V.

9.1.11. Example. Every inner product space is an inner product $\mathbb{C}$-module.

9.1.12. Proposition. *Let A be a C^*-algebra and V be a semi-inner product A-module. The semi-inner product $\langle\ |\ \rangle$ is conjugate linear in its first variable both literally and in the sense that $\langle va|\, w\rangle = a^*\langle v|\, w\rangle$ for all v, $w \in V$ and $a \in A$.*

9.1.13. Proposition (Schwarz inequality—for inner product A-modules)**.** *Let V be an inner product A-module where A is a C^*-algebra. Then*

$$\langle x|\, y\rangle^*\langle x|\, y\rangle \leq \|\langle x|\, x\rangle\|\, \langle y|\, y\rangle$$

for all x, $y \in V$.

Hint for proof. Show that no generality is lost in assuming that $\|\langle x|\, x\rangle\| = 1$. Consider the positive element $\langle xa - y|\, xa - y\rangle$ where $a = \langle x|\, y\rangle$. Use propositions 6.5.21 and 6.5.22.

9.1.14. Definition. For every element v of an inner product A-module (where A is a C^*-algebra) define

$$\|v\| := \|\langle v|\, v\rangle\|^{1/2}.$$

9.1.15. Proposition (Yet another Schwarz inequality)**.** *Let A be a C^*-algebra and V be an inner product A-module. Then for all v, $w \in V$*

$$\|\langle v|\, w\rangle\| \leq \|v\|\, \|w\|.$$

9.1.16. Corollary. *If v and w are elements of an inner product A-module (where A is a C^*-algebra), then $\|v + w\| \leq \|v\| + \|w\|$ and the map $x \mapsto \|x\|$ is a norm on V.*

9.1.17. Proposition. *If A is a C^*-algebra and V is an inner product A-module, then*

$$\|va\| \le \|v\| \, \|a\|$$

for all $v \in V$ and $a \in A$.

9.1.18. Definition. Let A be a C^*-algebra and V be an inner product A-module. If V is complete with respect to (the metric induced by) the norm defined in 9.1.14, then V is a HILBERT A-MODULE.

9.1.19. Example. For a and b in a C^*-algebra A define

$$\langle a | \, b \rangle := a^* b \,.$$

Then A is itself a Hilbert A-module. Any closed right ideal in A is also a Hilbert A-module.

9.1.20. Definition. Let V and W be Hilbert A-modules where A is a C^*-algebra. A mapping $T\colon V \to W$ is A-LINEAR if it is linear and if $T(va) = T(v)a$ holds for all $v \in V$ and $a \in A$. The mapping T is a HILBERT A-MODULE MORPHISM if it is bounded and A-linear.

Recall from proposition 5.1.6 that every Hilbert space operator has an adjoint. This is not true for Hilbert A-modules

9.1.21. Definition. Let V and W be Hilbert A-modules where A is a C^*-algebra. A function $T\colon V \to W$ is ADJOINTABLE if there exists a function $T^*\colon W \to V$ satisfying

$$\langle Tv | \, w \rangle = \langle v | \, T^* w \rangle$$

for all $v \in V$ and $w \in W$. The function T^*, if it exists, is the ADJOINT of T. Denote by $\mathfrak{L}(V, W)$ the family of adjointable maps from V to W. We shorten $\mathfrak{L}(V, V)$ to $\mathfrak{L}(V)$.

9.1.22. Proposition. *Let V and W be Hilbert A-modules where A is a C^*-algebra. If a function $T\colon V \to W$ is adjointable, then it is a Hilbert A-module morphism. Furthermore if T is adjointable, then so is its adjoint and $T^{**} = T$.*

9.1.23. Example. Let X be the unit interval $[0, 1]$ and let $Y = \{0\}$. With its usual topology X is a compact Hausdorff space and Y is a subspace of X. Let A be the C^*=algebra $\mathcal{C}(X)$ and J_0 be the ideal $\{f \in A\colon f(0) = 0\}$ (see proposition 3.2.9). Regard $V = A$ and $W = J_0$ as Hilbert A-modules (see example 9.1.19). Then the inclusion map $\iota\colon V \to W$ is a Hilbert A-module morphism which is not adjointable.

9.1.24. Proposition. *Let A be a C^*-algebra. The pair of maps $V \mapsto V$, $T \mapsto T^*$ is a contravariant functor from the category of Hilbert A-modules and adjointable maps into itself.*

9.1.25. Proposition. *Let A be a C^*-algebra and V be a Hilbert A-module. Then $\mathfrak{L}(V)$ is a unital C^*-algebra.*

9.1.26. Notation. Let V and W be Hilbert A-modules where A is a C^*-algebra. For $v \in V$ and $w \in W$ let

$$\Theta_{v,w}\colon W \to V \colon x \mapsto v \langle w | \, x \rangle \,.$$

(Compare this with 7.3.5.)

9.1.27. Proposition. *The map Θ defined above is sesquilinear.*

9.1.28. Proposition. *Let V and W be Hilbert A-modules where A is a C^*-algebra. For every $v \in V$ and $w \in W$ the map $\Theta_{v,w}$ is adjointable and $(\Theta_{v,w})^* = \Theta_{w,v}$.*

The next proposition generalizes proposition 7.3.9.

9.1.29. Proposition. *Let U, V, W, and Z be Hilbert A-modules where A is a C^*-algebra. If $S \in \mathfrak{L}(Z, W)$ and $T \in \mathfrak{L}(V, U)$, then*

$$T\Theta_{v,w} = \Theta_{Tv,w} \qquad \text{and} \qquad \Theta_{v,w} S = \Theta_{v,S^* w}$$

for all $v \in V$ and $w \in W$.

$$Z \xrightarrow{S} W \xrightarrow{\Theta_{v,w}} V \xrightarrow{T} U$$

9.1.30. Proposition. *Let U, V, and W be Hilbert A-modules where A is a C^*-algebra. Suppose $u \in U$; v, $v' \in V$; and $w \in W$. Then*

$$\Theta_{u,v}\Theta_{v',w} = \Theta_{u\langle v|\, v'\rangle,w} = \Theta_{u,w\langle v'|\, v\rangle}\,.$$

9.1.31. Notation. Let A be a C^*-algebra and V and W be Hilbert A-modules. We denote by $\mathfrak{K}(W,V)$ the closed linear span of $\{\Theta_{v,w} : v \in V \text{ and } w \in W\}$. As usual we shorten $\mathfrak{K}(V,V)$ to $\mathfrak{K}(V)$.

9.1.32. Proposition. *If A is a C^*-algebra and V is a Hilbert A-module, then $\mathfrak{K}(V)$ is an ideal in the C^*-algebra $\mathfrak{L}(V)$.*

The next example is intended as justification for the standard practice of identifying a C^*-algebra A with $\mathfrak{K}(A)$.

9.1.33. Example. If we regard a C^*-algebra A as an A-module (see example 9.1.19), then $\mathfrak{K}(A) \overset{*}{\cong} A$.

Hint for proof. As in corollary 5.3.19 define for each $a \in A$ the left multiplication operator $L_a : A \to A : x \mapsto ax$. Show that each such operator is adjointable and that the map $L : A \to \mathfrak{L}(A) : a \mapsto L_a$ is a $*$-isomorphism onto a C^*-subalgebra of $\mathfrak{L}(A)$. Then verify that $\mathfrak{K}(A)$ is the closure of the image under L of the span of products of elements of A.

9.1.34. Example. Let H be a Hilbert space regarded as a $\mathbb{C}$-module. Then $\mathfrak{K}(H)$ (as defined in 9.1.31) is the ideal of compact operators on H (see proposition 7.4.6).

Proof. See [25], example 2.27.

The preceding example has lead many researchers, when dealing with an arbitrary Hilbert module V, to refer to members of $\mathfrak{K}(V)$ as *compact operators*. This is dubious terminology since such operators certainly need not be compact. (For example, if we regard an infinite dimensional unital C^*-algebra A as an A-module, then $\Theta_{\mathbf{1},\mathbf{1}} = I_A$, but the identity operator on A is not compact—see example 7.4.4.)

The fact that in these notes rather limited use is made of Hilbert C^*-modules should not lead you to think that their study is specialized and/or of marginal interest. To the contrary, it is currently an important and vigorous research area having applications to fields as diverse as K-theory, graph C^*-algebras, quantum groups, quantum probability, vector bundles, non-commutative geometry, algebraic and geometric topology, operator spaces and algebras, and wavelets. Take a look at Michael Frank's webpage [13], *Hilbert C^*-modules and related subjects—a guided reference overview*, where he lists 1531 references (as of his 11.09.10 update) to books, papers, and theses dealing with such modules and categorizes them by application. There is an interesting graphic (on page 9) illustrating the growth of this field of mathematics. It covers material from the pioneering efforts in the 50's and early 60's (0–2 papers per year) to the time of this writing (about 100 papers per year).

9.2. Essential Ideals

9.2.1. Example. If A and B are C^*-algebras, then A (more precisely, $A \oplus \{\mathbf{0}\}$) is an ideal in $A \oplus B$.

9.2.2. Convention. As the preceding example suggests, it is conventional to regard A as a subset of $A \oplus B$. In the sequel we will do this without further mention.

9.2.3. Notation. For an element c of an algebra A let

$$I_c := \left\{ a_0c + cb_0 + \sum_{k=1}^{p} a_kcb_k : p \in \mathbb{N},\, a_0,\, \dots,\, a_p,\, b_0,\, \dots,\, b_p \in A \right\}$$

9.2.4. Proposition. *If c is an element of an algebra A, then I_c is an (algebraic) ideal in A.*

Notice that in the preceding proposition no claim is made that the algebraic ideal I_c must be proper. It may well be the case that $I_c = A$ (as, for example, when c is an invertible element of a unital algebra).

9.2.5. Definition. Let c be an element of a C^*-algebra A. Define J_c, the PRINCIPAL IDEAL containing c, to be the intersection of the family of all (closed) ideals of A which contain c. Clearly, J_c is the smallest ideal containing c.

9.2.6. Proposition. *In a C^*-algebra the closure of an algebraic ideal is an ideal.*

9.2.7. Example. The closure of a proper algebraic ideal in a C^*-algebra need not be a proper ideal. For example, l_c, the set of sequences of complex numbers which are eventually zero, is dense in the C^*-algebra $l_0 = \mathcal{C}_0(\mathbb{N})$. (But recall proposition 3.2.1.)

9.2.8. Proposition. *If c is an element of a C^*-algebra, then $J_c = \overline{I_c}$.*

9.2.9. Notation. We adopt a standard notational convention. If A and B are nonempty subsets of an algebra. By AB we mean the linear span of products of elements in A and elements in B; that is, $AB = \operatorname{span}\{ab \colon a \in A \text{ and } b \in B\}$. (Note that in definition 2.3.17 it makes no difference whether we take AJ to mean the set of products of elements in A with elements in J or the span of that set.)

9.2.10. Proposition. *If I and J are ideals in a C^*-algebra, then $\overline{IJ} = I \cap J$.*

A nonunital C^*-algebra A can be embedded as an ideal in a unital C^*-algebra in different ways. The smallest unital C^*-algebra containing A is its unitization $\widetilde{A}$ (see proposition 6.3.1). Of course there is no largest unital C^*-algebra in which A can be embedded as an ideal because if A is embedded as an ideal in a unital C^*-algebra B and C is *any* unital C^*-algebra, then A is an ideal in the still larger unital C^*-algebra $B \oplus C$. The reason this larger unitization is not of much interest is that the intersection of the ideal C with A is $\{\mathbf{0}\}$. This motivates the following definition.

9.2.11. Definition. An ideal J in a C^*-algebra A is ESSENTIAL if and only if $I \cap J \neq \mathbf{0}$ for every nonzero ideal I in A.

9.2.12. Example. A C^*-algebra A is an essential ideal in its unitization $\widetilde{A}$ if and only if A is *not* unital.

9.2.13. Example. If H is a Hilbert space the ideal of compact operators $\mathfrak{K}(H)$ is an essential ideal in the C^*-algebra $\mathfrak{B}(H)$.

9.2.14. Definition. Let J be an ideal in a C^*-algebra A. Then we define $J^\perp$, the ANNIHILATOR of J, to be $\big\{a \in A \colon Ja = \{\mathbf{0}\}\big\}$.

9.2.15. Proposition. *If J is an ideal in a C^*-algebra, then so is $J^\perp$.*

9.2.16. Proposition. *An ideal J in a C^*-algebra A is essential if and only if $J^\perp = \{\mathbf{0}\}$.*

9.2.17. Proposition. *If J is an ideal in a C^*-algebra, then $\big(J \oplus J^\perp\big)^\perp = \{\mathbf{0}\}$.*

9.2.18. Notation. Let f be a (real or) complex valued function on a set S. Then

$$Z_f := \{s \in S \colon f(s) = 0\}.$$

This is the ZERO SET of f.

Suppose that A is a nonunital commutative C^*-algebra. By the second *Gelfand-Naimark theorem* 6.3.11 there exists a noncompact locally compact Hausdorff space X such that $A = \mathcal{C}_0(X)$. (Here, of course, we are permitting ourselves a conventional abuse of language: for literal correctness the indicated equality should be an isometric $*$-isomorphism.) Now let Y be a compact

Hausdorff space in which X is an open subset and let $B = \mathcal{C}(Y)$. Then B is a unital commutative C^*-algebra. Regard A as embedded as an ideal in B by means of the map $\iota\colon A \to B\colon f \mapsto \widetilde{f}$ where

$$\widetilde{f}(y) = \begin{cases} f(y), & \text{if } y \in X; \\ 0, & \text{otherwise.} \end{cases}$$

Notice that the closed set X^c is $\bigcap\{Z_{\widetilde{f}} : f \in \mathcal{C}_0(X)\}$.

9.2.19. Proposition. *Let the notation be as in the preceding paragraph. Then the ideal A is essential in B if and only if the open subset X is dense in Y.*

Thus the property of an ideal being essential in the context of unitizations of nonunital commutative C^*-algebras corresponds exactly with the property of an open subspace being dense in the context of compactifications of noncompact locally compact Hausdorff spaces.

9.3. Compactifications and Unitizations

In definition 6.3.3 we called the object whose existence was proved in the preceding proposition 6.3.1 "the" unitization of a C^*-algebra. The definite article there is definitely misleading. Just as a topological space may have many different compactifications, a C^*-algebra may have many unitizations. If A is a nonunital commutative C^*-algebra, it is clear from corollary 6.6.15 that the unitization $\widetilde{A}$ is the smallest possible unitization of A. Similarly, in topology, if X is a noncompact locally compact Hausdorff space, then its one-point compactification is obviously the smallest possible compactification of X. Recall that in proposition 6.3.16 we established the fact that constructing the smallest unitization of A is "essentially" the same thing as constructing the smallest compactification of X.

It is sometimes convenient (as, for example, in the preceding paragraph) to take a "unitization" of an algebra that is already unital and sometimes convenient to take a "compactification" of a space that is already compact. Since there appears to be no universally accepted terminology, I introduce the following (definitely nonstandard, but I hope helpful) language.

9.3.1. Definition. Let A and B be C^*-algebras and X and Y be Hausdorff topological spaces. We will say that

(1) B is a UNITIZATION of A if B is unital and A is ($*$-isomorphic to) a C^*-subalgebra of B;
(2) B is an ESSENTIAL UNITIZATION of A if B is unital and A is ($*$-isomorphic to) an essential ideal of B;
(3) Y is a COMPACTIFICATION of X if it is compact and X is (homeomorphic to) a subspace of Y; and
(4) Y is an ESSENTIAL COMPACTIFICATION of X if it is compact and X is (homeomorphic to) a dense subspace of Y

Perhaps a few words are in order concerning the bits in parentheses in the preceding definition. It is seldom the case that a topological space X is literally a *subset* of a particular compactification of X or that a C^*-algebra A is a *subset* of a particular unitization of A. While certainly true that it is frequently convenient to regard one C^*-algebra as a subset of another when in fact the first is merely $*$-isomorphic to a subset of the second, there are also occasions when it clarifies matters to specify the actual embeddings involved. If the details of these distinctions are not entirely familiar, the next two definitions are intended to help.

9.3.2. Definition. Let A and B be C^*-algebras. We say that A is EMBEDDED in B if there exists an injective $*$-homomorphism $\iota\colon A \to B$; that is, if A is $*$-isomorphic to a C^*-subalgebra of B (see propositions 6.6.9 and 6.3.14). The injective $*$-homomorphism ι is an EMBEDDING of A into B. In this situation it is common practice to treat A and the range of ι as identical C^*-algebras. The pair (B, ι) is a UNITIZATION of A if B is a unital C^*-algebra and $\iota\colon A \to B$ is an embedding. The unitization (B, ι) is ESSENTIAL if the range of ι is an essential ideal in B.

9.3.3. Definition. Let X and Y be Hausdorff topological spaces. We say that X is EMBEDDED in Y if there exists a homeomorphism j from X to a subspace of Y. The homeomorphism j is a EMBEDDING of X into Y. As in C^*-algebras it is common practice to identify the range of j with the space X. The pair (Y, j) is a COMPACTIFICATION of X if Y is a compact Hausdorff space and $j\colon X \to Y$ is an embedding. The compactification (Y, j) is ESSENTIAL if the range of j is dense in Y.

We have discussed the smallest unitization of a C^*-algebra and the smallest compactification of a locally compact Hausdorff space. Now what about a largest, or even maximal, unital algebra containing A? Clearly there is no such thing, for if B is a unital algebra containing A, then so is $B \oplus C$ where C is any unital C^*-algebra. Similarly, there is no largest compact space containing X: if Y is a compact space containing X, then so is the topological disjoint union $Y \uplus K$ where K is any nonempty compact space. However, it does make sense to ask whether there is a maximal essential unitization of a C^*-algebra or a maximal essential compactification of a locally compact Hausdorff space. The answer is *yes* in both cases. The well-known *Stone-Čech compactification* $\beta(X)$ is maximal among essential compactifications of a noncompact locally compact Hausdorff space X. Details can be found in any good topology text. One readable standard treatment is [**30**], items 19.3–19.12. More sophisticated approaches make use of some functional analysis—see, for example, [**5**], chapter V, section 6. There turns out also to be a maximal essential unitization of a nonunital C^*-algebra A—it is called the *multiplier algebra* of A.

We say that an essential unitization M of a C^*-algebra A is *maximal* if any C^*-algebra that contains A as an essential ideal embeds in M. Here is a more formal statement.

9.3.4. Definition. An essential unitization (M, j) of a C^*-algebra A is said to be MAXIMAL if for every embedding $\iota\colon A \to B$ whose range is an essential ideal in B there exists a $*$-homomorphism $\phi\colon B \to M$ such that $\phi \circ \iota = j$.

9.3.5. Proposition. *In the preceding definition the $*$-homomorphism ϕ, if it exists must be injective.*

9.3.6. Proposition. *In the preceding definition the $*$-homomorphism ϕ, if it exists must be unique.*

Compare the following definition with 8.2.1.

9.3.7. Definition. Let A and B be C^*-algebras and V be a Hilbert A-module. A $*$-homomorphism $\phi\colon B \to \mathfrak{L}(V)$ is NONDEGENERATE if $\phi^{\to}(B)\,V$ is dense in V.

9.3.8. Proposition. *Let A, B, and J be C^*-algebras, V be a Hilbert B-module, and $\iota\colon J \to A$ be an injective $*$-homomorphism whose range is an ideal in A. If $\phi\colon J \to \mathfrak{L}(V)$ is a nondegenerate $*$-homomorphism, then there exists a unique extension of ϕ to a $*$-homomorphism $\overline{\phi}\colon A \to \mathfrak{L}(V)$ which satisfies $\overline{\phi} \circ \iota = \phi$.*

9.3.9. Proposition. *If A is a nonzero C^*-algebra, then $(\mathfrak{L}(A), L)$ is a maximal essential unitization of A. It is unique in the sense that if (M, j) is another maximal essential unitization of A, then there exists a $*$-isomorphism $\phi\colon M \to \mathfrak{L}(A)$ such that $\phi \circ j = L$.*

9.3.10. Definition. Let A be a C^*-algebra. We define the MULTIPLIER ALGEBRA of A, to be the family $\mathfrak{L}(A)$ of adjointable operators on A. From now on we denote this family by $M(A)$.

CHAPTER 10

FREDHOLM THEORY

10.1. The Fredholm Alternative

In 1903 Erik Ivar Fredholm published a seminal paper on integral equations in the journal *Acta Mathematica*. Among many important results was the theorem we know today as the *Fredholm alternative*. We state a version of this result in the language available to Fredholm at the time.

10.1.1. Proposition (Fredholm Alternative I)**.** *Let k be a continuous complex valued function on the unit square $[0,1]\times[0,1]$.* ***Either*** *the nonhomogeneous equations*

$$\lambda f(s) - \int_0^1 k(s,t)f(t)\,dt = g(s) \text{ and} \tag{1}$$

$$\overline{\lambda}h(s) - \int_0^1 \overline{k(t,s)}h(t)\,dt = j(s) \tag{2}$$

have solutions f and h for every given g and j, respectively, the solutions being unique, in which case the corresponding homogeneous equations

$$\lambda f(s) - \int_0^1 k(s,t)f(t)\,dt = 0 \text{ and} \tag{3}$$

$$\overline{\lambda}h(s) - \int_0^1 \overline{k(t,s)}h(t)\,dt = 0 \tag{4}$$

*have only the trivial solution; —**or else**—*
the homogeneous equations (3) and (4) have the same (nonzero) finite number of linearly independent solutions $f_1,\dots,f_n$ and $h_1,\dots,h_n$, respectively, in which case the nonhomogeneous equations (1) and (2) have a solution if and only if g and j satisfy

$$\int_0^1 h_k(t)\overline{g(t)}\,dt = 0 \text{ and} \tag{5}$$

$$\int_0^1 j(t)\overline{f_k(t)}\,dt = 0 \tag{6}$$

for $k=1,\dots,n$.

By 1906 David Hilbert had noticed that integration as such had very little to do with the correctness of this result. What was important, he discovered, was the compactness of the resulting integral operator. (In the early 20^{th} century *compactness* was called *complete continuity*. The term *Hilbert space* was used as early as 1911 in connections with the sequence space l_2. It was not until 1929 that John von Neumann introduced the notion of—and axioms defining—*abstract Hilbert spaces*.) So here is a somewhat updated version of *Fredholm alternative*.

10.1.2. Proposition (Fredholm Alternative II)**.** *If K is a compact Hilbert space operator, $\lambda\in\mathbb{C}$, and $T=\lambda I-K$, then* ***either*** *the nonhomogeneous equations*

$$Tf = g \text{ and} \tag{1'}$$

$$T^*h = j \tag{2'}$$

have solutions f and h for every given g and j, respectively, the solutions being unique, in which case the corresponding homogeneous equations

$$Tf = 0 \text{ and} \tag{3'}$$

$$T^*h = 0 \tag{4'}$$

*have only the trivial solution; —**or else**—*
the homogeneous equations (3') and (4') have the same (nonzero) finite number of linearly independent solutions $f_1, \dots, f_n$ and $h_1, \dots, h_n$, respectively, in which case the nonhomogeneous equations (1') and (2') have a solution if and only if g and j satisfy

$$h_k \perp g \text{ and} \tag{5'}$$

$$j \perp f_k \tag{6'}$$

for $k = 1, \dots, n$.

Notice that by making use of a few elementary facts concerning kernels and ranges of operators and orthogonality in Hilbert spaces, we can compress the statement of 10.1.2 quite a bit.

10.1.3. Proposition (Fredholm Alternative IIIa)**.** *If $T = \lambda I - K$ where K is a compact Hilbert space operator and $\lambda \in \mathbb{C}$, then*

(1) *T is injective if and only if it is surjective,*
(2) $\operatorname{ran} T^* = (\ker T)^\perp$*, and*
(3) $\dim \ker T = \dim \ker T^*$*.*

Also, conditions (1) and (2) hold for T^* as well as T.

10.2. The Fredholm Alternative – continued

10.2.1. Definition. An operator T on a Banach space is a RIESZ-SCHAUDER operator if it can be written in the form $T = S + K$ where S is invertible, K is compact, and $SK = KS$.

The material in section 7.1 and the preceding definition make it possible to generalize the version of the *Fredholm alternative* given in 10.1.3 to Banach spaces.

10.2.2. Proposition (Fredholm Alternative IIIb)**.** *If T is a Riesz-Schauder operator on a Banach space, then*

(1) *T is injective if and only if it is surjective,*
(2) $\operatorname{ran} T^* = (\ker T)^\perp$*, and*
(3) $\dim \ker T = \dim \ker T^* < \infty$*.*

Also, conditions (1) and (2) hold for T^* as well as T.

10.2.3. Proposition. *If M is a closed subspace of a Banach space, then $M^\perp \cong (B/M)^*$.*

PROOF. See [5], page 89.

10.2.4. Definition. If $T\colon V \to W$ is a linear map between vector spaces, then its COKERNEL is defined by

$$\operatorname{coker} T = W / \operatorname{ran} T\,.$$

Recall that the CODIMENSION of a subspace U of a vector space V is the dimension of V/U. Thus when T is a linear map $\dim \operatorname{coker} T = \operatorname{codim} \operatorname{ran} T$.

In the category **HIL** of Hilbert spaces and continuous linear maps the range of a morphism need not be an object of the category. Specifically the range of an operator need not be closed.

10.2.5. Example. The Hilbert space operator

$$T\colon l_2 \to l_2\colon (x_1, x_2, x_3, \dots) \mapsto (x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \dots)$$

is injective, self-adjoint, compact, and contractive, but its range, while dense in l_2, is not all of l_2.

Although it is incidental to our present purposes this is a convenient place to note the fact that the sum of two subspaces of a Hilbert space need not be a subspace.

10.2.6. Example. Let T be the operator on the Hilbert space l_2 defined in example 10.2.5, $M = l_2 \oplus \{\mathbf{0}\}$, and N be the graph of T. Then M and N are both (closed) subspaces of the Hilbert space $l_2 \oplus l_2$ but $M + N$ is not.

Proof. Verification of example 10.2.6 follows easily from the following result and example 10.2.5.

10.2.7. Proposition. *Let H be a Hilbert space, $T \in \mathfrak{B}(H)$, $M = H \oplus \{\mathbf{0}\}$, and N be the graph of T. Then*

(a) *The set N is a subspace of $H \oplus H$.*
(b) *The operator T is injective if and only if $M \cap N = \{ (\mathbf{0}, \mathbf{0}) \}$.*
(c) *The range of T is dense in H if and only if $M + N$ is dense in $H \oplus H$.*
(d) *The operator T is surjective if and only if $M + N = H \oplus H$.*

The good news for the theory of Fredholm operators is that operators with finite dimensional cokernels automatically have closed range.

10.2.8. Proposition. *If a bounded linear map $A \colon H \to K$ between Hilbert spaces has finite dimensional cokernel, then* $\operatorname{ran} A$ *is closed in K.*

We observe in *Fredholm alternative IIIb* 10.2.2 that condition (1) is redundant and also that (2) holds for *any* Banach space operator with closed range. This enables us to rephrase 10.2.2 more economically.

10.2.9. Proposition (Fredholm alternative IV)**.** *If T is a Riesz-Schauder operator on a Banach space, then*

(1) *T has closed range and*
(2) $\dim \ker T = \dim \ker T^* < \infty$.

10.3. Fredholm Operators

10.3.1. Definition. Let H be a Hilbert space and $\mathfrak{K}(H)$ be the ideal of compact operators in the C^*-algebra $\mathfrak{B}(H)$. Then the quotient algebra (see proposition 6.6.7) $\mathfrak{Q}(H) := \mathfrak{B}(H)/\mathfrak{K}(H)$ is the Calkin algebra. As usual the quotient map taking $\mathfrak{B}(H)$ onto $\mathfrak{B}(H)/\mathfrak{K}(H)$ is denoted by π so that if $T \in \mathfrak{B}(H)$ then $\pi(T) = [T] = T + \mathfrak{K}(H)$ is its corresponding element in the Calkin algebra. An element $T \in \mathfrak{B}(H)$ is a Fredholm operator if $\pi(T)$ is invertible in $\mathfrak{Q}(H)$. We denote the family of all Fredholm operators on H by $\mathfrak{F}(H)$.

10.3.2. Proposition. *If H is a Hilbert space, then $\mathfrak{F}(H)$ is a self-adjoint open subset of $\mathfrak{B}(H)$ which is closed under compact perturbations (that is, if T is Fredholm and K is compact, then $T+K$ is Fredholm).*

10.3.3. Theorem (Atkinson's theorem)**.** *A Hilbert space operator is Fredholm if and only if it has finite dimensional kernel and cokernel.*

Proof. See [1], theorem 3.3.2; [3], theorem I.8.3.6; [9], theorem 5.17; [16], theorem 2.1.4; or [29], theorem 14.1.1.

Interestingly the dimensions of the kernel and cokernel of a Fredholm operator are not very important quantities. Their difference however turns out to be very important.

10.3.4. Definition. If T is a Fredholm operator on a Hilbert space then its Fredholm index (or just index) is defined by

$$\operatorname{ind} T := \dim \ker T - \dim \operatorname{coker} T\,.$$

10.3.5. Example. Every invertible Hilbert space operator is Fredholm with index zero.

10.3.6. Example. The Fredholm index of the unilateral shift operator is -1.

10.3.7. Example. Every linear map $T\colon V \to W$ between finite dimensional vector spaces is Fredholm and $\operatorname{ind} T = \dim V - \dim W$.

10.3.8. Example. If T is a Fredholm operator on a Hilbert space, then $\operatorname{ind} T^* = -\operatorname{ind} T$.

10.3.9. Example. The index of any normal Fredholm operator is 0.

10.3.10. Lemma. *Let $S\colon U \to V$ and $T\colon V \to W$ be linear transformations between vector spaces. Then (there exist linear mappings such that) the following sequence is exact.*

$$\mathbf{0} \longrightarrow \ker S \longrightarrow \ker TS \longrightarrow \ker T \longrightarrow \operatorname{coker} S \longrightarrow \operatorname{coker} TS \longrightarrow \operatorname{coker} T \longrightarrow \mathbf{0}\,.$$

10.3.11. Lemma. *If $V_0, V_1, \dots, V_n$ are finite dimensional vector spaces and the sequence*

$$\mathbf{0} \longrightarrow V_0 \longrightarrow V_1 \longrightarrow \dots \longrightarrow V_n \longrightarrow \mathbf{0}$$

is exact, then

$$\sum_{k=0}^{n} (-1)^k \dim V_k = 0\,.$$

10.3.12. Proposition. *Let H be a Hilbert space. Then the set $\mathfrak{F}(H)$ of Fredholm operators on H is a semigroup under composition and the index function* ind *is an epimorphism from $\mathfrak{F}(H)$ onto the additive semigroup $\mathbb{Z}$ of integers.*

Proof. *Hint.* Use 10.3.10, 10.3.11, 10.3.5, 10.3.6, and 10.3.8.

10.4. The Fredholm Alternative – Concluded

The next result implies that every Fredholm operator of index zero is of the form invertible plus compact.

10.4.1. Proposition. *If T is a Fredholm operator of index zero on a Hilbert space, then there exists a finite rank partial isometry V such that $T - V$ is invertible.*

Proof. See [23], lemma 3.3.14 or [29], proposition 14.1.3.

10.4.2. Lemma. *If F is a finite rank operator on a Hilbert space, then $I + F$ is Fredholm with index zero.*

Proof. See [23], lemma 3.3.13 or [29], lemma 14.1.4.

10.4.3. Notation. Let H be a Hilbert space. For each integer n we denote the family of all Fredholm operators of index n on H by $\mathfrak{F}_n(H)$ or just $\mathfrak{F}_n$.

10.4.4. Proposition. *In every Hilbert space $\mathfrak{F}_0 + \mathfrak{K} = \mathfrak{F}_0$.*

Proof. See [9], lemma 5.20 or [29], 14.1.5.

10.4.5. Corollary (Fredholm alternative V)**.** *Every Riesz-Schauder operator on a Hilbert space is Fredholm of index zero.*

Proof. This follows immediately from the preceding proposition 10.4.4 and example 10.3.5. □

We have actually proved a stronger result: our final (quite general) version of the *Fredholm alternative*.

10.4.6. Corollary (Fredholm alternative VI)**.** *A Hilbert space operator is Riesz-Schauder if and only if it is Fredholm of index zero.*

Proof. Proposition 10.4.1 and corollary 10.4.5. □

10.4.7. Proposition. *If T is a Fredholm operator of index $n \in \mathbb{Z}$ on a Hilbert space H and K is a compact operator on H, then $T + K$ is also Fredholm of index n; that is*

$$\mathfrak{F}_n + \mathfrak{K} = \mathfrak{F}_n \,.$$

PROOF. See [16], proposition 2.1.6; [23], theorem 3.3.17; or [29], proposition 14.1.6.

10.4.8. Proposition. *Let H be a Hilbert space. Then $\mathfrak{F}_n(H)$ is an open subset of $\mathfrak{B}(H)$ for each integer n.*

PROOF. See [29], proposition 14.1.8.

10.4.9. Definition. A PATH in a topological space X is a continuous map from the interval $[0, 1]$ into X. Two points p and q in X are said to be CONNECTED BY A PATH (or HOMOTOPIC IN X) if there exists a path $f : [0, 1] \to X$ in X such that $f(0) = p$ and $f(1) = q$. In this case we write $p \sim_h q$.

10.4.10. Proposition. *The relation $\sim_h$ of homotopy equivalence defined above is an equivalence relation on the set of points of a topological space.*

10.4.11. Definition. If X is a topological space and $\sim_h$ is the relation of homotopy equivalence, then the resulting equivalence classes are the PATH COMPONENTS of X.

The next proposition identifies the path components of the set of Fredholm operators as the sets $\mathfrak{F}_n$ of operators with index n.

10.4.12. Proposition. *Operators S and T in the space $\mathfrak{F}(H)$ of Fredholm operators on a Hilbert space H are homotopic in $\mathfrak{F}(H)$ if and only if they have the same index.*

PROOF. See [29], corollary 14.1.9.

CHAPTER 11

EXTENSIONS

11.1. Essentially Normal Operators

11.1.1. Definition. Let T be an operator on a Hilbert space H. The ESSENTIAL SPECTRUM of T, denoted by $\sigma_e(T)$, is the spectrum of the image of T in the Calkin algebra $\mathfrak{Q}(H)$; that is,

$$\sigma_e(T) = \sigma_{\mathfrak{Q}(H)}(\pi(T))\,.$$

11.1.2. Proposition. *If T is an operator on a Hilbert space H, then*

$$\sigma_e(T) = \{\lambda \in \mathbb{C} \colon T - \lambda I \notin \mathfrak{F}(H)\}\,.$$

11.1.3. Proposition. *The essential spectrum of a self-adjoint Hilbert space operator T is the union of the accumulation points of the spectrum of T with the eigenvalues of T having infinite multiplicity. The members of $\sigma(T) \setminus \sigma_e(T)$ are the isolated eigenvalues of finite multiplicity.*

PROOF. See [16], proposition 2.2.2.

Here are two theorems from classical functional analysis.

11.1.4. Theorem (Weyl)**.** *If S and T are operators on a Hilbert space whose difference is compact, then their spectra agree except perhaps for eigenvalues.*

11.1.5. Theorem (Weyl-von Neumann)**.** *Let T be a self-adjoint operator on a separable Hilbert space H. For every $\epsilon > 0$ there exists a diagonalizable operator D such that $T - D$ is compact and $\|T - D\| < \epsilon$.*

PROOF. See [6], 38.1; [7], corollary II.4.2; or [16], 2.2.5.

11.1.6. Definition. Let H and K be Hilbert spaces. Operators $S \in \mathfrak{B}(H)$ and $T \in \mathfrak{B}(K)$ are ESSENTIALLY UNITARILY EQUIVALENT (or COMPALENT) if there exists a unitary map $U \colon H \to K$ such that $S - UTU^*$ is a compact operator on H. (We extend definitions 1.2.35 and 1.2.36 in the obvious way: $U \in \mathfrak{B}(H, K)$ is UNITARY if $U^*U = I_H$ and $UU^* = I_K$; and S and T are UNITARILY EQUIVALENT if there exists a unitary map $U \colon H \to K$ such that $S = UTU^*$.)

11.1.7. Proposition. *Self-adjoint operators S and T on separable Hilbert spaces are essentially unitarily equivalent if and only if they have the same essential spectrum.*

PROOF. See [16], proposition 2.2.4.

11.1.8. Definition. A Hilbert space operator T is ESSENTIALLY NORMAL if its *commutator* $[T, T^*] := TT^* - T^*T$ is compact. That is to say: T is essentially normal if its image $\pi(T)$ is a normal element of the Calkin algebra. The operator T is ESSENTIALLY SELF-ADJOINT if $T - T^*$ is compact; that is, if $\pi(T)$ is self-adjoint in the Calkin algebra.

11.1.9. Example. The unilateral shift operator S (see example 2.2.15) is essentially normal (but not normal).

11.2. Toeplitz Operators

11.2.1. Definition. Let $\zeta\colon \mathbb{T} \to \mathbb{T}\colon z \mapsto z$ be the identity function on the unit circle. Then $\{z^n\colon n \in \mathbb{Z}\}$ is an orthonormal basis for the Hilbert space $L_2(\mathbb{T})$ of (equivalence classes of) functions square integrable on $\mathbb{T}$ with respect to (suitably normalized) arc length measure. We denote by H^2 the subspace of $L_2(\mathbb{T})$ which is the closed linear span of $\{\zeta^n\colon n \geq 0\}$ and by P_+ the (orthogonal) projection on $L_2(\mathbb{T})$ whose range (and codomain) is H^2. The space H^2 is an example of a *Hardy space*. For every $\phi \in L_\infty(\mathbb{T})$ we define a mapping T_ϕ from the Hilbert space H^2 into itself by $T_\phi = P_+M_\phi$ (where M_ϕ is the multiplication operator defined in example 5.1.8). Such an operator is a TOEPLITZ OPERATOR and ϕ is its SYMBOL. Clearly T_ϕ is an operator on H^2 and $\|T_\phi\| \leq \|\phi\|_\infty$.

11.2.2. Example. The Toeplitz operator T_ζ acts on the basis vectors ζ^n of H^2 by $T_\zeta(\zeta^n) = \zeta^{n+1}$, so it is unitarily equivalent to S the unilateral shift.

11.2.3. Proposition. *The map $T\colon L_\infty(\mathbb{T}) \to \mathfrak{B}(H^2)\colon \phi \mapsto T_\phi$ is positive, linear, and involution preserving.*

11.2.4. Example. The Toeplitz operators T_ζ and $T_{\overline{\zeta}}$ show that the map T in the preceding proposition is *not* a representation of $L_\infty(\mathbb{T})$ on $\mathfrak{B}(H^2)$. (Compare this with example 8.2.4.)

11.2.5. Notation. Let $H^\infty := H^2 \cap L_\infty(\mathbb{T})$. This is another example of a *Hardy space*.

11.2.6. Proposition. *An essentially bounded function ϕ on the unit circle belongs to H^∞ if and only if the multiplication operator M_ϕ maps H^2 into H^2.*

Although (as we saw in example 11.2.4) the map T defined in proposition 11.2.3 is not in general multiplicative, it is multiplicative for a large class of functions.

11.2.7. Proposition. *If $\phi \in L_\infty(\mathbb{T})$ and $\psi \in H^\infty$, then $T_{\phi\psi} = T_\phi T_\psi$ and $T_{\overline{\psi}\phi} = T_{\overline{\psi}} T_\phi$.*

11.2.8. Proposition. *If the Toeplitz operator T_ϕ with symbol $\phi \in L_\infty(\mathbb{T})$ is invertible then the function ϕ is invertible.*

PROOF. See [9], proposition 7.6.

11.2.9. Theorem (Hartman-Wintner Spectral Inclusion Theorem)**.** *If $\phi \in L_\infty(\mathbb{T})$, then $\sigma(\phi) \subseteq \sigma(T_\phi)$ and $\rho(T_\phi) = \|T_\phi\| = \|\phi\|_\infty$.*

PROOF. See [9], corollary 7.7 or [20], theorem 3.5.7.

11.2.10. Corollary. *The mapping $T\colon L_\infty(\mathbb{T}) \to \mathfrak{B}(H^2)$ defined in proposition 11.2.3 is an isometry.*

11.2.11. Proposition. *A Toeplitz operator T_ϕ with symbol $\phi \in L_\infty(\mathbb{T})$ is compact if and only if $\phi = \mathbf{0}$.*

PROOF. See [20], theorem 3.5.8.

11.2.12. Proposition. *If $\phi \in \mathcal{C}(\mathbb{T})$ and $\psi \in L_\infty(\mathbb{T})$, then the semi-commutators $T_\phi T_\psi - T_{\phi\psi}$ and $T_\psi T_\phi - T_{\psi\phi}$ are compact.*

PROOF. See [1], proposition 4.3.1; [7], corollary V.1.4; or [20], lemma 3.5.9.

11.2.13. Corollary. *Every Toeplitz operator with continuous symbol is essentially normal.*

11.2.14. Definition. Suppose that H is a separable Hilbert space with basis $\{e_0, e_1, e_2, \dots\}$ and that T is an operator on H whose (infinite) matrix representation is $[t_{ij}]$. If the entries in this matrix depend only on the difference of the indices $i - j$ (that is, if each diagonal parallel to the main diagonal is a constant sequence), then the matrix is a TOEPLITZ MATRIX.

11.2.15. Proposition. *Let T be an operator on the Hilbert space H^2. The matrix representation of T with respect to the usual basis $\{\zeta^n\colon n \geq 0\}$ is a Toeplitz matrix if and only if $S^*TS = T$ (where S is the unilateral shift).*

Proof. See [1], proposition 4.2.3.

11.2.16. Proposition. *If T_ϕ is a Toeplitz operator with symbol $\phi \in L_\infty(\mathbb{T})$, then $S^*T_\phi S = T_\phi$. Conversely, if R is an operator on the Hilbert space H^2 such that $S^*RS = R$, then there exists a unique $\phi \in L_\infty(\mathbb{T})$ such that $R = T_\phi$.*

Proof. See [1], theorem 4.2.4.

11.2.17. Definition. The Toeplitz algebra $\mathfrak{T}$ is the C^*-subalgebra of $\mathfrak{B}(H^2)$ generated by the unilateral shift operator; that is, $\mathfrak{T} = C^*(S)$.

11.2.18. Proposition. *The set $\mathfrak{K}(H^2)$ of compact operators on H^2 is an ideal in the Toeplitz algebra $\mathfrak{T}$.*

11.2.19. Proposition. *The Toeplitz algebra comprises all compact perturbations of Toeplitz operators with continuous symbol. That is,*

$$\mathfrak{T} = \{T_\phi + K \colon \phi \in \mathcal{C}(\mathbb{T}) \text{ and } K \in \mathfrak{K}(H^2)\}\,.$$

Furthermore, if $T_\phi + K = T_\psi + L$ where $\phi,\ \psi \in \mathcal{C}(\mathbb{T})$ and $K,\ L \in \mathfrak{K}(H^2)$, then $\phi = \psi$ and $K = L$.

Proof. See [1], theorem 4.3.2 or [7], theorem V.1.5.

11.2.20. Proposition. *The map $\pi \circ T \colon \mathcal{C}(\mathbb{T}) \to Q(H^2) \colon \phi \mapsto \pi(T_\phi)$ is a unital $*$-monomorphism. So the map $\alpha \colon \mathcal{C}(\mathbb{T}) \to \mathfrak{T}/\mathfrak{K}(H^2) \colon \phi \mapsto \pi(T_\phi)$ establishes and isomorphism between the C^*-algebras $\mathcal{C}(\mathbb{T})$ and $\mathfrak{T}/\mathfrak{K}(H^2)$.*

Proof. See [16], proposition 2.3.3 or [20], theorem 3.5.11.

11.2.21. Corollary. *If $\phi \in \mathcal{C}(\mathbb{T})$, then $\sigma_e(T_\phi) = \operatorname{ran}\phi$.*

11.2.22. Proposition. *The sequence*

$$\mathbf{0} \longrightarrow \mathfrak{K}(H^2) \longrightarrow \mathfrak{T} \xrightarrow{\ \beta\ } \mathcal{C}(\mathbb{T}) \longrightarrow \mathbf{0}$$

is exact. It does not split.

Proof. The map β is defined by $\beta(R) := \alpha^{-1}\big(\pi(R)\big)$ for every $R \in \mathfrak{T}$ (where α is the isomorphism defined in the preceding proposition 11.2.20). See [1], remark 4.3.3; [7], theorem V.1.5; or [16], page 35.

11.2.23. Definition. The short exact sequence in the preceding proposition 11.2.22 is the Toeplitz extension of $\mathcal{C}(\mathbb{T})$ by $\mathfrak{K}(H^2)$.

11.2.24. Remark. A version of the diagram for the Toeplitz extension which appears frequently looks something like

$$\begin{tikzcd} \mathbf{0} \arrow[r] & \mathfrak{K}(H^2) \arrow[r] & \mathfrak{T} \arrow[r, shift left, "\beta"] & \mathcal{C}(\mathbb{T}) \arrow[l, shift left, "T"] \arrow[r] & \mathbf{0} \end{tikzcd} \tag{1}$$

where β is as in 11.2.22 and T is the mapping $\phi \mapsto T_\phi$. It is possible to misinterpret this diagram. It may suggest to the unwary that this is a split extension especially in as much as it is certainly true that $\beta \circ T = I_{\mathcal{C}(\mathbb{T})}$. The trouble, of course, is that this is not a diagram in the category **CSA** of C^*-algebras and $*$-homomorphisms. We have already seen in example 11.2.4 that the mapping T is not a $*$-homomorphism since it does not always preserve multiplication. Invertible elements in $\mathcal{C}(\mathbb{T})$ need not lift to invertible elements in the Toeplitz algebra $\mathfrak{T}$; so the $*$-epimorphism β does not have a right inverse in the category **CSA**.

Some authors deal with the problem by saying that the sequence (1) is semisplit (see, for example, Arveson [1], page 112). Others borrow a term from category theory where the word "section" means "right invertible". Davidson [7], for example, on page 134, refers to the mapping β as a "continuous section" and Douglas [9], on page 179, says it is an "isometrical cross section". While it is surely true that β has T as a right inverse in the category **SET** of sets and maps and even in the category of Banach spaces and bounded linear maps, it has no right inverse in **CSA**.

11.2.25. Proposition. *If ϕ is a function in $\mathcal{C}(\mathbb{T})$, then T_ϕ is Fredholm if and only if it is never zero.*

Proof. See [1], page 112, corollary 1; [7], theorem V.1.6; [9], theorem 7.2 6; or [20], corollary 3.5.12.

11.2.26. Proposition. *If ϕ is an invertible element of $\mathcal{C}(\mathbb{T})$, then there exists a unique integer n such that $\phi = \zeta^n \exp\psi$ for some $\psi \in \mathcal{C}(\mathbb{T})$.*

Proof. See [1], propositions 4.4.1 and 4.4.2 or [20], lemma 3.5.14.

11.2.27. Definition. The integer whose existence is asserted in the preceding proposition 11.2.26 is the winding number of the invertible function $\phi \in \mathcal{C}(\mathbb{T})$. It is denoted by $w(\phi)$.

11.2.28. Theorem (Toeplitz index theorem)**.** *If T_ϕ is a Toeplitz operator with a nowhere vanishing continuous symbol, then it is a Fredholm operator and*

$$\operatorname{ind}(T_\phi) = -w(\phi)\,.$$

Proof. Elementary proofs can be found in [1], theorem 4.4.3 and [20]. For those with a little background in homotopy of curves proposition 11.2.26, which leads to the definition above of *winding number*, can be bypassed. It is an elementary fact in homotopy theory that the fundamental group $\pi^1(\mathbb{C}\setminus 0)$ of the punctured plane is infinite cyclic. There is an isomorphism τ from $\pi^1(\mathbb{C}\setminus 0)$ to $\mathbb{Z}$ that associates the integer $+1$ with (the equivalence class containing) the function ζ. This allows us to associate with each invertible member ϕ of $\mathcal{C}(\mathbb{T})$ the integer corresponding under τ to its equivalence class. We call this integer the *winding number* of ϕ. Such a definition makes it possible to give a shorter more elegant proof of the *Toeplitz index theorem*. For a treatment in this vein see [9], theorem 7.26 or [16], theorem 2.3.2. For background concerning the fundamental group see [19], chapter two; [28], appendix A; or [30], sections 32 and 33.

11.2.29. Theorem (Wold decomposition)**.** *Every proper (that is, non-unitary) isometry on a Hilbert space is a direct sum of copies of the unilateral shift or else a direct sum of a unitary operator together with copies of the shift.*

Proof. See [7], theorem V.2.1; [14], problem (and solution) 149; or [20], theorem 3.5.17.

11.2.30. Theorem (Coburn)**.** *Let v be an isometry in a unital C^*-algebra A. Then there exists a unique unital $*$-homomorphism τ from the Toeplitz algebra $\mathfrak{T}$ to A such that $\tau(T_\zeta) = v$. Furthermore, if $vv^* \neq \mathbf{1}$, then τ is an isometry.*

Proof. See [20], theorem 3.5.18 or [7], theorem V.2.2.

11.3. Addition of Extensions

From now on all Hilbert spaces will be separable and infinite dimensional.

11.3.1. Proposition. *A Hilbert space operator T is essentially self-adjoint if and only if it is a compact perturbation of a self-adjoint operator.*

11.3.2. Proposition. *Two essentially self-adjoint Hilbert space operators T_1 and T_2 are essentially unitarily equivalent if and only if they have the same essential spectrum.*

The analogous result does not hold for essentially normal operators.

11.3.3. Example. The Toeplitz operators T_ζ and T_{ζ^2} are essentially normal with the same essential spectrum, but they are not essentially unitarily equivalent.

11.3.4. Definition. We now restrict our attention to a special class of extensions. If H is a Hilbert space and A is a C^*-algebra, we say that $(\mathfrak{E}, \phi)$ is an extension of $\mathfrak{K} = \mathfrak{K}(H)$ by A if $\mathfrak{E}$ is a unital C^*-subalgebra of $\mathfrak{B}(H)$ containing $\mathfrak{K}$ and ϕ is a unital $*$-homomorphism such that the sequence

$$\mathbf{0} \longrightarrow \mathfrak{K} \xrightarrow{\ \iota\ } \mathfrak{E} \xrightarrow{\ \phi\ } A \longrightarrow \mathbf{0} \tag{11.1}$$

(where ι is the inclusion mapping) is exact. In fact, we will be concerned almost exclusively with the case where $A = \mathcal{C}(X)$ for some compact metric space X. We will say that two extensions $(\mathfrak{E}, \phi)$ and $(\mathfrak{E}', \phi')$ are EQUIVALENT if there exists an isomorphism $\psi\colon \mathfrak{E} \to \mathfrak{E}'$ which makes the following diagram commute.

$$\begin{tikzcd} \mathbf{0} \arrow[r] & \mathfrak{K} \arrow[r, "\iota"] \arrow[d, "\psi|_{\mathfrak{K}}"'] & \mathfrak{E} \arrow[r, "\phi"] \arrow[d, "\psi"] & A \arrow[r] \arrow[d, equal] & \mathbf{0} \\ \mathbf{0} \arrow[r] & \mathfrak{K} \arrow[r, "\iota"] & \mathfrak{E}' \arrow[r, "\phi'"] & A \arrow[r] & \mathbf{0} \end{tikzcd} \tag{11.2}$$

Notice that this differs slightly from the definition of *strong equivalence* given in 6.2.9. We denote the family of all equivalence classes of such extensions by $\operatorname{Ext} A$. When $A = \mathcal{C}(X)$ we write $\operatorname{Ext} X$ rather than $\operatorname{Ext} \mathcal{C}(X)$.

11.3.5. Definition. If $U\colon H_1 \to H_2$ is a unitary mapping between Hilbert spaces, then the mapping

$$\mathrm{Ad}_U\colon \mathfrak{B}(H_1) \to \mathfrak{B}(H_2)\colon T \mapsto UTU^*$$

is called CONJUGATION by U.

It is clear that Ad_U is an isomorphism between the C^*-algebras $\mathfrak{B}(H_1)$ and $\mathfrak{B}(H_2)$. In particular, if U is a unitary operator on H_1, then Ad_U is an automorphism of both $\mathfrak{K}(H_1)$ and $\mathfrak{B}(H_1)$. Furthermore, conjugations are the only automorphisms of the C^*-algebra $\mathfrak{K}(H)$.

11.3.6. Proposition. *If H is a Hilbert space and $\phi\colon \mathfrak{K}(H) \to \mathfrak{K}(H)$ is an automorphism, then $\phi = \mathrm{Ad}_U$ for some unitary operator U on H*

PROOF. See [7], lemma V.6.1.

11.3.7. Proposition. *If H is a Hilbert space and A is a C^*-algebra, then extensions $(\mathfrak{E}, \phi)$ and $(\mathfrak{E}', \phi')$ in $\operatorname{Ext} A$ are equivalent if and only if there exists a unitary operator U in $\mathfrak{B}(H)$ such that $\mathfrak{E}' = U\mathfrak{E}U^*$ and $\phi = \phi' \mathrm{Ad}_U$.*

11.3.8. Example. Suppose that T is an essentially normal operator on a Hilbert space H. Let $\mathfrak{E}_T$ be the unital C^*-algebra generated by T and $\mathfrak{K}(H)$. Since $\pi(T)$ is a normal element of the Calkin algebra, the unital C^*-algebra $\mathfrak{E}_T/\mathfrak{K}(H)$ that it generates is commutative. Thus the *abstract spectral theorem* 5.4.7 gives us a C^*-algebra isomorphism $\Psi\colon \mathcal{C}(\sigma_e(T)) \to \mathfrak{E}_T/\mathfrak{K}(H)$. Let $\phi_T = \Psi^{-1} \circ \pi\big|_{\mathfrak{E}_T}$. Then the sequence

$$\mathbf{0} \longrightarrow \mathfrak{K}(H) \xrightarrow{\ \iota\ } \mathfrak{E}_T \xrightarrow{\ \phi_T\ } \mathcal{C}(\sigma_e(T)) \longrightarrow \mathbf{0}$$

is exact. This is the EXTENSION OF $\mathfrak{K}(H)$ DETERMINED BY T.

11.3.9. Proposition. *Let T and T' be essentially normal operators on a Hilbert space H. These operators are essentially unitarily equivalent if and only if the extensions they determine are equivalent.*

11.3.10. Proposition. *If $\mathfrak{E}$ is a C^*-algebra such that $\mathfrak{K}(H) \subseteq \mathfrak{E} \subseteq \mathfrak{B}(H)$ for some Hilbert space H, X is a nonempty compact subset of $\mathbb{C}$, and $(\mathfrak{E}, \phi)$ is an extension of $\mathfrak{K}(H)$ by $\mathcal{C}(X)$, then every element of $\mathfrak{E}$ is essentially normal.*

11.3.11. Definition. If $\phi_1\colon A_1 \to B$ and $\phi_2\colon A_2 \to B$ are unital $*$-homomorphisms between C^*-algebras, then a PULLBACK OF A_1 AND A_2 ALONG ϕ_1 AND ϕ_2, denoted by (P, π_1, π_2), is a C^*-algebra P together with a pair of unital $*$-homomorphisms $\pi_1\colon P \to A_1$ and $\pi_2\colon P \to A_2$ which satisfy the following two conditions:

(i) the diagram

$$\begin{tikzcd}
P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A_2 \arrow[d, "\phi_2"] \\
A_1 \arrow[r, "\phi_1"'] & B
\end{tikzcd}$$

commutes and

(ii) if $\rho_1 \colon Q \to A_1$ and $\rho_2 \colon Q \to A_2$ are unital $*$-homomorphisms of C^*-algebras such that the diagram

$$\begin{tikzcd}
Q \arrow[r, "\rho_2"] \arrow[d, "\rho_1"'] & A_2 \arrow[d, "\phi_2"] \\
A_1 \arrow[r, "\phi_1"'] & B
\end{tikzcd}$$

commutes, then there exists a unique unital $*$-homomorphism $\Psi \colon Q \to P$ which makes the diagram

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\Psi"] \arrow[drr, "\rho_2"] \arrow[ddr, "\rho_1"'] & & \\
& P \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & A_2 \arrow[d, "\phi_2"] \\
& A_1 \arrow[r, "\phi_1"'] & B
\end{tikzcd}$$

commute.

11.3.12. Proposition. *Let H be a Hilbert space, A be a unital C^*-algebra, and $\tau \colon A \to \mathfrak{Q}(H)$ be a unital $*$-monomorphism. Then there exists (uniquely up to isomorphism) a pullback $(\mathfrak{E}, \pi_1, \pi_2)$ of A and $\mathfrak{B}(H)$ along τ and π such that $(\mathfrak{E}, \pi_2)$ is an extension of $\mathfrak{K}(H)$ by A which makes the following diagram commute.*

$$\begin{tikzcd}
\mathbf{0} \arrow[r] & \mathfrak{K}(H) \arrow[r, "\iota"] \arrow[d, equal] & \mathfrak{E} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & A \arrow[r] \arrow[d, "\tau"] & \mathbf{0} \\
\mathbf{0} \arrow[r] & \mathfrak{K}(H) \arrow[r] & \mathfrak{B}(H) \arrow[r, "\pi"] & \mathfrak{Q}(H) \arrow[r] & \mathbf{0}
\end{tikzcd} \tag{11.3}$$

Proof. (Sketch.) Let $\mathfrak{E} = \{T \oplus a \in \mathfrak{B}(H) \oplus A \colon \tau(a) = \pi(T)\}$, $\pi_1 \colon \mathfrak{E} \to \mathfrak{B}(H) \colon T \oplus a \mapsto T$, and $\pi_2 \colon \mathfrak{E} \to \mathfrak{A} \colon T \oplus a \mapsto a$. The uniqueness (of any pullback) is proved using the usual "abstract nonsense". □

11.3.13. Proposition. *Let H be a Hilbert space, A be a unital C^*-algebra, and $(\mathfrak{E}, \phi)$ be an extension of $\mathfrak{K}(H)$ by A. Then there exists a unique $*$-monomorphism $\tau \colon A \to \mathfrak{Q}(H)$ which makes the diagram (11.3) commute.*

11.3.14. Definition. Let H be a Hilbert space and A be a unital C^*-algebra. Two unital $*$-monomorphisms τ_1 and τ_2 from A into $\mathfrak{Q}(H)$ are UNITARILY EQUIVALENT if there exists a unitary operator on H such that $\tau_2 = \mathrm{Ad}_U\,\tau_1$. Unitary equivalence of unital $*$-monomorphisms is of course an equivalence relation. The equivalence class containing τ_1 is denoted by $[\tau_1]$.

11.3.15. Proposition. *Let H be a Hilbert space and A be a unital C^*-algebra. Two extensions of $\mathfrak{K}(H)$ by A are equivalent if and only if their corresponding unital $*$-monomorphisms (see 11.3.13) are unitarily equivalent.*

11.3.16. Corollary. *If H is a Hilbert space and A a unital C^*-algebra there is a one-to-one correspondence between equivalence classes of extensions of $\mathfrak{K}(H)$ by A and unitary equivalence classes of unital $*$-monomorphisms from A into the Calkin algebra $\mathfrak{Q}(H)$.*

11.3.17. Convention. In light of the preceding corollary we will regard members of $\operatorname{Ext} A$ (or $\operatorname{Ext} X$) as either equivalence classes of extensions or unitary equivalence classes of $*$-monomorphisms, whichever seems the most convenient at the moment.

11.3.18. Proposition. *Every (separable infinite dimensional) Hilbert space H is isometrically isomorphic to $H \oplus H$. Thus the C^*-algebras $\mathfrak{B}(H)$ and $\mathfrak{B}(H \oplus H)$ are isomorphic.*

11.3.19. Definition. Let $\tau_1, \tau_2 \colon A \to \mathfrak{Q}(H)$ be unital $*$-monomorphisms (where A is a C^*-algebra and H is a Hilbert space). We define a unital $*$-monomorphism $\tau_1 \oplus \tau_2 \colon A \to \mathfrak{Q}(H)$ by

$$(\tau_1 \oplus \tau_2)(a) := \rho\big(\tau_1(a) \oplus \tau_2(a)\big)$$

for all $a \in A$ where (as in Douglas[10]) ν is the isomorphism established in 11.3.18 and ρ is the map which makes the following diagram commute.

$$\begin{tikzcd}
\mathfrak{B}(H) \oplus \mathfrak{B}(H) \arrow[r] \arrow[d, "\pi\oplus\pi"'] & \mathfrak{B}(H \oplus H) \arrow[r, "\nu"] & \mathfrak{B}(H) \arrow[d, "\pi"] \\
\mathfrak{Q}(H) \oplus \mathfrak{Q}(H) \arrow[rr, "\rho"'] & & \mathfrak{Q}(H)
\end{tikzcd} \tag{11.4}$$

We then define the obvious operation of addition on $\operatorname{Ext} A$:

$$[\tau_1] + [\tau_2] := [\tau_1 \oplus \tau_2]$$

for $[\tau_1]$, $[\tau_2] \in \operatorname{Ext} A$.

11.3.20. Proposition. *The operation of addition (given in 11.3.19) on $\operatorname{Ext} A$ is well defined and under this operation $\operatorname{Ext} A$ becomes a commutative semigroup.*

11.3.21. Definition. Let A be a unital C^*-algebra and $\mathbf{r} \colon A \to \mathfrak{B}(H)$ be a nondegenerate representation of A on some Hilbert space H. Let P be a Hilbert space projection and M be the range of P. Suppose that $P\mathbf{r}(a) - \mathbf{r}(a)P \in \mathfrak{K}(H)$ for every $a \in A$. Denote by P_M the projection P with its codomain set equal to its range; that is, $P_M \colon H \to M \colon x \mapsto Px$. Then for each $a \in A$ we define the ABSTRACT TOEPLITZ OPERATOR $T_a \in \mathfrak{B}(M)$ WITH SYMBOL a ASSOCIATED WITH the pair $(\mathbf{r}, P)$ by $T_a = P_M\mathbf{r}(a)\big|_M$.

11.3.22. Definition. Let notation be as in the preceding definition 11.3.21. Then we define the ABSTRACT TOEPLITZ EXTENSION τ_P ASSOCIATED WITH THE PAIR $(\mathbf{r}, P)$ by

$$\tau_P \colon A \to \mathfrak{Q}(M) \colon a \mapsto \pi(T_a)\,.$$

Notice that the (unital $*$-monomorphism associated with the concrete) Toeplitz extension defined in 11.2.23 is an example of an abstract Toeplitz extension, and also that, in general, abstract Toeplitz extensions need not be injective.

11.3.23. Definition. Let A be a unital C^*-algebra and H a Hilbert space. In the spirit of [16], definition 2.7.6, we will say that a unital $*$-monomorphism $\tau \colon A \to \mathfrak{Q}(H)$ is SEMISPLIT if there exists a unital $*$-monomorphism $\tau' \colon A \to \mathfrak{Q}(H)$ such that $\tau \oplus \tau'$ splits.

11.3.24. Proposition. *Suppose that A is a unital C^*-algebra and H is a Hilbert space. Then a unital $*$-monomorphism $\tau\colon A \to \mathfrak{Q}(H)$ is semisplit if and only if it is unitarily equivalent to an abstract Toeplitz extension.*

PROOF. See [16], proposition 2.7.10.

11.4. Tensor Products of C^*-algebras

Class Project for Winter Term
Presentation based on [29], Appendix T.

11.5. Completely Positive Maps

In example 5.3.7 it is shown how the set $\mathbf{M}_n$ of $n \times n$ matrices of complex numbers can be made into a C^*-algebra. We now generalize that example to the algebra $\mathbf{M}_n(A)$ of $n \times n$ matrices of elements of a C^*-algebra A.

11.5.1. Example. In example 2.3.15 it is asserted that if A is a C^*-algebra, then under the usual algebraic operations the set $\mathbf{M}_n(A)$ of $n \times n$ matrices of elements of A is an algebra. This algebra can be made into a $*$-algebra by taking conjugate transposition as involution. That is, define

$$[a_{ij}]^* := \left[a_{j\,i}{}^*\right]$$

where $a_{ij} \in A$ for $1 \le i, j \le n$.

Let H be a Hilbert space. For the moment it need not be infinite dimensional or separable. Denote by H^n its n-fold direct sum. That is,

$$H^n := \bigoplus_{k=1}^{n} H_k$$

where $H_k = H$ for $k = 1, \dots, n$. Let $[T_{jk}] \in \mathbf{M}_n(\mathfrak{B}(H))$. For $\mathbf{x} = x_1 \oplus \cdots \oplus x_n \in H^n$ define

$$\mathbf{T}\colon H^n \to H^n\colon \mathbf{x} \mapsto [T_{jk}]\mathbf{x} = \sum_{k=1}^{n} T_{1k}x_k \oplus \cdots \oplus \sum_{k=1}^{n} T_{nk}x_k\,.$$

Then $\mathbf{T} \in \mathfrak{B}(H^n)$. Furthermore, the map

$$\Psi\colon \mathbf{M}_n\big(\mathfrak{B}(H)\big) \to \mathfrak{B}(H^n)\colon [T_{jk}] \mapsto \mathbf{T}$$

is an isomorphism of $*$-algebras. Use Ψ to transfer the operator norm on $\mathfrak{B}(H^n)$ to $\mathbf{M}_n\big(\mathfrak{B}(H)\big)$. That is, define

$$\big\|\,[T_{jk}]\,\big\| := \|\mathbf{T}\|\,.$$

This makes $\mathbf{M}_n\big(\mathfrak{B}(H)\big)$ into a C^*-algebra isomorphic to $\mathfrak{B}(H^n)$.

Now suppose that A is an arbitrary C^*-algebra. Version III of the *Gelfand-Naimark theorem* (see 8.3.8) allows us to identify A with a C^*-subalgebra of $\mathfrak{B}(H)$ for some Hilbert space H and, consequently, $\mathbf{M}_n(A)$ with a C^*-subalgebra of $\mathbf{M}_n\big(\mathfrak{B}(H)\big)$. With this identification $\mathbf{M}_n(A)$ becomes a C^*-algebra. (Notice that by corollary 5.3.15 norms on C^*-algebras are unique; so it is clear that the norm on $\mathbf{M}_n(A)$ is independent of the particular way in which A is represented as a C^*-subalgebra of operators on some Hilbert space.)

11.5.2. Example. Let $k \in \mathbb{N}$ and $A = \mathbf{M}_k$. Then for every $n \in \mathbb{N}$

$$\mathbf{M}_n(A) = \mathbf{M}_n(\mathbf{M}_k) \cong \mathbf{M}_{nk}\,.$$

The isomorphism is the obvious one: just delete the inner brackets. For example, the isomorphism from $\mathbf{M}_2(\mathbf{M}_2)$ to $\mathbf{M}_4$ is given by

$$\begin{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} & \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ \\ \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} & \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{bmatrix}$$

11.5.3. Definition. A mapping $\phi \colon A \to B$ between C^*-algebras is POSITIVE if it takes positive elements to positive elements; that is, if $\phi(a) \in B^+$ whenever $a \in A^+$.

11.5.4. Example. Every $*$-homomorphism between C^*-algebras is positive. (See proposition 6.5.19.)

11.5.5. Proposition. *Every positive linear map between C^*-algebras is bounded.*

11.5.6. Notation. Let A and B be C^*-algebras and $n \in \mathbb{N}$. A linear map $\phi \colon A \to B$ induces a linear map $\phi^{(n)} := \phi \otimes \mathrm{id}_{\mathbf{M}_n} \colon A \otimes \mathbf{M}_n \to B \otimes \mathbf{M}_n$. As we have seen $A \otimes \mathbf{M}_n$ can be identified with $\mathbf{M}_n(A)$. Thus we may think of $\phi^{(n)}$ as the map from $\mathbf{M}_n(A)$ to $\mathbf{M}_n(B)$ defined by $\phi^{(n)}\big([a_{jk}]\big) = \big[\phi(a_{jk})\big]$.

It is easy to see that if ϕ preserves multiplication, then so does each $\phi^{(n)}$ and if ϕ preserves involution, then so does each $\phi^{(n)}$. Positivity, however, is not always preserved as the next example 11.5.8 shows.

11.5.7. Definition. In $\mathbf{M}_n$ the STANDARD MATRIX UNITS are the matrices e^{jk} $(1 \le j, k \le n)$ whose entry in the j^{th} row and k^{th} column is 1 and whose other entries are all 0.

11.5.8. Example. Let $A = \mathbf{M}_2$. Then $\phi \colon A \to A \colon a \mapsto a^t$ (where a^t is the transpose of a) is a positive mapping. The map $\phi^{(2)} \colon \mathbf{M}_2(A) \to \mathbf{M}_2(A)$ is not positive. To see this let e^{11}, e^{12}, e^{21}, and e^{22} be the standard matrix units for A. Then

$$\phi^{(2)}\left(\begin{bmatrix} e^{11} & e^{12} \\ e^{21} & e^{22} \end{bmatrix}\right) = \begin{bmatrix} \phi(e^{11}) & \phi(e^{12}) \\ \phi(e^{21}) & \phi(e^{22}) \end{bmatrix} = \begin{bmatrix} e^{11} & e^{21} \\ e^{12} & e^{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is not positive.

11.5.9. Definition. A linear mapping $\phi \colon A \to B$ between unital C^*-algebras is n-POSITIVE (for $n \in \mathbb{N}$) if $\phi^{(n)}$ is positive. It is COMPLETELY POSITIVE if it is n-positive for every $n \in \mathbb{N}$.

11.5.10. Example. Every $*$-homomorphism between C^*-algebras is completely positive.

11.5.11. Proposition. *Let A be a unital C^*-algebra and $a \in A$. Then $\|a\| \le 1$ if and only if $\begin{bmatrix} \mathbf{1} & a \\ a^* & \mathbf{1} \end{bmatrix}$ is positive in $\mathbf{M}_2(A)$.*

11.5.12. Proposition. *Let A be a unital C^*-algebra and a, $b \in A$. Then $a^*a \le b$ if and only if $\begin{bmatrix} \mathbf{1} & a \\ a^* & b \end{bmatrix}$ is positive in $\mathbf{M}_2(A)$.*

11.5.13. Proposition. *Every unital 2-positive map between unital C^*-algebras is contractive.*

11.5.14. Proposition (Kadison's inequality)**.** *If $\phi \colon A \to B$ is a unital 2-positive map between unital C^*-algebras, then*

$$\big(\phi(a)\big)^* \phi(a) \le \phi(a^*a)$$

for every $a \in A$.

11.5.15. Definition. It is easy to see that if $\phi\colon A \to B$ is a bounded linear map between C^*-algebras, then $\phi^{(n)}$ is bounded for each $n \in \mathbb{N}$. If $\|\phi\|_{\mathrm{cb}} := \sup\{\|\phi^{(n)}\| \colon n \in \mathbb{N}\} < \infty$, then ϕ is COMPLETELY BOUNDED.

11.5.16. Proposition. *If $\phi\colon A \to B$ is a completely positive unital linear map between unital C^*-algebras, then ϕ is completely bounded and*

$$\|\phi(\mathbf{1})\| = \|\phi\| = \|\phi\|_{\mathrm{cb}}\,.$$

PROOF. See [22].

11.5.17. Theorem (Stinespring's dilation theorem)**.** *Let A be a unital C^*-algebra, H be a Hilbert space, and $\phi\colon A \to \mathfrak{B}(H)$ be a unital linear map. Then ϕ is completely positive if and only if there exists a Hilbert space H_0, an isometry $V\colon H \to H_0$, and a nondegenerate representation $\mathbf{r}\colon A \to \mathfrak{B}(H_0)$ such that $\phi(a) = V^*\mathbf{r}(a)V$ for every $a \in A$.*

PROOF. See [16], theorem 3.1.3 or [22], theorem 4.1.

11.5.18. Notation. Let A be a C^*-algebra, $f\colon A \to \mathfrak{B}(H_0)$, and $g\colon A \to \mathfrak{B}(H)$, where H_0 and H are Hilbert spaces. We write $g \lesssim f$ if there exists an isometry $V\colon H \to H_0$ such that $g(a) - V^*f(a)V \in \mathfrak{K}(H)$ for every $a \in A$. The relation $\lesssim$ is a preordering but not a partial ordering; that is, it is reflexive and transitive but not antisymmetric.

11.5.19. Theorem (Voiculescu)**.** *Let A be a separable unital C^*-algebra, $\mathbf{r}\colon A \to \mathfrak{B}(H_0)$ be a nondegenerate representation, and $\phi\colon A \to \mathfrak{B}(H)$ (where H and H_0 are Hilbert spaces). If $\phi(a) = \mathbf{0}$ whenever $\mathbf{r}(a) \in \mathfrak{K}(H_0)$, then $\phi \lesssim \mathbf{r}$.*

PROOF. The proof is long and complicated but "elementary". See [16], sections 3.5–3.6.

11.5.20. Proposition. *Let A be a separable unital C^*-algebra, H be a Hilbert space, and τ_1, $\tau_2\colon A \to \mathfrak{Q}(H)$ be unital $*$-monomorphisms. If τ_2 splits, then $\tau_1 + \tau_2$ is unitarily equivalent to τ_1.*

PROOF. See [16], theorem 3.4.7.

11.5.21. Corollary. *Suppose that A is a separable unital C^*-algebra, H is a Hilbert space, and $\tau\colon A \to \mathfrak{Q}(H)$ is a split unital $*$-monomorphism. Then $[\tau]$ is an additive identity in* $\operatorname{Ext} A$.

Note that a unital $*$-monomorphism $\tau\colon A \to \mathfrak{Q}(H)$ from a separable unital C^*-algebra to the Calkin algebra of a Hilbert space H is semisplit if and only if $[\tau]$ has an additive inverse in $\operatorname{Ext} A$. The next proposition says that this happens if and only if τ has a completely positive lifting to $\mathfrak{B}(H)$.

11.5.22. Proposition. *Let A be a separable unital C^*-algebra and H be a Hilbert space. A $*$-monomorphism $\tau\colon A \to \mathfrak{Q}(H)$ is semisplit if and only if there exists a unital $*$-homomorphism $\widetilde{\tau}\colon A \to \mathfrak{B}(H)$ such that the diagram*

$$\begin{tikzcd}
 & \mathfrak{B}(H) \arrow[d, "\pi"] \\
A \arrow[ur, "\widetilde{\tau}"] \arrow[r, "\tau"'] & \mathfrak{Q}(H)
\end{tikzcd}$$

commutes.

PROOF. See [16], theorem 3.1.5.

11.5.23. Definition. A C^*-algebra A is NUCLEAR if for every C^*-algebra B there is a unique C^*-norm on the algebraic tensor product $A \odot B$.

11.5.24. Example. Every finite dimensional C^*-algebra is nuclear.

Proof. See [20], theorem 6.3.9.

11.5.25. Example. The C^*-algebra $\mathbf{M}_n$ is nuclear.

Proof. See [3], II.9.4.2.

11.5.26. Example. If H is a Hilbert space, the C^*-algebra $\mathfrak{K}(H)$ of compact operators is nuclear.

Proof. See [20], example 6.3.2.

11.5.27. Example. If X is a compact Hausdorff space, then the C^*-algebra $\mathcal{C}(X)$ is nuclear.

Proof. See [22], proposition 12.9.

11.5.28. Example. Every commutative C^*-algebra is nuclear.

Proof. See [12], theorem 7.4.1 or [29], theorem T.6.20.

The conclusion of the next result is known as the *completely positive lifting property*.

11.5.29. Proposition. *Let A be a nuclear separable unital C^*-algebra and J be a separable ideal in a unital C^*-algebra B. Then for every completely positive map $\phi\colon A \to B/J$ there exists a completely positive map $\widetilde{\phi}\colon A \to B$ which makes the diagram*

$$\begin{tikzcd} & B \arrow[d] \\ A \arrow[ur, "\widetilde{\phi}"] \arrow[r, "\phi"'] & B/J \end{tikzcd}$$

commute.

Proof. See [16], theorem 3.3.6.

11.5.30. Corollary. *If A is a nuclear separable unital C^*-algebra, then* $\operatorname{Ext} A$ *is an Abelian group.*

CHAPTER 12

K-THEORY

12.1. Projections on Matrix Algebras

Recall from definition 10.4.9 that points p and q in a topological space X are HOMOTOPIC IN X if there exists a continuous function $f\colon[0,1]\to X\colon t\mapsto f_t$ such that $f_0=p$ and $f_1=q$. In this case we write $p\sim_h q$ in X (or just $p\sim_h q$ when the space X is clear from context).

12.1.1. Proposition. *The relation $\sim_h$ between points in a topological space is an equivalence relation. The equivalence classes are the* PATH COMPONENTS *of X.*

12.1.2. Notation. In a unital C^*-algebra A the path component of the space of unitary operators containing the identity is denoted by $\mathfrak{U}_0(A)$. That is, $\mathfrak{U}_0(A)=\{u\in\mathfrak{U}(A)\colon u\sim_h \mathbf{1}\ \text{ in } \mathfrak{U}(A)\}$.

12.1.3. Example. If h is a self-adjoint element of a unital C^*-algebra A, then $\exp(ih)\in\mathfrak{U}_0(A)$.

12.1.4. Proposition. *If u is a unitary element in a unital C^*-algebra A whose spectrum is not all of $\mathbb{T}$, then $u\in\mathfrak{U}_0(A)$.*

12.1.5. Proposition. *Let u_1, u_2, u_3, and u_4 be unitary elements in a unital C^*-algebra A. If $u_1\sim_h u_2$ and $u_3\sim_h u_4$, then $u_1u_3\sim_h u_2u_4$.*

12.1.6. Proposition. *If u_1 and u_2 are unitary elements in a unital C^*-algebra A such that $\|u_1-u_2\|<2$, then $u_1\sim_h u_2$.*

12.1.7. Example. If u_1 and u_2 are unitary elements in a unital C^*-algebra A, then

$$\begin{bmatrix} u_1 & \mathbf{0}\\ \mathbf{0} & u_2\end{bmatrix}\sim_h\begin{bmatrix} u_1u_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{bmatrix}$$

in $\mathfrak{U}\big(\mathbf{M}_2(A)\big)$.

12.1.8. Proposition. *If A is a unital C^*-algebra, then $\mathfrak{U}_0(A)$ is a normal subgroup of $\mathfrak{U}(A)$ which is both open and closed in $\mathfrak{U}(A)$. Furthermore, $u\in\mathfrak{U}_0(A)$ if and only if $u=\exp(ih_1)\cdots\exp(ih_n)$ for some finite collection of self-adjoint elements $h_1,\dots,h_n$ in A.*

PROOF. See [27], proposition 2.1.6.

12.1.9. Proposition (Polar decomposition)**.** *If a is an invertible element of a unital C^*-algebra, then $u:=a|a|^{-1}$ (exists and) is unitary. Clearly then, $a=u|a|$.*

PROOF. See [27], proposition 2.1.8.

12.1.10. Proposition. *If p and q are projections in a unital C^*-algebra, then $\|2p-\mathbf{1}\|=1$ and $\|p-q\|\le 1$.*

12.1.11. Notation. If p and q are projections in a C^*-algebra A, we write $p\sim_u q$ (p is UNITARILY EQUIVALENT to q) if there exists a unitary element $u\in\widetilde{A}$ such that $q=upu^*$.

12.1.12. Proposition. *The relation $\sim_u$ is an equivalence relation on the family $\mathcal{P}(A)$ of projections in a C^*-algebra A.*

12.1.13. Notation. If p and q are projections in a C^*-algebra A, we write $p\sim q$ (p is MURRAY-VON NEUMANN EQUIVALENT to q) is there exists an element $v\in A$ such that $v^*v=p$ and $vv^*=q$. (Note that such a v is automatically a partial isometry.)

12.1.14. Proposition. *The relation $\sim$ of Murray-von Neumann equivalence is an equivalence relation on the family $\mathcal{P}(A)$ of projections in a C^*-algebra A.*

12.1.15. Proposition. *If p and q are projections in a C^*-algebra, then*

$$p \sim_h q \implies p \sim_u q \implies p \sim q\,.$$

12.1.16. Proposition. *If p and q are projections in a unital C^*-algebra A, then $p \sim_u q$ if and only if there exists an element $u \in \mathfrak{U}(A)$ such that $q = upu^*$.*

12.1.17. Definition. An element s of a unital C^*-algebra is an ISOMETRY if $s^*s = \mathbf{1}$.

12.1.18. Example. If p and q are projections in a C^*-algebra, then

$$p \sim q \;\not\Longrightarrow\; p \sim_u q\,.$$

For example, if s is a nonunitary isometry, then $s^*s \sim ss^*$, but $s^*s \nsim_u ss^*$.

12.1.19. Notation. If a_1, a_2, $\dots a_n$ are elements of a C^*-algebra A, then $\operatorname{diag}(a_1, a_2, \dots, a_n)$ is the diagonal matrix in $\mathbf{M}_n(A)$ whose main diagonal consists of the elements a_1, $\dots$, a_n. We also use the notations for block matrices. For example if a is an $m \times m$ matrix and b is an $n \times n$ matrix, then $\operatorname{diag}(a, b)$ is the $(m+n) \times (m+n)$ matrix $\begin{bmatrix} a & \mathbf{0} \\ \mathbf{0} & b \end{bmatrix}$.

12.1.20. Proposition. *Let p and q be projections in a C^*-algebra A. Then*

$$p \sim q \implies \operatorname{diag}(p, \mathbf{0}) \sim_u \operatorname{diag}(q, \mathbf{0}) \text{ in } \mathbf{M}_2(A).$$

12.1.21. Example. If p and q are projections in a C^*-algebra, then

$$p \sim_u q \;\not\Longrightarrow\; p \sim_h q\,.$$

An example of this phenomenon is not easy to come by. To see what is involved look at [27], examples 2.2.9 and 11.3.4.

12.1.22. Proposition. *Let p and q be projections in a C^*-algebra A. Then*

$$p \sim_u q \implies \operatorname{diag}(p, \mathbf{0}) \sim_h \operatorname{diag}(q, \mathbf{0}) \text{ in } \mathbf{M}_2(A).$$

12.1.23. Notation. When A is a C^*-algebra and $n \in \mathbb{N}$ we let

$$\mathcal{P}_n(A) = \mathcal{P}(\mathbf{M}_n(A)) \quad \text{and}$$

$$\mathcal{P}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(A).$$

We now extend Murray-von Neumann equivalence to matrices of different sizes.

12.1.24. Definition. If A is a C^*-algebra, $p \in \mathcal{P}_m(A)$, and $q \in \mathcal{P}_n(A)$, we set $p \sim_\circ q$ if there exists $v \in \mathbf{M}_{n,m}(A)$ such that

$$v^*v = p \qquad \text{and} \qquad vv^* = q.$$

Note: if $m = n$, then $\sim_\circ$ is just Murray-von Neumann equivalence $\sim$. (Here, $\mathbf{M}_{n,m}(A)$ is the set of $n \times m$ matrices with entries belonging to A.)

12.1.25. Proposition. *The relation $\sim_\circ$ defined above is an equivalence relation on $\mathcal{P}_\infty(A)$.*

12.1.26. Definition. For each C^*-algebra A we define a binary operation $\oplus$ on $\mathcal{P}_\infty(A)$ by

$$p \oplus q = \operatorname{diag}(p, q)\,.$$

Thus if $p \in \mathcal{P}_m(A)$ and $q \in \mathcal{P}_n(A)$, then $p \oplus q \in \mathcal{P}_{m+n}(A)$.

In the next proposition $\mathbf{0}_n$ is the additive identity in $\mathcal{P}_n(A)$.

12.1.27. Proposition. *Let A be a C^*-algebra and $p \in \mathcal{P}_\infty(A)$. Then $p = p \oplus \mathbf{0}_n$ for every $n \in \mathbb{N}$.*

12.1.28. Proposition. *Let A be a C^*-algebra and p, p', q, $q' \in \mathcal{P}_\infty(A)$. If $p \sim_\circ p'$ and $q \sim_\circ q'$, then $p \oplus q \sim_\circ p' \oplus q'$.*

12.1.29. Proposition. *Let A be a C^*-algebra and p, $q \in \mathcal{P}_\infty(A)$. Then $p \oplus q \sim_\circ q \oplus p$.*

12.1.30. Proposition. *Let A be a C^*-algebra and p, $q \in \mathcal{P}_n(A)$ for some n. If $p \perp q$, then $p + q$ is a projection in $\mathbf{M}_n(A)$ and $p + q \sim_\circ p \oplus q$.*

12.1.31. Proposition. *If A is a C^*-algebra, then $\mathcal{P}_\infty(A)$ is a commutative semigroup under the operation $\oplus$.*

12.1.32. Notation. If A is a C^*-algebra and $p \in \mathcal{P}_\infty(A)$, let $[p]_{\mathcal{D}}$ be the equivalence class containing p determined by the equivalence relation $\sim_\circ$. Also let $\mathcal{D}(A) := \{\, [p]_{\mathcal{D}} \colon p \in \mathcal{P}_\infty(A)\,\}$.

12.1.33. Definition. Let A be a C^*-algebra. Define a binary operation $+$ on $\mathcal{D}(A)$ by

$$[p]_{\mathcal{D}} + [q]_{\mathcal{D}} := [p \oplus q]_{\mathcal{D}}$$

where p, $q \in \mathcal{P}_\infty(A)$.

12.1.34. Proposition. *The operation $+$ defined in 12.1.33 is well defined and makes $\mathcal{D}(A)$ into a commutative semigroup.*

12.1.35. Example. Making use of properties of the trace function on $\mathbf{M}_n$ it is not difficult to see that

$$\mathcal{D}(\mathbb{C}) = \mathbb{Z}^+ = \{0, 1, 2, \dots\}\,.$$

12.1.36. Example. If H is a Hilbert space, then

$$\mathcal{D}(\mathfrak{B}(H)) = \mathbb{Z}^+ \cup \{\infty\}.$$

(Use the usual addition on $\mathbb{Z}^+$ and let $n + \infty = \infty + n = \infty$ whenever $n \in \mathbb{Z}^+ \cup \{\infty\}$.)

12.1.37. Example. For the C^*-algebra $\mathbb{C} \oplus \mathbb{C}$ we have

$$\mathcal{D}(\mathbb{C} \oplus \mathbb{C}) = \mathbb{Z}^+ \oplus \mathbb{Z}^+\,.$$

12.2. The Grothendieck Construction

12.2.1. Definition. Let $(S, +)$ be a commutative semigroup. Define a relation $\sim$ on $S \times S$ by

$$(a, b) \sim (c, d) \text{ if there exists } k \in S \text{ such that } a + d + k = b + c + k.$$

12.2.2. Proposition. *The relation $\sim$ defined above is an equivalence relation.*

12.2.3. Notation. For the equivalence relation $\sim$ defined in 12.2.1 the equivalence class containing the pair (a, b) will be denote by $\langle a, b\rangle$ rather than by $[(a, b)]$.

12.2.4. Definition. Let $(S, +)$ be a commutative semigroup. On $G(S)$ define a binary operation (also denoted by $+$) by:

$$\langle a, b\rangle + \langle c, d\rangle := \langle a + c, b + d\rangle\,.$$

12.2.5. Proposition. *The operation $+$ defined above is well defined and under this operation $G(S)$ becomes and Abelian group.*

The Abelian group $(G(S), +)$ is called the Grothendieck group of S.

12.2.6. Definition. An injective additive map $\phi \colon S \to G$ from a semigroup S into a group G is an embedding of S into G.

12.2.7. Proposition. *For a semigroup S and an arbitrary $a \in S$ define a mapping*

$$\gamma_S \colon S \to G(S) \colon s \mapsto \langle s + a, a\rangle\,.$$

The mapping γ_S, called the Grothendieck map, *is well defined and is a semigroup homomorphism. It is an embedding if and only if the semigroup S has the cancellation property.*

Saying that the Grothendieck map is well defined means that its definition is independent of the choice of a. We frequently write just γ for γ_S.

12.2.8. Example. Both $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ are commutative semigroups under addition. They have the same Grothendieck group

$$G(\mathbb{N}) = G(\mathbb{Z}^+) = \mathbb{Z}\,.$$

12.2.9. Example. Let S be the commutative additive semigroup $\mathbb{Z}^+ \cup \{\infty\}$ (see example 12.1.36). Then $G(S) = \{\mathbf{0}\}$.

12.2.10. Example. Let $\mathbb{Z}_0$ be the (commutative) multiplicative semigroup of nonzero integers. Then $G(\mathbb{Z}_0) = \mathbb{Q}_0$, the Abelian multiplicative group of nonzero rational numbers.

12.2.11. Proposition. *If S is a commutative semigroup, then*

$$G(S) = \{\gamma(x) - \gamma(y) \colon x, y \in S\}\,.$$

12.2.12. Proposition (Universality of the Grothendieck map)**.** *Let S be a commutative (additive) semigroup and $G(S)$ be its Grothendieck group. If H is an Abelian group and $\phi\colon S \to H$ is an additive map, then there exists a unique group homomorphism $\psi\colon G(S) \to H$ such that the following diagram commutes.*

$$\begin{tikzcd}
S \arrow[r, "\gamma"] \arrow[dr, "\phi"'] & \Box G(S) \arrow[d, dashed, "\Box\psi"] & G(S) \arrow[d, dashed, "\psi"] \\
& \Box H & H
\end{tikzcd} \tag{12.1}$$

In the preceding diagram $\Box G$ and $\Box H$ are just G and H regarded as semigroups and $\Box\psi$ is the corresponding semigroup homomorphism. In other words, the forgetful functor $\Box$ "forgets" only about identities and inverses but not about the operation of addition. Thus the triangle on the left is a commutative diagram in the category of semigroups and semigroup homomorphisms.

12.2.13. Proposition. *Let $\phi\colon S \to T$ be a homomorphism of commutative semigroups. Then the map $\gamma_T \circ \phi\colon S \to G(T)$ is additive. By proposition 12.2.12 there exists a unique group homomorphism $G(\phi)\colon G(S) \to G(T)$ such that the following diagram commutes.*

$$\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[d, "\gamma_S"'] & T \arrow[d, "\gamma_T"] \\
G(S) \arrow[r, "G(\phi)"'] & G(T)
\end{tikzcd}$$

12.2.14. Proposition (Functorial property of the Grothendieck construction)**.** *The pair of maps $S \mapsto G(S)$, which takes semigroups to their corresponding Grothendieck groups, and $\phi \mapsto G(\phi)$, which takes semigroup homomorphisms to group homomorphism (as defined in 12.2.13) is a covariant functor from the category of commutative semigroups and semigroup homomorphisms to the category of Abelian groups and group homomorphisms.*

One slight advantage of the rather pedantic inclusion of a forgetful functor in diagram (12.1) is that it makes it possible to regard the Grothendieck map $\gamma\colon S \mapsto \gamma_S$ as a natural transformation of functors.

12.2.15. Corollary (Naturality of the Grothendieck map)**.** *Let $\Box$ be the forgetful functor on Abelian groups which "forgets" about identities and inverses but not the group operation as in 12.2.12. Then $\Box G$ is a covariant functor from the category of semigroups and semigroup homomorphisms to* *itself. Furthermore, the Grothendieck map $\gamma\colon S \mapsto \gamma_S$ is a natural transformation from the identity functor to the functor $\Box G$.*

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "\gamma_S"'] & T \arrow[d, "\gamma_T"] \\ \Box G(S) \arrow[r, "\Box G(\phi)"'] & \Box G(T) \end{tikzcd}$$

12.3. $K_0(A)$—the Unital Case

12.3.1. Definition. Let A be a unital C^*-algebra. Let $K_0(A) := G(\mathcal{D}(A))$, the Grothendieck group of the semigroup $\mathcal{D}(A)$ defined in 12.1.32 and 12.1.33.

12.3.2. Definition. For a unital C^*-algebra A define

$$[\ \ ]_0\colon \mathcal{P}_\infty(A) \to K_0(A)\colon p \mapsto \gamma_{\mathcal{D}(A)}\big([p]_{\mathcal{D}}\big)\,.$$

12.3.3. Definition. Let A be a unital C^*-algebra and p, $q \in \mathcal{P}_\infty(A)$. We say that P is STABLY EQUIVALENT to q and write $p \sim_s q$ if there exists a projection $r \in \mathcal{P}_\infty(A)$ such that $p \oplus r \sim_\circ q \oplus r$.

12.3.4. Proposition. *Stable equivalence $\sim_s$ is an equivalence relation on $\mathcal{P}_\infty(A)$.*

12.3.5. Proposition. *Let A be a unital C^*-algebra and p, $q \in \mathcal{P}_\infty(A)$. Then $p \sim_s q$ if and only if $p \oplus \mathbf{1}_n \sim_\circ q \oplus \mathbf{1}_n$ for some $n \in \mathbb{N}$.*

12.3.6. Proposition (Standard Picture of $K_0(A)$ when A is unital)**.** *If A is a unital C^*-algebra, then*

$$\begin{aligned} K_0(A) &= \{[p]_0 - [q]_0\colon p, q \in \mathcal{P}_\infty(A)\} \\ &= \{[p]_0 - [q]_0\colon n \in \mathbb{N} \text{ and } p, q \in \mathcal{P}_n(A)\}\,. \end{aligned}$$

12.3.7. Proposition. *Let A be a unital C^*-algebra and p, $q \in \mathcal{P}_\infty(A)$. Then $[p \oplus q]_0 = [p]_0 + [q]_0$.*

12.3.8. Proposition. *Let A be a unital C^*-algebra and p, $q \in \mathcal{P}_n(A)$. If $p \sim_h q$ in $\mathcal{P}_n(A)$, then $[p]_0 = [q]_0$.*

12.3.9. Proposition. *Let A be a unital C^*-algebra and p, $q \in \mathcal{P}_n(A)$. If $p \perp q$ in $\mathcal{P}_n(A)$, then $p + q \in \mathcal{P}_n(A)$ and $[p+q]_0 = [p]_0 + [q]_0$.*

12.3.10. Proposition. *Let A be a unital C^*-algebra and p, $q \in \mathcal{P}_\infty(A)$. Then $[p]_0 = [q]_0$ if and only if $p \sim_s q$.*

PROOF. See [27], proposition 3.1.7.

In the next proposition the forgetful functor $\Box$ is the one described in proposition 12.2.12.

12.3.11. Proposition (Universal Property of K_0—Unital Case)**.** *Let A be a unital C^*-algebra, G be an Abelian group, and $\nu\colon \mathcal{P}_\infty(A) \to \Box G$ be a semigroup homomorphism that satisfies*

(i) *$\nu(\mathbf{0}_A) = \mathbf{0}_G$ and*
(ii) *if $p \sim_h q$ in $\mathcal{P}_n(A)$ for some n, then $\nu(p) = \nu(q)$.*

Then there exists a unique group homomorphism $\widetilde{\nu}\colon K_0(A) \to G$ such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{P}_\infty(A) \arrow[r, "{[\ \ ]_0}"] \arrow[dr, "\nu"'] & \Box K_0(A) \arrow[d, dashed, "\Box\widetilde{\nu}"] & K_0(A) \arrow[d, dashed, "\widetilde{\nu}"] \\ & \Box G & G \end{tikzcd}$$

Proof. See [27], proposition 3.1.8.

12.3.12. Definition. A $*$-homomorphism $\phi\colon A \to B$ between C^*-algebras extends, for every $n \in \mathbb{N}$, to a $*$-homomorphism $\phi\colon \mathbf{M}_n(A) \to \mathbf{M}_n(B)$ and also (since $*$-homomorphisms take projections to projections) to a $*$-homomorphism ϕ from $\mathcal{P}_\infty(A)$ to $\mathcal{P}_\infty(B)$. For such a $*$-homomorphism ϕ define

$$\nu\colon \mathcal{P}_\infty(A) \to K_0(B)\colon p \mapsto [\phi(p)]_0 .$$

Then ν is a semigroup homomorphism satisfying conditions (i) and (ii) of proposition 12.3.11 according to which there exists a unique group homomorphism $K_0(\phi)\colon K_0(A) \to K_0(B)$ such that $K_0(\phi)\big([p]_0\big) = \nu(p)$ for every $p \in \mathcal{P}_\infty(A)$.

12.3.13. Proposition. *The pair of maps $A \mapsto K_0(A)$, $\phi \mapsto K_0(\phi)$ is a covariant functor from the category of unital C^*-algebras and $*$-homomorphisms to the category of Abelian groups and group homomorphisms. Furthermore, for all $*$-homomorphisms $\phi\colon A \to B$ between unital C^*-algebras the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{P}_\infty(A) \arrow[r, "\phi"] \arrow[d, "{[\ ]_0}"'] & \mathcal{P}_\infty(B) \arrow[d, "{[\ ]_0}"] \\
K_0(A) \arrow[r, "K_0(\phi)"'] & K_0(B)
\end{tikzcd}$$

12.3.14. Notation. For C^*-algebras A and B let $\operatorname{Hom}(A,B)$ be the family of all $*$-homomorphisms from A to B.

12.3.15. Definition. Let A and B be C^*-algebras and $a \in A$. For ϕ, $\psi \in \operatorname{Hom}(A,B)$ let

$$d_a(\phi,\psi) = \|\phi(a) - \psi(a)\| .$$

Then d_a is a pseudometric on $\operatorname{Hom}(A,B)$. The weak topology generated by the family $\{d_a\colon a \in A\}$ is the POINT-NORM TOPOLOGY on $\operatorname{Hom}(A,B)$.

12.3.16. Definition. Let A and B be C^*-algebras and ϕ_0, $\phi_1 \in \operatorname{Hom}(A,B)$. A HOMOTOPY from ϕ_0 to ϕ_1 is a $*$-homomorphism from A to $\mathcal{C}([0,1],B)$ such that $\phi_0 = E_0 \circ \phi$ and $\phi_1 = E_1 \circ \phi$. (Here, E_t is the evaluation functional at $t \in [0,1]$.) We say that ϕ_0 and ϕ_1 are HOMOTOPIC if there exists a homotopy from ϕ_0 to ϕ_1, in which case we write $\phi_0 \sim_h \phi_1$.

12.3.17. Proposition. *Two $*$-homomorphisms ϕ_0, $\phi_1\colon A \to B$ between C^*-algebras are homotopic if and only if there exists a point-norm continuous path from ϕ_0 to ϕ_1 in* $\operatorname{Hom}(A,B)$.

12.3.18. Definition. We say that C^*-algebras are HOMOTOPICALLY EQUIVALENT if there exist $*$-homomorphisms $\phi\colon A \to B$ and $\psi\colon B \to A$ such that $\psi \circ \phi \sim_h \mathrm{id}_A$ and $\phi \circ \psi \sim_h \mathrm{id}_B$. A C^*-algebra is CONTRACTIBLE if it is homotopically equivalent to $\{\mathbf{0}\}$.

12.3.19. Proposition. *Let A and B be unital C^*-algebras. If $\phi \sim_h \psi$ in* $\operatorname{Hom}(A,B)$*, then $K_0(\phi) = K_0(\psi)$.*

Proof. See [27], proposition 3.2.6.

12.3.20. Proposition. *If unital C^*-algebras A and B are homotopically equivalent, then $K_0(A) \cong K_0(B)$.*

Proof. See [27], proposition 3.2.6.

12.3.21. Proposition. *If A is a unital C^*-algebra, then the split exact sequence*

$$\begin{tikzcd}
\mathbf{0} \arrow[r] & A \arrow[r, "\iota"] & \widetilde{A} \arrow[r, shift left, "\pi"] & \mathbb{C} \arrow[l, shift left, "\lambda"] \arrow[r] & \mathbf{0}
\end{tikzcd} \tag{12.2}$$

induces another split exact sequence

$$\mathbf{0} \longrightarrow K_0(A) \xrightarrow{K_0(\iota)} K_0(\widetilde{A}) \underset{K_0(\lambda)}{\overset{K_0(\pi)}{\rightleftarrows}} K_0(\mathbb{C}) \longrightarrow \mathbf{0}\,. \tag{12.3}$$

PROOF. See [27], lemma 3.2.8.

12.3.22. Example. For every $n \in \mathbb{N}$, $K_0(\mathbf{M}_n) \cong \mathbb{Z}$.

12.3.23. Example. If H is a Hilbert space, then $K_0(\mathfrak{B}(H)) \cong \mathbf{0}$.

12.3.24. Definition. Recall that a topological space X is CONTRACTIBLE if there is a point a in the space and a continuous function $f\colon [0,1] \times X \to X$ such that $f(1,x) = x$ and $f(0,x) = a$ for every $x \in X$.

12.3.25. Example. If X is a contractible compact Hausdorff space, then $K_0(\mathcal{C}(X)) \cong \mathbb{Z}$.

12.4. $K_0(A)$—the Nonunital Case

12.4.1. Definition. Let A be a nonunital C^*-algebra. Recall that the split exact sequence (12.2) for the unitization of A induces a split exact sequence (12.3) between the corresponding K_0 groups. Define

$$K_0(A) = \ker(K_0(\pi))\,.$$

12.4.2. Proposition. *For a nonunital C^*-algebra A the mapping $[\ \ ]_0\colon \mathcal{P}_\infty(A) \to K_0(\widetilde{A})$ may be regarded as a mapping from $\mathcal{P}_\infty(A)$ into $K_0(A)$.*

12.4.3. Proposition. *For both unital and nonunital C^*-algebras the sequence*

$$\mathbf{0} \longrightarrow K_0(A) \longrightarrow K_0(\widetilde{A}) \longrightarrow K_0(\mathbb{C}) \longrightarrow \mathbf{0}$$

is exact.

12.4.4. Proposition. *For both unital and nonunital C^*-algebras the group $K_0(A)$ is (isomorphic to) $\ker(K_0(\pi))$.*

12.4.5. Proposition. *If $\phi\colon A \to B$ is a $*$-homomorphism between C^*-algebras, then there exists a unique $*$-homomorphism $K_0(\phi)$ which makes the following diagram commute.*

$$\begin{tikzcd}
K_0(A) \arrow[r] \arrow[d, dotted, "K_0(\phi)"'] & K_0(\widetilde{A}) \arrow[r, "K_0(\pi_A)"] \arrow[d, "K_0(\widetilde{\phi})"] & K_0(\mathbb{C}) \arrow[d, equal] \\
K_0(B) \arrow[r] & K_0(\widetilde{B}) \arrow[r, "K_0(\pi_B)"'] & K_0(\mathbb{C})
\end{tikzcd}$$

12.4.6. Proposition. *The pair of maps $A \mapsto K_0(A)$, $\phi \mapsto K_0(\phi)$ is a covariant functor from the category* **CSA** *of C^*-algebras and $*$-homomorphisms to the category of Abelian groups and group homomorphisms.*

In propositions 12.3.19 and 12.3.20 we asserted the homotopy invariance of the functor K_0 for unital C^*-algebras. We now extend the result to arbitrary C^*-algebras.

12.4.7. Proposition. *Let A and B be C^*-algebras. If $\phi \sim_h \psi$ in $\mathrm{Hom}(A,B)$, then $K_0(\phi) = K_0(\psi)$.*

PROOF. See [27], proposition 4.1.4.

12.4.8. Proposition. *If C^*-algebras A and B are homotopically equivalent, then $K_0(A) \cong K_0(B)$.*

PROOF. See [27], proposition 4.1.4.

12.4.9. Definition. Let π and λ be the $*$-homomorphisms in the split exact sequence (12.2) for the unitization of as C^*-algebra A. Define the SCALAR MAPPING $s\colon \widetilde{A} \to \widetilde{A}$ for $\widetilde{A}$ by $s := \lambda \circ \pi$. Every member of $\widetilde{A}$ can be written in the form $a + \alpha \mathbf{1}_{\widetilde{A}}$ for some $a \in A$ and $\alpha \in \mathbb{C}$. Notice that $s(a + \alpha \mathbf{1}_{\widetilde{A}}) = \alpha \mathbf{1}_{\widetilde{A}}$ and that $x - s(x) \in A$ for every $x \in \widetilde{A}$. For each natural number n the scalar mapping induces a corresponding map $s = s_n\colon \mathbf{M}_n(\widetilde{A}) \to \mathbf{M}_n(\widetilde{A})$. An element $x \in \mathbf{M}_n(\widetilde{A})$ is a SCALAR ELEMENT of $\mathbf{M}_n(\widetilde{A})$ if $s(x) = x$.

12.4.10. Proposition (Standard Picture of $K_0(A)$ for arbitrary A)**.** *If A is a C^*-algebra, then*

$$K_0(A) = \{\, [p]_0 - [s(p)]_0 \colon p, q \in \mathcal{P}_\infty(\widetilde{A})\}\,.$$

12.5. Exactness and Stability Properties of the K_0 Functor

12.5.1. Definition. A covariant functor F from a category $\mathbf{A}$ to a category $\mathbf{B}$ is EXACT if it takes exact sequences in $\mathbf{A}$ to exact sequences in $\mathbf{B}$. It is SPLIT EXACT if it takes split exact sequences to split exact sequences. And it is HALF EXACT provided that whenever the sequence

$$\mathbf{0} \longrightarrow A_1 \xrightarrow{\ j\ } A_2 \xrightarrow{\ k\ } A_3 \longrightarrow \mathbf{0}$$

is exact in $\mathbf{A}$, then

$$F(A_1) \xrightarrow{\ F(j)\ } F(A_2) \xrightarrow{\ F(k)\ } F(A_3)$$

is exact in $\mathbf{B}$.

12.5.2. Proposition. *The functor K_0 is half exact.*

PROOF. See [27], proposition 4.3.2.

12.5.3. Proposition. *The functor K_0 is split exact.*

PROOF. See [27], proposition 4.3.3.

12.5.4. Proposition. *The functor K_0 preserves direct sums. That is, if A and B are C^*-algebras, then $K_0(A \oplus B) = K_0(A) \oplus K_0(B)$.*

12.5.5. Example. If A is a C^*-algebra, then $K_0(\widetilde{A}) = K_0(A) \oplus \mathbb{Z}$.

Despite being both split exact and half exact the functor K_0 is not exact. Each of the next two examples is sufficient to demonstrate this.

12.5.6. Example. The sequence

$$\mathbf{0} \longrightarrow \mathcal{C}_0\big((0,1)\big) \xrightarrow{\ \iota\ } \mathcal{C}\big([0,1]\big) \xrightarrow{\ \psi\ } \mathbb{C} \oplus \mathbb{C} \longrightarrow \mathbf{0}$$

where $\psi(f) = \big(f(0), f(1)\big)$, is clearly exact; but $K_0(\psi)$ is not surjective.

12.5.7. Example. If H is a Hilbert space the exact sequence

$$\mathbf{0} \longrightarrow \mathfrak{K}(H) \xrightarrow{\ \iota\ } \mathfrak{B}(H) \xrightarrow{\ \pi\ } \mathfrak{Q}(H) \longrightarrow \mathbf{0}$$

associated with the Calkin algebra $\mathfrak{Q}(H)$ is exact but $K_0(\iota)$ is not injective. (This example requires a fact we have not yet derived: $K_0(\mathfrak{K}(H)) \cong \mathbb{Z}$.)

Next is an important stability property of the functor K_0.

12.5.8. Proposition. *If A is a C^*-algebra, then $K_0(A) \cong K_0(\mathbf{M}_n(A))$.*

PROOF. See [27], proposition 4.3.8.

12.6. Inductive Limits

12.6.1. Definition. In any category an INDUCTIVE SEQUENCE is a pair (A, ϕ) where $A = (A_j)$ is a sequence of objects and $\phi = (\phi_j)$ is a sequence of morphisms such that $\phi_j \colon A_j \to A_{j+1}$ for each j. An INDUCTIVE LIMIT (or DIRECT LIMIT) of the sequence (A, ϕ) is a pair (L, μ) where L is an object and $\mu = (\mu_n)$ is a sequence of morphisms $\mu_j \colon A_j \to L$ which satisfy

(i) $\mu_j = \mu_{j+1} \circ \phi_j$ for each $j \in \mathbb{N}$, and
(ii) if (M, λ) is a pair where M is an object and $\lambda = (\lambda_j)$ is a sequence of morphisms $\lambda_j \colon A_j \to M$ satisfying $\lambda_j = \lambda_{j+1} \circ \phi_j$, then there exists a unique morphism $\psi \colon L \to M$ such that $\lambda_j = \psi \circ \mu_j$ for each $j \in \mathbb{N}$.

Abusing language we usually say that L is the inductive limit of the sequence (A_j) and write $L = \varinjlim A_j$.

$$\begin{tikzcd}
 & & & & L \arrow[dd, dashed, "\psi"] \\
\cdots \arrow[r] & A_j \arrow[r, "\phi_j"] \arrow[urrr, "\mu_j"] \arrow[drrr, "\lambda_j"'] & A_{j+1} \arrow[r] \arrow[urr, "\mu_{j+1}"] \arrow[drr, "\lambda_{j+1}"] & \cdots & \\
 & & & & M
\end{tikzcd}$$

12.6.2. Proposition. *Inductive limits (if they exist in a category) are unique (up to isomorphism).*

12.6.3. Proposition. *Every inductive sequence (A, ϕ) of C^*-algebras has an inductive limit. (And so does every inductive sequence of Abelian groups.)*

PROOF. See [3], II.8.2.1; [27], proposition 6.2.4; or [29], appendix L.

12.6.4. Proposition. *If (L, μ) is the inductive limit of an inductive sequence (A, ϕ) of C^*-algebras, then $L = \overline{\bigcup \mu_n{}^{\to}(A_n)}$ and $\|\mu_m(a)\| = \lim_{n\to\infty}\|\phi_{n,m}(a)\| = \inf_{n\geq m}\|\phi_{n,m}(a)\|$ for all $m \in \mathbb{N}$ and $a \in A$.*

PROOF. See [27], proposition 6.2.4.

12.6.5. Definition. An APPROXIMATELY FINITE DIMENSIONAL C^*-ALGEBRA (an AF-algebra for short) is the inductive limit of a sequence of finite dimensional C^*-algebras.

12.6.6. Example. If (A_n) is an increasing sequence of C^*-subalgebras of a C^*-algebra D (with $\iota_n \colon A_n \to A_{n+1}$ being the inclusion map for each n), then (A, ι) is an inductive sequence of C^*-algebras whose inductive limit is (B, j) where $B = \overline{\bigcup_{n=1}^{\infty} A_n}$ and $j_n \colon A_n \to B$ is the inclusion map for each n.

12.6.7. Example. The sequence $\mathbf{M}_1 = \mathbb{C} \xrightarrow{\phi_1} \mathbf{M}_2 \xrightarrow{\phi_2} \mathbf{M}_3 \xrightarrow{\phi_3} \dots$ (where $\phi_n(a) = \operatorname{diag}(a, 0)$ for each $n \in \mathbb{N}$ and $a \in \mathbf{M}_n$) is an inductive sequence of C^*-algebras whose inductive limit is the C^*-algebra $\mathfrak{K}(H)$ of compact operators on a Hilbert space H.

PROOF. See [27], section 6.4.

12.6.8. Example. The sequence $\mathbb{Z} \xrightarrow{\mathbf{1}} \mathbb{Z} \xrightarrow{\mathbf{2}} \mathbb{Z} \xrightarrow{\mathbf{3}} \mathbb{Z} \xrightarrow{\mathbf{4}} \dots$ (where $\mathbf{n}\colon \mathbb{Z} \to \mathbb{Z}$ satisfies $\mathbf{n}(1) = n$) is an inductive sequence of Abelian groups whose inductive limit is the set $\mathbb{Q}$ of rational numbers.

12.6.9. Example. The sequence $\mathbb{Z} \xrightarrow{\mathbf{2}} \mathbb{Z} \xrightarrow{\mathbf{2}} \mathbb{Z} \xrightarrow{\mathbf{2}} \mathbb{Z} \xrightarrow{\mathbf{2}} \dots$ is an inductive sequence of Abelian groups whose inductive limit is the set of dyadic rational numbers.

The next result is referred to the *continuity property of* K_0.

12.6.10. Proposition. *If (A, ϕ) is an inductive sequence of C^*-algebras, then*

$$K_0\big(\varinjlim A_n\big) = \varinjlim K_0(A_n)\,.$$

PROOF. See [27], theorem 6.3.2.

12.6.11. Example. If H is a Hilbert space, then $K_0(\mathfrak{K}(H)) \cong \mathbb{Z}$.

12.7. Bratteli Diagrams

12.7.1. Proposition. *Nonzero algebra homomorphisms from M_k into M_n exist only if $n \geq k$, in which case they are precisely the mappings of the form*

$$a \mapsto u\ \mathrm{diag}(a, a, \dots, a, \mathbf{0})\, u^*$$

where u is a unitary matrix. Here there are m copies of a and $\mathbf{0}$ is the $r \times r$ zero matrix where $n = mk + r$. The number m is the MULTIPLICITY *of ϕ.*

PROOF. See [24], corollary 1.3.

12.7.2. Example. An example of a homomorphism from M_2 into M_7 is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & b & 0 & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{diag}(A, A, \mathbf{0})$$

where $m = 2$ and $r = 3$.

12.7.3. Proposition. *Every finite dimensional C^*-algebra A is isomorphic to a direct sum of matrix algebras*

$$A \simeq M_{k_1} \oplus \cdots \oplus M_{k_r}.$$

Suppose A and B are finite dimensional C^*-algebras, so that

$$A \simeq M_{k_1} \oplus \cdots \oplus M_{k_r} \qquad \text{and} \qquad B \simeq M_{n_1} \oplus \cdots \oplus M_{n_s}$$

and suppose that $\phi\colon A \to B$ is a unital *-homomorphism. Then ϕ is determined (up to unitary equivalence in B) by an $s \times r$ matrix $\mathbf{m} = \big[m_{ij}\big]$ of positive integers such that

$$\mathbf{m}\,\mathbf{k} = \mathbf{n}. \tag{12.4}$$

Here $\mathbf{k} = (k_1, \dots, k_r)$, $\mathbf{n} = (n_1, \dots, n_s)$, and the number m_{ij} is the multiplicity of the map

$$M_{k_j} \longrightarrow A \xrightarrow{\ \phi\ } B \longrightarrow M_{n_i}$$

PROOF. See [7], theorem III.1.1.

12.7.4. Definition. Let ϕ be as in the preceding proposition 12.7.3. A BRATTELI DIAGRAM for ϕ consists of two rows (or columns) of vertices labeled by the k_j and the n_i together with m_{ij} edges connecting k_j to n_i (for $1 \leq j \leq r$ and $1 \leq i \leq s$).

12.7.5. Example. Suppose $\phi\colon \mathbb{C}\oplus\mathbb{C}\to M_4\oplus M_3$ is given by

$$(\lambda,\mu)\mapsto\left(\begin{bmatrix}\lambda&0&0&0\\0&\lambda&0&0\\0&0&\lambda&0\\0&0&0&\mu\end{bmatrix},\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\mu\end{bmatrix}\right).$$

Then $m_{11}=3$, $m_{12}=1$, $m_{21}=2$, and $m_{22}=1$. Notice that

$$\mathbf{m}\,\mathbf{k}=\begin{bmatrix}3&1\\2&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}4\\3\end{bmatrix}=\mathbf{n}.$$

A Bratteli diagram for ϕ is

$$\begin{tikzcd}
1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[dr, shift left] \arrow[dr, shift right] & 4 \\
1 \arrow[ur] \arrow[r] & 3
\end{tikzcd}$$

12.7.6. Example. Suppose $\phi\colon \mathbb{C}\oplus M_2\to M_3\oplus M_5\oplus M_2$ is given by

$$(\lambda,b)\mapsto\left(\begin{bmatrix}\lambda&0\\0&b\end{bmatrix},\begin{bmatrix}\lambda&0&0\\0&b&0\\0&0&b\end{bmatrix},b\right).$$

Then $m_{11}=1$, $m_{12}=1$, $m_{21}=1$, $m_{22}=2$, $m_{31}=0$, and $m_{32}=1$. Notice that

$$\mathbf{m}\,\mathbf{k}=\begin{bmatrix}1&1\\1&2\\0&1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\5\\2\end{bmatrix}=\mathbf{n}.$$

A Bratteli diagram for ϕ is

$$\begin{tikzcd}
 & 3 \\
1 \arrow[ur] \arrow[dr] & \\
 & 5 \\
2 \arrow[uuur] \arrow[ur, shift left] \arrow[ur, shift right] \arrow[dr] & \\
 & 2
\end{tikzcd}$$

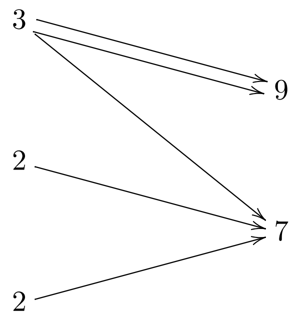
12.7.7. Example. Suppose $\phi\colon M_3\oplus M_2\oplus M_2\to M_9\oplus M_7$ is given by

$$(a,b,c)\mapsto\left(\begin{bmatrix}a&\mathbf{0}&\mathbf{0}\\\mathbf{0}&a&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\end{bmatrix},\begin{bmatrix}a&\mathbf{0}&\mathbf{0}\\\mathbf{0}&b&\mathbf{0}\\\mathbf{0}&\mathbf{0}&c\end{bmatrix}\right).$$

Then $m_{11} = 2$, $m_{12} = 0$, $m_{13} = 0$, $m_{21} = 1$, $m_{22} = 1$, and $m_{23} = 1$. Notice that this time something peculiar happens: we have $\mathbf{m}\,\mathbf{k} \neq \mathbf{n}$:

$$\mathbf{m}\,\mathbf{k} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix} \leq \begin{bmatrix} 9 \\ 7 \end{bmatrix} = \mathbf{n}.$$

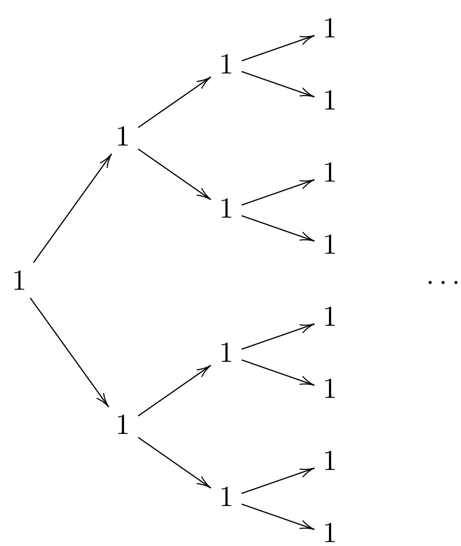
What's the problem here? Well, the result stated earlier was for *unital* *-homomorphisms, and this ϕ is *not* unital. In general, this will be the best we can expect: $\mathbf{m}\,\mathbf{k} \leq \mathbf{n}$.
Here is the resulting Bratteli diagram for ϕ:



12.7.8. Example. If K is the Cantor set, then $\mathcal{C}(K)$ is an AF-algebra. To see this write K as the intersection of a decreasing family of closed subsets K_j each of which consists of 2^j disjoint closed subintervals of $[0,1]$. For each $j \geq 0$ let A_j be the subalgebra of functions in $\mathcal{C}(K)$ which are constant on each of the intervals making up K_j. Thus $A_j \simeq \mathbb{C}^{2^j}$ for each $j \geq 0$. The imbedding

$$\phi_j \colon A_j \to A_{j+1} \colon (a_1, a_2, \dots, a_{2^j}) \mapsto (a_1, a_1, a_2, a_2, \dots, a_{2^j}, a_{2^j})$$

splits each minimal projection into the sum of two minimal projections. For ϕ_0 the corresponding matrix $\mathbf{m}_0$ of "partial multiplicities" is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; the matrix $\mathbf{m}_1$ corresponding to ϕ_1 is $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$; and so on. Thus the Bratteli diagram for the inductive limit $\mathcal{C}(K) = \varinjlim A_j$ is:



12.7.9. Example. Here is an example of a so-called CAR-algebra (CAR = Canonical Anticommutation Relations). For $j \geq 0$ let $A_j = M_{2^j}$ and

$$\phi_j \colon A_j \longrightarrow A_{j+1} \colon \mathbf{a} \mapsto \begin{bmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{a} \end{bmatrix}.$$

The multiplicity "matrix" $\mathbf{m}$ for each ϕ_j is just the 1×1 matrix $[2]$. We see that for each j

$$\mathbf{m}\,\mathbf{k}(j) = [2]\,[2^j] = [2^{j+1}] = \mathbf{n}(j).$$

This leads to the following Bratteli diagram

$$1 \rightrightarrows 2 \rightrightarrows 4 \rightrightarrows 8 \rightrightarrows 16 \rightrightarrows 32 \rightrightarrows \cdots$$

for the inductive limit $C = \varinjlim A_j$.

However the range of ϕ_j is contained in a subalgebra $B_{j+1} \simeq M_{2^j} \oplus M_{2^j}$ of A_{j+1}. (Take $B_0 = A_0 = \mathbb{C}$.) Thus for $j \in \mathbb{N}$ we may regard ϕ_j as a mapping from B_j to B_{j+1}:

$$\phi_j \colon (\mathbf{b}, \mathbf{c}) \mapsto \left(\begin{bmatrix} \mathbf{b} & 0 \\ 0 & \mathbf{c} \end{bmatrix}, \begin{bmatrix} \mathbf{b} & 0 \\ 0 & \mathbf{c} \end{bmatrix} \right).$$

Now the multiplicity matrix $\mathbf{m}$ for each ϕ_j is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and we see that for each j

$$\mathbf{m}\,\mathbf{k}(j) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{j-1} \\ 2^{j-1} \end{bmatrix} = \begin{bmatrix} 2^j \\ 2^j \end{bmatrix} = \mathbf{n}(j).$$

Then, since $C = \varinjlim A_j = \varinjlim B_j$, we have a second (quite different) Bratteli diagram for the AF-algebra C.

$$\begin{tikzcd}
 & 1 \arrow[r] \arrow[dr] & 2 \arrow[r] \arrow[dr] & 4 \arrow[r] \arrow[dr] & 8 \arrow[r] \arrow[dr] & 16 \arrow[r] \arrow[dr] & \cdots \\
1 \arrow[ur] \arrow[dr] & & & & & & \\
 & 1 \arrow[r] \arrow[uur] & 2 \arrow[r] \arrow[uur] & 4 \arrow[r] \arrow[uur] & 8 \arrow[r] \arrow[uur] & 16 \arrow[r] \arrow[uur] & \cdots
\end{tikzcd}$$

12.7.10. Example. This is the FIBONACCI ALGEBRA. For $j \in \mathbb{N}$ define sequences (p_j) and (q_j) by the familiar recursion relations:

$$\begin{aligned} p_1 &= q_1 = 1, \\ p_{j+1} &= p_j + q_j, \text{ and} \\ q_{j+1} &= p_j. \end{aligned}$$

For all $j \in \mathbb{N}$ let $A_j = M_{p_j} \oplus M_{q_j}$ and

$$\phi_j \colon A_j \to A_{j+1} \colon (a, b) \mapsto \left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, a \right).$$

The multiplicity matrix $\mathbf{m}$ for each ϕ_j is $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and for each j

$$\mathbf{m}\,\mathbf{k}(j) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_j \\ q_j \end{bmatrix} = \begin{bmatrix} p_j + q_j \\ p_j \end{bmatrix} = \begin{bmatrix} p_{j+1} \\ q_{j+1} \end{bmatrix} = \mathbf{n}(j).$$

The resulting Bratteli diagram for $F = \varinjlim A_j$ is

$$\begin{tikzcd}
 & 1 \arrow[r] \arrow[ddr] & 2 \arrow[r] \arrow[ddr] & 3 \arrow[r] \arrow[ddr] & 5 \arrow[r] \arrow[ddr] & 8 \arrow[r] \arrow[ddr] & \cdots \\
1 \arrow[ur] \arrow[dr] & & & & & & \\
 & 1 \arrow[uur] & 1 \arrow[uur] & 2 \arrow[uur] & 3 \arrow[uur] & 5 \arrow[uur] & \cdots
\end{tikzcd}$$

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