Companion to Functional Analysis

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Contents

PREFACE

Paul Halmos famously remarked in his beautiful *Hilbert Space Problem Book* [**22**] that "The only way to learn mathematics is to do mathematics." Halmos is certainly not alone in this belief. The current set of notes is an activity-oriented companion to the study of functional analysis. It is intended as a pedagogical companion for the beginner, an introduction to some of the main ideas in functional analysis, a compendium of problems I think are useful in learning the subject, and an annotated reading/reference list.

The great majority of the results in beginning functional analysis are straightforward and can be verified by the thoughtful student. Indeed, that is the main point of these notes—to convince the beginner that the subject is accessible. In the material that follows there are numerous indicators that suggest activity on the part of the reader: words such as "proposition", "example", "exercise", and "corollary", if not followed by a proof or a reference to a proof, are invitations to verify the assertions made. Of course, there are a few theorems where, in my opinion, the time and effort which go into proving them is greater than the benefits derived. In such cases, when I have no improvements to offer to the standard proofs, instead of repeating them I give references.

These notes were written for a year long course in functional analysis for graduate students at Portland State University. During the year students are asked to choose, in addition to these notes, other sources of information for the course, either printed texts or online documents, which suit their individual learning styles. As a result the material that is intended to be covered during the Fall quarter, the first 6–8 chapters, is relatively complete. After that, when students have found other sources they like, the notes become sketchier.

There are of course a number of advantages and disadvantages in consigning a document to electronic life. One advantage is the rapidity with which links implement cross-references. Hunting about in a book for *lemma 3.14.23* can be time-consuming (especially when an author engages in the entirely logical but utterly infuriating practice of numbering lemmas, propositions, theorems, corollaries, and so on, separately). A perhaps more substantial advantage is the ability to correct errors, add missing bits, clarify opaque arguments, and remedy infelicities of style in a timely fashion. The correlative disadvantage is that a reader returning to the web page after a short time may find everything (pages, definitions, theorems, sections) numbered differently. (LaTeXis an amazing tool.) I will change the date on the title page to inform the reader of the date of the last nontrivial update (that is, one that affects numbers or cross-references).

The most serious disadvantage of electronic life is impermanence. In most cases when a web page vanishes so, for all practical purposes, does the information it contains. For this reason (and the fact that I want this material to be freely available to anyone who wants it) I am making use of a "Share Alike" license from *Creative Commons*. It is my hope that anyone who finds this material useful will correct what is wrong, add what is missing, and improve what is clumsy. For more information on creative commons licenses see http://creativecommons.org/. Concerning the text itself, please send corrections, suggestions, complaints, and all other comments to the author at

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Greek Letters

Upper case	Lower case	English name (approximate pronunciation)
A	α	Alpha (AL-fuh)
B	β	Beta (BAY-tuh)
Γ	γ	Gamma (GAM-uh)
Δ	δ	Delta (DEL-tuh)
E	ϵ or ε	Epsilon (EPP-suh-lon)
Z	ζ	Zeta (ZAY-tuh)
H	η	Eta (AY-tuh)
Θ	θ	Theta (THAY-tuh)
I	ι	Iota (eye-OH-tuh)
K	κ	Kappa (KAP-uh)
Λ	λ	Lambda (LAM-duh)
M	μ	Mu (MYOO)
N	ν	Nu (NOO)
Ξ	ξ	Xi (KSEE)
O	o	Omicron (OHM-ih-kron)
Π	π	Pi (PIE)
P	ρ	Rho (ROH)
Σ	σ	Sigma (SIG-muh)
T	τ	Tau (TAU)
Υ	υ	Upsilon (OOP-suh-lon)
Φ	ϕ	Phi (FEE or FAHY)
X	χ	Chi (KHAY)
Ψ	ψ	Psi (PSEE or PSAHY)
Ω	ω	Omega (oh-MAY-guh)

Fraktur Fonts

In these notes Fraktur fonts are used (most often for families of sets and families of operators). Below are the Roman equivalents for each letter. When writing longhand or presenting material on a blackboard it is usually best to substitute script English letters.

Fraktur Upper case	Fraktur Lower case	Roman Lower Case
$\mathfrak{A}$	$\mathfrak{a}$	a
$\mathfrak{B}$	$\mathfrak{b}$	b
$\mathfrak{C}$	$\mathfrak{c}$	c
$\mathfrak{D}$	$\mathfrak{d}$	d
$\mathfrak{E}$	$\mathfrak{e}$	e
$\mathfrak{F}$	$\mathfrak{f}$	f
$\mathfrak{G}$	$\mathfrak{g}$	g
$\mathfrak{H}$	$\mathfrak{h}$	h
$\mathfrak{I}$	$\mathfrak{i}$	i
$\mathfrak{J}$	$\mathfrak{j}$	j
$\mathfrak{K}$	$\mathfrak{k}$	k
$\mathfrak{L}$	$\mathfrak{l}$	l
$\mathfrak{M}$	$\mathfrak{m}$	m
$\mathfrak{N}$	$\mathfrak{n}$	n
$\mathfrak{O}$	$\mathfrak{o}$	o
$\mathfrak{P}$	$\mathfrak{p}$	p
$\mathfrak{Q}$	$\mathfrak{q}$	q
$\mathfrak{R}$	$\mathfrak{r}$	r
$\mathfrak{S}$	$\mathfrak{s}$	s
$\mathfrak{T}$	$\mathfrak{t}$	t
$\mathfrak{U}$	$\mathfrak{u}$	u
$\mathfrak{V}$	$\mathfrak{v}$	v
$\mathfrak{W}$	$\mathfrak{w}$	w
$\mathfrak{X}$	$\mathfrak{x}$	x
$\mathfrak{Y}$	$\mathfrak{y}$	y
$\mathfrak{Z}$	$\mathfrak{z}$	z

CHAPTER 1

LINEAR ALGEBRA AND THE SPECTRAL THEOREM

The emphasis in beginning analysis courses is on the behavior of individual (real or complex valued) functions. In a *functional analysis* course the focus is shifted to *spaces* of such functions and certain classes of mappings between these spaces. It turns out that a concise, perhaps overly simplistic, but certainly not misleading, description of such material is *infinite dimensional linear algebra*. From an analyst's point of view the greatest triumphs of linear algebra were, for the most part, theorems about operators on finite dimensional vector spaces (principally $\mathbb{R}^n$ and $\mathbb{C}^n$). However, the vector spaces of scalar-valued mappings defined on various domains are typically infinite dimensional. Whether one sees the process of generalizing the marvelous results of finite dimensional linear algebra to the infinite dimensional realm as a depressingly unpleasant string of abstract complications or as a rich source of astonishing insights, fascinating new questions, and suggestive hints towards applications in other fields depends entirely on one's orientation towards mathematics in general.

In light of the preceding remarks it may be helpful to begin our journey into this new terrain with a brief review of some of the classical successes of linear algebra. It is not unusual for students who successfully complete a beginning linear algebra to have at the end only the vaguest idea of what the course was about. Part of the blame for this may be laid on the many elementary texts which make an unholy conflation of two quite different, if closely related subjects: the study of *vector spaces* and the linear maps between them on the one hand, and *inner product spaces* and their linear maps on the other. Vector spaces have none of the geometric/topological notions of distance or length or perpendicularity or open sets or angle between vectors; there is only addition and scalar multiplication. Inner product spaces are vector spaces endowed with these additional structures. Let's look at them separately.

1.1. Vector Spaces and the Decomposition of Diagonalizable Operators

1.1.1. Convention. In these notes all vector spaces will be assumed to be vector spaces over the field $\mathbb{C}$ of complex numbers (in which case it is called a *complex vector space*) or the field $\mathbb{R}$ of real numbers (in which case it is a *real vector space*). No other fields will appear. When $\mathbb{K}$ appears it can be taken to be either $\mathbb{C}$ or $\mathbb{R}$. A SCALAR is a member of $\mathbb{K}$; that is, either a *complex number* or a *real number*.

1.1.2. Definition. The triple $(V, +, M)$ is a (COMPLEX) VECTOR SPACE if $(V, +)$ is an Abelian group and $M\colon \mathbb{C} \to \operatorname{Hom}(V)$ is a unital ring homomorphism (where $\operatorname{Hom}(V)$ is the ring of group homomorphisms on V).

To check on the meanings of any the terms in the preceding definition, take a look at the first three sections of the first chapter of my linear algebra notes [14].

1.1.3. Exercise. The definition of *vector space* found in many elementary texts is something like the following: a *vector space* is a set V together with operations of addition and scalar multiplication which satisfy the following axioms:

(1) if x, $y \in V$, then $x + y \in V$;
(2) $(x + y) + z = x + (y + z)$ for every x, y, $z \in V$ (associativity);
(3) there exists $\mathbf{0} \in V$ such that $x + \mathbf{0} = x$ for every $x \in V$ (existence of additive identity);
(4) for every $x \in V$ there exists $-x \in V$ such that $x+(-x) = \mathbf{0}$ (existence of additive inverses);

(5) $x + y = y + x$ for every x, $y \in V$ (commutativity);
(6) if $\alpha \in \mathbb{C}$ and $x \in V$, then $\alpha x \in V$;
(7) $\alpha(x + y) = \alpha x + \alpha y$ for every $\alpha \in \mathbb{C}$ and every x, $y \in V$;
(8) $(\alpha + \beta)x = \alpha x + \beta x$ for every α, $\beta \in \mathbb{C}$ and every $x \in V$;
(9) $(\alpha\beta)x = \alpha(\beta x)$ for every α, $\beta \in \mathbb{C}$ and every $x \in V$; and
(10) $1\,x = x$ for every $x \in V$.

Verify that this definition is equivalent to the one given above in 1.1.2.

1.1.4. Definition. A subset M of a vector space V is a VECTOR SUBSPACE of V if it is a vector space under the operations it inherits from V.

1.1.5. Proposition. *A nonempty subset of M of a vector space V is a vector subspace of V if and only if it is closed under addition and scalar multiplication. (That is: if $\mathbf{x}$ and $\mathbf{y}$ belong to M, so does $\mathbf{x} + \mathbf{y}$; and if $\mathbf{x}$ belongs to M and $\alpha \in \mathbb{K}$, then $\alpha\mathbf{x}$ belongs to M.)*

1.1.6. Definition. A vector y is a LINEAR COMBINATION of vectors $x_1, \dots, x_n$ if there exist scalars $\alpha_1, \dots \alpha_n$ such that $y = \sum_{k=1}^{n} \alpha_k x_k$. The linear combination $\sum_{k=1}^{n} \alpha_k x_k$ is TRIVIAL if all the coefficients $\alpha_1, \dots \alpha_n$ are zero. If at least one α_k is different from zero, the linear combination is NONTRIVIAL.

1.1.7. Definition. If A is a nonempty subset of a vector space V, then $\operatorname{span} A$, the SPAN of A, is the set of all linear combinations of elements of A.

1.1.8. Definition. A subset A of a vector space is LINEARLY DEPENDENT if the zero vector $\mathbf{0}$ can be written as a nontrivial linear combination of elements of A; that is, if there exist vectors $x_1, \dots, x_n \in A$ and scalars $\alpha_1, \dots, \alpha_n$, **not all zero,** such that $\sum_{k=1}^{n} \alpha_k x_k = \mathbf{0}$. A subset of a vector space is LINEARLY INDEPENDENT if it is not linearly dependent.

Technically, it is a *set* of vectors that is linearly dependent or independent. Nevertheless, these terms are frequently used as if they were properties of the vectors themselves. For instance, if $S = \{x_1, \dots, x_n\}$ is a finite set of vectors in a vector space, you may see the assertions "the set S is linearly independent" and "the vectors $x_1, \dots x_n$ are linearly independent" used interchangeably.

1.1.9. Definition. A set B of vectors in a vector space V is a HAMEL BASIS for V if it is linearly independent and spans B.

1.1.10. Proposition. *Let A be a linearly independent subset of a vector space V. Then there exists a Hamel basis for V which contains A.*

Hint for proof. Show first that a linearly independent subset of V is a basis for V if and only if it is a maximal linearly independent subset. Then order the set of linearly independent subsets of V which contain A by inclusion and apply *Zorn's lemma.* (If you are not familiar with *Zorn's lemma*, see section 1.7 of my linear algebra notes [14].)

1.1.11. Corollary. *Every vector space has a basis.*

1.1.12. Definition. A function $T \colon V \to W$ between vector spaces is LINEAR if $T(u+v) = Tu+Tv$ for all u, $v \in V$ and $T(\alpha v) = \alpha T v$ for all $\alpha \in \mathbb{K}$ and $v \in V$. Linear functions are frequently called *linear transformations* or *linear maps.* When $V = W$ we say that the linear map T is an OPERATOR on V. Depending on context we denote the identity operator $x \mapsto x$ on V by id_V or I_V or just I. Recall that if $T \colon V \to W$ is a linear map, then the KERNEL of T, denoted by $\ker T$, is $T^{\leftarrow}(\{0\}) := \{x \in V : Tx = \mathbf{0}\}$. Also, the RANGE of T, denoted by $\operatorname{ran} T$, is $T^{\rightarrow}(V) := \{Tx \colon x \in V\}$.

1.1.13. Definition. A linear map $T \colon V \to W$ between vector spaces is INVERTIBLE (or is an ISOMORPHISM) if there exists a linear map $T^{-1} \colon W \to V$ such that $T^{-1}T = \mathrm{id}_V$ and $TT^{-1} = \mathrm{id}_W$.

Recall that if a linear map is invertible its inverse is unique. Recall also that for an operator T on a finite dimensional vector space the following are equivalent:

(a) T is an isomorphism;
(b) T is injective;
(c) the kernel of T is $\{0\}$; and
(d) T is surjective.

1.1.14. Definition. Two operators R and T on a vector space V are SIMILAR if there exists an invertible operator S on V such that $R = STS^{-1}$.

1.1.15. Proposition. *If V is a vector space, then similarity is an equivalence relation on $\mathfrak{L}(V)$.*

Let V and W be finite dimensional vector spaces with ordered bases. Suppose that V is n-dimensional with ordered basis $\{e^1, e^2, \dots, e^n\}$ and W is m-dimensional. Recall from beginning linear algebra that if $T\colon V \to W$ is linear, then its *matrix representation* $[T]$ (taken with respect to the ordered bases in V and W) is the $m \times n$-matrix $[T]$ whose k^{th} column $(1 \le k \le n)$ is the column vector Te^k in W. The point here is that the action of T on a vector x can be *represented* as multiplication of x by the matrix $[T]$; that is,

$$Tx = [T]x.$$

Perhaps this equation requires a little interpretation. The left side is the function T evaluated at x, the result of this evaluation being thought of as a column vector in W; the right side is an $m \times n$ matrix multiplied by an $n \times 1$ matrix (that is, a column vector). So the asserted equality is of two $m \times 1$ matrices (column vectors).

1.1.16. Definition. Let V be a finite dimensional vector space and $B = \{e^1, \dots, e^n\}$ be a basis for V. An operator T on V is DIAGONAL if there exist scalars $\alpha_1, \dots, \alpha_n$ such that $Te^k = \alpha_k e^k$ for each $k \in \mathbb{N}_n$. Equivalently, T is diagonal if its matrix representation $[T] = [T_{ij}]$ has the property that $T_{ij} = 0$ whenever $i \ne j$.

Asking whether a particular operator on some finite dimensional vector space is diagonal is, strictly speaking, nonsense. As defined, the operator property of being diagonal is definitely *not* a vector space concept. It makes sense only for a vector space *for which a basis has been specified.* This important, if obvious, fact seems to go unnoticed in many beginning linear algebra courses, due, I suppose, to a rather obsessive fixation on $\mathbb{R}^n$ in such courses. Here is the relevant *vector space* property.

1.1.17. Definition. An operator T on a finite dimensional vector space V is DIAGONALIZABLE if there exists a basis for V with respect to which T is diagonal. Equivalently, an operator on a finite dimensional vector space *with basis* is diagonalizable if it is similar to a diagonal operator.

1.1.18. Definition. Let M and N be subspaces of a vector space V. If $M \cap N = \{\mathbf{0}\}$ and $M + N = V$, then V is the (INTERNAL) DIRECT SUM of M and N. In this case we write

$$V = M \oplus N\,.$$

We say that M and N are COMPLEMENTARY VECTOR SUBSPACES and that each is a (vector space) COMPLEMENT of the other. The CODIMENSION of the subspace M is the dimension of its complement N.

1.1.19. Example. Let $\mathcal{C} = \mathcal{C}[-1,1]$ be the vector space of all continuous real valued functions on the interval $[-1,1]$. A function f in $\mathcal{C}$ is EVEN if $f(-x) = f(x)$ for all $x \in [-1,1]$; it is ODD if $f(-x) = -f(x)$ for all $x \in [-1,1]$. Let $\mathcal{C}_o = \{f \in \mathcal{C}\colon f \text{ is odd }\}$ and $\mathcal{C}_e = \{f \in \mathcal{C}\colon f \text{ is even }\}$. Then $\mathcal{C} = \mathcal{C}_o \oplus \mathcal{C}_e$.

1.1.20. Proposition. *If M is a subspace of a vector space V, then there exists a subspace N of V such that $V = M \oplus N$.*

1.1.21. Proposition. *Let $T\colon V \to W$ be a linear transformation between vector spaces. Then*

(a) *T has a left inverse if and only if it is injective; and*

(b) *T has a right inverse if and only if it is surjective.*

1.1.22. Proposition. *Let V be a vector space and suppose that $V = M \oplus N$. Then for every $v \in V$ there exist unique vectors $m \in M$ and $n \in N$ such that $v = m + n$.*

1.1.23. Definition. Let V be a vector space and suppose that $V = M \oplus N$. We know from 1.1.22 that for each $\mathbf{v} \in V$ there exist unique vectors $\mathbf{m} \in M$ and $\mathbf{n} \in N$ such that $\mathbf{v} = \mathbf{m} + \mathbf{n}$. Define a function $E_{MN} \colon V \to V$ by $E_{MN} v = n$. The function E_{MN} is the PROJECTION OF V ALONG M ONTO N. (Frequently we write E for E_{MN}. But keep in mind that E depends on both M and N.)

1.1.24. Proposition. *Let V be a vector space and suppose that $V = M \oplus N$. If E is the projection of V along M onto N, then*

(a) *E is linear;*
(b) *$E^2 = E$ (that is, E is* IDEMPOTENT*);*
(c) *$\operatorname{ran} E = N$; and*
(d) *$\ker E = M$.*

1.1.25. Proposition. *Let V be a vector space and suppose that $E \colon V \to V$ is a function which satisfies*

(a) *E is linear, and*
(b) *$E^2 = E$.*

Then

$$V = \ker E \oplus \operatorname{ran} E$$

and E is the projection of V along $\ker E$ onto $\operatorname{ran} E$.

It is important to note that an obvious consequence of the last two propositions is that a function $T \colon V \to V$ from a finite dimensional vector space into itself is a projection if and only if it is linear and idempotent.

1.1.26. Proposition. *Let V be a vector space and suppose that $V = M \oplus N$. If E is the projection of V along M onto N, then $I - E$ is the projection of V along N onto M.*

As we have just seen, if E is a projection on a vector space V, then the identity operator on V can be written as the sum of two projections E and $I - E$ whose corresponding ranges form a direct sum decomposition of the space $V = \operatorname{ran} E \oplus \operatorname{ran}(I - E)$. We can generalize this to more than two projections.

1.1.27. Definition. Suppose that on a vector space V there exist projection operators $E_1, \dots, E_n$ such that

(a) $I_V = E_1 + E_2 + \dots + E_n$ and
(b) $E_i E_j = 0$ whenever $i \neq j$.

Then we say that $I_V = E_1 + E_2 + \dots + E_n$ is a RESOLUTION OF THE IDENTITY.

1.1.28. Proposition. *If $I_V = E_1 + E_2 + \dots + E_n$ is a resolution of the identity on a vector space V, then $V = \bigoplus_{k=1}^{n} \operatorname{ran} E_k$.*

1.1.29. Example. Let P be the plane in $\mathbb{R}^3$ whose equation is $x - z = 0$ and L be the line whose equations are $y = 0$ and $x = -z$. Let E be the projection of $\mathbb{R}^3$ along L onto P and F be the projection of $\mathbb{R}^3$ along P onto L. Then

$$[E] = \begin{bmatrix} \frac12 & 0 & \frac12 \\ 0 & 1 & 0 \\ \frac12 & 0 & \frac12 \end{bmatrix} \qquad \text{and} \qquad [F] = \begin{bmatrix} \frac12 & 0 & -\frac12 \\ 0 & 0 & 0 \\ -\frac12 & 0 & \frac12 \end{bmatrix}.$$

1.1.30. Definition. A complex number λ is an EIGENVALUE of an operator T on a vector space V if $\ker(T - \lambda I_V)$ contains a nonzero vector. Any such vector is an EIGENVECTOR of T associated with λ and $\ker(T - \lambda I_V)$ is the EIGENSPACE of T associated with λ. The set of all eigenvalues of the operator T is its POINT SPECTRUM and is denoted by $\sigma_p(T)$.

If M is an $n \times n$ matrix, then $\det(M - \lambda I_n)$ (where I_n is the $n \times n$ identity matrix) is a polynomial in λ of degree n. This is the CHARACTERISTIC POLYNOMIAL of M. A standard way of computing the eigenvalues of an operator T on a finite dimensional vector space is to find the zeros of the characteristic polynomial of its matrix representation. It is an easy consequence of the multiplicative property of the determinant function that the characteristic polynomial of an operator T on a vector space V is independent of the basis chosen for V and hence of the particular matrix representation of T that is used.

1.1.31. Example. The eigenvalues of the operator on (the real vector space) $\mathbb{R}^3$ whose matrix representation is $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ are -2 and $+2$, the latter having (both algebraic and geometric) multiplicity 2. The eigenspace associated with the negative eigenvalue is $\operatorname{span}\{(1, 0, -1)\}$ and the eigenspace associated with the positive eigenvalue is $\operatorname{span}\{(1, 0, 1), (0, 1, 0)\}$.

The central fact asserted by the finite dimensional vector space version of the *spectral theorem* is that every diagonalizable operator on such a space can be written as a linear combination of projection operators where the coefficients of the linear combination are the eigenvalues of the operator and the ranges of the projections are the corresponding eigenspaces. Thus if T is a diagonalizable operator on a finite dimensional vector space V, then V has a basis consisting of eigenvectors of T.

Here is a formal statement of the theorem.

1.1.32. Theorem (Spectral Theorem: finite dimensional vector space version)**.** *Suppose that T is a diagonalizable operator on a finite dimensional vector space V. Let $\lambda_1, \dots, \lambda_n$ be the (distinct) eigenvalues of T. Then there exists a resolution of the identity $I_V = E_1 + \dots + E_n$, where for each k the range of the projection E_k is the eigenspace associated with λ_k, and furthermore*

$$T = \lambda_1 E_1 + \dots + \lambda_n E_n\,.$$

PROOF. A proof of this theorem can be found in [27] on page 212.

1.1.33. Example. Let T be the operator on (the real vector space) $\mathbb{R}^2$ whose matrix representation is $\begin{bmatrix} -7 & 8 \\ -16 & 17 \end{bmatrix}$.

(a) The characteristic polynomial for T is $c_{T}(\lambda) = \lambda^2 - 10\lambda + 9$.

(b) The eigenspace M_1 associated with the eigenvalue 1 is $\operatorname{span}\{(1, 1)\}$.

(c) The eigenspace M_2 associated with the eigenvalue 9 is $\operatorname{span}\{(1, 2)\}$.

(d) We can write T as a linear combination of projection operators. In particular,

$$T = 1 \cdot E_1 + 9 \cdot E_2 \text{ where } [E_1] = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \text{ and } [E_2] = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}.$$

(e) Notice that the sum of $[E_1]$ and $[E_2]$ is the identity matrix and that their product is the zero matrix.

(f) The matrix $S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ diagonalizes $[T]$. That is, $S^{-1}\,[T]\,S = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$.

(g) A matrix representing $\sqrt{T}$ is $\begin{bmatrix} -1 & 2 \\ -4 & 5 \end{bmatrix}$.

1.1.34. Exercise. Let T be the operator on $\mathbb{R}^3$ whose matrix representation is $\begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 2 \\ 1 & -1 & 0 \end{bmatrix}$.

(a) Find the characteristic and minimal polynomials of T.

(b) What can be concluded from the form of the minimal polynomial?

(c) Find a matrix which diagonalizes T. What is the diagonal form of T produced by this matrix?
(d) Find (the matrix representation of) $\sqrt{T}$.

1.1.35. Definition. A vector space operator T is NILPOTENT if $T^n = \mathbf{0}$ for some $n \in \mathbb{N}$.

An operator on a finite dimensional vector space need not be diagonalizable. If it is not, how close to diagonalizable is it? Here is one answer.

1.1.36. Theorem. *Let T be an operator on a finite dimensional vector space V. Suppose that the minimal polynomial for T factors completely into linear factors*

$$m_T(x) = (x - \lambda_1)^{r_1} \dots (x - \lambda_k)^{r_k}$$

where $\lambda_1, \dots \lambda_k$ are the (distinct) eigenvalues of T. For each j let $W_j = \ker(T - \lambda_j I)^{r_j}$ and E_j be the projection of V onto W_j along $W_1 + \dots + W_{j-1} + W_{j+1} + \dots + W_k$. Then

$$V = W_1 \oplus W_2 \oplus \dots \oplus W_k,$$

each W_j is invariant under T, and $I = E_1 + \dots + E_k$. Furthermore, the operator

$$D = \lambda_1 E_1 + \dots + \lambda_k E_k$$

is diagonalizable, the operator

$$N = T - D$$

is nilpotent, and N commutes with D.

PROOF. See [27], pages 222–223.

1.1.37. Definition. Since, in the preceding theorem, $T = D + N$ where D is diagonalizable and N is nilpotent, we say that D is the DIAGONALIZABLE PART of T and N is the NILPOTENT PART of T.

1.1.38. Exercise. Let T be the operator on $\mathbb{R}^3$ whose matrix representation is $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$.

(a) Find the characteristic and minimal polynomials of T.
(b) Find the eigenspaces of T.
(c) Find the diagonalizable part D and nilpotent part N of T.
(d) Find a matrix which diagonalizes D. What is the diagonal form of D produced by this matrix?
(e) Show that D commutes with N.

1.2. Normal Operators on an Inner Product Space

1.2.1. Definition. Let V be a vector space. A function s which associates to each pair of vectors x and y in V a scalar $s(x, y)$ is an SEMI-INNER PRODUCT on V provided that for every x, y, $z \in V$ and $\alpha \in \mathbb{K}$ the following four conditions are satisfied:

(a) $s(x + y, z) = s(x, z) + s(y, z)$;
(b) $s(\alpha x, y) = \alpha s(x, y)$;
(c) $s(x, y) = \overline{s(y, x)}$; and
(d) $s(x, x) \geq 0$.

If, in addition, the function s satisfies

(e) For every nonzero x in V we have $s(x, x) > 0$.

then s is an INNER PRODUCT on V. We will usually write the inner product of two vectors x and y as $\langle x, y \rangle$ rather than $s(x, y)$.
The overline in (c) denotes complex conjugation (so is redundant in the case of a real vector space). Conditions (a) and (b) show that a semi-inner product is linear in its first variable. Conditions (a)

and (b) of proposition 1.2.3 say that a complex inner product is CONJUGATE LINEAR in its second variable. When a scalar valued function of two variables on a complex semi-inner product space is linear in one variable and conjugate linear in the other, it is often called SESQUILINEAR. (The prefix "sesqui-" means "one and a half".) We will also use the term *sesquilinear* for a bilinear function on a real semi-inner product space. Taken together conditions (a)–(d) say that the inner product is a *positive definite conjugate symmetric sesquilinear form.*

1.2.2. Notation. If V is a vector space which has been equipped with an semi-inner product and $x \in V$ we introduce the abbreviation

$$\|x\| := \sqrt{s(x,x)}$$

which is read *the norm of* x or *the length of* x. (This somewhat optimistic terminology is justified, at least when s is an inner product, in proposition 1.2.16 below.)

1.2.3. Proposition. *If x, y, and z are vectors in a space with semi-inner product s and $\alpha \in \mathbb{K}$, then*

(a) $s(x, y+z) = s(x,y) + s(x,z)$,
(b) $s(x, \alpha y) = \overline{\alpha} s(x,y)$, *and*
(c) $s(x,x) = 0$ *if and only if* $x = \mathbf{0}$.

1.2.4. Example. For vectors $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ belonging to $\mathbb{K}^n$ define

$$\langle x, y \rangle = \sum_{k=1}^{n} x_k \overline{y_k}\,.$$

Then $\mathbb{K}^n$ is an inner product space.

1.2.5. Example. Let l_2 be the set of all square summable sequences of complex numbers. (A sequence $x = (x_k)_{k=1}^{\infty}$ is SQUARE SUMMABLE if $\sum_{k=1}^{\infty} |x_k|^2 < \infty$.) (The vector space operations are defined pointwise.) For vectors $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ belonging to l_2 define

$$\langle x, y \rangle = \sum_{k=1}^{\infty} x_k \overline{y_k}\,.$$

Then l_2 is an inner product space. (It must be shown, among other things, that the series in the preceding definition actually converges.)

1.2.6. Example. For $a < b$ let $\mathcal{C}([a,b])$ be the family of all continuous complex valued functions on the interval $[a,b]$. For every f, $g \in \mathcal{C}([a,b])$ define

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\,dx.$$

Then $\mathcal{C}([a,b])$ is an inner product space.

Here is the single most useful fact about semi-inner products. It is an inequality attributed variously to Cauchy, Schwarz, Bunyakowsky, or combinations of the three.

1.2.7. Theorem. *Let s be a semi-inner product defined on a vector space V. Then* Schwarz inequality

$$|s(x,y)| \le \|x\|\,\|y\|.$$

holds for all vectors x and y.

Hint for proof. Fix x, $y \in V$. For every $\alpha \in \mathbb{K}$ we know that

$$0 \le s(x - \alpha y, x - \alpha y). \tag{1.1}$$

Expand the right hand side of (1.1) into four terms and write $s(y,x)$ in polar form: $s(y,x) = re^{i\theta}$, where $r > 0$ and $\theta \in \mathbb{R}$. Then in the resulting inequality consider those α of the form $e^{-i\theta}t$ where

$t \in \mathbb{R}$. Notice that now the right side of (1.1) is a quadratic polynomial in t. What can you say about its discriminant?

1.2.8. Proposition. *Let V be a vector space with semi-inner product s and let $z \in V$. Then $s(z,z) = 0$ if and only if $s(z,y) = 0$ for all $y \in V$.*

1.2.9. Proposition. *Let V be a vector space on which a semi-inner product s has been defined. Then the set $L := \{z \in V : s(z,z) = 0\}$ is a vector subspace of V and the quotient vector space V/L can be made into an inner product space by defining*

$$\langle\, [x], [y]\,\rangle := s(x,y)$$

for all $[x]$, $[y] \in V/L$.

The next proposition shows that an inner product is continuous in its first variable. Conjugate symmetry then guarantees continuity in the second variable.

1.2.10. Proposition. *If (x_n) is a sequence in an inner product space V which converges to a vector $a \in V$, then $\langle x_n, y\rangle \to \langle a, y\rangle$ for every $y \in V$.*

1.2.11. Example. If (a_k) is a square summable sequence of real numbers, then the series $\sum_{k=1}^{\infty} k^{-1}a_k$ converges absolutely.

1.2.12. Exercise. Let $0 < a < b$.

(a) If f and g are continuous real valued functions on the interval $[a,b]$, then

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \le \int_a^b \big(f(x)\big)^2\,dx \cdot \int_a^b \big(g(x)\big)^2\,dx.$$

(b) Use part (a) to find numbers M and N (depending on a and b) such that

$$M(b-a) \le \ln\left(\frac{b}{a}\right) \le N(b-a).$$

(c) Use part (b) to show that $2.5 \le e \le 3$.

Hint for proof. For (b) try $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{\sqrt{x}}$. Then try $f(x) = \dfrac{1}{x}$ and $g(x) = 1$. For (c) try $a = 1$ and $b = 3$. Then try $a = 2$ and $b = 5$.

1.2.13. Definition. Let V be a vector space. A function $\|\ \|\colon V \to \mathbb{R}\colon x \mapsto \|x\|$ is a NORM on V if

(a) $\|x + y\| \le \|x\| + \|y\|$ for all x, $y \in V$;
(b) $\|\alpha x\| = |\alpha|\,\|x\|$ for all $x \in V$ and $\alpha \in \mathbb{K}$; and
(c) if $\|x\| = 0$, then $x = \mathbf{0}$.

The expression $\|x\|$ may be read as "the *norm* of x" or "the *length* of x". If the function $\|\ \|$ satisfies (a) and (b) above (but perhaps not (c)) it is a SEMINORM on V.

A vector space on which a norm has been defined is a NORMED LINEAR SPACE (or NORMED VECTOR SPACE). A vector in a normed linear space which has norm 1 is a UNIT VECTOR.

1.2.14. Exercise. Show why it is clear from the definition that $\|\mathbf{0}\| = 0$ and that norms (and seminorms) can take on only positive values.

1.2.15. Example. Let $\mathcal{C}(X)$ be the family of all continuous complex valued functions on a compact Hausdorff space X. Under the UNIFORM NORM

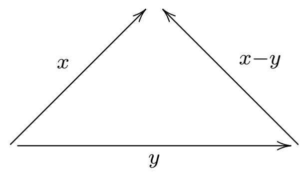
$$\|f\|_u := \sup\{|f(x)| : 0 \le x \le 1\}$$

$\mathcal{C}(X)$ is a normed linear space. If the space X is not compact we can make the set $\mathcal{C}_b(X)$ of all *bounded* continuous complex valued functions on X into a normed linear space in the same way.

Every inner product space is a normed linear space.

1.2.16. Proposition. *Let V be an inner product space. The map $x \mapsto \|x\|$ defined on V in 1.2.2 is a norm on V.*

Every normed linear space is a metric space. More precisely, a norm on a vector space induces a metric d, which is defined by $d(x, y) = \|x - y\|$. That is, the distance between two vectors is the length of their difference.



If no other metric is specified we always regard a normed linear space as a metric space under this induced metric. Thus every metric (and hence every topological) concept makes sense in a (semi)normed linear space.

1.2.17. Proposition. *Let V be a normed linear space. Define $d\colon V \times V \to \mathbb{R}$ by $d(x, y) = \|x - y\|$. Then d is a metric on V. If V is only a seminormed space, then d is a pseudometric.*

When there is a topology on a vector space, in particular in normed linear spaces, we reserve the word "operator" for those linear mappings from the space into itself which are continuous. We are usually not made aware of this conflicting terminology in elementary linear algebra because that subject focuses primarily on finite dimensional vector and inner product spaces where the question is moot: on finite dimensional normed linear spaces all linear maps are automatically continuous (see proposition 3.3.6).

1.2.18. Definition. An OPERATOR on a normed linear space V is a *continuous* linear map from V into itself. (Notice the difference between this definition and the one in 1.1.12.)

1.2.19. Definition. Vectors x and y in an inner product space V are ORTHOGONAL (or PERPENDICULAR) if $\langle x, y\rangle = 0$. In this case we write $x \perp y$. Subsets A and B of V are ORTHOGONAL if $a \perp b$ for every $a \in A$ and $b \in B$. In this case we write $A \perp B$.

1.2.20. Definition. If M and N are subspaces of an inner product space V we use the notation $V = M \oplus N$ to indicate not only that V is the (vector space) direct sum of M and N but also that M and N are orthogonal. Thus we say that V is the (INTERNAL) ORTHOGONAL DIRECT SUM of M and N.

1.2.21. Proposition. *Let a be a vector in an inner product space V. Then $a \perp x$ for every $x \in V$ if and only if $a = 0$.*

1.2.22. Proposition. *In an inner product space $x \perp y$ if and only if $\|x + \alpha y\| = \|x - \alpha y\|$ for all scalars α.*

1.2.23. Proposition (The Pythagorean theorem)**.** *If $x \perp y$ in an inner product space, then*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 .$$

1.2.24. Definition. Let V and W be inner product spaces. For (v, w) and (v', w') in $V \times W$ and $\alpha \in \mathbb{C}$ define

$$(v, w) + (v', w') = (v + v', w + w')$$

and

$$\alpha(v, w) = (\alpha v, \alpha w) .$$

This results in a vector space, which is the *(external) direct sum* of V and W. To make it into an inner product space define

$$\langle (v,w),(v',w')\rangle = \langle v,v'\rangle + \langle w,w'\rangle.$$

This makes the direct sum of V and W into an inner product space. It is the (EXTERNAL ORTHOGONAL) DIRECT SUM of V and W and is denoted by $V \oplus W$.

Notice that the same notation $\oplus$ is used for both internal and external direct sums and for both vector space direct sums (see definition 1.1.18) and orthogonal direct sums. So when we see the symbol $V \oplus W$ it is important to know which category we are in: vector spaces or inner product spaces, especially as it is common practice to omit the word "orthogonal" as a modifier to "direct sum" even in cases when it is intended.

1.2.25. Example. In $\mathbb{R}^2$ let M be the x-axis and L be the line whose equation is $y = x$. If we think of $\mathbb{R}^2$ as a (real) vector space, then it is correct to write $\mathbb{R}^2 = M \oplus L$. If, on the other hand, we regard $\mathbb{R}^2$ as a (real) inner product space, then $\mathbb{R}^2 \neq M \oplus L$ (because M and L are not perpendicular).

1.2.26. Proposition. *Let V be an inner product space. The inner product on V, regarded as a map from $V \oplus V$ into $\mathbb{C}$, is continuous. So is the norm, regarded as a map from V into $\mathbb{R}$.*

Concerning the proof of the preceding proposition, notice that the maps $(v,v') \mapsto \|v\| + \|v'\|$, $(v,v') \mapsto \sqrt{\|v\|^2 + \|v'\|^2}$, and $(v,v') \mapsto \max\{\|v\|,\|v'\|\}$ are all norms on $V \oplus V$. Which one is induced by the inner product on $V \oplus V$? Why does it not matter which one we use in proving that the inner product is continuous?

1.2.27. Proposition (The parallelogram law)**.** *If x and y are vectors in an inner product space, then*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2\,.$$

While every inner product induces a norm not every norm comes from an inner product.

1.2.28. Example. There is no inner product on space $\mathcal{C}([0,1])$ which induces the uniform norm.

Hint for proof. Use the preceding proposition.

1.2.29. Proposition (The polarization identity)**.** *If x and y are vectors in a complex inner product space, then*

$$\langle x,y\rangle = \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2)\,.$$

What is the correct identity for a *real* inner product space?

1.2.30. Notation. Let V be an inner product space, $x \in V$, and A, $B \subseteq V$. If $x \perp a$ for every $a \in A$, we write $x \perp A$; and if $a \perp b$ for every $a \in A$ and $b \in B$, we write $A \perp B$. We define $A^\perp$, the ORTHOGONAL COMPLEMENT of A, to be $\{x \in V : x \perp A\}$. Since the phrase "orthogonal complement of A" is a bit unwieldy, especially when used repeatedly, the symbol "$A^\perp$" is usually pronounced "A perp". We write $A^{\perp\perp}$ for $\left(A^\perp\right)^\perp$.

1.2.31. Proposition. *If A is a subset of an inner product space V, then $A^\perp$ is a closed linear subspace of V.*

1.2.32. Proposition (Gram-Schmidt orthonormalization)**.** *If (v^k) is a linearly independent sequence in an inner product space V, then there exists an orthonormal sequence (e^k) in V such that* $\operatorname{span}\{v^1,\dots,v^n\} = \operatorname{span}\{e^1,\dots,e^n\}$ *for every $n \in \mathbb{N}$.*

1.2.33. Corollary. *If M is a subspace of a finite dimensional inner product space V, then $V = M \oplus M^\perp$.*

It will be convenient to say of sequences that they *eventually* have a certain property or that they *frequently* have that property.

1.2.34. Definition. Let (x_n) be a sequence of elements of a set S and P be some property that members of S may possess. We say that the sequence (x_n) EVENTUALLY has property P if there exists $n_0 \in \mathbb{N}$ such that x_n has property P for every $n \geq n_0$. (Another way to say the same thing: x_n has property P for all but finitely many n.)

1.2.35. Example. Denote by l_c the vector space consisting of all sequences of complex numbers which are eventually zero; that is, the set of all sequences with only finitely many nonzero entries. They are also referred to as the sequences with *finite support*. The vector space operations are defined pointwise. We make the space l_c into an inner product space by defining $\langle a, b \rangle = \langle\, (a_n), (b_n)\, \rangle := \sum_{k=1}^{\infty} a_n \overline{b_n}$.

1.2.36. Definition. Let (x_n) be a sequence of elements of a set S and P be some property that members of S may possess. We say that the sequence (x_n) FREQUENTLY has property P if for every $k \in \mathbb{N}$ there exists $n \geq k$ such that x_n has property P. (An equivalent formulation: x_n has property P for infinitely many n.)

1.2.37. Example. Let $V = l_2$ be the inner product space of all square-summable sequences of complex numbers (see example 1.2.5) and $M = l_c$ (see example 1.2.35). Then the conclusion of 1.2.33 fails.

1.2.38. Definition. A LINEAR FUNCTIONAL on a vector space V is a linear map from V into its scalar field. The set of all linear functionals on V is the (ALGEBRAIC) DUAL SPACE of V. We will use the notation $V^{\#}$ for the algebraic dual space.

1.2.39. Theorem (Riesz-Fréchet Theorem)**.** *If $f \in V^{\#}$ where V is a finite dimensional inner product space, then there exists a unique vector a in V such that*

$$f(x) = \langle x, a \rangle$$

for all x in V.

We will prove shortly (in 4.6.2)that every continuous linear functional on an *arbitrary* inner product space has the above representation. The finite dimensional version stated here is a special case, since every linear map on a finite dimensional inner product space is continuous (see proposition 3.3.6).

1.2.40. Definition. Let $T\colon V \to W$ be a linear transformation between complex inner product spaces. If there exists a function $T^*\colon W \to V$ which satisfies

$$\langle T\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, T^*\mathbf{w} \rangle$$

for all $\mathbf{v} \in V$ and $\mathbf{w} \in W$, then T^* is the ADJOINT (or CONJUGATE TRANSPOSE, or HERMITIAN CONJUGATE) of T.

1.2.41. Proposition. *If $T\colon V \to W$ is a linear map between finite dimensional inner product spaces, then T^* exists.*

Hint for proof. The functional $\phi\colon V \times W \to \mathbb{C}\colon (v, w) \mapsto \langle Tv, w \rangle$ is sesquilinear. Fix $w \in W$ and define $\phi_w\colon V \to \mathbb{C}\colon v \mapsto \phi(v, w)$. Then $\phi_w \in V^{\#}$. Use the *Riesz-Fréchet theorem* (1.2.39).

1.2.42. Proposition. *If $T\colon V \to W$ is a linear map between finite dimensional inner product spaces, then the function T^* defined above is linear and $T^{**} = T$.*

1.2.43. Theorem (The fundamental theorem of linear algebra)**.** *If $T\colon V \to W$ is a linear map between finite dimensional inner product spaces, then*

$$\ker T^* = (\operatorname{ran} T)^{\perp} \quad \textit{and} \quad \operatorname{ran} T^* = (\ker T)^{\perp}.$$

1.2.44. Definition. An operator U on an inner product space is UNITARY if $UU^* = U^*U = I$, that is if $U^* = U^{-1}$.

1.2.45. Definition. Two operators R and T on an inner product space V are UNITARILY EQUIVALENT if there exists a unitary operator U on V such that $R = U^*TU$.

1.2.46. Proposition. *If V is an inner product space, then unitary equivalence is in fact an equivalence relation on $\mathfrak{L}(V)$.*

1.2.47. Definition. An operator T on a finite dimensional inner product space V is UNITARILY DIAGONALIZABLE if there exists an orthonormal basis for V with respect to which T is diagonal. Equivalently, an operator on a finite dimensional inner product space *with basis* is diagonalizable if it is unitarily equivalent to a diagonal operator.

1.2.48. Definition. An operator T on an inner product space is SELF-ADJOINT (or HERMITIAN) if $T^* = T$.

1.2.49. Definition. A projection P in an inner product space is an ORTHOGONAL PROJECTION if it is self-adjoint. If M is the range of an orthogonal projection we will adopt the notation P_M for the projection rather than the more cumbersome $E_{M^\perp M}$.

CAUTION. A projection on a vector space or a normed linear space is linear and idempotent, while an orthogonal projection on an inner product space is linear, idempotent, and self-adjoint. This otherwise straightforward situation is somewhat complicated by a common tendency to refer to orthogonal projections simply as "projections". In fact, later in these notes we will adopt this very convention. In inner product spaces $\oplus$ usually indicates orthogonal direct sum and "projection" usually means "orthogonal projection". In many elementary linear algebra texts, where everything happens in $\mathbb{R}^n$, it can be quite exasperating trying to divine whether on any particular page the author is treating $\mathbb{R}^n$ as a vector space or as an inner product space.

1.2.50. Proposition. *If P is an orthogonal projection on an inner product space V, then we have the orthogonal direct sum decomposition $V = \ker P \oplus \operatorname{ran} P$.*

1.2.51. Definition. If $I_V = P_1 + P_2 + \cdots + P_n$ is a resolution of the identity in an inner product space V and each P_k is an orthogonal projection, then we say that $I = P_1 + P_2 + \cdots + P_n$ is an ORTHOGONAL RESOLUTION OF THE IDENTITY.

1.2.52. Proposition. *If $I_V = P_1 + P_2 + \cdots + P_n$ is an orthogonal resolution of the identity on an inner product space V, then $V = \bigoplus_{k=1}^n \operatorname{ran} P_k$.*

1.2.53. Definition. An operator N on an inner product space is NORMAL if $NN^* = N^*N$.

Two great triumphs of linear algebra are the *spectral theorem* for operators on a (complex) finite dimensional inner product space (see 1.2.54), which gives a simply stated necessary and sufficient condition for an operator to be unitarily diagonalizable, and theorem 1.2.55, which gives a complete classification of those operators.

1.2.54. Theorem (Spectral Theorem for Finite Dimensional Complex Inner Product Spaces)**.** *Let T be an operator on a finite dimensional inner product space V with (distinct) eigenvalues $\lambda_1, \dots, \lambda_n$. Then T is unitarily diagonalizable if and only if it is normal. If T is normal, then there exists an orthogonal resolution of the identity $I_V = P_1 + \cdots + P_n$, where for each k the range of the orthogonal projection P_k is the eigenspace associated with λ_k, and furthermore*

$$T = \lambda_1 P_1 + \cdots + \lambda_n P_n\,.$$

PROOF. See [39], page 227.

1.2.55. Theorem. *Two normal operators on a finite dimensional inner product space are unitarily equivalent if and only if they have the same eigenvalues each with the same multiplicity; that is, if and only if they have the same characteristic polynomial.*

PROOF. See [27], page 357.

Much of the remainder of this course is about the definitely nontrivial adventure of finding appropriate generalizations of the preceding two results to the infinite dimensional setting and the astonishing landscapes which come into view along the way.

1.2.56. Exercise. Let $N = \dfrac{1}{3}\begin{bmatrix} 4+2i & 1-i & 1-i \\ 1-i & 4+2i & 1-i \\ 1-i & 1-i & 4+2i \end{bmatrix}$.

(a) The matrix N is normal.

(b) Thus according to the *spectral theorem* N can be written as a linear combination of orthogonal projections. Explain clearly how to do this (and carry out the computation).

CHAPTER 2

A VERY BRIEF DIGRESSION ON THE LANGUAGE OF CATEGORIES

In mathematics we study things (objects) and certain mappings between them (morphisms). To mention just a few, sets and functions, groups and homomorphisms, topological spaces and continuous maps, vector spaces and linear transformations, and Hilbert spaces and bounded linear maps. These examples come from different branches of mathematics—set theory, group theory, topology, linear algebra, and functional analysis, respectively. But these different areas have many things in common: in many field terms like product, coproduct, subobject, quotient, pullback, isomorphism, and projective limit appear. Category theory is an attempt to unify and formalize some of these common concepts. In a sense, category theory is the study of what different branches of mathematics have in common. Perhaps a better description is: categorical language tells us "how things work", not "what they are".

In these notes, indeed any text at this level, ubiquitously uses the language of sets without assuming a detailed prior study of axiomatic set theory. Similarly, we will cheerfully use the *language* of categories without first embarking on a foundationally satisfactory study of category theory (which itself is a large and important area of research). Sometimes textbooks make learning even the language of categories challenging by leaning heavily on one's algebraic background. Just as most people are comfortable using the language of sets (whether or not they have made a serious study of set *theory*), nearly everyone should find the use of categorical language both convenient and enlightening without elaborate prerequisites.

For those who wish to delve more deeply into the subject I can recommend a very gentle entrée to the world of categories which appears as Chapter 3 of Semadeni's beautiful book [**43**]. A classic text written by one of the founders of the subject is [**31**]. A more recent text is [**32**].

By pointing to unifying principles the language of categories often provides striking insight into "the way things work" in mathematics. Equally importantly, one gains in efficiency by not having to go through essentially the same arguments over and over again in just slightly different contexts.

For the minute we do little more than define "object" and "morphism", and give a few examples.

2.1. Objects and Morphisms

2.1.1. Definition. Let $\mathfrak{A}$ be a class, whose members we call OBJECTS. For every pair (S, T) of objects we associate a set $\mathfrak{Mor}(S, T)$, whose members we call MORPHISMS (or ARROWS) from S to T. We assume that $\mathfrak{Mor}(S, T)$ and $\mathfrak{Mor}(U, V)$ are disjoint unless $S = U$ and $T = V$.

We suppose further that there is an operation $\circ$ (called COMPOSITION) that associates with every $\alpha \in \mathfrak{Mor}(S, T)$ and every $\beta \in \mathfrak{Mor}(T, U)$ a morphism $\beta \circ \alpha \in \mathfrak{Mor}(S, U)$ in such a way that:

(a) $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$ whenever $\alpha \in \mathfrak{Mor}(S, T)$, $\beta \in \mathfrak{Mor}(T, U)$, and $\gamma \in \mathfrak{Mor}(U, V)$;

(b) for every object S there is a morphism $I_S \in \mathfrak{Mor}(S, S)$ satisfying $\alpha \circ I_S = \alpha$ whenever $\alpha \in \mathfrak{Mor}(S, T)$ and $I_S \circ \beta = \beta$ whenever $\beta \in \mathfrak{Mor}(R, S)$.

Under these circumstances the class $\mathfrak{A}$, together with the associated families of morphisms, is a CATEGORY.

We will reserve the notation $S \xrightarrow{\alpha} T$ for a situation in which S and T are objects in some category and α is a morphism belonging to $\mathfrak{Mor}(S, T)$. As is the case with groups and vector spaces we usually omit the composition symbol $\circ$ and write $\beta\alpha$ for $\beta \circ \alpha$.

2.1.2. Example. The category **SET** has sets for objects and functions (maps) as morphisms.

2.1.3. Example. The category **AbGp** has Abelian groups for objects and group homomorphisms as morphisms.

2.1.4. Example. The category **VEC** has vector spaces for objects and linear transformations as morphisms.

2.1.5. Example. The category **TOP** has topological spaces for objects and continuous functions as morphisms.

2.1.6. Definition. Let $(A, +, M)$ be a vector space over $\mathbb{K}$ which is equipped with another binary operation $A \times A \to A$ where $(x, y) \mapsto x \cdot y$ in such a way that $(A, +, \cdot\,)$ is a ring. (The notation $x \cdot y$ is usually shortened to xy.)If additionally the equations

$$\alpha(xy) = (\alpha x)y = x(\alpha y) \tag{2.1}$$

hold for all x, $y \in A$ and $\alpha \in \mathbb{K}$, then $(A, +, M, \cdot\,)$ is an ALGEBRA over the field $\mathbb{K}$ (sometimes referred to as a LINEAR ASSOCIATIVE ALGEBRA). We abuse terminology in the usual way by writing such things as, "Let A be an algebra." We say that an algebra A is UNITAL if its underlying ring $(A, +, \cdot\,)$ has a multiplicative identity; that is, if there exists an element $\mathbf{1}_A \neq \mathbf{0}$ in A such that $\mathbf{1}_A \cdot x = x \cdot \mathbf{1}_A = x$ for all $x \in A$. And it is COMMUTATIVE if its ring is; that is, if $xy = yx$ for all x, $y \in A$.

A subset B of an algebra A is a SUBALGEBRA of A if it is an algebra under the operations it inherits from A. A subalgebra B of a unital algebra A is a UNITAL SUBALGEBRA if it contains the multiplicative identity of A.

CAUTION. To be a unital subalgebra it is *not* enough for B to have a multiplicative identity of its own; it must contain the identity of A. Thus, an algebra can be both unital and a subalgebra of A without being a unital subalgebra of A.

A map $f\colon A \to B$ between algebras is an (ALGEBRA) HOMOMORPHISM if it is a linear map between A and B as vector spaces which preserves multiplication (that is, $f(xy) = f(x)f(y)$ for all x, $y \in A$). In other words, an algebra homomorphism is a linear ring homomorphism. It is a UNITAL (ALGEBRA) HOMOMORPHISM if it preserves identities; that is, if both A and B are unital algebras and $f(\mathbf{1}_A) = 1_B$. The KERNEL of an algebra homomorphism $f\colon A \to B$ is, of course, $\{a \in A\colon f(a) = \mathbf{0}\}$.

If f^{-1} exists and is also an algebra homomorphism, then f is an ISOMORPHISM from A to B. If an isomorphism from A to B exists, then A and B are ISOMORPHIC.

2.1.7. Example. The category **ALG** has algebras for objects and algebra homomorphisms for morphisms.

The preceding examples are examples of *concrete categories*—that is, categories in which the objects are sets (together, usually, with additional structure) and the morphism are functions (usually preserving this extra structure). In these notes the categories of interest to us are concrete ones. Here (for those who are curious) is a more formal definition of *concrete category.*

2.1.8. Definition. A category $\mathfrak{A}$ together with a function $|\ \ |$ which assigns to each object A in $\mathfrak{A}$ a set $|A|$ is a CONCRETE category if the following conditions are satisfied:

(a) every morphism $A \xrightarrow{\ f\ } B$ is a function from $|A|$ to $|B|$;
(b) each identity morphism I_A in $\mathfrak{A}$ is the identity function on $|A|$; and
(c) composition of morphisms $A \xrightarrow{\ f\ } B$ and $B \xrightarrow{\ g\ } C$ agrees with composition of the functions $f\colon |A| \to |B|$ and $g\colon |B| \to |C|$.

If A is an object in a concrete category $\mathfrak{A}$, then $|A|$ is the UNDERLYING SET of A.

Although it is true that the categories of interest in these notes are concrete categories, it may nevertheless be interesting to see an example of a category that is *not* concrete.

2.1.9. Example. Let G be a monoid (that is, a semigroup with identity). Consider a category $\mathbf{C}$ having exactly one object, which we call $\star$. Since there is only one object there is only one family of morphisms $\mathfrak{Mor}(\star, \star)$, which we take to be G. Composition of morphisms is defined to be the monoid multiplication. That is, $a \circ b := ab$ for all $a, b \in G$. Clearly composition is associative and the identity element of G is the identity morphism. So $\mathbf{C}$ is a category.

2.1.10. Definition. In any concrete category we will call an injective morphism a MONOMORPHISM and a surjective morphism an EPIMORPHISM.

CAUTION. The definitions above reflect the original Bourbaki use of the term and are the ones most commonly adopted by mathematicians outside of category theory itself where "monomorphism" means "left cancellable" and "epimorphism" means "right cancellable". (Notice that the terms *injective* and *surjective* may not make sense when applied to morphisms in a category that is not concrete.)

A morphism $B \xrightarrow{g} C$ is LEFT CANCELLABLE if whenever morphisms $A \xrightarrow{f_1} B$ and $A \xrightarrow{f_2} B$ satisfy $gf_1 = gf_2$, then $f_1 = f_2$. Mac Lane suggested calling left cancellable morphisms MONIC morphisms. The distinction between monic morphisms and monomorphisms turns out to be slight. In these notes almost all of the morphisms we encounter are monic if and only if they are monomorphisms. As an easy exercise prove that any injective morphism in a (concrete) category is monic. The converse sometimes fails.

In the same vein Mac Lane suggested calling a *right cancellable* morphism (that is, a morphism $A \xrightarrow{f} B$ such that whenever morphisms $B \xrightarrow{g_1} C$ and $B \xrightarrow{g_2} C$ satisfy $g_1 f = g_2 f$, then $g_1 = g_2$) an EPIC morphism. Again it is an easy exercise to show that in a (concrete) category any epimorphism is epic. The converse, however, fails in some rather common categories.

2.1.11. Definition. The terminology for inverses of morphisms in categories is essentially the same as for functions. Let $S \xrightarrow{\alpha} T$ and $T \xrightarrow{\beta} S$ be morphisms in a category. If $\beta \circ \alpha = I_S$, then β is a LEFT INVERSE of α and, equivalently, α is a RIGHT INVERSE of β. We say that the morphism α is an ISOMORPHISM (or is INVERTIBLE) if there exists a morphism $T \xrightarrow{\beta} S$ which is both a left and a right inverse for α. Such a function is denoted by α^{-1} and is called the INVERSE of α.

In any concrete category one can inquire whether every bijective morphism (that is, every map which is both a monomorphism and an epimorphism) is an isomorphism. The answer is often a trivial *yes* (as in **SET**, **AbGp**, and **VEC**) or a trivial *no* (for example, in the category **POSET** of partially ordered sets and order preserving maps. But on occasion the answer turns out to be a fascinating and deep result (see for example the *open mapping theorem* 6.4.3).

2.1.12. Example. In the category **SET** every bijective morphism is an isomorphism.

2.1.13. Example. The category **LAT** has lattices as objects and lattice homomorphisms as morphisms. (A LATTICE is a partially ordered set in which every pair of elements has an infimum and a supremum and a LATTICE HOMOMORPHISM is a map between lattices which preserves infima and suprema.) In this category bijective morphisms are isomorphisms.

2.1.14. Example. If in the category $\mathbf{C}$ of example 2.1.9 the monoid G is a group then every morphism in $\mathbf{C}$ is an isomorphism.

2.2. Functors

2.2.1. Definition. If $\mathbf{A}$ and $\mathbf{B}$ are categories a COVARIANT FUNCTOR F from $\mathbf{A}$ to $\mathbf{B}$ (written $\mathbf{A} \xrightarrow{F} \mathbf{B}$) is a pair of maps: an OBJECT MAP F which associates with each object S in $\mathbf{A}$ an object $F(S)$ in $\mathbf{B}$ and a MORPHISM MAP (also denoted by F) which associates with each morphism $f \in \mathfrak{Mor}(S, T)$ in $\mathbf{A}$ a morphism $F(f) \in \mathfrak{Mor}(F(S), F(T))$ in $\mathbf{B}$, in such a way that

(a) $F(g \circ f) = F(g) \circ F(f)$ whenever $g \circ f$ is defined in $\mathbf{A}$; and
(b) $F(\mathrm{id}_S) = \mathrm{id}_{F(S)}$ for every object S in $\mathbf{A}$.

The definition of a CONTRAVARIANT FUNCTOR $\mathbf{A} \xrightarrow{F} \mathbf{B}$ differs from the preceding definition only in that, first, the morphism map associates with each morphism $f \in \mathfrak{Mor}(S,T)$ in $\mathbf{A}$ a morphism $F(f) \in \mathfrak{Mor}(F(T), F(S))$ in $\mathbf{B}$ and, second, condition (a) above is replaced by

(a') $F(g \circ f) = F(f) \circ F(g)$ whenever $g \circ f$ is defined in $\mathbf{A}$.

2.2.2. Example. A FORGETFUL FUNCTOR is a functor that maps objects and morphisms from a category $\mathbf{C}$ to a category $\mathbf{C}'$ with less structure or fewer properties. For example, if V is a vector space, the functor F which "forgets" about the operation of scalar multiplication on vector spaces would map V into the category of Abelian groups. (The Abelian group $F(V)$ would have the same set of elements as the vector space V and the same operation of addition, but it would have no scalar multiplication.) A linear map $T\colon V \to W$ between vector spaces would be taken by the functor F to a group homomorphism $F(T)$ between the Abelian groups $F(V)$ and $F(W)$.

Forgetful functor can "forget" about properties as well. If G is an object in the category of Abelian groups, the functor which "forgets" about commutativity in Abelian groups would take G into the category of groups.

It was mentioned in the preceding section that all the categories that are of interest in these notes are concrete categories (ones in which the objects are sets with additional structure and the morphisms are maps which preserve, in some sense, this additional structure). We will have several occasions to use a special type of forgetful functor—one which forgets about all the structure of the objects except the underlying set and which forgets any structure preserving properties of the morphisms. If A is an object in some concrete category $\mathbf{C}$, we denote by $|A|$ its underlying set. And if $A \xrightarrow{f} B$ is a morphism in $\mathbf{C}$ we denote by $|f|$ the map from $|A|$ to $|B|$ regarded simply as a function between sets. It is easy to see that $|\ \ |$, which takes objects in $\mathbf{C}$ to objects in $\mathbf{SET}$ (the category of sets and maps) and morphisms in $\mathbf{C}$ to morphisms in $\mathbf{SET}$, is a covariant functor.

In the category $\mathbf{VEC}$ of vector spaces and linear maps, for example, $|\ \ |$ causes a vector space V to "forget" about both its addition and scalar multiplication ($|V|$ is just a set). And if $T\colon V \to W$ is a linear transformation, then $|T|\colon |V| \to |W|$ is just a map between sets—it has "forgotten" about preserving the operations.

2.2.3. Notation. Let $f\colon S \to T$ be a function between sets. Then we define $f^{\to}(A) = \{f(x)\colon x \in A\}$ and $f^{\leftarrow}(B) = \{x \in S\colon f(x) \in B\}$. We say that $f^{\to}(A)$ is the IMAGE OF A UNDER f and that $f^{\leftarrow}(B)$ is the PREIMAGE OF B UNDER f.

2.2.4. Definition. A partially ordered set is ORDER COMPLETE if every nonempty subset has a supremum (that is, a least upper bound) and an infimum (a greatest lower bound).

2.2.5. Definition. Let S be a set. Then the POWER SET of S, denoted by $\mathfrak{P}(S)$, is the family of all subsets of S.

2.2.6. Example (The power set functors)**.** Let S be a nonempty set.

(a) The power set $\mathfrak{P}(S)$ of S partially ordered by $\subseteq$ is order complete.
(b) The class of order complete partially ordered sets and order preserving maps is a category.
(c) For each function f between sets let $\mathfrak{P}(f) = f^{\to}$. Then $\mathfrak{P}$ is a covariant functor from the category of sets and functions to the category of order complete partially ordered sets and order preserving maps.
(d) For each function f between sets let $\mathfrak{P}(f) = f^{\leftarrow}$. Then $\mathfrak{P}$ is a contravariant functor from the category of sets and functions to the category of order complete partially ordered sets and order preserving maps.

2.2.7. Definition. Let $T\colon V \to W$ be a linear map between vector spaces. For every $g \in W^{\#}$ let $T^{\#}(g) = g\,T$. Notice that $T^{\#}(g) \in V^{\#}$. The map $T^{\#}$ from the vector space $W^{\#}$ into the vector space $V^{\#}$ is the (vector space) ADJOINT map of T.

2.2.8. Example. The pair of maps $V \mapsto V^{\#}$ (taking a vector space to its algebraic dual) and $T \mapsto T^{\#}$ (taking a linear map to its dual) is a contravariant functor from the category **VEC** of vector spaces and linear maps to itself.

2.2.9. Example. Let X and Y be topological spaces and $\phi\colon X \to Y$ be continuous. Define $\mathcal{C}\phi$ on $\mathcal{C}(Y)$ by

$$\mathcal{C}\phi(g) = g \circ \phi$$

for all $g \in \mathcal{C}(Y)$. Then the pair of maps $X \mapsto \mathcal{C}(X)$ and $\phi \mapsto \mathcal{C}\phi$ is a contravariant functor from the category of topological spaces and continuous maps to the category of unital algebras and unital algebra homomorphisms.

CHAPTER 3

NORMED LINEAR SPACES

3.1. Norms

In the world of analysis the predominant denizens are function spaces, vector spaces of real or complex valued functions. To be of interest to an analyst such a space should come equipped with a topology. Often the topology is a metric topology, which in turn frequently comes from a norm (see proposition 1.2.17).

3.1.1. Example. For $x = (x_1, \dots, x_n) \in \mathbb{K}^n$ let $\|x\| = \left(\sum_{k=1}^n |x_k|^2\right)^{1/2}$. This is the USUAL NORM (or EUCLIDEAN NORM) on $\mathbb{K}^n$; unless the contrary is explicitly stated, $\mathbb{K}^n$ when regarded as a normed linear space will always be assumed to possess this norm. It is clear that this norm induces the usual (Euclidean) metric on $\mathbb{K}^n$.

3.1.2. Example. For $x = (x_1, \dots, x_n) \in \mathbb{K}^n$ let $\|x\|_1 = \sum_{k=1}^n |x_k|$. The function $x \mapsto \|x\|_1$ is easily seen to be a norm on $\mathbb{K}^n$. It is sometimes called the 1-NORM on $\mathbb{K}^n$. It induces the so-called *taxicab metric* on $\mathbb{R}^2$.

The next four examples are actually a single example. The first two are special cases of the third, which is in turn a special case of the fourth.

3.1.3. Example. For $x = (x_1, \dots, x_n) \in \mathbb{K}^n$ let $\|x\|_\infty = \max\{\, |x_k| \colon 1 \le k \le n\}$. This defines a norm on $\mathbb{K}^n$; it is the UNIFORM NORM on $\mathbb{K}^n$. It induces the uniform metric on $\mathbb{K}^n$. An alternative notation for $\|x\|_\infty$ is $\|x\|_U$.

3.1.4. Example. The set l_∞ of all bounded sequences of complex numbers is clearly a vector space under pointwise operations of addition and scalar multiplication. For $x = (x_1, x_2, \dots,) \in l_\infty$ let $\|x\|_\infty = \sup\{\, |x_k| \colon 1 \le k\}$. This defines a norm on l_∞; it is the UNIFORM NORM on l_∞.

3.1.5. Example. Let S be a nonempty set. If f is a bounded K-valued function on S, let

$$\|f\|_u := \sup\{|f(x)| \colon x \in S\}\,.$$

This is a norm on the vector space $\mathcal{B}(S)$ of all bounded complex valued functions on S and is called the UNIFORM NORM. Clearly this norm gives rise to the uniform metric on $\mathcal{B}(S)$. Notice that examples 3.1.3 and 3.1.4 are special cases of this one. (Let $S = \mathbb{N}_n := \{1, 2, \dots, n\}$ or $S = \mathbb{N}$.)

3.1.6. Example. It is easy to make a substantial generalization of the preceding example by replacing the field $\mathbb{K}$ by an arbitrary normed linear space. Suppose then that S is a nonempty set and V is a normed linear space. For f in the vector space $\mathcal{B}(S, V)$ of all bounded V-valued functions on S, let

$$\|f\|_u := \sup\{\|f(x)\| \colon x \in S\}\,.$$

Then this is a norm and it is called the UNIFORM NORM on $\mathcal{B}(S, V)$. (Why is the word "easy" in the first sentence of this example appropriate?)

3.1.7. Definition. Let S be a set, V be a normed linear space, and $\mathcal{F}(S, V)$ be the vector space of all V-valued functions on S. Consider a sequence (f_n) of functions in $\mathcal{F}(S, V)$. If there is a function g in $\mathcal{F}(S, V)$ such that

$$\sup\{\|f_n(x) - g(x)\| \colon x \in S\} \to 0 \quad \text{as } n \to \infty,$$

then we say that the sequence (f_n) CONVERGES UNIFORMLY to g and write $f_n \to g\,(\text{unif})$. The function g is the UNIFORM LIMIT of the sequence (f_n). Notice that if g and all the f_n's belong to $\mathcal{B}(S,V)$, then uniform convergence of (f_n) to g is just convergence of (f_n) to g with respect to the uniform metric.

There are many ways in which sequences of functions converge. Arguably the two most common modes of convergence are uniform convergence, which we have just discussed, and pointwise convergence.

3.1.8. Definition. Let S be a set, V be a normed linear space, and (f_n) be a sequence in $\mathcal{F}(S,V)$. If there is a function g such that

$$f_n(x) \to g(x) \qquad \text{for all } x \in S,$$

then (fn) CONVERGES POINTWISE to g. In this case we write

$$f_n \to g\,(\text{ptws}).$$

The function g is the POINTWISE LIMIT of the f_n's.

If (f_n) is an increasing sequence of real (or extended real) valued functions and $f_n \to g\,(\text{ptws})$, we write $f_n \uparrow g\,(\text{ptws})$. And if (f_n) is decreasing and has g as a pointwise limit, we write $f_n \downarrow g\,(\text{ptws})$.

The most obvious connection, familiar from any real analysis course, between these two types of convergence is that uniform convergence implies pointwise convergence.

3.1.9. Proposition. *Let S be a set and V be a normed linear space. If a sequence (f_n) in $\mathcal{F}(S,V)$ converges uniformly to a function g in $\mathcal{F}(S,V)$, then (f_n) converges pointwise to g.*

The converse is not true.

3.1.10. Example. For each $n \in \mathbb{N}$ let $f_n\colon [0,1] \to \mathbb{R}\colon x \mapsto x^n$. Then the sequence (f_n) converges pointwise on $[0,1]$, but not uniformly.

3.1.11. Example. For each $n \in \mathbb{N}$ and each $x \in \mathbb{R}^+$ let $f_n(x) = \frac{1}{n}\,x$. Then on each interval of the form $[0,a]$ where $a > 0$ the sequence (f_n) converges uniformly to the constant function 0. On the interval $[0,\infty)$ it converges pointwise to 0, but not uniformly.

Recall also from real analysis that a uniform limit of bounded functions must itself be bounded.

3.1.12. Proposition. *Let S be a set, V be a normed linear space, and (f_n) be a sequence in $\mathcal{B}(S,V)$ and g be a member of $\mathcal{F}(S,V)$. If $f_n \to g$ (unif), then g is bounded.*

And a uniform limit of continuous functions is continuous.

3.1.13. Proposition. *Let X be a topological space, V be a normed linear space, and (f_n) be a sequence of continuous V-valued functions on X and g be a member of $\mathcal{F}(X,V)$. If $f_n \to g$ (unif), then g is continuous.*

3.1.14. Proposition. *If x and y are elements of a vector space V equipped with a norm $\|\ \|$, then*

$$\big|\, \|x\| - \|y\| \,\big| \le \|x - y\|\,.$$

3.1.15. Corollary. *The norm on a normed linear space is a uniformly continuous function.*

3.1.16. Notation. Let M be a metric space, $a \in M$, and $r > 0$. We denote $\{x \in M \colon d(x,a) < r\}$, the open ball of radius r about a, by $B_r(a)$. Similarly, we denote $\{x \in M \colon d(x,a) \le r\}$, the CLOSED BALL of radius r about a, by $C_r(a)$ and $\{x \in M \colon d(x,a) = r\}$, the SPHERE of radius r about a, by $S_r(a)$. We will use B_r, C_r, and S_r as abbreviations for $B_r(0)$, $C_r(0)$, and $S_r(0)$, respectively.

3.1.17. Proposition. *If V is a normed linear space, if $x \in V$, and if $r,\ s > 0$, then*

(a) $B_r(\mathbf{0}) = -B_r(\mathbf{0})$;
(b) $B_{rs}(\mathbf{0}) = rB_s(\mathbf{0})$;

(c) $x + B_r(\mathbf{0}) = B_r(x)$; *and*
(d) $B_r(\mathbf{0}) + B_r(\mathbf{0}) = 2\,B_r(\mathbf{0})$. *(Is it true in general that* $A + A = 2A$ *when* A *is a subset of a vector space?)*

3.1.18. Definition. If a and b are vectors in the vector space V, then the CLOSED SEGMENT between a and b, denoted by $[a, b]$, is $\{(1-t)a + tb \colon 0 \le t \le 1\}$.

CAUTION. Notice that there is a slight conflict between this notation for closed *segments*, when applied to the vector space $\mathbb{R}$ of real numbers, and the usual notation for closed *intervals* in $\mathbb{R}$. In $\mathbb{R}$ the closed segment $[a, b]$ is the same as the closed interval $[a, b]$ provided that $a \le b$. If $a > b$, however, the closed segment $[a, b]$ is the same as the segment $[b, a]$, it contains all numbers c such that $b \le c \le a$, whereas the closed interval $[a, b]$ is empty.

3.1.19. Definition. A subset C of a vector space V is CONVEX if the closed segment $[a, b]$ is contained in C whenever a, $b \in C$.

3.1.20. Example. In a normed linear space every open ball is a convex set. And so is every closed ball.

3.1.21. Proposition. *The intersection of a family of convex subsets of a vector space is convex.*

3.1.22. Definition. Let V be a vector space. Recall that a *linear combination* of a finite set $\{x_1, \dots, x_n\}$ of vectors in V is a vector of the form $\sum_{k=1}^{n} \alpha_k x_k$ where $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. If $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$, then the linear combination is *trivial*; if at least one α_k is different from zero, the linear combination is *nontrivial*. A linear combination $\sum_{k=1}^{n} \alpha_k x_k$ of the vectors $x_1, \dots, x_n$ is a CONVEX COMBINATION if $\alpha_k \ge 0$ for each k $(1 \le k \le n)$ and if $\sum_{k=1}^{n} \alpha_k = 1$.

3.1.23. Definition. Let A be a nonempty subset of a vector space V. We define the CONVEX HULL of A, denoted by $\operatorname{co}(A)$, to be the smallest convex subset of V which contain A.

3.1.24. Exercise. Use proposition 3.1.21 to show that definition 3.1.23 makes sense. Then show that a "constructive characterization" is equivalent; that is, prove that the convex hull of A is the set of all convex combinations of elements of A.

3.1.25. Proposition. *If* $T \colon V \to W$ *is a linear map between vector spaces and* C *is a convex subset of* V*, then* $T^{\to}(C)$ *is a convex subset of* W*.*

3.1.26. Proposition. *In a normed linear space the closure of every convex set is convex.*

3.1.27. Proposition. *Let* V *be a normed linear space. For each* $a \in V$ *the map* $T_a \colon V \to V \colon x \mapsto x + a$ *(called* TRANSLATION *by* a*) is a homeomorphism.*

3.1.28. Corollary. *If* U *is a nonempty open subset of a normed linear space* V*, then* $U - U$ *contains a neighborhood of* 0*.*

3.1.29. Proposition. *If* (x_n) *is a sequence in a normed linear space and* $x_n \to a$*, then* $\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k \to a$*.*

In proposition 1.2.16 we showed how an inner product on a vector space induces a norm on that space. It is reasonable to ask if all norms can be generated in this fashion from an inner product. The answer is *no*. The next proposition gives a very simple necessary and sufficient condition for a norm to arise from an inner product.

3.1.30. Proposition. *Let* V *be a normed linear space. There exists an inner product on* V *which induces the norm on* V *if and only if the norm satisfies the* parallelogram law *1.2.27.*

Hint for proof. To prove that if a norm satisfies the *parallelogram law* then it is induced by an inner product, use the equation given in the *polarization identity* (proposition 1.2.29) as a definition.

Prove first that $\langle y, x\rangle = \overline{\langle x, y\rangle}$ for all x, $y \in V$ and that $\langle x, x\rangle > 0$ whenever $x \neq 0$. Next, for arbitrary $z \in V$, define a function $f\colon V \to \mathbb{C}\colon x \mapsto \langle x, z\rangle$. Prove that

$$f(x+y) + f(x-y) = 2f(x) \tag{$*$}$$

for all x, $y \in V$. Use this to prove that $f(\alpha x) = \alpha f(x)$ for all $x \in V$ and $\alpha \in \mathbb{R}$. (Start with α being a natural number.) Then show that f is additive. (If u and v are arbitrary elements of V let $x = u + v$ and $y = u - v$. Use $(*)$.) Finally prove that $f(\alpha x) = \alpha f(x)$ for complex α by showing that $f(ix) = i\, f(x)$ for all $x \in V$.

3.1.31. Proposition. *Let V be a vector space with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Let $\mathfrak{T}_k$ be the topology induced on V by $\|\cdot\|_k$ (for $k = 1, 2$). If there exists a constant $\alpha > 0$ such that $\|x\|_1 \leq \alpha\|x\|_2$ for every $x \in V$, then $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$.*

3.1.32. Definition. Two norms on a vector space V are EQUIVALENT if there exist constants α, $\beta > 0$ such that for all $x \in V$

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1 .$$

3.1.33. Proposition. *If $\|\ \ \|_1$ and $\|\ \ \|_2$ are equivalent norms on a vector space V, then they induce the same topology on V.*

Proposition 3.1.33 gives us an easily verifiable sufficient condition for two norms to induce identical topologies on a vector space. Thus, if we are trying to show, for example, that some subset of a normed linear space is open it may very well be the case that the proof can be simplified by replacing the given norm with an equivalent one.

Similarly, suppose that we are attempting to verify that a function f between two normed linear spaces is continuous. Since continuity is defined in terms of open sets and equivalent norms produce exactly the same open sets (see proposition 3.1.33), we are free to replace the norms on the domain of f and the codomain of f with any equivalent norms we please. This process can sometimes simplify arguments significantly. (This possibility of simplification, incidentally, is one major advantage of giving a topological definition of continuity in the first place.)

3.1.34. Definition. Let $x = (x_n)$ a sequence of vectors in a normed linear space V. The infinite series $\sum_{k=1}^{\infty} x_k$ is CONVERGENT if there exists a vector $b \in V$ such that $\|b - s_n\| \to 0$ as $n \to \infty$, where $s_n = \sum_{k=1}^{n} x_k$ is the n^{th} partial sum of the sequence x. The series is ABSOLUTELY CONVERGENT if $\sum_{k=1}^{\infty}\|x_k\| < \infty$.

3.1.35. Exercise. Let l_1 be the set of all sequences $x = (x_n)$ of complex numbers such that the series $\sum x_k$ is absolutely convergent. Make l_1 into a vector space with the usual pointwise definition of addition and scalar multiplication. For every $x \in l_1$ define

$$\|x\|_1 := \sum_{k=1}^{\infty} |x_k|$$

and

$$\|x\|_\infty := \sup\{|x_k|\colon k \in \mathbb{N}\}.$$

(These are, respectively, the 1-norm and the *uniform* norm.) Show that both $\|\ \ \|_1$ and $\|\ \ \|_\infty$ are norms on l_1. Then prove or disprove:

(a) If a sequence (x^i) of vectors in the normed linear space $\big(l_1, \|\ \ \|_1\big)$ converges, then the sequence also converges in $\big(l_1, \|\ \ \|_\infty\big)$.

(b) If a sequence (x^i) of vectors in the normed linear space $\big(l_1, \|\ \ \|_\infty\big)$ converges, then the sequence also converges in $\big(l_1, \|\ \ \|_1\big)$.

3.2. Bounded Linear Maps

Analysts work with objects having both algebraic and topological structure. In the preceding section we examined vector spaces endowed with a topology derived from a norm. If these are the objects under consideration, what morphisms are likely to be of greatest interest? The plausible, and correct, answer is *maps which preserve both topological and algebraic structures*, that is, continuous linear maps. The resulting category is denoted by $\mathbf{NLS_\infty}$. The isomorphisms in this category are, topologically, homeomorphisms. It is clear that one might choose instead to study a category in which the mappings satisfy a stronger condition: the isomorphisms are *isometries*, that is they preserve distances (equivalently, they preserve norms). In this category, denoted by $\mathbf{NLS_1}$, the morphisms are CONTRACTIVE LINEAR MAPS, that is, linear maps $T\colon V \to W$ between normed linear spaces such that $\|Tx\| \le \|x\|$ for all $x \in V$. It seems appropriate to refer to $\mathbf{NLS_\infty}$ as the *topological category* of normed linear spaces and to $\mathbf{NLS_1}$ as the *geometric category* of normed linear spaces. Many authors use the unmodified term "isomorphism" for an isomorphism in the topological category $\mathbf{NLS_\infty}$ and "isometric isomorphism" for an isomorphism in the geometric category $\mathbf{NLS_1}$. In these notes we will focus on the topological category. The isometric theory of normed linear spaces although important is somewhat more specialized.

3.2.1. Exercise. Verify the unproved assertions in the preceding paragraph. In particular, prove that

(a) $\mathbf{NLS_\infty}$ is a category.
(b) An isomorphism in $\mathbf{NLS_\infty}$ is both a vector space isomorphism and a homeomorphism.
(c) A linear map between normed linear spaces is an isometry if and only if it is norm preserving. (*Definitions.* For $k = 1, 2$ let V_k be a normed linear space, $\|\ \ \|_k$ be the norm on V_k, d_k be the metric on V_k induced by $\|\ \ \|_k$, and $f\colon V_1 \to V_2$. Then f is an ISOMETRY (or an ISOMETRIC map) if $d_2(f(x), f(y)) = d_1(x, y)$ for all x, $y \in V_1$. It is NORM PRESERVING if $\|f(x)\|_2 = \|x\|_1$ for all $x \in V_1$.)
(d) $\mathbf{NLS_1}$ is a category.
(e) An isomorphism in $\mathbf{NLS_1}$ is both an isometry and a vector space isomorphism.
(f) The category $\mathbf{NLS_1}$ is a subcategory of $\mathbf{NLS_\infty}$ in the sense that every morphism of the former belongs to the latter.

Next we consider a very useful condition which, for linear maps, turns out to be equivalent to continuity.

3.2.2. Definition. A linear transformation $T\colon V \to W$ between normed linear spaces is BOUNDED if $T(B)$ is a bounded subset of W whenever B is a bounded subset of V. In other words, a bounded linear map takes bounded sets to bounded sets. We denote by $\mathfrak{B}(V, W)$ the family of all bounded linear transformations from V into W. A bounded linear map from a space V into itself is often called a (bounded linear) OPERATOR. The family of all operators on a normed linear space V is denoted by $\mathfrak{B}(V)$. The class of normed linear spaces together with the bounded linear maps between them constitute a category. We verify below that this is just the category $\mathbf{NLS_\infty}$.

CAUTION. It is extremely important to realize that a bounded linear map will *not*, in general, be a bounded function in the usual sense of a *bounded function* on some set (see examples 3.1.5 and 3.1.6). The use of "bounded" in these two conflicting senses may be unfortunate, but it is well established.

3.2.3. Example. The linear map $T\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto 3x$ is a bounded linear map (since it maps bounded subsets of $\mathbb{R}$ to bounded subsets of $\mathbb{R}$), but, regarded just as a function, T is not bounded (since its range is not a bounded subset of $\mathbb{R}$).

The following observation my help reduce confusion.

3.2.4. Proposition. *A linear transformation, unless it is the constant map that takes every vector to zero,* cannot *be a bounded function.*

One of the most striking aspects of linearity is that for linear maps the concepts of continuity, continuity at a single point, and uniform continuity coalesce. And in fact they are exactly the same thing as boundedness.

3.2.5. Theorem. *Let $T\colon V \to W$ be a linear transformation between normed linear spaces. Then the following are equivalent:*

(a) *T is bounded.*
(b) *The image of the closed unit ball under T is bounded.*
(c) *T is uniformly continuous on V.*
(d) *T is continuous on V.*
(e) *T is continuous at $\mathbf{0}$.*
(f) *There exists a number $M > 0$ such that $\|Tx\| \le M\|x\|$ for all $x \in V$.*

3.2.6. Proposition. *Let $T\colon V \to W$ be a bounded linear transformation between normed linear spaces. Then the following four numbers (exist and) are equal.*

(a) $\sup\{\|Tx\|\colon \|x\| \le 1\}$
(b) $\sup\{\|Tx\|\colon \|x\| = 1\}$
(c) $\sup\{\|Tx\|\,\|x\|^{-1}\colon x \ne \mathbf{0}\}$
(d) $\inf\{M > 0\colon \|Tx\| \le M\|x\| \text{ for all } x \in V\}$

3.2.7. Definition. If T is a bounded linear map, then $\|T\|$, called the NORM of T, is defined to be any one of the four expressions in the previous exercise.

3.2.8. Example. If V and W are normed linear spaces, then the function

$$\|\ \ \|\colon \mathfrak{B}(V,W) \to \mathbb{R}\colon T \mapsto \|T\|$$

is, in fact, a norm.

3.2.9. Example. The family $\mathfrak{B}(V,W)$ of all bounded linear maps between normed linear spaces is itself a normed linear space.

3.2.10. Proposition. *The family $\mathfrak{B}(V,W)$ of all bounded linear maps between normed linear spaces is complete (with respect to the metric induced by its norm) whenever W is complete.*

3.2.11. Proposition. *Let U, V, and W be normed linear spaces. If $S \in \mathfrak{B}(U,V)$ and $T \in \mathfrak{B}(V,W)$, then $TS \in \mathfrak{B}(U,W)$ and $\|TS\| \le \|T\|\|S\|$.*

3.2.12. Example. On any normed linear space V the *identity operator*

$$\mathrm{id}_V = I_V = I\colon V \to V\colon v \mapsto v$$

is bounded and $\|I_V\| = 1$. The *zero operator*

$$\mathbf{0}_V = \mathbf{0}\colon V \to V\colon v \mapsto \mathbf{0}$$

is also bounded and $\|0_V\| = 0$.

3.2.13. Example. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3\colon (x,y) \mapsto (3x, x+2y, x-2y)$. Then $\|T\| = \sqrt{11}$.

Many students just finishing a beginning calculus course feel that differentiation is a “nicer” operation than integration—probably because the rules for integration seem more complicated than those for differentiation. However, when we regard them as linear maps exactly the opposite is true.

3.2.14. Example. As a linear map on the space of differentiable functions on $[0,1]$ (with the uniform norm) integration is bounded; differentiation is not (and so is continuous nowhere!).

3.3. Finite Dimensional Spaces

In this section are listed a few standard and useful facts about finite dimensional spaces. Proofs of these results are quite easy to find. Consult, for example, [4], section 4.4; [7], section III.3; [30], sections 2.4–5; or [42], pages 16–18.

3.3.1. Proposition. *If V is an n-dimensional normed linear space over $\mathbb{K}$, then there exists a map from V onto $\mathbb{K}^n$ which is both a homeomorphism and a vector space isomorphism.*

3.3.2. Corollary. *Every finite dimensional normed linear space is complete.*

3.3.3. Corollary. *Every finite dimensional vector subspace of a normed linear space is closed.*

3.3.4. Corollary. *Any two norms on a finite dimensional vector space are equivalent.*

3.3.5. Proposition. *The closed unit ball in a finite dimensional normed linear space is compact if and only if the space is finite dimensional.*

3.3.6. Proposition. *If V and W are normed linear spaces and V is finite dimensional, then every linear map $T: V \to W$ is continuous.*

Let T be a bounded linear map between (perhaps infinite dimensional) normed linear spaces. Just as in the finite dimensional case, the kernel and range of T are objects of great importance. In both the finite and infinite dimensional case they are vector subspaces of the spaces which contain them. Additionally, in both cases the kernel of T is closed. (Under a continuous function the inverse image of a closed set is closed.) There is, however, a major difference between the two cases. In the finite dimensional case the range of T must be closed (by 3.3.3). As we will see in example 5.2.14 linear maps between infinite dimensional spaces need not have closed range.

3.4. Quotients of Normed Linear Spaces

3.4.1. Definition. Let A be an object in a concrete category $\mathbf{C}$. A surjective morphism $A \xrightarrow{\pi} B$ in $\mathbf{C}$ is a QUOTIENT MAP for A if a function $g: B \to C$ (in $\mathbf{SET}$) is a morphism (in $\mathbf{C}$) whenever $g \circ \pi$ is a morphism. An object B in $\mathbf{C}$ is a QUOTIENT OBJECT for A if it is the range of some quotient map for A.

3.4.2. Example. In the category $\mathbf{VEC}$ of vector spaces and linear maps every surjective linear map is a quotient map.

3.4.3. Example. In the category $\mathbf{TOP}$ not every epimorphism is a quotient map.

Hint for proof. Consider the identity map on the reals with different topologies on the domain and codomain.

The next item, which should be familiar from linear algebra, shows how a particular quotient object can be generated by "factoring out a subspace".

3.4.4. Definition. Let M be a subspace of a vector space V. Define an equivalence relation $\sim$ on V by

$$x \sim y \qquad \text{if and only if} \qquad y - x \in M.$$

For each $x \in V$ let $[x]$ be the equivalence class containing x. Let V/M be the set of all equivalence classes of elements of V. For $[x]$ and $[y]$ in V/M define

$$[x] + [y] := [x + y]$$

and for $\alpha \in \mathbb{R}$ and $[x] \in V/M$ define

$$\alpha[x] := [\alpha x].$$

Under these operations V/M becomes a vector space. It is the QUOTIENT SPACE of V by M. The notation V/M is usually read "V mod M". The linear map

$$\pi: V \to V/M : x \mapsto [x]$$

is called the QUOTIENT MAP.

3.4.5. Exercise. Verify the assertions made in definition 3.4.4. In particular, show that $\sim$ is an equivalence relation, that addition and scalar multiplication of the set of equivalence classes are well defined, that under these operations V/M is a vector space, that the function called here the "quotient map" is in fact a quotient map in the sense of 3.4.1, and that this quotient map is linear.

The following result is called the *fundamental quotient theorem* or the first isomorphism theorem for vector spaces.

3.4.6. Theorem. *Let V and W be vector spaces and $M \preceq V$. If $T \in \mathfrak{L}(V, W)$ and $\ker T \supseteq M$, then there exists a unique $\widetilde{T} \in \mathfrak{L}(V/M\,, W)$ which makes the following diagram commute.*

$$\begin{tikzcd} V \arrow[d, "\pi"'] \arrow[dr, "T"] & \\ V/M \arrow[r, "\widetilde{T}"'] & W \end{tikzcd}$$

Furthermore, $\widetilde{T}$ is injective if and only if $\ker T = M$; and $\widetilde{T}$ is surjective if and only if T is.

3.4.7. Corollary. *If $T\colon V \to W$ is a linear map between vector spaces, then $\operatorname{ran} T \cong V/\ker T$.*

3.4.8. Proposition. *Let V be a normed linear space and M be a closed subspace of V. Then the map*

$$\|\ \ \|\colon V/M \to \mathbb{R}\colon [x] \mapsto \inf\{\|u\|\colon u \sim x\}$$

is a norm on V/M. It is called the QUOTIENT NORM *on V/M. Furthermore, the quotient map*

$$\pi\colon V \to V/M\colon x \mapsto [x]$$

is a bounded linear surjection with $\|\pi\| \le 1$.

3.4.9. Theorem (Fundamental quotient theorem for $\mathbf{NLS_\infty}$)**.** *Let V and W be normed linear spaces and M be a closed subspace of V. If T is a bounded linear map from V to W and $\ker T \supseteq M$, then there exists a unique bounded linear map $\widetilde{T}\colon V/M \to W$ which makes the following diagram commute.*

$$\begin{tikzcd} V \arrow[d, "\pi"'] \arrow[dr, "T"] & \\ V/M \arrow[r, "\widetilde{T}"'] & W \end{tikzcd}$$

Furthermore: $\|\widetilde{T}\| = \|T\|$; $\widetilde{T}$ is injective if and only if $\ker T = M$; and $\widetilde{T}$ is surjective if and only if T is.

3.5. Products of Normed Linear Spaces

Products and coproducts, like quotients, are best described in terms of what they *do*.

3.5.1. Definition. Let A_1 and A_2 be objects in a category $\mathbf{C}$. We say that the triple (P, π_1, π_2), where P is an object and $\pi_k\colon P \to A_k$ $(k = 1, 2)$ are morphisms, is a PRODUCT of A_1 and A_2 if for every object B and every pair of morphisms $f_k\colon B \to A_k$ $(k = 1, 2)$ there exists a unique morphism $g\colon B \to P$ such that $f_k = \pi_k \circ g$ for $k = 1, 2$.

It is conventional to say, “Let P be a product of ...” for, “Let (P, π_1, π_2) be a product of ...”. The product of A_1 and A_2 is often written as $A_1 \times A_2$ or as $\prod_{k=1,2} A_k$.

In a particular category products may or may not exist. It is an interesting and elementary fact that whenever they exist they are unique (up to isomorphism), so that we may unambiguously speak of *the* product of two objects. When we say that a categorical object satisfying some condition(s) is *unique up to isomorphism* we mean, of course, that any two objects satisfying the condition(s) must be isomorphic. We will often use the phrase “essentially unique” for “unique up to isomorphism.”

3.5.2. Proposition. *In any category products (if they exist) are essentially unique.*

3.5.3. Example. In the category **SET** the product of two sets A_1 and A_2 exists and is in fact the usual Cartesian product $A_1 \times A_2$ together with the usual coordinate projections $\pi_k \colon A_1 \times A_2 \to A_k \colon (a_1, a_2) \mapsto a_k$.

3.5.4. Example. If V_1 and V_2 are vector spaces we make the Cartesian product $V_1 \times V_2$ into a vector space as follows. Define addition by

$$(u, v) + (w, x) := (u + w, v + x)$$

and scalar multiplication by

$$\alpha(u, v) := (\alpha u, \alpha v)\,.$$

This makes $V_1 \times V_2$ into a vector space and that this space together with the usual coordinate projections $\pi_k \colon V_1 \times V_2 \to V_k \colon (v_1, v_2) \mapsto v_k$ $(k = 1, 2)$ is a product in the category **VEC**. It is usually called the DIRECT SUM of V and W and is denoted by $V \oplus W$.

It often possible and desirable to take the product of an arbitrary family of objects in a category. Following is a generalization of definition 3.5.1.

3.5.5. Definition. Let $\big(A_\lambda\big)_{\lambda\in\Lambda}$ be an indexed family of objects in a category **C**. We say that the object P together with an indexed family $\big(\pi_\lambda\big)_{\lambda\in\Lambda}$ of morphisms $\pi_\lambda \colon P \to A_\lambda$ is a PRODUCT of the objects A_λ if for every object B and every indexed family $\big(f_\lambda\big)_{\lambda\in\Lambda}$ of morphisms $f_\lambda \colon B \to A_\lambda$ there exists a unique map $g \colon B \to P$ such that $f_\lambda = \pi_\lambda \circ g$ for every $\lambda \in \Lambda$.

A category in which arbitrary products exist is said to be PRODUCT COMPLETE. Many of the categories we encounter in these notes are product complete.

3.5.6. Definition. Let $\big(S_\lambda\big)_{\lambda\in\Lambda}$ be an indexed family of sets. The CARTESIAN PRODUCT of the indexed family, denoted by $\prod_{\lambda\in\Lambda} S_\lambda$ or just $\prod S_\lambda$, is the set of all functions $f \colon \Lambda \to \bigcup S_\lambda$ such that $f(\lambda) \in S_\lambda$ for each $\lambda \in \Lambda$. The maps $\pi_\lambda \colon \prod S_\lambda \to S_\lambda \colon f \mapsto f(\lambda)$ are the canonical COORDINATE PROJECTIONS. In many cases the notation f_λ is more convenient than $f(\lambda)$. (See, for example, example 3.5.8 below.)

3.5.7. Example. A very important special case of the preceding definition occurs when all of the sets S_λ are identical: say $S_\lambda = A$ for every $\lambda \in \Lambda$. In this case the Cartesian product comprises all the functions which map Λ into A. That is, $\prod_{\lambda\in\Lambda} S_\lambda = A^\Lambda$. Notice also that in this case the coordinate projections are *evaluation maps*. For each λ the coordinate projection π_λ takes each point f in the product (that is, each function f from Λ into A) to $f(\lambda)$ its value at λ. Briefly, each coordinate projection is an evaluation map at some point.

3.5.8. Example. What is $\mathbb{R}^n$? It is just the set of all n-tuples of real numbers. That is, it is the set of functions from $\mathbb{N}_n = \{1, 2, \dots, n\}$ into $\mathbb{R}$. In other words $\mathbb{R}^n$ is just shorthand for $\mathbb{R}^{\mathbb{N}_n}$. In $\mathbb{R}^n$ one usually writes x_j for the j^{th} coordinate of a vector x rather than $x(j)$.

3.5.9. Example. In the category **SET** the product of an indexed family of sets exists and is in fact the Cartesian product of these sets together with the canonical coordinate projections.

3.5.10. Example. If $\big(V_\lambda\big)_{\lambda\in\Lambda}$ is an indexed family of vector spaces we make the Cartesian product $\prod V_\lambda$ into a vector space as follows. Define addition and scalar multiplication pointwise: for f, $g \in \prod A_\lambda$ and $\alpha \in \mathbb{R}$

$$(f+g)(\lambda) := f(\lambda) + g(\lambda)$$

and

$$(\alpha f)(\lambda) := \alpha f(\lambda)\,.$$

This makes $\prod V_\lambda$ into a vector space, which is sometimes called the DIRECT PRODUCT of the spaces V_λ, and this space together with the canonical coordinate projections (which are certainly linear maps) is a product in the category **VEC**.

3.5.11. Example. Let V and W be normed linear spaces. On the Cartesian product $V \times W$ define

$$\|(x,y)\|_2 = \sqrt{\|x\|^2 + \|y\|^2},$$
$$\|(x,y)\|_1 = \|x\| + \|y\|,$$

and

$$\|(x,y)\|_u = \max\{\|x\|, \|y\|\}\,.$$

The first of these is the EUCLIDEAN NORM (or 2-NORM) on $V \times W$, the second is the 1-NORM, and the last is the UNIFORM NORM. Verifying that these are all norms on $V \times W$ is quite similar to the arguments required in examples 3.1.1, 3.1.2, and 3.1.3. That they are equivalent norms is a consequence of the following inequalities and proposition 3.1.33.

$$\|x\| + \|y\| \le \sqrt{2}\sqrt{\|x\|^2 + \|y\|^2} \le 2\max\{\|x\|, \|y\|\} \le 2(\|x\| + \|y\|)$$

3.5.12. Convention. In the preceding example we defined three (equivalent) norms on the product space $V \times W$. We will take the first of these $\|\ \ \|_1$ as the PRODUCT NORM on $V \times W$. Thus whenever V and W are normed linear spaces, unless the contrary is specified we will regard the product $V \times W$ as a normed linear space under this norm. Usually we write just $\|(x,y)\|$ instead of $\|(x,y)\|_1$. The product of the normed linear spaces V and W is usually denoted by $V \oplus W$ and is called the DIRECT SUM of V and W. (In the special case where $V = W = \mathbb{R}$, what metric does the product norm induce on $\mathbb{R}^2$?)

3.5.13. Example. Show that product $V \oplus W$ of normed linear spaces is in fact a product in the category $\mathbf{NLS_\infty}$ of normed linear spaces and bounded linear maps.

3.5.14. Proposition. *Let $\big((x_n, y_n)\big)$ be a sequence in the direct sum $V \oplus W$ of two normed linear spaces. Prove that $\big((x_n, y_n)\big)$ converges to a point (a,b) in $V \oplus W$ if and only if $x_n \to a$ in V and $y_n \to b$ in W.*

3.5.15. Proposition. *Addition is a continuous operation on a normed linear space V. That is, the map*

$$A\colon V \oplus V \to V\colon (x,y) \mapsto x + y$$

is continuous.

3.5.16. Exercise. Give a *very* short proof (no ϵ's or δ's or open sets) that if (x_n) and (y_n) are sequences in a normed linear space which converge to a and b, respectively, then $x_n + y_n \to a + b$.

3.5.17. Proposition. *Scalar multiplication is a continuous operation on a normed linear space V in the sense that the map*

$$S\colon \mathbb{K} \times V \to V\colon (\alpha, x) \mapsto \alpha x$$

is continuous.

3.5.18. Proposition. *If B and C are subsets of a normed linear space and α is a scalar, then*

(a) $\overline{\alpha B} = \alpha \overline{B}$*; and*

(b) $\overline{B} + \overline{C} \subseteq \overline{B + C}$.

3.5.19. Example. If B and C are closed subsets of a normed linear space, then it does not necessarily follow that $B + C$ is closed (and therefore $\overline{B} + \overline{C}$ and $\overline{B + C}$ need not be equal).

3.5.20. Proposition. *Let C_1 and C_2 be compact subsets of a normed linear space V. Then*

(a) *$C_1 \times C_2$ is a compact subset of $V \times V$, and*
(b) *$C_1 + C_2$ is a compact subset of V.*

3.5.21. Example. Let X be a topological space. Then the family $\mathcal{C}(X)$ of all continuous complex valued functions on X is a vector space under the usual pointwise operations of addition and scalar multiplication.

3.5.22. Example. Let X be a topological space. We denote by $\mathcal{C}_b(X)$ the family of all bounded continuous complex valued functions on X. It is a normed linear space under the uniform norm. In fact, it is a subspace of $\mathcal{B}(X)$ (see example 3.1.5).

3.5.23. Definition. Let A be an algebra on which a norm has been defined. Suppose that additionally the SUBMULTIPLICATIVE property

$$\|xy\| \leq \|x\| \, \|y\|$$

is satisfied for all x, $y \in A$. Then A is a NORMED ALGEBRA. We make one further requirement: if the algebra A is unital, then

$$\|\mathbf{1}\| = 1 \, .$$

In this case A is a UNITAL NORMED ALGEBRA.

3.5.24. Example. In example 3.1.5 we showed that the family $\mathcal{B}(S)$ of bounded real valued functions on a set S is a normed linear space under the uniform norm. It is also a commutative unital normed algebra.

3.5.25. Proposition. *Multiplication is a continuous operation on a normed algebra V in the sense that the map*

$$M \colon V \times V \to V \colon (x, y) \mapsto xy$$

is continuous.

Hint for proof. If you can prove that multiplication on the real numbers is continuous you can almost certainly prove that it is continuous on arbitrary normed algebras.

3.5.26. Example. The family $\mathcal{C}_b(X)$ of bounded continuous real valued functions on a topological space X is, under the usual pointwise operations and the uniform norm, a commutative unital normed algebra.

3.5.27. Example. The family $\mathfrak{B}(V)$ of all (bounded linear) operators on a normed linear space V is a unital normed algebra.

3.5.28. Definition. In 1.2.38 we defined the algebraic dual $V^{\#}$ of a vector space V. Of great importance in normed spaces is the study of the *continuous* members of $V^{\#}$, the *continuous linear functionals*, also known as *bounded linear functionals*. The set V^* of all such functionals on the normed space V is the DUAL SPACE of V. To distinguish it from the algebraic dual and the order dual, it is sometimes called the NORM DUAL or the TOPOLOGICAL DUAL of V.

3.5.29. Example. If S is a set and $a \in S$, then the evaluation functional

$$E_a \colon \mathcal{B}(S) \to \mathbb{R} \colon f \mapsto f(a)$$

is a bounded linear functional on the space $\mathcal{B}(S)$ of all bounded real valued functions on S. If S happens to be a topological space then we may also regard E_a as a member of the dual space of $\mathcal{C}_b(S)$. (In either case, what is $\|E_a\|$?)

3.6. Coproducts of Normed Linear Spaces

Coproducts are like products—except all the arrows are reversed.

3.6.1. Definition. Let A_1 and A_2 be objects in a category $\mathbf{C}$. The triple (P, j_1, j_2), (P is an object and $j_k\colon A_k \to P$, $k = 1, 2$ are morphisms) is a COPRODUCT of A_1 and A_2 if for every object B and every pair of morphisms $g_k\colon A_k \to B$ (k = 1,2) there exists a unique morphism $h\colon P \to B$ such that $g_k = h \circ j_k$ for $k = 1, 2$.

3.6.2. Proposition. *In any category coproducts (if they exist) are essentially unique.*

3.6.3. Definition. Let A and B be sets. When A and B are disjoint we will often use the notation $A \uplus B$ instead of $A \cup B$ (to emphasize the disjointness of A and B). When $C = A \uplus B$ we say that C is the DISJOINT UNION of A and B. Similarly we frequently choose to write the union of a *pairwise disjoint* family $\mathfrak{A}$ of sets as $\biguplus \mathfrak{A}$. And the notation $\biguplus_{\lambda\in\Lambda} A_\lambda$ may be used to denote the union of a *pairwise disjoint* indexed family $\{A_\lambda \colon \lambda \in \Lambda\}$ of sets. When $C = \biguplus \mathfrak{A}$ or $C = \biguplus_{\lambda\in\Lambda} A_\lambda$ we say that C is the DISJOINT UNION of the appropriate family.

3.6.4. Definition. In the preceding definition we introduced the notation $A \uplus B$ for the union of two disjoint sets A and B. We now extend this notation somewhat and allow ourselves to take the *disjoint union* of sets A and B even if they are not disjoint. We "make them disjoint" by identifying (in the obvious way) the set A with the set $A' = \{(a, 1) \colon a \in A\}$ and B with the set $B' = \{(b, 2) \colon b \in B\}$. Then we denote the union of A' and B', which *are* disjoint, by $A \uplus B$ and call it the DISJOINT UNION of A and B.

In general, if $\big(A_\lambda\big)_{\lambda\in\Lambda}$ is an indexed family of sets, its DISJOINT UNION, $\biguplus\limits_{\lambda\in\Lambda} A_\lambda$ is defined to be $\bigcup\{(A_\lambda, \lambda) \colon \lambda \in \Lambda\}$.

3.6.5. Example. In the category $\mathbf{SET}$ the coproduct of two sets A_1 and A_2 exists and is their disjoint union $A_1 \uplus A_2$ together with the obvious inclusion maps $\iota_k\colon A_k \to A_1 \uplus A_2$ ($k = 1, 2$).

It is interesting to observe that while the product and coproduct of a finite collection of objects in the category $\mathbf{SET}$ are quite different, in the more complex category $\mathbf{VEC}$ they turn out to be exactly the same thing.

3.6.6. Example. If V_1 and V_2 are vector spaces make the Cartesian product $V_1 \times V_2$ into a vector space as in example 3.5.4. This space together with the obvious injections is a coproduct in the category $\mathbf{VEC}$.

It often possible and desirable to take the coproduct of an arbitrary family of objects in a category. Following is a generalization of definition 3.6.1.

3.6.7. Definition. Let $\big(A_\lambda\big)_{\lambda\in\Lambda}$ be an indexed family of objects in a category $\mathbf{C}$. We say that the object C together with an indexed family $\big(\iota_\lambda\big)_{\lambda\in\Lambda}$ of morphisms $\iota_\lambda\colon A_\lambda \to C$ is a COPRODUCT of the objects A_λ if for every object B and every indexed family $\big(f_\lambda\big)_{\lambda\in\Lambda}$ of morphisms $f_\lambda\colon A_\lambda \to B$ there exists a unique map $g\colon C \to B$ such that $f_\lambda = g \circ \iota_\lambda$ for every $\lambda \in \Lambda$. The usual notation for the coproduct of the objects A_λ is $\coprod_{\lambda\in\Lambda} A_\lambda$.

3.6.8. Example. In the category $\mathbf{SET}$ the coproduct of an indexed family of sets exists and is the disjoint union of these sets.

3.6.9. Definition. Let S be a set and V be a vector space. The SUPPORT of a function $f\colon S \to V$ is $\{s \in S \colon f(s) \neq \mathbf{0}\}$.

3.6.10. Example. If $\big(V_\lambda\big)_{\lambda\in\Lambda}$ is an indexed family of vector spaces we make the Cartesian product into a vector space as in example 3.5.10. The set of functions f belonging to $\prod V_\lambda$ which have finite support (that is, which are nonzero only finitely often) is clearly a subspace of $\prod V_\lambda$. This subspace is the DIRECT SUM of the spaces V_λ. It is denoted by $\bigoplus_{\lambda\in\Lambda} V_\lambda$. This space together with the obvious injective maps (which are linear) is a coproduct in the category $\mathbf{VEC}$.

3.6.11. Exercise. What are the coproducts in the category $\mathbf{NLS_\infty}$?

3.6.12. Exercise. Identify the product and coproduct of two spaces V and W in the category $\mathbf{NLS_1}$ of normed linear spaces and linear contractions.

CHAPTER 4

HILBERT SPACES

For the most elementary facts about Hilbert spaces I can think of no better introduction than the first two chapters of Halmos's classic text [21]. Another good introduction is [50]. You may also find chapters 3 and 9 of [30] and chapter 4 of [20] helpful. For a start towards more serious explorations into the mysteries of Hilbert space look at [7], [8], and [23].

4.1. Definition and Examples

A Hilbert space is a *complete inner product space*. In more detail:

4.1.1. Definition. In proposition 1.2.16 we showed how an inner product on a vector space induces a norm on the space and in proposition 1.2.17 how a norm in turn induces a metric. If an inner product space is complete with respect to this metric it is a HILBERT SPACE. Similarly, a BANACH SPACE is a complete normed linear space and a BANACH ALGEBRA is a complete normed algebra.

4.1.2. Example. Under the inner product defined in example 1.2.4 $\mathbb{K}^n$ is a Hilbert space.

4.1.3. Example. Under the inner product defined in example 1.2.5 l_2 is a Hilbert space.

4.1.4. Example. Under the inner product defined in example 1.2.6 $\mathcal{C}([a,b])$ is *not* a Hilbert space.

4.1.5. Example. The inner product space l_c of all sequences (a_n) of complex numbers which are eventually zero (see example 1.2.35) is not a Hilbert space.

4.1.6. Example. Let μ be a positive measure on a σ-algebra $\mathfrak{A}$ of subsets of a set S. A complex valued function f on S is MEASURABLE if the inverse image under f of every Borel set (equivalently, every open set) in $\mathbb{C}$ belongs to $\mathfrak{A}$. We define an equivalence relation $\sim$ on the family of measurable complex valued functions by setting $f \sim g$ whenever f and g differ on a set of measure zero, that is, whenever $\mu(\{x \in S \colon f(x) \neq g(x)\}) = 0$. We adopt conventional notation and denote the equivalence class containing f by f itself (rather than something more logical such as $[f]$). We denote the family of (equivalence classes of) measurable complex valued functions on S by $\mathcal{M}(S,\mu)$. A function $f \in \mathcal{M}(S,\mu)$ is SQUARE INTEGRABLE if $\int_S |f(x)|^2\, d\mu(x) < \infty$. We denote the family of (equivalence classes of) square integrable functions on S by $L_2(S,\mu)$. For every $f,\ g \in L_2(S,\mu)$ define

$$\langle f, g\rangle = \int_S f(x)\overline{g(x)}\, d\mu(x)\,.$$

With this inner product (and the obvious pointwise vector space operations) $L_2(S,\mu)$ is a Hilbert space.

4.1.7. Example. Let λ be Lebesgue measure on the interval $[0,1]$ and H be the set of all absolutely continuous functions on $[0,1]$ such that f' belongs to $L^2([0,1],\lambda)$ and $f(0)=0$. For f and g in H define

$$\langle f, g\rangle = \int_0^1 f'(t)\overline{g'(t)}\, dt.$$

This is an inner product on H under which H becomes a Hilbert space.

4.1.8. Example. If H and K are Hilbert spaces, then their (external orthogonal) direct sum $H \oplus K$ (as defined in 1.2.24) is also a Hilbert space.

4.2. Nets

Nets (sometimes called *generalized sequences*) are useful for characterizing many properties of general topological spaces. In such spaces they are used in basically the same way as sequences are used in metric spaces.

4.2.1. Definition. Let $\leq$ be a relation on a nonempty set S.

(a) If the relation $\leq$ is reflexive and transitive, it is a PREORDERING.
(b) If $\leq$ is a preordering and is also antisymmetric, it is a PARTIAL ORDERING.
(c) Elements x and y in a preordered set are COMPARABLE if either $x \leq y$ or $y \leq x$.
(d) If $\leq$ is a partial ordering with respect to which any two elements are comparable, it is a LINEAR ORDERING (or a TOTAL ORDERING).
(e) If the relation $\leq$ is a preordering (respectively, partial ordering, linear ordering) on S, then the pair $(S, \leq)$ is a PREORDERED SET (respectively, PARTIALLY ORDERED SET, LINEARLY ORDERED SET).
(f) A linearly ordered subset of a partially ordered set $(S, \leq)$ is a CHAIN in S.

We may write $b \geq a$ as a substitute for $a \leq b$. The notation $a < b$ (or equivalently, $b > a$) means that $a \leq b$ and $a \neq b$.

4.2.2. Definition. A preordered set in which every pair of elements has an upper bound is a DIRECTED SET.

4.2.3. Example. If S is a set, then $\operatorname{Fin} S$ is a directed set under inclusion $\subseteq$.

4.2.4. Definition. Let S be a set and Λ be a directed set. A mapping $x\colon \Lambda \to S$ is a NET in S (or a *net of members of* S). The value of x at $\lambda \in \Lambda$ is usually written x_λ (rather than $x(\lambda)$) and the net x itself is often denoted by $\big(x_\lambda\big)_{\lambda\in\Lambda}$ or just $\big(x_\lambda\big)$.

4.2.5. Example. The most obvious examples of nets are sequences. These are functions whose domain is the (preordered) set $\mathbb{N}$ of natural numbers.

4.2.6. Definition. A net $x = \big(x_\lambda\big)$ is said to EVENTUALLY have some specified property if there exists $\lambda_0 \in \Lambda$ such that x_λ has that property whenever $\lambda \geq \lambda_0$; and it has the property FREQUENTLY if for every $\lambda_0 \in \Lambda$ there exists $\lambda \in \Lambda$ such that $\lambda \geq \lambda_0$ and x_λ has the property.

4.2.7. Definition. A NEIGHBORHOOD of a point in a topological space is any set which contains an open set containing the point.

4.2.8. Definition. A net x in a topological space X CONVERGES to a point $a \in X$ if it is eventually in every neighborhood of a. In this case a is the LIMIT of x and we write

$$x_\lambda \xrightarrow[\lambda\in\Lambda]{} a$$

or just $x_\lambda \to a$. When limits are unique we may also use the notation $\lim_{\lambda\in\Lambda} x_\lambda = a$ or, more simply, $\lim x_\lambda = a$.

The point a in X is a CLUSTER POINT of the net x if x is frequently in every neighborhood of a.

4.2.9. Definition. Let $(X, \mathfrak{T})$ be a topological space. A family $\mathfrak{B} \subseteq \mathfrak{T}$ is a BASE for $\mathfrak{T}$ if each member of $\mathfrak{T}$ is a union of members of $\mathfrak{B}$. In other words, a family $\mathfrak{B}$ of open sets is a base for $\mathfrak{T}$ if for each open set U there exists a subfamily $\mathfrak{B}'$ of $\mathfrak{B}$ such that $U = \bigcup \mathfrak{B}'$.

In practice it is often more convenient to specify a base for a topology than to specify the topology itself. It is important to realize, however, that there may be many different bases for the same topology. Once a particular base has been chosen we refer to its members as *basic open sets.*

4.2.10. Definition. Let $(X, \mathfrak{T})$ be a topological space. A subfamily $\mathfrak{S} \subseteq \mathfrak{T}$ is a SUBBASE for the topology $\mathfrak{T}$ if the family of all finite intersections of members of $\mathfrak{S}$ is a base for $\mathfrak{T}$.

4.2.11. Example. Let X be a topological space and $a \in X$. A BASE for the family of neighborhoods of the point a is a family $\mathfrak{B}_a$ of neighborhoods of a with the property that every neighborhood of a contains at least one member of $\mathfrak{B}_a$. In general, there are many choices for a neighborhood base at a point. Once such a base is chosen we refer to its members as *basic neighborhoods* of a.

Let $\mathfrak{B}_a$ be a base for the family of neighborhoods at a. Order $\mathfrak{B}_a$ by containment (the reverse of inclusion); that is, for U, $V \in \mathfrak{B}_a$ set

$$U \preceq V \text{ if and only if } U \supseteq V\,.$$

This makes $\mathfrak{B}_a$ into a directed set. Now (using the axiom of choice) choose one element x_U from each set $U \in \mathfrak{B}_a$. Then (x_U) is a net in X and $x_U \to a$.

The next two propositions assure us that in order for a net to converge to a point a in a topological space it is sufficient that it be eventually in every basic (or even subbasic) neighborhood of a.

4.2.12. Proposition. *Let $\mathfrak{B}$ be a base for the topology on a topological space X and a be a point of X. A net (x_λ) converges to a if and only if it is eventually in every neighborhood of a which belongs to $\mathfrak{B}$.*

4.2.13. Proposition. *Let $\mathfrak{S}$ be a subbase for the topology on a topological space X and a be a point of X. A net (x_λ) converges to a if and only if it is eventually in every neighborhood of a which belongs to $\mathfrak{S}$.*

4.2.14. Example. Let $J = [a, b]$ be a fixed interval in the real line, $\mathbf{x} = (x_0, x_1, \dots, x_n)$ be an $(n+1)$-tuple of points of J, and $\mathbf{t} = (t_1, \dots, t_n)$ be an n-tuple. The pair $(\mathbf{x}; \mathbf{t})$ is a PARTITION WITH SELECTION of the interval J if:

(a) $x_{k-1} < x_k$ for $1 \le k \le n$;
(b) $t_k \in [x_{k-1}, x_k]$ for $1 \le k \le n$;
(c) $x_0 = a$; and
(d) $x_n = b$.

The idea is that $\mathbf{x}$ *partitions* the interval into subintervals and t_k is the point *selected* from the k^{th} subinterval. If $\mathbf{x} = (x_0, x_1, \dots, x_n)$, then $\{\mathbf{x}\}$ denotes the *set* $\{x_0, x_1, \dots, x_n\}$. Let $P = (\mathbf{x}; \mathbf{s})$ and $Q = (\mathbf{y}; \mathbf{t})$ be partitions (with selections) of the interval J. We write $P \preccurlyeq Q$ and say that Q is a REFINEMENT of P if $\{\mathbf{x}\} \subseteq \{\mathbf{y}\}$. Under the relation $\preccurlyeq$ the family $\mathfrak{P}$ of partitions with selection on J is a directed (but *not* partially ordered) set.

Now suppose f is a bounded function on the interval J and $P = (\mathbf{x}; \mathbf{t})$ is a partition of J into n subintervals. Let $\Delta x_k := x_k - x_{k-1}$ for $1 \le k \le n$. Then define

$$S_f(P) := \sum_{k=1}^{n} f(t_k)\,\Delta x_k\,.$$

Each such sum $S_f(P)$ is a RIEMANN SUM of f on J. Notice that since $\mathfrak{P}$ is a directed set, S_f is a net of real numbers. If the net S_f of Riemann sums converges we say that the function f is RIEMANN INTEGRABLE. The limit of the net S_f (when it exists) is the RIEMANN INTEGRAL of f over the interval J and is denoted by $\int_a^b f(x)\,dx$.

4.2.15. Example. In a topological space X with the indiscrete topology (where the only open sets are X and the empty set) every net in the space converges to every point of the space.

4.2.16. Example. In a topological space X with the discrete topology (where every subset of X is open) a net in the space converges if and only if it is eventually constant.

4.2.17. Definition. A HAUSDORFF topological space is one in which every pair of distinct points can be separated by open sets.

As example 4.2.15 shows nets in general topological spaces need not have unique limits. A space does not require very restrictive assumptions however to make this pathology go away. For limits to be unique it is sufficient to require that a space be Hausdorff. Interestingly, it turns out that this condition is also necessary.

4.2.18. Proposition. *If X is a topological space, then the following are equivalent:*

(a) *X is Hausdorff.*
(b) *No net in X can converge to more than one point.*
(c) *The diagonal in $X \times X$ is closed.*

(The DIAGONAL of a set S is $\{(s,s)\colon s \in S\} \subseteq S \times S$.)

4.2.19. Example. Recall from beginning calculus that the n^{th} PARTIAL SUM of an infinite series $\sum_{k=1}^{\infty} x_k$ is defined to be $s_n = \sum_{k=1}^{n} x_k$. The series is said to CONVERGE if the sequence (s_n) of partial sums converges and, if the series converges, its *sum* is the limit of the sequence (s_n) of partial sums. Thus, for example, the infinite series $\sum_{k=1}^{\infty} 2^{-k}$ converges and its sum is 1.

The idea of *summing* an infinite series depends heavily on the fact that the partial sums of the series are linearly ordered by inclusion. The "partial sums" of a net of numbers need not be partially ordered, so we need a new notion of "summation", sometimes referred to as *unordered summation.*

4.2.20. Notation. If A is a set we denote by $\operatorname{Fin} A$ the family of all finite subsets of A. (Note that this is a directed set under the relation $\subseteq$.)

4.2.21. Definition. Let $A \subseteq \mathbb{R}$. For every $F \in \operatorname{Fin} A$ define S_F to be the sum of the numbers in F, that is, $S_F = \sum F$. Then $S = \big(S_F\big)_{F \in \operatorname{Fin} A}$ is a net in $\mathbb{R}$. If this net converges, the set A of numbers is said to be SUMMABLE; the limit of the net is the SUM of A and is denoted by $\sum A$.

4.2.22. Definition. A net (x_λ) of real numbers is INCREASING if $\lambda \le \mu$ implies $x_\lambda \le x_\mu$. A net of real numbers is BOUNDED if its range is.

4.2.23. Example. Let $f(n) = 2^{-n}$ for every $n \in \mathbb{N}$, and let $x\colon \operatorname{Fin}(\mathbb{N}) \to \mathbb{R}$ be defined by $x(A) = \sum_{n \in A} f(n)$.

(a) The net x is increasing.
(b) The net x is bounded.
(c) The net x converges to 1.
(d) The set of numbers $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\}$ is summable and its sum is 1.

4.2.24. Example. In a metric space every convergent sequence is bounded. Give an example to show that a convergent net in a metric space (even in $\mathbb{R}$) need not be bounded.

It is familiar from beginning calculus that bounded increasing sequences in $\mathbb{R}$ converge. The preceding example is a special case of a more general observation: bounded increasing nets in $\mathbb{R}$ converge.

4.2.25. Proposition. *Every bounded increasing net $\big(x_\lambda\big)_{\lambda \in \Lambda}$ of real numbers converges and*

$$\lim x_\lambda = \sup\{x_\lambda : \lambda \in \Lambda\}\,.$$

4.2.26. Example. Let (x_k) be a sequence of distinct positive real numbers and $A = \{x_k\colon k \in \mathbb{N}\}$. Then $\sum A$ (as defined above) is equal to the sum of the series $\sum_{k=1}^{\infty} x_k$.

4.2.27. Example. The set $\big\{\frac{1}{k}\colon k \in \mathbb{N}\big\}$ is not summable and the infinite series $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge.

4.2.28. Example. The set $\big\{(-1)^k \frac{1}{k}\colon k \in \mathbb{N}\big\}$ is not summable but the infinite series $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ does converge.

4.2.29. Proposition. *Every summable subset of real numbers is countable.*

Hint for proof. If A is a summable set of real numbers and $n \in \mathbb{N}$, how many members of A can be larger than $1/n$?

In proposition 4.2.18 we have already seen one example of a way in which nets characterize a topological property: a space is Hausdorff if and only if limits of nets in the space are unique (when they exist). Here is another topological property—continuity at a point—that can be conveniently characterized in terms of nets.

4.2.30. Proposition. *Let X and Y be topological spaces. A function $f\colon X \to Y$ is continuous at a point a in X if and only if $f\big(x_\lambda\big) \to f(a)$ in Y whenever $\big(x_\lambda\big)$ is a net converging to a in X.*

In metric spaces we characterize interiors and closures of sets in terms of sequences. For general topological spaces we must replace sequences by nets.

4.2.31. Proposition. *A point a in a topological space X is in the interior of a subset S of X if and only if every net in X that converges to a is eventually in S.*

Hint for proof. One direction is easy. In the other use exercise 4.2.12 and the sort of net constructed in example 4.2.11.

4.2.32. Proposition. *A point b in a topological space X belongs to the closure of a subset S of X if and only if some net in S converges to b.*

4.2.33. Proposition. *A subset A of a topological space is closed if and only if the limit of every convergent net in A belongs to A.*

4.3. Unordered Summation

Defining *unordered summation* is no more difficult in normed linear spaces than in $\mathbb{R}$. We generalize definition 4.2.21.

4.3.1. Definition. Let V be a normed linear space and $A \subseteq V$. For every $F \in \operatorname{Fin} A$ define

$$s_F = \sum F$$

where $\sum F$ is the sum of the (finitely many) vectors belonging to F. Then $s = \big(s_F\big)_{F\in\operatorname{Fin} A}$ is a net in V. If this net converges, the set A is said to be SUMMABLE; the limit of the net is the SUM of A and is denoted by $\sum A$. If $\{\|a\|\colon a \in A\}$ is summable, we say that the set A is ABSOLUTELY SUMMABLE.

To accommodate sums in which the same vector appears more than once, we need to make use of indexed families of vectors. They require a slightly different approach. Suppose, for example, that $A = \big(x_i\big)_{i\in I}$ where I is an arbitrary index set. Then for each $F \in \operatorname{Fin} I$

$$s_F = \sum_{i\in F} x_i.$$

As above s is a net in V. If it converges $\big(x_i\big)_{i\in I}$ is summable and its limit is the sum of the indexed family, and is denoted by $\sum_{i\in I} x_i$. An alternative way of saying that $\big(x_i\big)_{i\in I}$ is summable is to say that the "series" $\sum_{i\in I} x_i$ CONVERGES or that the "sum" $\sum_{i\in I} x_i$ EXISTS. An indexed family $\big(x_i\big)_{i\in I}$ is ABSOLUTELY SUMMABLE if the indexed family $\big(\|x_i\|\big)_{i\in I}$ is summable. An alternative way of saying that $\big(x_i\big)_{i\in I}$ is absolutely summable is to say that the "series" $\sum_{i\in I} x_i$ is ABSOLUTELY CONVERGENT.

Sequences, as usual, are treated as a separate case. Let (a_k) be a sequence in V. If the infinite series $\sum_{k=1}^\infty a_k$ (that is, the sequence of partial sums $s_n = \sum_{k=1}^n a_k$) converges to a vector b in

V, then we say that the sequence (a_k) is SUMMABLE or, equivalently, that the series $\sum_{k=1}^{\infty} a_k$ is a CONVERGENT SERIES. The vector b is called the SUM of the series $\sum_{k=1}^{\infty} a_k$ and we write

$$\sum_{k=1}^{\infty} a_k = b\,.$$

It is clear that a necessary and sufficient condition for a series $\sum_{k=1}^{\infty} a_k$ to be convergent or, equivalently, for the sequence (a_k) to be summable, is that there exist a vector b in V such that

$$\left\| b - \sum_{k=1}^{n} a_k \right\| \to 0 \qquad \text{as} \qquad n \to \infty. \tag{4.1}$$

4.3.2. Proposition. *A normed linear space V is complete if and only if every absolutely summable sequence in V is summable.*

Hint for proof. Let (x_n) be a Cauchy sequence in a normed linear space in which every absolutely summable sequence is summable. Find a subsequence $\left(x_{n_k}\right)$ of (x_n) such that $\|x_n - x_m\| < 2^{-k}$ whenever m, $n \geq n_k$. If we let $y_1 = x_{n_1}$ and $y_j = x_{n_j} - x_{n_{j-i}}$ for $j > 1$, then (y_j) is absolutely summable.

4.3.3. Proposition. *If $\left(x_i\right)_{i\in I}$ and $\left(y_i\right)_{i\in I}$ are summable indexed families in a normed linear space and α is a scalar, then $\left(\alpha x_i\right)_{i\in I}$ and $\left(x_i + y_i\right)_{i\in I}$ are summable; and*

$$\sum_{i\in I} \alpha x_i = \alpha \sum_{i\in I} x_i$$

and

$$\sum_{i\in I} x_i + \sum_{i\in I} y_i = \sum_{i\in I} (x_i + y_i).$$

4.3.4. Remark. One frequently finds some version of the following "proposition" in textbooks.

> If $\sum_{i\in I} x_i = u$ and $\sum_{i\in I} x_i = v$ in a Hilbert (or Banach) space, then $u = v$.

(I once saw a 6 line proof of this.) Of which result is this a trivial consequence?

4.3.5. Proposition. *If $\left(x_i\right)_{i\in I}$ is a summable indexed family of vectors in a Hilbert space H and y is a vector in H, then $\left(\langle x_i, y\rangle\right)_{i\in I}$ is a summable indexed family of scalars and*

$$\sum_{i\in I} \langle x_i, y\rangle = \left\langle \sum_{i\in I} x_i, y \right\rangle.$$

The following is a generalization of example 4.1.3.

4.3.6. Example. Let I be an arbitrary nonempty set and $l^2(I)$ be the set of all complex valued functions x on I such that $\sum_{i\in I} |x_i|^2 < \infty$. For all x, $y \in l^2(I)$ define

$$\langle x, y\rangle = \sum_{i\in I} x(i)\overline{y(i)}.$$

(a) If $x \in l^2(I)$, then $x(i) = 0$ for all but countably many i.
(b) The map $(x, y) \mapsto \langle x, y\rangle$ is an inner product on the vector space $l^2(I)$.
(c) The space $l^2(I)$ is a Hilbert space.

4.3.7. Proposition. *Let (x_n) be a sequence in a Hilbert space. If the sequence regarded as an indexed family is summable, then the infinite series $\sum_{k=1}^{\infty} x_k$ converges.*

4.3.8. Example. The converse of proposition 4.3.7 does not hold.

The direct sum of two Hilbert spaces is a Hilbert space.

4.3.9. Proposition. *If H and K are Hilbert spaces, then their (orthogonal) direct sum (as defined in 1.2.24) is a Hilbert space.*

We can also take the product of more than two Hilbert spaces—even an infinite family of them.

4.3.10. Proposition. *Let $\big(H_i\big)_{i\in I}$ be an indexed family of Hilbert spaces. A function $x\colon I \to \bigcup H_i$ belongs to $H = \bigoplus\limits_{i\in I} H_i$ if $x(i) \in H_i$ for every i and if $\sum_{i\in I}\|x(i)\|^2 < \infty$. Defining $\langle\ ,\ \rangle$ on H by $\langle x, y\rangle := \sum_{i\in I}\langle x(i), y(i)\rangle$ makes H into an Hilbert space.*

4.3.11. Definition. The Hilbert space defined in the preceding proposition is the (EXTERNAL ORTHOGONAL) DIRECT SUM of the Hilbert spaces H_i.

4.3.12. Exercise. Is the direct sum defined above (with appropriate morphisms) a product in the category **HSp** of Hilbert spaces and bounded linear maps? Is it a coproduct?

4.4. Hilbert space Geometry

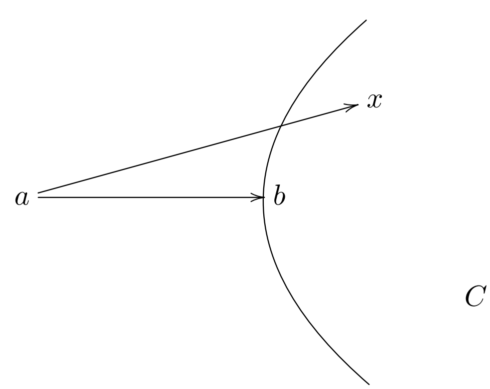
4.4.1. Convention. In the context of Hilbert spaces the word "subspace" will always mean *closed vector subspace*. The reason for this is that we want a subspace of a Hilbert space to be a Hilbert space in its own right; that is to say, a subspace of a Hilbert space should be a subobject in the category of Hilbert spaces (and appropriate morphisms). To indicate that M is a subspace of H we write $M \preccurlyeq H$. A (not necessarily closed) vector subspace of a Hilbert space is also called a *linear subspace* or a *linear manifold*.

4.4.2. Definition. Let A be a nonempty subset of a Banach (or Hilbert) space B. We define the CLOSED LINEAR SPAN of A (denoted by $\bigvee A$) to be the smallest subspace of B containing A

The preceding definition should provoke instant suspicion. How do we know there *is* a "smallest" subspace of B which contains A? One fairly obvious candidate (let's call it the "abstract" one) for such a subspace is the intersection of *all* the subspaces containing A. (Of course, this might also be nonsense—*unless* we know that there is at least one subspace containing A and that the intersection of the family of subspaces containing A is itself a subspace containing A.) Another plausible candidate (we'll call it the "constructive" one) is the closure of the span of A. (But *is* the closure of a subspace a subspace? And is it the smallest one containing A?) Finally, even if these two approaches both make sense, are they equivalent definitions? That is, do they both define the same collection of objects? (Notice how similar all this is to exercise 3.1.24 on convex hulls.)

4.4.3. Proposition. *The preceding "abstract" and "constructive" definitions of* closed linear span *make sense. And they are equivalent.*

4.4.4. Theorem (Minimizing Vector Theorem)**.** *If C is a nonempty closed convex subset of a Hilbert space H and $a \in C^c$, then there exists a unique $b \in C$ such that $\|b - a\| \le \|x - a\|$ for every $x \in C$.*



Hint for proof. For the existence part let $d = \inf\{\|x - a\| : x \in C\}$. Choose a sequence (y_n) of vectors in C such that $\|y_n - a\| \to d$. To prove that the sequence (y_n) is Cauchy, apply the *parallelogram law* 1.2.27 to the vectors $y_m - a$ and $y_n - a$ for $m, n \in \mathbb{N}$. The uniqueness part is similar: apply the *parallelogram law* to the vectors $b - a$ and $c - a$ where b and c are two vectors satisfying the conclusion of the theorem.

4.4.5. Example. The vector space $\mathbb{R}^2$ under the uniform metric is a Banach space. To see that in this space the *minimizing vector theorem* does not hold take C to be the closed unit ball about the origin and a to be the point $(2, 0)$.

4.4.6. Example. The sets

$$C_1 = \left\{ f \in \mathcal{C}([0,1],\mathbb{R}) \colon \int_0^{1/2} f - \int_{1/2}^1 f = 1 \right\} \qquad \text{and} \qquad C_2 = \left\{ f \in L_1([0,1],\mathbb{R}) \colon \int_0^1 f = 1 \right\}$$

are examples that show that neither the existence nor the uniqueness claims of the *minimizing vector theorem* necessarily holds in a Banach space.

4.4.7. Theorem (Vector Decomposition Theorem)**.** *Let H be a Hilbert space and M be a (closed linear) subspace of H. Then for every $x \in H$ there exist unique vectors $y \in M$ and $z \in M^\perp$ such that $x = y + z$.*

Hint for proof. Assume $x \notin M$ (otherwise the result is trivial). Explain how we know that there exists a unique vector $y \in M$ such that

$$\|x - y\| \le \|x - m\| \tag{4.2}$$

for all $m \in M$. Explain why it is enough, for the existence part of the proof, to prove that $x - y$ is perpendicular to every *unit* vector $m \in M$. Explain why it is true that for every unit vector $m \in M$

$$\|x - y\|^2 \le \|x - (y + \lambda m)\|^2 \tag{4.3}$$

where $\lambda := \langle x - y, m \rangle$.

4.4.8. Corollary. *If M is a subspace of a Hilbert space H, then $H = M \oplus M^\perp$.*

4.4.9. Example. The preceding result says that every subspace of a Hilbert space is a direct summand. This need not true if M is assumed to be just a linear subspace of the space. For example, notice that $M = l_c$ (see example 4.1.5) is a linear subspace of the Hilbert space l_2 (see example 4.1.3) but $M^\perp = \{\mathbf{0}\}$.

4.4.10. Corollary. *A vector subspace A of a Hilbert space H is dense in H if and only if $A^\perp = \{0\}$.*

4.4.11. Proposition. *Every proper subspace of a Hilbert space has empty interior.*

4.4.12. Proposition. *If a Hamel basis for a Hilbert space is not finite, then it is uncountable.*

Hint for proof. Suppose the space has a countably infinite basis $\{e^1, e^2, e^3, \dots\}$. For each $n \in \mathbb{N}$ let $M_n = \operatorname{span}\{e^1, e^2, \dots, e^n\}$. Apply the *Baire category theorem* to $\bigcup_{n=1}^\infty M_n$ keeping in mind the preceding proposition.

4.4.13. Proposition. *Let H be a Hilbert space. Then the following hold:*

(a) *if $A \subseteq H$, then $A \subseteq A^{\perp\perp}$;*
(b) *if $A \subseteq B \subseteq H$, then $B^\perp \subseteq A^\perp$;*
(c) *if M is a subspace of H, then $M = M^{\perp\perp}$; and*
(d) *if $A \subseteq H$, then $\bigvee A = A^{\perp\perp}$.*

4.4.14. Proposition. *Let M and N be subspaces of a Hilbert space. Then*

(a) $(M + N)^\perp = (M \cup N)^\perp = M^\perp \cap N^\perp$*, and*
(b) $(M \cap N)^\perp = M^\perp + N^\perp$.

4.4.15. Proposition. *Let V be a normed linear space. There exists an inner product on V which induces the norm on V if and only if the norm satisfies the* parallelogram law *1.2.27.*

Hint for proof. We say that an inner product $\langle \cdot\,,\cdot \rangle$ on V induces the norm $\|\cdot\|$ if $\|x\| = \sqrt{\langle x, x\rangle}$ holds for every x in V.) To prove that if a norm satisfies the *parallelogram law* then it is induced by an inner product, use the equation in the *polarization identity* 1.2.29 as a definition. Prove first that $\langle y, x\rangle = \overline{\langle x, y\rangle}$ for all x, $y \in V$ and that $\langle x, x\rangle > 0$ whenever $x \neq 0$. Next, for arbitrary $z \in V$, define a function $f\colon V \to \mathbb{C}\colon x \mapsto \langle x, z\rangle$. Prove that

$$f(x+y) + f(x-y) = 2f(x) \tag{$*$}$$

for all x, $y \in V$. Use this to prove that $f(\alpha x) = \alpha f(x)$ for all $x \in V$ and $\alpha \in \mathbb{R}$. (Start with α being a natural number.) Then show that f is additive. (If u and v are arbitrary elements of V let $x = u + v$ and $y = u - v$. Use $(*)$.) Finally prove that $f(\alpha x) = \alpha f(x)$ for complex α by showing that $f(ix) = i\,f(x)$ for all $x \in V$.

4.5. Orthonormal Sets and Bases

4.5.1. Definition. A nonempty subset E of a Hilbert space is ORTHONORMAL if $e \perp f$ for every pair of distinct vectors e and f in E and $\|e\| = 1$ for every $e \in E$.

4.5.2. Proposition. *In an inner product space every orthonormal set is linearly independent.*

4.5.3. Proposition (Bessel's inequality)**.** *Let H be a Hilbert space and E be an orthonormal subset of H. For every $x \in H$*

$$\sum_{e\in E} \|\langle x, e\rangle\|^2 \leq \|x\|^2\,.$$

4.5.4. Proposition. *Let E be an orthonormal set in a Hilbert space H. Then the following are equivalent.*

(a) *E is a maximal orthonormal set.*
(b) *If $x \perp E$, then $x = 0$ (E is* total*).*
(c) *$\bigvee E = H$ (E is a* complete orthonormal set*).*
(d) *$x = \displaystyle\sum_{e\in E} \langle x, e\rangle e$ for all $x \in H$.* (Fourier expansion)
(e) *$\langle x, y\rangle = \displaystyle\sum_{e\in E} \langle x, e\rangle\langle e, y\rangle$ for all $x, y \in H$.* (Parseval's identity.)
(f) *$\|x\|^2 = \displaystyle\sum_{e\in E} |\langle x, e\rangle|^2$ for all $x \in H$.* (Parseval's identity.)

4.5.5. Definition. If H is a Hilbert space, then an orthonormal set E satisfying any one (hence all) of the conditions in proposition 4.5.4 is called an ORTHONORMAL BASIS for H (or a COMPLETE ORTHONORMAL SET in H, or a HILBERT SPACE BASIS for H). Notice that this is a very different thing from the usual (Hamel) basis for a vector space (see 1.1.9).

4.5.6. Proposition. *If E is an orthonormal subset of a Hilbert space H, then there exists an orthonormal basis for H which contains E.*

4.5.7. Example. For each $n \in \mathbb{N}$ let e^n be the sequence in l_2 whose n^{th} coordinate is 1 and all the other coordinates are 0. Then $\{e^n \colon n \in \mathbb{N}\}$ is an orthonormal basis for l_2. This is the USUAL ORTHONORMAL BASIS for l_2.

4.5.8. Exercise. A function $f\colon \mathbb{R} \to \mathbb{C}$ of the form

$$f(t) = a_0 + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt)$$

where $a_0, \dots, a_n, b_1, \dots, b_n$ are complex numbers is a TRIGONOMETRIC POLYNOMIAL.

(a) Explain why complex valued 2π-periodic functions on the real line $\mathbb{R}$ are often identified with complex valued functions on the unit circle $\mathbb{T}$.
(b) Some authors say that a *trigonometric polynomial* is a function of the form

$$f(t) = \sum_{k=-n}^{n} c_k e^{ikt}$$

where $c_1, \ldots, c_n$ are complex numbers. Explain carefully why this is exactly the same as the preceding definition.
(c) Justify the use of the term *trigonometric polynomial.*
(d) Prove that every continuous 2π-periodic function on $\mathbb{R}$ is the uniform limit of a sequence of trigonometric polynomials.

4.5.9. Proposition. *Consider the Hilbert space $L_2([0, 2\pi], \mathbb{C})$ with inner product*

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\,dx.$$

For every integer n (positive, negative, or zero) let $\mathbf{e}^n(x) = e^{inx}$ for all $x \in [0, 2\pi]$. Then the set of these functions $\mathbf{e}^n$ for $n \in \mathbb{Z}$ is an orthonormal basis for $L_2([0, 2\pi], \mathbb{C})$.

4.5.10. Example. The sum of the infinite series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ is $\dfrac{\pi^2}{6}$.

Hint for proof. Let $f(x) = x$ for $0 \le x \le 2\pi$. Regard f as a member of $L_2([0, 2\pi])$. Find $\|f\|$ in two ways.

The next proposition allows us to define the notion of the *dimension* of a Hilbert space. (In finite dimensional spaces it agrees with the vector space concept.)

4.5.11. Proposition. *Any two orthonormal bases of a Hilbert space have the same cardinality (that is, they are in one-to-one correspondence).*

Proof. See [21], page 29, Theorem 1.

4.5.12. Definition. The DIMENSION of a Hilbert space is the cardinality of any orthonormal basis for the space.

4.5.13. Proposition. *A Hilbert space is separable if and only if it has a countable orthonormal basis.*

Proof. See [30], page 171, Theorem 3.6-4.

4.6. The Riesz-Fréchet Theorem

4.6.1. Example. Let H be a Hilbert space and $a \in H$. Define $\psi_a \colon H \to \mathbb{C} \colon x \mapsto \langle x, a\rangle$. Then $\psi_a \in H^*$.

Now we generalize theorem 1.2.39 to the infinite dimensional setting. This result is sometimes called the *Riesz representation theorem* (which invites confusion with the more substantial result about representing certain linear functionals as measures) or the *little Riesz representation theorem.* This theorem says that the *only* continuous linear functionals on a Hilbert space are the functions ψ_a defined in 4.6.1. In other words, given a continuous linear functional f on a Hilbert space there is a unique vector a in the space such that the action of f on a vector x can be represented simply by taking the inner product of x with a.

4.6.2. Theorem (Riesz-Fréchet Theorem)**.** *If $f \in H^*$ where H is a Hilbert space, then there exists a unique vector a in H such that $f = \psi_a$. Furthermore, $\|a\| = \|\psi_a\|$.*

Hint for proof. For the case $f \neq 0$, choose a unit vector z in $(\ker f)^\perp$. Notice that for every $x \in H$ the vector $x - \frac{f(x)}{f(z)}\, z$ belongs to $\ker f$.

4.6.3. Corollary. *The kernel of every nonzero bounded linear functional on a Hilbert space has codimension* 1.

Hint for proof. The CODIMENSION of a subspace of a Hilbert space is the dimension of its orthogonal complement. This is really a corollary to the *proof* of the *Riesz-Fréchet theorem*. In our proof we found that the vector representing the bounded linear functional f was a (nonzero) multiple of a unit vector in $(\ker f)^\perp$. What would happen if there were two linearly independent vectors in $(\ker f)^\perp$?

4.6.4. Example. Let H be the collection of all absolutely continuous complex valued functions f on $[0, 1]$ such that $f(0) = 0$ and $f' \in L_2([0, 1], \lambda)$. Define an inner product on H by

$$\langle f, g\rangle := \int_0^1 f'(t)\, \overline{g'(t)}\, dt.$$

We already know that H is a Hilbert space (see example 4.1.7) Fix $t \in (0, 1]$. Define

$$E_t \colon H \to \mathbb{C} \colon f \mapsto f(t).$$

Then E_t is a bounded linear functional on H. What is its $\|E_t\|$? What vector g in H represents the functional E_t?

4.6.5. Exercise. One afternoon your pal Fred R. Dimm comes to you with a problem. "Look," he says, "On $L_2 = L_2(\mathbb{R}, \mathbb{R})$ the evaluation functional E_0, which evaluates each member of L_2 at 0, is clearly a bounded linear functional. So by the Riesz representation theorem there should be some function g in L_2 such that $f(0) = \langle f, g\rangle = \int_{-\infty}^{\infty} fg$ for all f in L_2. But that's a property of the 'δ-function', which I was told doesn't exist. What am I doing wrong?" Give Freddy some (helpful) advice.

4.6.6. Exercise. Later the same afternoon Freddy's back. After considering the advice you gave him (in the preceding problem) he has revised his question by letting the domain of E_0 be the set of bounded continuous real valued functions defined on $\mathbb{R}$. Patiently you explain to Freddy that there are now at least two things wrong his invocation of the 'δ-function'. What are they?

4.6.7. Exercise. The nightmare continues. It's 11 P. M. and Freddy's back again. This time (happily having given up on δ-functions) he wants to apply the representation theorem to the functional "evaluation at zero" on the space $l_2(\mathbb{Z})$. Can he do it? Explain.

4.7. Strong and Weak Topologies on Hilbert Spaces

We recall briefly the definition and crucial facts about *weak topologies*. For a more detailed introduction to this class of topologies see any good textbook on topology or section 11.4 of my notes [13].

4.7.1. Definition. Suppose that S is a set, that for every $\alpha \in A$ (where A is an arbitrary index set) X_α is a topological space, and that $f_\alpha \colon S \to X_\alpha$ for every $\alpha \in A$. Let

$$\mathfrak{S} = \{ f_\alpha{}^{\leftarrow}(U_\alpha) \colon U_\alpha \overset{\circ}{\subseteq} X_\alpha \text{ and } \alpha \in A \}\,.$$

Use the family $\mathfrak{S}$ as a subbase for a topology on S. This topology is called the WEAK TOPOLOGY induced by (or determined by) the functions f_α.

4.7.2. Proposition. *Under the weak topology (defined above) on a set S each of the functions f_α is continuous; in fact the weak topology is the weakest topology on S under which these functions are continuous.*

4.7.3. Proposition. *Let X be a topological space with the weak topology determined by a family $\mathcal{F}$ of functions. Prove that a function $g\colon W \to X$, where W is a topological space, is continuous if and only if $f \circ g$ is continuous for every $f \in \mathcal{F}$.*

4.7.4. Definition. Let $\big(A_\lambda\big)_{\lambda\in\Lambda}$ be an indexed family of topological spaces. The weak topology on the Cartesian product $\prod A_\lambda$ induced by the family of coordinate projections π_λ (see definition 3.5.6) is called the PRODUCT TOPOLOGY.

4.7.5. Proposition. *The product of a family of Hausdorff spaces is Hausdorff.*

4.7.6. Example. Let Y be a topological space with the weak topology (see definition 4.7.1) determined by an indexed family $(f_\alpha)_{\alpha\in A}$ of functions where for each $\alpha \in A$ the codomain of f_α is a topological space X_α. Then a net $\big(y_\lambda\big)_{\lambda\in\Lambda}$ in Y converges to a point $a \in Y$ if and only if $f_\alpha(y_\lambda) \xrightarrow[\lambda\in\Lambda]{} f_\alpha(a)$ for every $\alpha \in A$.

4.7.7. Example. If $Y = \prod_{\alpha\in A} X_\alpha$ is a product of nonempty topological spaces with the product topology (see definition 4.7.4), then a net $\big(y_\lambda\big)$ in Y converges to $a \in Y$ if and only if $\big(y_\lambda\big)_\alpha \to a_\alpha$ for every $\alpha \in A$.

4.7.8. Example. If V and W are normed linear spaces, then the product topology on $V \times W$ is the same as the topology induced by the product norm on $V \oplus W$.

The next example illustrates the inadequacy of sequences when dealing with general topological spaces.

4.7.9. Example. Let $\mathcal{F} = \mathcal{F}([0,1])$ be the set of all functions $f\colon [0,1] \to \mathbb{R}$ and give $\mathcal{F}$ the product topology, that is, the weak topology determined by the evaluation functionals

$$E_x\colon \mathcal{F} \to [0,1]\colon f \mapsto f(x)$$

where $x \in [0,1]$. Thus a basic open neighborhood of a point $g \in \mathcal{F}$ is determined by a finite set of points $A \in \operatorname{Fin}[0.1]$ and a number $\epsilon > 0$:

$$U(g\,;\,A\,;\,\epsilon) := \{f \in \mathcal{F}\colon |f(t) - g(t)| < \epsilon \text{ for all } t \in A\}.$$

(What is usual name for this topology?) Let $\mathcal{G}$ be the set of all functions in $\mathcal{F}$ having finite support. Then

(a) the constant function $\mathbf{1}$ belongs to $\overline{\mathcal{G}}$,
(b) there is a net of functions in $\mathcal{G}$ which converges to $\mathbf{1}$, but
(c) no sequence in $\mathcal{G}$ converges to $\mathbf{1}$.

4.7.10. Definition. A net (x_λ) in a Hilbert space H is said to CONVERGE WEAKLY to a point a in H if $\langle x_\lambda, y\rangle \longrightarrow \langle a, y\rangle$ for every $y \in H$. In this case we write $x_\lambda \xrightarrow{w} a$. In a Hilbert space a net is said to CONVERGE STRONGLY (or CONVERGE IN NORM) if it converges with respect to the topology induced by the norm. This is the usual type of convergence in a Hilbert space. If we wish to emphasize a distinction between modes of convergence we may write $x_\lambda \xrightarrow{s} a$ for strong convergence.

4.7.11. Exercise. Explain why the topology on a Hilbert space generated by the weak convergence of nets is in fact a weak topology in the usual topological sense of the term.

When we say that a subset of a Hilbert space H is WEAKLY CLOSED we mean that it is closed with respect to the weak topology on H; a set is WEAKLY COMPACT if it is compact in the weak topology on H; and so on. A function $f\colon H \to H$ is WEAKLY CONTINUOUS if it is continuous as a map from H with the weak topology into H with the weak topology.

4.7.12. Exercise. Let H be a Hilbert space and (x_λ) be a net in H.

(a) If $x_\lambda \xrightarrow{s} a$ in H, then $x_\lambda \xrightarrow{w} a$ in H

(b) The strong (norm) topology on H is stronger (bigger) than the weak topology.
(c) Show by example that the converse of (a) does not hold.
(d) Is the norm on H weakly continuous?
(e) If $x_\lambda \xrightarrow{w} a$ in H and if $\|x_\lambda\| \longrightarrow \|a\|$, then $x_\lambda \xrightarrow{s} a$ in H.
(f) If a linear map $T\colon H \to H$ is (strongly) continuous, then it is weakly continuous.

4.7.13. Exercise. Let H be an infinite dimensional Hilbert space. Consider the following subsets of H:

(a) the open unit ball;
(b) the closed unit ball;
(c) the unit sphere;
(d) a closed linear subspace.

Determine for each of these sets whether it is each of the following: strongly closed, weakly closed, strongly open, weakly open, strongly compact, weakly compact. (One part of this is too hard at the moment: *Alaoglu's theorem* 6.3.7 will tell us that the closed unit ball is weakly compact.)

4.7.14. Exercise. Let $\{e^n\colon n \in \mathbb{N}\}$ be the usual basis for l_2. For each $n \in \mathbb{N}$ let $a_n = \sqrt{n}\, e^n$. Which of the following are correct?

(a) 0 is a weak accumulation point of the set $\{a_n\colon n \in \mathbb{N}\}$.
(b) 0 is a weak cluster point of the sequence (a_n).
(c) 0 is the weak limit of a subsequence of (a_n).

4.8. Completion of an Inner Product Space

A little review of the topic of completion of metric spaces may be useful here. Occasionally it may be tempting to think that the completion of a metric space M is a complete metric space of which some dense subset is homeomorphic to M. The problem with this characterization is that it may very well produce two "completions" of a metric space which are not even homeomorphic.

4.8.1. Example. Let M be the interval $(0, \infty)$ with its usual metric. There exist two complete metric spaces N_1 and N_2 such that

(a) N_1 and N_2 are *not* homeomorphic,
(b) M is homeomorphic to a dense subset of N_1, and
(c) M is homeomorphic to a dense subset of N_2.

Here is the correct definition.

4.8.2. Definition. Let M and N be metric spaces. We say that N is a COMPLETION of M if N is complete and M is isometric to a dense subset of N.

In the construction of the real numbers from the rationals one approach is to treat the set of rationals as a metric space and define the set of reals to be its completion. One standard way of producing the completion involves equivalence classes of Cauchy sequences of rational numbers. This technique extends to metric spaces: an elementary way of completing an arbitrary metric space starts with defining an equivalence relation on the set of Cauchy sequences of elements of the space. Here is a slightly less elementary but considerably simpler approach.

4.8.3. Proposition. *Every metric space M is isometric to subspace of $\mathcal{B}(M)$.*

Hint for proof. If (M, d) is a metric space fix $a \in M$. For each $x \in M$ define $\phi_x\colon M \to \mathbb{R}$ by $\phi_x(u) = d(x, u) - d(u, a)$. Show first that $\phi_x \in \mathcal{B}(M)$ for every $x \in M$. Then show that $\phi\colon M \to \mathcal{B}(M)$ is an isometry.

4.8.4. Corollary. *Every metric space has a completion.*

Much of mathematics involves the construction of new objects from old ones—things such as products, coproducts, quotients, completions, compactifications, and unitizations.

Often it is possible—and highly desirable—to characterize such a construction by means of a diagram which describes what the constructed object "does" rather than telling what it "is" or how it is constructed. Such a diagram is a UNIVERSAL MAPPING DIAGRAM and it describes the UNIVERSAL PROPERTY of the object being constructed. In particular it is usually possible to characterize such a construction by the existence of a unique morphism having some particular property. Because this morphism and its corresponding property uniquely characterize the construction in question, they are referred to as a *universal morphism* and a *universal property*, respectively. The following definition is one way of describing the action of such a morphism. If this is your first meeting with the concept of *universal* don't be alarmed by its rather abstract nature. Few people, I think, feel comfortable with this definition until they have encountered a half dozen or more examples in different contexts. (There exist even more technical definitions for this important concept. For example, *an initial or terminal object in a comma category*. To unravel this, if you are interested, search for "universal property" in Wikipedia [46].)

4.8.5. Definition. Let $\mathbf{A} \xrightarrow{\mathsf{F}} \mathbf{B}$ be a functor between categories $\mathbf{A}$ and $\mathbf{B}$ and B be an object in $\mathbf{B}$. A pair (A, u) with A an object in $\mathbf{A}$ and u a $\mathbf{B}$-morphism from B to $\mathsf{F}(A)$ is a UNIVERSAL MORPHISM (or UNIVERSAL ARROW for B (with respect to the functor F) if for every object A' in $\mathbf{A}$ and every $\mathbf{B}$-morphism $B \xrightarrow{f} \mathsf{F}(A')$ there exists a unique $\mathbf{A}$-morphism $A \xrightarrow{\tilde{f}} A'$ such that the following diagram commutes.

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[dr, "f"'] & \mathsf{F}(A) \arrow[d, dashed, "\mathsf{F}(\tilde{f})"] & A \arrow[d, dashed, "\tilde{f}"] \\ & \mathsf{F}(A') & A' \end{tikzcd} \tag{4.4}$$

In this context the object A is often referred to as a UNIVERSAL OBJECT in $\mathbf{A}$.

The next proposition shows that the completion of a metric space is universal. In this result the categories of interest are the category of all metric spaces and uniformly continuous maps and its subcategory consisting of all complete metric spaces and uniformly continuous maps. Here the forgetful functor $|\ \ |$ just forgets about completeness of objects (and does not alter morphisms).

4.8.6. Proposition. *Let M be a metric space and $\overline{M}$ be its completion. If N is a complete metric space and $f\colon M \to N$ is uniformly continuous, then there exists a unique uniformly continuous function $\overline{f}\colon \overline{M} \to N$ which makes the following diagram commute.*

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[dr, "f"'] & |\overline{M}| \arrow[d, dashed, "|\overline{f}|"] & \overline{M} \arrow[d, dashed, "\overline{f}"] \\ & |N| & N \end{tikzcd}$$

Universal constructions in any category whatever produce objects that are unique up to isomorphism.

4.8.7. Proposition. *Universal objects in a category are essentially unique.*

The following consequence of propositions 4.8.6 and 4.8.7 allows us to speak of *the* completion of a metric space.

4.8.8. Corollary. *Metric space completions are unique (up to isometry).*

Now, what about the completion of inner product spaces? As we have seen, every inner product space is a metric space; so it has a (metric space) completion. The question is: Is this completion a Hilbert space? The answer is, happily, *yes*.

4.8.9. Theorem. *Let V be an inner product space and H be its metric space completion. Then there exists an inner product on H which*

(a) *is an extension of the one on V and*
(b) *induces the metric on H.*

CHAPTER 5

HILBERT SPACE OPERATORS

5.1. Invertible Linear Maps and Isometries

5.1.1. Definition. A bounded linear map $T\colon V \to W$ between normed linear spaces is INVERTIBLE if there exists a bounded linear map $T^{-1}\colon W \to V$ such that $T^{-1}T = I_V$ and $TT^{-1} = I_W$.

5.1.2. Definition. Let M and N be metric spaces with metrics d and ρ, respectively. A function $f\colon M \to N$ is an ISOMETRY if $\rho(f(s), f(t)) = d(s,t)$ for all s, $t \in M$.

5.1.3. Definition. A linear map $L\colon V \to W$ between two normed linear spaces is NORM PRESERVING if $\|Lv\| = \|v\|$ for all $v \in V$.

5.1.4. Definition. A linear map $T\colon H \to K$ between two inner product spaces is INNER PRODUCT PRESERVING if $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all x, $y \in H$.

5.1.5. Proposition. *Let $T\colon H \to K$ be a linear map between inner product spaces. Then the following are equivalent:*

(a) *T is inner product preserving;*
(b) *T is norm preserving; and*
(c) *T is an isometry.*

5.1.6. Definition. A linear map $T\colon H \to K$ is called an isomorphism if it is surjective and satisfies any (hence all) of the three conditions in proposition 5.1.5. (Notice that isometries are automatically injective.)

CAUTION. There are (at least) two important categories in which the objects are Hilbert spaces:

(a) $\mathbf{HSp}$ = Hilbert spaces and bounded linear maps, and
(b) $\mathbf{HSp_1}$ = Hilbert spaces and linear contractions.

In any category it is usual to define *isomorphism* to be an *invertible morphism*. Since it is the (topological) category of Hilbert spaces and bounded linear maps that we encounter nearly exclusively in these notes it might seem reasonable to apply the word "isomorphism" to the isomorphisms in this category—in other words, to invertible maps; while an isomorphism in the more restrictive (geometric) category of Hilbert spaces and linear contractions might reasonably be called an *isometric isomorphism*. But this is not the convention. When most mathematicians think of a "Hilbert space isomorphism" they think of a map which preserves all the Hilbert space structure—including the inner product. Thus (invoking 5.1.5) the word "isomorphism" when applied to maps between Hilbert spaces has come to mean *isometric isomorphism*. And consequently, the isomorphisms in the more common category $\mathbf{HSp}$ are called *invertible (bounded linear) maps*.

Recall also: We reserve the word "operator" for bounded linear maps from a Hilbert space *into itself*.

5.1.7. Example. Define an operator T on the Hilbert space $H = L_2([0,\infty))$ by

$$Tf(t) = \begin{cases} f(t-1), & \text{if } t \geq 1 \\ 0, & \text{if } 0 \leq t < 1. \end{cases}$$

Then T is an isometry but not an isometric isomorphism. If, on the other hand, $H = L_2(\mathbb{R})$ and T maps each $f \in H$ to the function $t \mapsto f(t-1)$, then T is an (isometric) isomorphism.

5.1.8. Example. Let H be the Hilbert space defined in example 4.1.7. Then the differentiation mapping $D\colon f \mapsto f'$ from H into $L_2([0,1])$ is an isometric isomorphism. What is its inverse?

5.1.9. Example. Let $\{(X_i, \mu_i)\colon i \in I\}$ be a family of σ-finite measure spaces. Make the disjoint union $X = \biguplus_{i\in I} X_i$ into a measure space (X,μ) in the usual way. Then $L_2(X,\mu)$ is isometrically isomorphic to the direct sum of the Hilbert spaces $L_2(X_i,\mu_i)$.

5.1.10. Example. Let μ be Lebesgue measure on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and ν be Lebesgue measure on $\mathbb{R}$. Let $(Uf)(x) = \dfrac{f(\arctan x)}{\sqrt{1+x^2}}$ for all $f \in L_2(\mu)$ and all $x \in \mathbb{R}$. Then U is an isometric isomorphism between $L_2([-\frac{\pi}{2}, \frac{\pi}{2}], \mu)$ and $L_2(\mathbb{R}, \nu)$.

5.1.11. Proposition. *Every inner product preserving surjection from one Hilbert space into another is automatically linear.*

5.1.12. Proposition. *If $T\colon H \to K$ is a bounded linear map between Hilbert spaces, then*

$$\|T\| = \sup\{|\langle Tx, y\rangle|\colon \|x\| = \|y\| = 1\}\,.$$

Hint for proof. Show first that $\|x\| = \sup\{|\langle x, y\rangle|\colon \|y\| = 1\}$.

5.1.13. Definition. A CURVE in a Hilbert space H is a continuous map from $[0,1]$ into H. A curve is SIMPLE if it is injective. If c is a curve and $0 \le a < b \le 1$, then the CHORD of c corresponding to the interval $[a,b]$ is the vector $c(b) - c(a)$. Two chords are NON-OVERLAPPING if their associated parameter intervals have at most an endpoint in common.

Halmos, in [22], illustrates the roominess of infinite dimensional Hilbert spaces by means of the following elegant example.

5.1.14. Example. In every infinite dimensional Hilbert space there exists a simple curve which makes a right-angle turn at each point, in the sense that every pair of non-overlapping chords are perpendicular.

Hint for proof. Consider characteristic functions in $L_2([0,1])$.

5.2. Operators and their Adjoints

5.2.1. Proposition. *Let S, $T \in \mathfrak{B}(H,K)$ where H and K are Hilbert spaces. If $\langle Sx, y\rangle = \langle Tx, y\rangle$ for every $x \in H$ and $y \in K$, then $S = T$.*

5.2.2. Definition. Let T be an operator on a Hilbert space H. Then Q_T, the QUADRATIC FORM ASSOCIATED WITH T, is the scalar valued function defined by

$$Q_T(x) := \langle Tx, x\rangle$$

for all $x \in H$.

An operator on complex Hilbert spaces is zero if and only if its associated quadratic forms is.

5.2.3. Proposition. *If H is a* complex *Hilbert space and $T \in \mathfrak{B}(H)$, then $T = \mathbf{0}$ if and only if $Q_T = 0$.*

Hint for proof. In the hypothesis $Q_T(z) = 0$ for all $z \in H$ replace z first by $y + x$ and then by $y + ix$.

The preceding proposition is one of the few we will encounter that does not hold for both real and complex Hilbert spaces.

5.2.4. Example. Proposition 5.2.3 is not true if in the hypothesis the word "real" is substituted for "complex".

5.2.5. Definition. A function $T\colon V \to W$ between complex vector spaces is CONJUGATE LINEAR if it is additive ($T(x+y) = Tx + Ty$) and satisfies $T(\alpha x) = \overline{\alpha}\, Tx$ for all $x \in V$ and $\alpha \in \mathbb{C}$. A complex valued function of two variables $\phi\colon V \times W \to \mathbb{C}$ is a SESQUILINEAR FUNCTIONAL if it is linear in its first variable and conjugate linear in its second variable.

If H and K are normed linear spaces, a sesquilinear functional ϕ on $H \times K$ is BOUNDED if there exists a constant $M > 0$ such that

$$|\phi(x,y)| \le M\|x\|\|y\|$$

for all $x \in H$ and $y \in K$.

5.2.6. Proposition. *If $\phi\colon H \oplus K \to \mathbb{C}$ is a bounded sesquilinear functional on the direct sum of two Hilbert spaces, then the following numbers (exist and) are equal:*

(a) $\sup\{|\phi(x,y)|\colon \|x\| \le 1, \|y\| \le 1\}$

(b) $\sup\{|\phi(x,y)|\colon \|x\| = \|y\| = 1\}$

(c) $\sup\left\{\dfrac{|\phi(x,y)|}{\|x\|\,\|y\|}\colon x, y \ne 0\right\}$

(d) $\inf\{M > 0\colon |\phi(x,y)| \le M\|x\|\,\|y\| \textit{ for all } x,\, y \in H\}$.

Hint for proof. The proof is virtually identical to the corresponding result for linear maps (see 3.2.5).

5.2.7. Definition. Let $\phi\colon H \oplus K \to \mathbb{C}$ be a bounded sesquilinear functional on the direct sum of two Hilbert spaces. We define $\|\phi\|$, the *norm* of ϕ, to be any of the (equal) expressions in the preceding result.

5.2.8. Proposition. *Let $T\colon H \to K$ be a bounded linear map between Hilbert spaces. Then*

$$\phi\colon H \oplus K \to \mathbb{C}\colon (x,y) \mapsto \langle Tx, y\rangle$$

is a bounded sesquilinear functional on $H \oplus K$ and $\|\phi\| = \|T\|$.

5.2.9. Proposition. *Let $\phi\colon H \times K \to \mathbb{C}$ be a bounded sesquilinear functional on the product of two Hilbert spaces. Then there exist unique bounded linear maps $T \in \mathfrak{B}(H,K)$ and $S \in \mathfrak{B}(K,H)$ such that*

$$\phi(x,y) = \langle Tx, y\rangle = \langle x, Sy\rangle$$

for all $x \in H$ and $y \in K$. Also, $\|T\| = \|S\| = \|\phi\|$.

Hint for proof. Show that for every $x \in H$ the map $y \mapsto \overline{\phi(x,y)}$ is a bounded linear functional on K.

5.2.10. Definition. Let $T\colon H \to K$ be a bounded linear map between Hilbert spaces. The mapping $(x,y) \mapsto \langle Tx, y\rangle$ from $H \oplus K$ into $\mathbb{C}$ is a bounded sesquilinear functional. By the preceding proposition there exists an unique bounded linear map $T^*\colon K \to H$ called the ADJOINT of T such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

for all $x \in H$ and $y \in K$. Also, $\|T\| = \|T^*\|$.

5.2.11. Example. Recall from example 4.1.3 that the family l_2 of all square summable sequences of complex numbers is a Hilbert space. Let

$$S\colon l_2 \to l_2\colon (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots)\,.$$

Then S is an operator on l_2,called the UNILATERAL SHIFT OPERATOR, and $\|S\| = 1$. The adjoint S^* of the unilateral shift is given by

$$S^*\colon l_2 \to l_2\colon (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots).$$

5.2.12. Example. Let $\big(e^n\big)_{n=1}^{\infty}$ be an orthonormal basis for a separable Hilbert space H and $\big(\alpha_n\big)_{n=1}^{\infty}$ be a bounded sequence of complex numbers. Let $Te^n = \alpha_n e^n$ for every n. Then T can be extended uniquely to a operator on H. What is the norm of this operator? What is its adjoint?

5.2.13. Definition. The operator discussed in the preceding example is known as a DIAGONAL OPERATOR. A common notation for this operator is $\operatorname{diag}(\alpha_1, \alpha_2, \dots)$.

5.2.14. Example. The range of the diagonal operator $\operatorname{diag}\big(1, \frac{1}{2}, \frac{1}{3}, \dots\big)$ on the Hilbert space l_2 is not closed.

5.2.15. Example. Let $(S, \mathcal{A}, \mu)$ be a sigma-finite positive measure space and $L_2(S, \mu)$ be the Hilbert space of all (equivalence classes of) complex valued functions on S which are square integrable with respect to μ. Let ϕ be an essentially bounded complex valued μ-measurable function on S. Define M_ϕ on $L_2(S, \mu)$ by $M_\phi(f) := \phi f$. Then M_ϕ is an operator on $L_2(S, \mu)$; it is called a MULTIPLICATION OPERATOR. Its norm is given by $\|M_\phi\| = \|\phi\|_\infty$ and its adjoint by ${M_\phi}^* = M_{\overline{\phi}}$.

It will be convenient to expand definition 1.2.45 slightly.

5.2.16. Definition. Operators S and T on Hilbert spaces H and K, respectively, are said to be UNITARILY EQUIVALENT if there is an (isometric) isomorphism $U\colon H \to K$ such that $T = USU^{-1}$.

5.2.17. Example. Let μ be Lebesgue measure on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and ν be Lebesgue measure on $\mathbb{R}$. Let $\phi(x) = x$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\psi(x) = \arctan x$ for $x \in \mathbb{R}$. Then the multiplication operators M_ϕ and M_ψ (on $L_2([-\frac{\pi}{2}, \frac{\pi}{2}], \mu)$ and $L_2(\mathbb{R}, \nu)$, respectively) are unitarily equivalent.

5.2.18. Example. Let Let M_ϕ be a multiplication operator on $L_2(S, \mu)$ where $(S, \mathcal{A}, \mu)$ is a σ-finite measure space as in example 5.2.15. Then M_ϕ is idempotent if and only if ϕ is the characteristic function of some set in $\mathcal{A}$.

One may regard the SPECTRAL THEOREM in beginning linear algebra as saying: **Every normal operator on a finite dimensional inner product space is unitarily equivalent to a multiplication operator.** What is truly remarkable is the way in which this generalizes to infinite dimensional Hilbert spaces—a subject for much later discussion.

5.2.19. Exercise. Let $\mathbb{N}_3 = \{1, 2, 3\}$ and μ be counting measure on $\mathbb{N}_3$.

(a) Identify the Hilbert space $L_2(\mathbb{N}_3, \mu)$.
(b) Identify the multiplication operators on $L_2(\mathbb{N}_3, \mu)$.
(c) Let T be the linear operator on $\mathbb{C}^3$ whose matrix representation is

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Use the operator T to illustrate the preceding formulation of the spectral theorem.

5.2.20. Example. Let $L_2 = L_2([0, 1], \lambda)$ be the Hilbert space of all (equivalence classes of) complex valued functions on $[0, 1]$ which are square integrable with respect to Lebesgue measure λ. If $k\colon [0, 1] \times [0, 1] \to \mathbb{C}$ is a bounded Borel measurable function, define K on L_2 by

$$(Kf)(x) = \int_0^1 k(x, y) f(y)\, dy\,.$$

Then K is an INTEGRAL OPERATOR and its KERNEL is the function k. (This is another use of the word "kernel"; it has nothing whatever to do with the more common use of the word: $\ker K = K^{\leftarrow}(\{\mathbf{0}\})$—see definition 1.1.12). The adjoint K^* of K is also an integral operator on L_2 and its kernel is the function k^* defined on $[0, 1] \times [0, 1]$ by $k^*(x, y) = \overline{k(y, x)}$.

Hint for proof. Verify that for $f \in L_2$

$$|(Kf)(x)| \le \|k\|_\infty^{1/2} \left[\int_0^1 |k(x,y)||f(y)|^2\,dy\right]^{1/2}.$$

5.2.21. Example. Let $H = L_2([0,1])$ be the real Hilbert space of all (equivalence classes of) real valued functions on $[0,1]$ which are square integrable with respect to Lebesgue measure. Define V on H by

$$Vf(x) = \int_0^x f(t)\,dt\,. \tag{5.1}$$

Then V is an operator on H. This is a VOLTERRA OPERATOR and is an example of an integral operator. (What is its kernel k?)

5.2.22. Exercise. Let V be the *Volterra operator* defined above.

(a) Show that V is injective.
(b) Compute V^*.
(c) What is $V + V^*$? What is its range?

5.2.23. Proposition. *If S and T are operators on a Hilbert space H and $\alpha \in \mathbb{C}$, then*

(a) $(S+T)^* = S^* + T^*$;
(b) $(\alpha T)^* = \overline{\alpha} T^*$;
(c) $T^{**} = T$; *and*
(d) $(TS)^* = S^* T^*$.

5.2.24. Proposition. *Let T be an operator on a Hilbert space H. Then*

(a) $\|T^*\| = \|T\|$ *and*
(b) $\|T^*T\| = \|T\|^2$.

Hint for proof. Notice that since $T = T^{**}$ to prove (a) it is sufficient to show that $\|T\| \le \|T^*\|$. To this end observe that $\|Tx\|^2 = \langle T^*Tx, x\rangle \le \|T^*\|\|T\|$ whenever $x \in H$ and $\|x\| \le 1$. From this conclude that $\|T\|^2 \le \|T^*T\| \le \|T^*\|\|T\|$.

The rather innocent looking condition (b) in the preceding proposition will turn out to be *the* property of fundamental interest when we study the spectral theory of Hilbert space operators.

5.2.25. Proposition. *If T is an operator on a Hilbert space, then*

(a) $\ker T^* = (\operatorname{ran} T)^\perp$,
(b) $\overline{\operatorname{ran} T^*} = (\ker T)^\perp$,
(c) $\ker T = (\operatorname{ran} T^*)^\perp$, *and*
(d) $\overline{\operatorname{ran} T} = (\ker T^*)^\perp$.

Compare the preceding result with theorem 1.2.43.

5.2.26. Definition. An operator T on a normed linear space V is BOUNDED AWAY FROM ZERO (or BOUNDED BELOW) if there exists a number $\delta > 0$ such that $\|Tx\| \ge \delta\|x\|$ for all $x \in V$.

5.2.27. Example. Clearly if an operator is bounded away from zero, then it is injective. The operator $T\colon x \mapsto (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$ defined on the Hilbert space l_2 shows that being bounded away from zero is a strictly stronger condition than being injective.

5.2.28. Proposition. *If a Hilbert space operator is bounded away from zero, it has closed range.*

5.2.29. Proposition. *An operator on a Hilbert space is invertible if and only if it is bounded away from zero and has dense range.*

5.2.30. Proposition. *The pair of maps $H \mapsto H$ and $T \mapsto T^*$ taking every Hilbert space to itself and every bounded linear map between Hilbert spaces to its adjoint is a contravariant functor from the category* **HSp** *of Hilbert spaces and bounded linear maps to itself.*

5.2.31. Proposition. *Let H be a Hilbert space, $T \in \mathfrak{B}(H)$, $M = H \oplus \{0\}$, and N be the graph of T.*

(a) *Then N is a closed linear subspace of $H \oplus H$.*
(b) *T is injective if and only if $M \cap N = \{(0,0)\}$.*
(c) *$\operatorname{ran} T$ is dense in H if and only if $M + N$ is dense in $H \oplus H$.*
(d) *T is surjective if and only if $M + N = H \oplus H$.*

5.2.32. Example. There exists a Hilbert space H with subspaces M and N such that $M + N$ is not closed.

5.2.33. Proposition. *Let H and K be Hilbert spaces and V be a vector subspace of H. Every bounded linear map $T \colon V \to K$ can be extended to a bounded linear map from $\overline{V}$, the closure of V, without increasing its norm.*

5.2.34. Exercise. Let $\mathcal{E}$ be an orthonormal basis for a Hilbert space H. Discuss linear transformations T on H such that $T(e) = 0$ for every $e \in \mathcal{E}$. Give a nontrivial example.

5.3. Algebras with Involution

5.3.1. Definition. An INVOLUTION on an algebra A is a map $x \mapsto x^*$ from A into A which satisfies

(a) $(x + y)^* = x^* + y^*$,
(b) $(\alpha x)^* = \overline{\alpha}\, x^*$,
(c) $x^{**} = x$, and
(d) $(xy)^* = y^* x^*$

for all x, $y \in A$ and $\alpha \in \mathbb{C}$. An algebra on which an involution has been defined is a $*$-ALGEBRA (pronounced "star algebra"). If a is an element of a $*$-algebra, then a^* is called the ADJOINT of a.

An algebra homomorphism ϕ between $*$-algebras which preserves involution (that is, such that $\phi(a^*) = (\phi(a))^*$) is a $*$-HOMOMORPHISM (pronounced "star homomorphism". A $*$-homomorphism $\phi \colon A \to B$ between unital algebras is said to be UNITAL if $\phi(\mathbf{1}_A) = \mathbf{1}_B$. In the category of $*$-algebras and $*$-homomorphisms, the isomorphisms (called for emphasis $*$-ISOMORPHISMS) are the bijective $*$-homomorphisms.

5.3.2. Example. In the algebra $\mathbb{C}$ of complex numbers the map $z \mapsto \overline{z}$ of a number to its complex conjugate is an involution.

5.3.3. Example. The map of an $n \times n$ matrix to its conjugate transpose is an involution on the unital algebra M_n.

5.3.4. Example. Let X be a compact Hausdorff space. The map $f \mapsto \overline{f}$ of a function to its complex conjugate is an involution in the algebra $\mathcal{C}(X)$.

5.3.5. Example. The map $T \mapsto T^*$ of a Hilbert space operator to its adjoint is an involution in the algebra $\mathfrak{B}(H)$ (see proposition 5.2.23).

5.3.6. Proposition. *Let a and b be elements of a $*$-algebra. Then a commutes with b if and only if a^* commutes with b^*.*

5.3.7. Proposition. *In a unital $*$-algebra $\mathbf{1}^* = \mathbf{1}$.*

5.3.8. Proposition. *If a $*$-algebra A has a left multiplicative identity e, then A is unital and $e = \mathbf{1}_A$.*

5.3.9. Proposition. *Let a be an element of a unital $*$-algebra. Then a^* is invertible if and only if a is. And when a is invertible we have*

$$(a^*)^{-1} = \left(a^{-1}\right)^*.$$

5.4. Self-Adjoint Operators

5.4.1. Definition. An element a of a $*$-algebra A is SELF-ADJOINT (or HERMITIAN) if $a^* = a$. It is NORMAL if $a^*a = aa^*$. And it is UNITARY if $a^*a = aa^* = \mathbf{1}$. The set of all self-adjoint elements of A is denoted by $\mathfrak{H}(A)$, the set of all normal elements by $\mathfrak{N}(A)$, and the set of all unitary elements by $\mathfrak{U}(A)$.

5.4.2. Proposition. *Let a be an element of a complex $*$-algebra. Then there exist unique self-adjoint elements u and v such that $a = u + iv$.*

Hint for proof. Think of the special case of writing a complex number in terms of its real and imaginary parts.

5.4.3. Definition. Let S be a subset of a $*$-algebra A. Then $S^* = \{s^* : s \in S\}$. The subset S is SELF-ADJOINT if $S^* = S$.

A nonempty self-adjoint subalgebra of A is a $*$-SUBALGEBRA (or a SUB-$*$-ALGEBRA).

CAUTION. The preceding definition does *not* say that the elements of a self-adjoint subset of a $*$-algebra are themselves self-adjoint.

5.4.4. Proposition. *An operator T on a complex Hilbert space H is self-adjoint if and only if its associated quadratic form Q_T is real valued.*

Hint for proof. Use the same trick as in 5.2.3. In the hypothesis that $Q_T(z)$ is always real, replace z first by $y + x$ and then by $y + ix$.

5.4.5. Definition. Let T be an operator on a Hilbert space H. The NUMERICAL RANGE of T, denoted by $W(T)$, is the range of the restriction to the unit sphere of H of the quadratic form associated with T. That is,

$$W(T) := \{Q_T(x) : x \in H \text{ and } \|x\| = 1\}.$$

The NUMERICAL RADIUS of T, denoted by $w(T)$, is $\sup\{|\lambda| : \lambda \in W(T)\}$.

5.4.6. Proposition. *If T is a self-adjoint Hilbert space operator, then $\|T\| = w(T)$.*

Hint for proof. The hard part is showing that $\|T\| \leq w(T)$. To this end show first that

$$Q_T(x + y) - Q_T(x - y) = 4\Re\langle Tx, y\rangle \tag{5.2}$$

for all vectors x and y in the Hilbert space, where Q_T is the quadratic form associated with T and $\Re z$ is the real part of the complex number z. Use this to verify that if $\|x\| = \|y\| = 1$, then

$$\Re\langle Tx, y\rangle \leq w(T). \tag{5.3}$$

Write the complex number $\langle Tx, y\rangle$ in polar form $re^{i\theta}$. Clearly if we replace x by $e^{-i\theta}x$ in inequality (5.3) the resulting inequality still holds whenever $\|x\| = \|y\| = 1$.

Finally notice that if $\|x\| = 1$ and y is any vector in H such that $Tx = \|Tx\|\, y$, then

$$\|Tx\| = \langle Tx, y\rangle\,. \tag{5.4}$$

Then the desired result follows immediately.

In proposition 5.2.3 we saw that an operator T on a *complex* Hilbert space is zero if and only if its associated quadratic form is. It is an easy corollary of the preceding proposition that this equivalence also holds for self-adjoint operators on a real Hilbert space.

5.4.7. Corollary. *Let T be a self-adjoint operator on a Hilbert space. Then $T = \mathbf{0}$ if and only if its associated quadratic form Q_T is zero.*

5.4.8. Definition. An operator T on a Hilbert space H is POSITIVE if it is self-adjoint and its associated quadratic form is positive ($Q_T(x) \geq 0$ for every $x \in H$).

5.4.9. Exercise. Let H be a complex Hilbert space and T be a positive operator on H. Define

$$\langle x, y\rangle_1 := \langle Tx, y\rangle \qquad \text{and} \qquad \|x\|_1 \equiv \sqrt{\langle x, x\rangle_1}$$

for all $x \in H$.

(a) Prove that $\|\ \ \|_1$ is a norm if and only if $\ker T = \{0\}$.
(b) Suppose $\ker T = \{0\}$. Show that the norms $\|\ \ \|$ and $\|\ \ \|_1$ induce the same topology on H if and only if T is invertible in $\mathfrak{B}(H)$. *Hint.* When there are two topologies on H, it is frequently useful to consider the identity operator on H.
(c) Suppose T is invertible in $\mathfrak{B}(H)$. For $S \in \mathfrak{B}(H)$ find an expression for the adjoint of S with respect to the inner product $\langle\ ,\ \rangle_1$.

5.5. Projections

5.5.1. Convention. At the moment our attention is focused primarily on operators on Hilbert spaces. In this context the term *projection* is always taken to mean *orthogonal projection* (see definition 1.2.49). Thus a Hilbert space operator is called a projection if $P^2 = P$ and $P^* = P$.

We generalize the definition of "(orthogonal) projection" from Hilbert spaces to $*$-algebras.

5.5.2. Definition. A PROJECTION in a $*$-algebra A is an element p of the algebra which is idempotent ($p^2 = p$) and self-adjoint ($p^* = p$). The set of all projections in A is denoted by $\mathcal{P}(A)$. In the case of $\mathfrak{B}(H)$, the bounded operators on a Hilbert space, we write $\mathcal{P}(H)$, or, if H is understood, just $\mathcal{P}$, for the more informative $\mathcal{P}(\mathfrak{B}(H))$.

5.5.3. Proposition. *Every operator on a Hilbert space that is an isometry on the orthogonal complement of its kernel has closed range.*

Hint for proof. Notice first that if $M = (\ker T)^\perp$, where T is the operator in question, then $\operatorname{ran} T = T^\to(M)$.

5.5.4. Proposition. *Let P be a projection on a Hilbert space H. Then*

(a) $Px = x$ *if and only if* $x \in \operatorname{ran} P$*;*
(b) $\ker P = (\operatorname{ran} P)^\perp$*; and*
(c) $H = \ker P \oplus \operatorname{ran} P$*.*

In part (c) of the preceding proposition the symbol $\oplus$ stands of course for *orthogonal* direct sum (see the paragraph following 1.2.49).

5.5.5. Proposition. *Let M be a subspace of a Hilbert space H. If $P \in \mathfrak{B}(H)$, if $Px = x$ for every $x \in M$, and if $Px = \mathbf{0}$ for every $x \in M^\perp$, then P is the projection of H onto M.*

In the remainder of this section it is important to keep firmly in mind that the spaces $\mathfrak{B}(H)$ of Hilbert space operators are *examples* of $*$-algebras, and that (orthogonal) projections on a Hilbert space are *examples* of projections in $*$-algebras. Thus on the one hand, everything that is claimed for such projections in $*$-algebras holds for Hilbert space projections. For lack of a better name we will refer to these results as documenting the ABSTRACT structure of projections. On the other hand, Hilbert space projections have additional structure which makes no sense in general $*$-algebras: they have *domains* and *ranges*, and they *act on vectors*. When we investigate such matters we will say we are dealing with the SPATIAL (or CONCRETE) structure of projections. Notice how these two views alternate in the remainder of this section.

Our first result gives necessary and sufficient conditions for the sum of projections to be a projection.

5.5.6. Proposition. *Let p and q be projections in a $*$-algebra. Then the following are equivalent:*

(a) $pq = \mathbf{0}$*;*
(b) $qp = \mathbf{0}$*;*

(c) *$qp = -pq$;*
(d) *$p + q$ is a projection.*

5.5.7. Definition. Let p and q be projections in a $*$-algebra. If any of the conditions in the preceding result holds, then we say that p and q are ORTHOGONAL and write $p \perp q$. (Thus for operators on a Hilbert space we could correctly, if unpleasantly, speak of *orthogonal orthogonal projections*!)

Next is the corresponding spatial result for the sum of two projections.

5.5.8. Proposition. *Let P and Q be projections on a Hilbert space H. Then $P \perp Q$ if and only if $\operatorname{ran} P \perp \operatorname{ran} Q$. In this case $P + Q$ is a projection whose kernel is $\ker P \cap \ker Q$ and whose range is $\operatorname{ran} P + \operatorname{ran} Q$.*

5.5.9. Example. On a Hilbert space (orthogonal) projections need not commute. For example let P be the projection of the (real) Hilbert space R^2 onto the line $y = x$ and Q be the projection of R^2 onto the x-axis. Then $PQ \neq QP$.

The product of two projections is a projection if and only if they commute.

5.5.10. Proposition. *Let p and q be projections in a $*$-algebra. Then pq is a projection if and only if $pq = qp$.*

5.5.11. Proposition. *Let P and Q be projections on a Hilbert space H. If $PQ = QP$, then PQ is a projection whose kernel is $\ker P + \ker Q$ and whose range is $\operatorname{ran} P \cap \operatorname{ran} Q$.*

5.5.12. Proposition. *Let p and q be projections in a $*$-algebra. Then the following are equivalent:*

(a) *$pq = p$;*
(b) *$qp = p$;*
(c) *$q - p$ is a projection.*

5.5.13. Definition. Let p and q be projections in a $*$-algebra. If any of the conditions in the preceding result holds, then we write $p \preceq q$.

5.5.14. Proposition. *Let P and Q be projections on a Hilbert space H. Then the following are equivalent:*

(a) *$P \preceq Q$;*
(b) *$\|Px\| \leq \|Qx\|$ for all $x \in H$; and*
(c) *$\operatorname{ran} P \subseteq \operatorname{ran} Q$.*

In this case $Q - P$ is a projection whose kernel is $\operatorname{ran} P + \ker Q$ and whose range is $\operatorname{ran} Q \ominus \operatorname{ran} P$.

Notation: Let H, M, and N be subspaces of a Hilbert space. The assertion $H = M \oplus N$, may be rewritten as $M = H \ominus N$ (or $N = H \ominus M$).

5.5.15. Proposition. *The operation $\preceq$ defined in 5.5.13 for projections on a $*$-algebra A is a partial ordering on $\mathcal{P}(A)$. If p is a projection in A, then $\mathbf{0} \preceq p \preceq \mathbf{1}$.*

5.5.16. Proposition. *Suppose p and q are projections on a $*$-algebra A. If $pq = qp$, then the infimum of p and q, which we denote by $p \curlywedge q$, exists with respect to the partial ordering $\preceq$ and $p \curlywedge q = pq$. The infimum $p \curlywedge q$ may exist even when p and q do not commute. A necessary and sufficient condition that $p \perp q$ hold is that both $p \curlywedge q = \mathbf{0}$ and $pq = qp$ hold.*

5.5.17. Proposition. *Suppose p and q are projections on a $*$-algebra A. If $p \perp q$, then the supremum of p and q, which we denote by $p \curlyvee q$, exists with respect to the partial ordering $\preceq$ and $p \curlyvee q = p + q$. The supremum $p \curlyvee q$ may exist even when p and q are not orthogonal.*

5.6. Normal Operators

5.6.1. Proposition. *An operator T on a Hilbert space H is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in H$.*

Hint for proof. Use corollary 5.4.7.

5.6.2. Corollary. *If T is a normal operator on a Hilbert space, then $\ker T = \ker T^*$.*

5.6.3. Proposition. *If T is a normal operator on a Hilbert space, then $\|T^2\| = \|T\|^2$.*

Hint for proof. Substitute Tx for x in proposition 5.6.1. Use proposition 5.2.24.

5.6.4. Proposition. *If T is a normal operator on a Hilbert space H, then*

$$H = \overline{\operatorname{ran} T} \oplus \ker T\,.$$

5.6.5. Corollary. *A normal operator on a Hilbert space has dense range if and only if it is injective.*

5.6.6. Example. The *unilateral shift* operator (5.2.11) is an example of a Hilbert space operator that is injective but does not have dense range. Its adjoint has dense range (in fact, is surjective) but is not injective.

5.6.7. Corollary. *A normal operator on a Hilbert space is invertible if and only if it is bounded away from zero.*

5.7. Operators of Finite Rank

5.7.1. Definition. The RANK of a linear map is the (vector space) dimension of its range. Thus an operator of FINITE RANK is one which has finite dimensional range. We denote by $\mathfrak{FR}(V)$ the collection of finite rank operators on a vector space V.

5.7.2. Example. For vectors x and y in a Hilbert space H define

$$x \otimes y \colon H \to H \colon z \mapsto \langle z, y \rangle x\,.$$

If x and y are nonzero vectors in H, then $x \otimes y$ is a rank-one operator on H.

The next two propositions identify the adjoints and composites of these rank-one operators.

5.7.3. Proposition. *If u and v are vectors in a Hilbert space, then*

$$(u \otimes v)^* = v \otimes u\,.$$

5.7.4. Proposition. *If u, v, x, and y are vectors in a Hilbert space, then*

$$(u \otimes v)(x \otimes y) = \langle x, v \rangle (u \otimes y)\,.$$

5.7.5. Proposition. *If x is a vector in a Hilbert space H, then $x \otimes x$ is a rank-one projection if and only if x is a unit vector.*

5.7.6. Proposition. *Suppose that $\{e^1, e^2, e^3, \dots, e^n\}$ is an orthonormal set in a Hilbert space H. Let $M = \operatorname{span}\{e^1, \dots, e^n\}$. Then $P = \sum_{k=1}^n e^k \otimes e^k$ is the orthogonal projection of H onto M.*

5.7.7. Proposition. *If u and v are vectors in a Hilbert space and T is an operator on H, then*

$$T\,(u \otimes v) = (Tu) \otimes v\,.$$

5.7.8. Proposition. *If u and v are vectors in a Hilbert space and T is an operator on H, then*

$$(u \otimes v)\,T = u \otimes T^*v\,.$$

We will see shortly that the family of all finite rank operators on a Hilbert space is a minimal $*$-ideal in the $*$-algebra $\mathfrak{B}(H)$. In preparation for this we review a few basic facts about ideals in algebras.

5.7.9. Definition. An LEFT IDEAL in an algebra A is a vector subspace J of A such that $AJ \subseteq J$. (For RIGHT IDEALS, of course, we require $JA \subseteq J$.) We say that J is an IDEAL if it is a two-sided ideal, that is, both a left and a right ideal. A PROPER ideal is an ideal which is a proper subset of A.

The ideals $\{0\}$ and A are often referred to as the TRIVIAL IDEALS of A. The algebra A is SIMPLE if it has no nontrivial ideals.

A MAXIMAL ideal is a proper ideal that is properly contained in no other proper ideal. We denote the family of all maximal ideals in an algebra A by $\operatorname{Max} A$. A MINIMAL ideal is a nonzero ideal that properly contains no other nonzero ideal.

5.7.10. Convention. Whenever we refer to an *ideal* in an algebra we understand it to be a two-sided ideal (unless the contrary is stated).

5.7.11. Proposition. *No invertible element in a unital algebra can belong to a proper ideal.*

5.7.12. Proposition. *Every proper ideal in a unital algebra A is contained in a maximal ideal. Thus, in particular,* $\operatorname{Max} A$ *is nonempty whenever A is a unital algebra.*

Hint for proof. Zorn's lemma.

5.7.13. Proposition. *Let a be an element of a commutative algebra A. If A is unital and a is not invertible, then aA is a proper ideal in A.*

5.7.14. Definition. The ideal aA in the preceding proposition is the PRINCIPAL IDEAL generated by a.

5.7.15. Definition. Let J be an ideal in an algebra A. Define an equivalence relation $\sim$ on A by

$$a \sim b \qquad \text{if and only if} \qquad b - a \in J.$$

For each $a \in A$ let $[a]$ be the equivalence class containing a. Let A/J be the set of all equivalence classes of elements of A. For $[a]$ and $[b]$ in A/J define

$$[a] + [b] := [a + b] \qquad \text{and} \qquad [a][b] := [ab]$$

and for $\alpha \in \mathbb{C}$ and $[a] \in A/J$ define

$$\alpha[a] := [\alpha a]\,.$$

Under these operations A/J becomes an algebra. It is the QUOTIENT ALGEBRA of A by J. The notation A/J is usually read "A mod J". The surjective algebra homomorphism

$$\pi \colon A \to A/J \colon a \mapsto [a]$$

is called the QUOTIENT MAP.

5.7.16. Proposition. *The preceding definition makes sense and the claims made therein are correct. Furthermore, if the ideal J is proper, then the quotient algebra A/J is unital if A is.*

Hint for proof. You will need to show that:

(a) $\sim$ is an equivalence relation.
(b) Addition and multiplication of equivalence classes is well defined.
(c) Multiplication of an equivalence class by a scalar is well defined.
(d) A/J is an algebra.
(e) The "quotient map" π really is a surjective algebra homomorphism.

(At what point is it necessary that we factor out an ideal and not just a subalgebra?)

5.7.17. Definition. In an algebra A with involution a $*$-IDEAL is a self-adjoint ideal in A.

5.7.18. Proposition. *Suppose that J is a $*$-ideal in a $*$-algebra A. In the quotient algebra A/J, as developed in 5.7.15, define $[a]^* := [a^*]$ for every $[a] \in A/J$. This is well-defined and makes A/J into a $*$-algebra.*

5.7.19. Proposition. *Let H be a Hilbert space. Then the family $\mathfrak{FR}(H)$ of all finite rank operators on H is a $*$-ideal in the $*$-algebra $\mathfrak{B}(H)$.*

5.7.20. Proposition. *Let T be an operator of rank-n on a Hilbert space H. Then there exist vectors $u^1, \dots, u^n, v^1, \dots, v^n$ in H such that*

$$T = \sum_{k=1}^{n} u^k \otimes v^k \,.$$

Hint for proof. Let $\{e^1, \dots, e^n\}$ be an orthonormal basis for the range of T. For an arbitrary vector $x \in H$ write out the *Fourier expansion* of Tx (see proposition 4.5.4(d)).

5.7.21. Corollary. *Every rank-one operator on a Hilbert space is of the form $u \otimes v$ for some nonzero vectors u and v in H.*

5.7.22. Corollary. *Every finite rank operator on a Hilbert space is a sum of (finitely many) rank-one operators.*

5.7.23. Proposition. *If H is a Hilbert space, then the family $\mathfrak{FR}(H)$ of finite rank operators on H is a minimal ideal in $\mathfrak{B}(H)$.*

Hint for proof. Let $\mathfrak{J}$ be a nonzero ideal in $\mathfrak{B}(H)$. We need to show that $\mathfrak{J}$ contains $\mathfrak{FR}(H)$. By proposition 5.7.20 it suffices to prove that $\mathfrak{J}$ contains every operator of the form $u \otimes v$. To this end let T be any nonzero member of $\mathfrak{J}$, let y be a unit vector in the range of T, and let x be any vector in H such that $y = Tx$. Consider the operator $(u \otimes y)T(x \otimes v)$.

Although the set of finite rank operators is an ideal in the Banach algebra of operators on a Hilbert space, it is not a closed ideal and, as a consequence, turns out not to be an appropriate set to factor out in hopes of getting a quotient object in the category of Banach algebras. The closure of the ideal of finite rank operators on a Hilbert space is the ideal of *compact operators*. In a sense it would be a natural next step to examine the properties of this class of operators, among which we find some of the most important operators in applications, integral operators, for example. However, a smoother and more perspicuous exposition is possible with an expanded collection of tools at our disposal. So for the next few chapters we will concentrate on developing some of the most useful tools in functional analysis.

CHAPTER 6

BANACH SPACES

6.1. The Hahn-Banach Theorems

It should be said at the beginning that referring to "the" *Hahn-Banach theorem* is something of a misnomer. A writer who claims to be using the *Hahn-Banach theorem* may be using any of a dozen or so versions of the theorem or its corollaries. What do the various versions have in common? Basically they assure us that Banach spaces have a rich enough supply of bounded linear functionals to make their dual spaces useful tools in studying the spaces themselves. That is, they guarantee an interesting and productive duality theory. Below we develop four versions of the result and three of its corollaries. We will succumb to usual practice: when in the sequel we make use of any of these results we will say we are using "the" *Hahn-Banach theorem*. Incidentally, in the name Banach, the "a"'s are pronounced as in *car*, the "ch" as in *chaos*, and the accent is on the first syllable (BA-nach).

6.1.1. Definition. A real valued function f on a real vector space V is a SUBLINEAR FUNCTIONAL if $f(x+y) \leq f(x) + f(y)$ and $f(\alpha x) = \alpha f(x)$ for all x, $y \in V$ and $\alpha \geq 0$.

6.1.2. Example. A norm on a real vector space is a sublinear functional.

Our first version of the *Hahn-Banach theorem* is pure linear algebra and applies only to *real* vector spaces. It allows us to extend certain linear functionals on such spaces from subspaces to the whole space.

6.1.3. Theorem (Hahn-Banach theorem I)**.** *Let M be a subspace of a real vector space V and p be a sublinear functional on V. If f is a linear functional on M such that $f \leq p$ on M, then f has an extension g to all of V such that $g \leq p$ on V.*

Hint for proof. In this theorem the notation $h \leq p$ means that $\operatorname{dom} h \subseteq \operatorname{dom} p = V$ and that $h(x) \leq p(x)$ for all $x \in \operatorname{dom} h$. In this case we say that h is DOMINATED by p. We write $f \preccurlyeq g$ if g is an extension of f (that is, if $f(x) = g(x)$ for all $x \in \operatorname{dom} f$).

Let $\mathcal{S}$ be the family of all real valued linear functionals on subspaces of V which are extensions of f and are dominated by p. The family $\mathcal{S}$ is partially ordered by $\preccurlyeq$. Use *Zorn's lemma* to produce a maximal element g of $\mathcal{S}$. The proof is finished if you can show that $\operatorname{dom} g = V$.

To this end, suppose to the contrary that there is a vector $c \in V$ which does not belong to $D = \operatorname{dom} g$. Prove first that

$$g(y) - p(y-c) \leq -g(x) + p(x+c) \tag{6.1}$$

for all x, $y \in D$. Conclude that there is a number γ which lies between the supremum of all the numbers on the left side of (6.1) and the infimum of those on the right.

Let E be the span of $D \cup \{c\}$. Define h on E by $h(x + \alpha c) = g(x) + \alpha\gamma$ for $x \in D$ and $\alpha \in \mathbb{R}$. Now show $h \in \mathcal{S}$. (Take $\alpha > 0$ and work separately with $h(x + \alpha c)$ and $h(x - \alpha c)$.)

The next theorem is the version of theorem 6.1.3 for spaces with complex scalars. It is sometimes referred to as the *Bohnenblust-Sobczyk-Suhomlinov theorem.*

6.1.4. Theorem (Hahn-Banach theorem II)**.** *Let V be a complex vector space, p be a seminorm on V, and M be a subspace of V. If $f \colon M \to \mathbb{C}$ is a linear functional such that $|f| \leq p$, then f has an extension g which is a linear functional on all of V satisfying $|g| \leq p$.*

Hint for proof. This result is essentially a straightforward corollary to the real version of the *Hahn-Banach theorem* given above in 6.1.3. Let $V_{\mathbb{R}}$ be the real vector space associated with V (same vectors but using only $\mathbb{R}$ as the scalar field). Notice that if u is the real part of f (that is, $u(x) = \Re(f(x))$ for each $x \in M$), then u is an $\mathbb{R}$-linear functional on M (that is, $f(x+y) = f(x) + f(y)$ and $f(\alpha x) = \alpha f(x)$ for all x, $y \in M$ and all $\alpha \in \mathbb{R}$) and $f(x) = u(x) - i\,u(ix)$. And conversely, if u is a $\mathbb{R}$-linear functional on some vector space, then $f(x) := u(x) - i\,u(ix)$ defines a (complex valued) linear functional on that space. Notice also that if u is the real part of f, then $|u(x)| \le p$ for all x if and only if $|f(x)| \le p$ for all x. With these observations in mind just apply 6.1.3 to the real part of f on $V_{\mathbb{R}}$.

Next is perhaps the most often used version of the *Hahn-Banach theorem.*

6.1.5. Theorem (Hahn-Banach theorem III)**.** *Let M be a vector subspace of a (real or complex) normed linear space V. Then every continuous linear functional f on M has an extension g to all of V such that $\|g\| = \|f\|$.*

Hint for proof. Consider the function $p\colon V \to \mathbb{R}\colon x \mapsto \|f\|\|x\|$.

It is sometimes useful, when proving a theorem, to consider the possibility of more "natural" or even more "obvious" proofs. On occasion one is rewarded by an enlightening simplification and clarification of complex material. And sometimes one is fooled.

6.1.6. Exercise. Your buddy Fred R. Dimm thinks he has discovered a really simple direct proof of the version of the Hahn-Banach theorem given in proposition 6.1.5: A continuous linear functional f defined on a (linear) subspace M of a normed linear space V can be extended to a continuous linear functional g on all of V without increasing its norm. He says that all one needs to do is let N be any vector space complement of M and let B be a (Hamel) basis for N. Define $g(e) = 0$ for every vector $e \in B$ and $g = f$ on M. Then extend g by linearity to all of V. Surely, Fred says, g is linear by construction and its norm has not increased since we have made it zero where it was not previously defined. Show Fred the error of his ways.

6.1.7. Theorem (Hahn-Banach theorem IV)**.** *Let V be a (real or complex) normed linear space, M be a vector subspace of V, and suppose that $y \in V$ is such that $d := \operatorname{dist}(y, M) > 0$. Then there exists a functional $g \in V^*$ such that $g^{\rightarrow}(M) = \{0\}$, $g(y) = d$, and $\|g\| = 1$.*

Hint for proof. Let $N := \operatorname{span}(M \cup \{y\})$ and define $f\colon N \to \mathbb{K}\colon x + \alpha y \mapsto \alpha d$ $(x \in M,\ \alpha \in \mathbb{K})$.

The most useful special case of the preceding theorem is when $M = \{\mathbf{0}\}$ and $y \ne \mathbf{0}$. This tells us that the only way for $f(x)$ to vanish for every $f \in V^*$ is for x to be the zero vector.

6.1.8. Corollary. *Let V be a normed linear space and $x \in V$. If $f(x) = 0$ for every $f \in V^*$, then $x = \mathbf{0}$.*

6.1.9. Definition. A family $\mathcal{G}(S)$ of scalar valued functions on a set S SEPARATES points of S if for every pair x and y of distinct points in S there exists a function $f \in \mathcal{G}(S)$ such that $f(x) \ne f(y)$.

6.1.10. Corollary. *If V is a normed linear space, then V^* separates points of V.*

6.1.11. Corollary. *Let V be a normed linear space and $x \in V$. Then there exists $f \in V^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.*

6.1.12. Proposition. *Let V be a normed linear space, $\{x_1, \dots, x_n\}$ be a linearly independent subset of V, and $\alpha_1, \dots, \alpha_n$ be scalars. Then there exists an f in V^* such that $f(x_k) = \alpha_k$ for $1 \le k \le n$.*

Hint for proof. For $x = \sum_{k=1}^{n} \beta_k x_k$ let $g(x) = \sum_{k=1}^{n} \alpha_k \beta_k$.

6.1.13. Proposition. *Let V be a normed linear space and $\mathbf{0} \ne x \in V$. Then*

$$\|x\| = \max\{|f(x)|\colon f \in V^* \text{ and } \|f\| \le 1\}\,.$$

The use of max instead of sup in the preceding proposition is intentional. There exists an f in the closed unit ball of V^* such that $f(x) = \|x\|$.

6.1.14. Proposition. *Let V be a normed linear space. If V^* is separable, then so is V.*

6.2. Natural Transformations

6.2.1. Definition. Let $\mathbf{A}$ and $\mathbf{B}$ be categories and F, $G\colon \mathbf{A} \to \mathbf{B}$ be covariant functors. A NATURAL TRANSFORMATION from F to G is a map τ which assigns to each object A in $\mathbf{A}$ a morphism $\tau_A\colon F(A) \to G(A)$ in $\mathbf{B}$ in such a way that for every morphism $\alpha\colon A \to A'$ in $\mathbf{A}$ the following diagram commutes.

$$\begin{tikzcd} F(A) \arrow[r, "F(\alpha)"] \arrow[d, "\tau_A"'] & F(A') \arrow[d, "\tau_{A'}"] \\ G(A) \arrow[r, "G(\alpha)"'] & G(A') \end{tikzcd}$$

We denote such a transformation by $\tau\colon F \to G$. (The definition of a natural transformation between two contravariant functors should be obvious: just reverse the horizontal arrows in the preceding diagram.

A natural transformation $\tau\colon F \to G$ is a NATURAL EQUIVALENCE if each morphism τ_A is an isomorphism in $\mathbf{B}$.

6.2.2. Example. Recall that if V is a normed linear space its *dual space* V^* is the set of all continuous linear functionals on V. On V^* addition and scalar multiplication are defined pointwise. The norm is the usual operator norm: if $f \in V^*$, then

$$\|f\| := \inf\{M > 0\colon |f(x)| \le M\|x\| \text{for all } x \in V\} = \sup\{|f(x)|\colon \|x\| \le 1\}.$$

Under the metric induced by this norm the dual space V^* is complete and thus is a Banach space. If $T\colon V \to W$ is a continuous linear map between normed linear spaces, $T^*\colon W^* \to V^*$ is defined as for vector spaces: $T^*(g) = g \circ T$ for every $g \in W^*$. The map T^*, called the ADJOINT of T, is a continuous linear map. The dual V^{**} of the dual V^* of a normed linear space V is the SECOND DUAL of V and $T^{**} := \big(T^*\big)^*$ maps V^{**} into W^{**}. If V is a normed linear space, then the map $\tau_V\colon V \to V^{**}\colon x \mapsto x^{**}$ where $x^{**}(f) = f(x)$ for all $f \in V^*$ is called the NATURAL EMBEDDING of V into V^{**}. (It is not altogether obvious that this mapping is injective. We will show that it is in proposition 6.2.5.)

On the category $\mathbf{BAN}_\infty$ of Banach spaces and bounded linear transformations let I be the identity functor and $(\,\cdot\,)^{**}$ be the second dual functor. Then the *natural embedding* τ is a natural transformation from I to $(\,\cdot\,)^{**}$.

6.2.3. Example. Explain exactly what is meant when people say that the isomorphism

$$\mathcal{C}(X \uplus Y) \cong \mathcal{C}(X) \times \mathcal{C}(Y)$$

is a natural equivalence. The categories involved are $\mathbf{CpH}$ (compact Hausdorff spaces and continuous maps) and $\mathbf{BAN}_\infty$ (Banach spaces and bounded linear transformations).

6.2.4. Example. On the category $\mathbf{CpH}$ of compact Hausdorff spaces and continuous maps let I be the identity functor and let $\mathcal{C}^*$ be the functor which takes a compact Hausdorff space X to the Banach space $(C(X))^*$ and takes a continuous function $\phi\colon X \to Y$ between compact Hausdorff spaces to the contractive linear transformation $\mathcal{C}^*\phi = (\mathcal{C}\phi)^*$. Define the *evaluation map* E by

$$E_X\colon X \to \mathcal{C}^*(X)\colon x \mapsto E_X(x)$$

where $(E_X(x))(f) = f(x)$ for all $f \in \mathcal{C}(X)$.

(a) Notice that this definition does make sense. If $x \in X$, then $E_X(x)$ really does belong to $\mathcal{C}^*(X)$.

(b) Show that the map E makes the following diagram commute

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "E_X"'] & Y \arrow[d, "E_Y"] \\ \mathcal{C}^*(X) \arrow[r, "\mathcal{C}^*(\phi)"'] & \mathcal{C}^*(Y) \end{tikzcd}$$

The preceding example may seem a bit disappointing. It seems very much as if we are attempting to establish a natural transformation between the identity functor (on **CpH**) and the $\mathcal{C}^*$ functor. The problem, of course, is that the two functors have different target categories. This is a question we will deal with later. The evaluation map $E_X\colon X \to \mathcal{C}^*(X)$ is certainly not surjective. Notice, for example, that every functional of the form $E_X(x)$ preserves multiplication (as well as addition and scalar multiplication). A consequence of this turns out to be that such functionals all live on the unit sphere of $\mathcal{C}^*(X)$. With a new topology (the so-called *weak-star topology*—see 6.3.5) on $\mathcal{C}^*(X)$ the range of E_X turns out to be a compact Hausdorff space which is naturally equivalent to X itself. Even more remarkably, we will eventually show that the range of E_X can be regarded as the maximal ideal space of the algebra $\mathcal{C}(X)$.

6.2.5. Proposition. *If V is a normed linear space the natural embedding $\tau_V\colon V \to V^{**}\colon x \mapsto x^{**}$ (defined in example 6.2.2) is an isometric linear map.*

6.2.6. Definition. A Banach space B is REFLEXIVE if the natural embedding $\tau_B\colon B \to B^{**}$ (defined in the preceding proposition) is onto.

Example 6.2.2 shows that in the category of Banach spaces and bounded linear maps, the mapping $\tau\colon B \mapsto \tau_B$ is a natural transformation from the identity functor to the second dual functor. With the *Hahn-Banach theorem* we can say more.

6.2.7. Proposition. *In the category of reflexive Banach spaces and bounded linear maps the mapping $\tau\colon B \mapsto \tau_B$ is a natural equivalence between the identity and second dual functors.*

6.2.8. Example. Every Hilbert space is reflexive.

6.2.9. Exercise. Let V and W be normed linear spaces, U be a nonempty convex subset of V, and $f\colon U \to W$. Without using any form of the *mean value theorem* show that if $df = \mathbf{0}$ on U, then f is constant on U.

6.2.10. Example. Let l_∞ be the Banach space of all bounded sequences of real numbers and c be the subspace of l_∞ comprising all sequences $x = (x_n)$ such that $\lim_{n\to\infty} x_n$ exists. Define

$$f\colon c \to \mathbb{R}\colon x \mapsto \lim_{n\to\infty} x_n.$$

Then f is a continuous linear functional on c and $\|f\| = 1$. By the *Hahn-Banach theorem* f has an extension to $\widehat{f} \in {l_\infty}^*$ with $\|\widehat{f}\| = 1$. For every subset A of $\mathbb{N}$ define a sequence $\big(x_n^A\big)$ by $x_n^A = \begin{cases} 1, & \text{if } n \in A; \\ 0, & \text{if } n \notin A. \end{cases}$ Next define $\mu\colon \mathfrak{P}(\mathbb{N}) \to \mathbb{R}\colon A \mapsto \widehat{f}\big(x_n^A\big)$. Then μ is a finitely additive set function but is not countably additive.

6.3. Banach Space Duality

6.3.1. Notation. Let V be a normed linear space, $M \subseteq V$, and $F \subseteq V^*$. Then we define

$$M^\perp := \{f \in V^*\colon f(x) = 0 \text{ for all } x \in M\}$$

$$F_\perp := \{x \in V\colon f(x) = 0 \text{ for all } f \in F\}$$

One may read $M^\perp$ as "M perp" or "M upper perp"; and $N^\perp$ is "N perp" or "N lower perp". The set $M^\perp$ is the ANNIHILATOR of M and $F_\perp$ is the PRE-ANNIHILATOR of F.

6.3.2. Proposition (Annihilators)**.** *Let B be a Banach space, $M \subseteq B$, and $F \subseteq B^*$. Then*

(a) *If $M \subseteq N \subseteq B$, then $N^\perp \subseteq M^\perp$.*
(b) *If $F \subseteq G \subseteq B^*$, then $G_\perp \subseteq F_\perp$.*
(c) *$M^\perp$ is a closed linear subspace of B^*.*
(d) *$F_\perp$ is a closed linear subspace of B.*
(e) *$M \subseteq M^\perp{}_\perp$.*
(f) *$F \subseteq F_\perp{}^\perp$.*
(g) *$M^\perp{}_\perp = \bigvee M$.*
(h) *$M^\perp = B^*$ if and only if $M = \{0\}$.*
(i) *$F_\perp = B$ if and only if $F = \{0\}$.*
(j) *If M is a linear subspace of B, then $M^\perp = \{0\}$ if and only if M is dense in B.*
(k) *If B is reflexive, then $F_\perp{}^\perp = \bigvee F$.*

Hint for proof. For (k) show that $\tau(F_\perp) = F^\perp$ and that therefore $F_\perp{}^\perp = F^\perp{}_\perp$.

6.3.3. Proposition. *Let $T \in \mathfrak{B}(B, C)$ where B and C are Banach spaces. Then*

(a) $\ker T^* = (\operatorname{ran} T)^\perp$.
(b) $\ker T = (\operatorname{ran} T^*)_\perp$.
(c) $\overline{\operatorname{ran} T} = (\ker T^*)_\perp$.
(d) $\overline{\operatorname{ran} T^*} \subseteq (\ker T)^\perp$.

The next example is of course a trivial consequence of proposition 3.3.5, but a simple direct proof would be nice.

6.3.4. Example. The closed unit ball in l_2 is not compact.

The usual norm topology on normed linear spaces turns out to be much too large for some applications. As we have seen it has so many open sets that the closed unit ball is not compact (except in the finite dimensional case). On the other hand, too few open sets (the indiscrete topology, as an extreme example) would destroy duality theory by failing to provide a plentiful supply of continuous linear functionals. The so-called *weak-star topology* (defined below) on the dual of a normed linear space turns out to be just the right size. It has enough open sets to guarantee a vigorous duality theory and few enough open sets to make the closed unit ball compact (see *Alaoglu's theorem* 6.3.7).

6.3.5. Definition. Let V be a normed linear space. The WEAK TOPOLOGY on V is the weak topology induced by the elements of V^*. (So, of course, the weak topology on the dual space V^* is the weak topology induced by the elements of V^{**}.) When a net (x_λ) converges weakly to a vector a in V we write $x_\lambda \xrightarrow{w} a$. The w^*-TOPOLOGY (pronounced *weak star topology*) on the dual space V^* is the weak topology induced by the elements of $\operatorname{ran}\tau$ where τ is the natural injection of V into V^{**}. When a net (f_λ) converges weakly to a vector g in V^* we write $f_\lambda \xrightarrow{w^*} g$. Notice that when a Banach space B is reflexive (in particular, for Hilbert spaces) the weak and weak star topologies on B^* are identical.

6.3.6. Proposition. *Let $f \in V^*$ where V is a Banach space. For every $\epsilon > 0$ and every finite subset $F \subseteq V$ define*

$$U(f; F; \epsilon) = \{g \in V^* \colon |f(x) - g(x)| < \epsilon \text{ for all } x \in F\}.$$

The family of all such sets is a base for the w^*-topology on V^*.

The next result, *Alaoglu's theorem*, asserting the w^*-compactness of the closed unit ball in the dual of a normed linear space V is one of the most useful theorems in functional analysis. Thus at first it may seem somewhat surprising that it is so hard to find a complete and carefully explained proof of this result. The standard proof is in a sense very simple; it is basically just recognizing that the members of a family $\mathcal{F}$ of scalar valued functions can be regarded either

(i) as functions defined on an uncountable product of compact subsets of the scalar field, or alternatively

(ii) as members of the closed unit ball of V^*.

The crux of the proof is convincing yourself that the convergence of a net of such functions in the product space is exactly "the same thing" as its w^*-convergence in the closed unit ball of V^*. Trying simultaneously to identify and distinguish between two sets of functions is notationally challenging. So my best advice is to work through it yourself until you see how ingenious, but basically simple, the argument is.

6.3.7. Theorem (Alaoglu's theorem)**.** *The closed unit ball of a normed linear space is compact in the w^*-topology.*

Hint for proof. Let V be a normed linear space and denote by $[V^*]_1$ the closed unit ball of its dual space. For each $x \in V$ define $D_x = \{\lambda \in \mathbb{K} \colon |\lambda| \le \|x\|\}$. The product $\mathbf{P} = \prod\limits_{x\in V} D_x$ is compact by *Tychonoff's theorem*. Consider the function

$$\Phi\colon [V^*]_1 \to \mathbf{P}\colon f \mapsto \big(f(x)\big)_{x\in V}$$

where $[V^*]_1$ has the w^*-topology and $\mathbf{P}$ has the product topology (see 4.7.4). Here we have written $\big(f(x)\big)_{x\in V}$ for the indexed family of numbers (or generalized sequence) rather than the more usual $\big(f_x\big)_{x\in V}$. (For more on indexed families see [13], section 7.2.) It is easy to see that Φ is injective. To show that it is continuous let $\big(f_\lambda\big)_{\lambda\in\Lambda}$ be a net in $[V^*]_1$ and g be a function in $[V^*]_1$ such that $f_\lambda \xrightarrow{\ w^*\ } g$. Notice that this happens if and only if $\pi_x(f_\lambda) \to \pi_x(g)$ for every $x \in V$ (where π_x is a coordinate projection on $\mathbf{P}$—see 3.5.6). Finally, let $\big(f_\lambda\big)_{\lambda\in\Lambda}$ be a net in $[V^*]_1$ such that $\big(\Phi(f_\lambda)\big)_{\lambda\in\Lambda}$ converges in $\mathbf{P}$. Show that the net $\big(f_\lambda(x)\big)_{\lambda\in\Lambda}$ converges in $\mathbb{K}$ for every $x \in V$. Let $g(x) = \lim\limits_\lambda f_\lambda(x)$ for all $x \in V$ and prove that g is linear, that $\|g\| \le 1$, that $f_\lambda \xrightarrow{\ w^*\ } g$ in $[V^*]_1$, and that $\Phi(f_\lambda) \to \Phi(g)$ in $\mathbf{P}$.

To the best of my knowledge the most reader-friendly version of this standard proof can be found in [5], Theorem 15.11. A much shorter, much easier notationally, much slicker, and much less edifying proof is available in [37], theorem 2.5.2. I find it less edifying because of its reliance on universal nets. A net in a set S is UNIVERSAL if for every subset T of S the net is either eventually in T or eventually in its complement. Despite the fact that every net can be shown to have a universal subnet, even to find a singe nontrivial example of a universal net (that is, one that is not eventually constant) requires the *axiom of choice*.

6.3.8. Convention. In the context of Banach spaces we will adopt the same convention for the use of the word "subspace" we used for Hilbert spaces. The word "subspace" will always mean *closed vector subspace*. To indicate that M is a subspace of a Banach space B we write $M \preccurlyeq B$.

6.3.9. Definition. A sequence of Banach spaces and continuous linear maps

$$\cdots \longrightarrow B_{n-1} \xrightarrow{\ T_n\ } B_n \xrightarrow{\ T_{n+1}\ } B_{n+1} \longrightarrow \cdots$$

is said to be EXACT AT B_n if $\operatorname{ran} T_n = \ker T_{n+1}$. A sequence is EXACT if it is exact at each of its constituent Banach spaces. A sequence of Banach spaces and continuous linear maps of the form

$$0 \longrightarrow A \xrightarrow{\ S\ } B \xrightarrow{\ T\ } C \longrightarrow 0 \tag{6.2}$$

is a SHORT EXACT SEQUENCE. (Here $\mathbf{0}$ denotes the trivial 0-dimensional Banach space, and the unlabeled arrows are the obvious maps.)

The preceding definitions were for the category $\mathbf{BAN}_\infty$ of Banach spaces and continuous linear maps. Of course the notion of an *exact sequence* makes sense in many situations—for example, in the categories of Banach algebras, C^*-algebras, Hilbert spaces, vector spaces, Abelian groups, and modules, among others.

6.3.10. Proposition. *Consider the following diagram in the category of Banach spaces and continuous linear maps*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "j"] \arrow[d, "f"] & B \arrow[r, "k"] \arrow[d, "g"] & C \arrow[r] \arrow[d, dashed, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "j'"'] & B' \arrow[r, "k'"'] & C' \arrow[r] & 0
\end{tikzcd}$$

If the rows are exact and the left square commutes, then there exists a unique continuous linear map $h\colon C \to C'$ *which makes the right square commute.*

6.3.11. Proposition (The Five Lemma)**.** *Suppose that in the following diagram of Banach spaces and continuous linear maps*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "j"] \arrow[d, "f"] & B \arrow[r, "k"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "j'"'] & B' \arrow[r, "k'"'] & C' \arrow[r] & 0
\end{tikzcd}$$

the rows are exact and the squares commute. Then the following hold.

(a) *If g is surjective, so is h.*
(b) *If f is surjective and g is injective, then h is injective.*
(c) *If f and h are surjective, so is g.*
(d) *If f and h are injective, so is g.*

6.3.12. Proposition. *If M is a subspace of a Banach space B, then the quotient normed linear space B/M is also a Banach space.*

6.3.13. Definition. A functor from the category of Banach spaces and continuous linear maps into itself is EXACT if it takes short exact sequences to short exact sequences.

6.3.14. Example. The functor $B \mapsto B^*$, $T \mapsto T^*$ from the category of Banach spaces and continuous linear maps into itself is exact.

6.3.15. Example. Let B be a Banach space and $\tau_B : x \mapsto x^{**}$ be the natural embedding from B into B^{**} (see example 6.2.2). Then the quotient space $B^{**}/\operatorname{ran}\tau_B$ is a Banach space.

6.3.16. Notation. We will denote by $\overline{B}$ the quotient space $B^{**}/\operatorname{ran}\tau_B$ in the preceding example.

6.3.17. Proposition. *If $T\colon B \to C$ is a continuous linear map between Banach spaces, then there exists a unique continuous linear map $\overline{T}\colon \overline{B} \to \overline{C}$ such that $\overline{T}(\phi + \operatorname{ran}\tau_B) = (T^{**}\phi) + \operatorname{ran}\tau_C$ for all $\phi \in B^{**}$.*

6.3.18. Example. The pair of maps $B \mapsto \overline{B}$, $T \mapsto \overline{T}$ from the category $\mathbf{BAN}_\infty$ into itself is an exact covariant functor.

6.3.19. Proposition. *In the category of Banach spaces and continuous linear maps, if the sequence*

$$0 \longrightarrow A \xrightarrow{\quad S \quad} B \xrightarrow{\quad T \quad} C \longrightarrow 0$$

is exact, then the following diagram commutes and has exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, "S"'] & A^{**} \arrow[r] \arrow[d, "S^{**}"] & \overline{A} \arrow[r] \arrow[d, "\overline{S}"] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d, "T"'] & B^{**} \arrow[r] \arrow[d, "T^{**}"] & \overline{B} \arrow[r] \arrow[d, "\overline{T}"] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[d] & C^{**} \arrow[r] \arrow[d] & \overline{C} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

6.3.20. Corollary. *Every subspace of a reflexive Banach space is reflexive.*

6.3.21. Corollary. *If M is a subspace of a reflexive Banach space B, then B/M is reflexive.*

6.3.22. Corollary. *Let M be a subspace of a Banach space B. If both M and B/M are reflexive, then so is B.*

6.3.23. Definition. Let $T \in \mathfrak{L}(B, C)$, where B and C are Banach spaces. Define $\operatorname{coker} T$, the COKERNEL of T, to be $C/\operatorname{ran} T$.

6.3.24. Proposition. *Let A, B, and C be Banach spaces. If $S \in \mathfrak{B}(A, B)$, $T \in \mathfrak{B}(B, C)$, and TS all have closed range, then the sequence*

$$\mathbf{0} \longrightarrow \ker S \longrightarrow \ker TS \longrightarrow \ker T \longrightarrow \operatorname{coker} S \longrightarrow \operatorname{coker} TS \longrightarrow \operatorname{coker} T \longrightarrow \mathbf{0}$$

is exact.

Unmasking the identity of the dual of a Banach space is frequently a bit challenging. Here are a few important examples.

6.3.25. Example. Let c_0 be the space of all sequences of complex numbers that converge to 0. Then c_0 is a Banach space and its dual is l_1, the space of all absolutely summable sequences of complex numbers (see exercise 3.1.35.

6.3.26. Corollary. *The space l_1 is a Banach space.*

6.3.27. Example. Let c be the space of all sequences (a_n) of complex numbers such that $\lim_{n\to\infty} a_n$ exists. Then c is a Banach space and its dual is l_1.

6.3.28. Notation. Let X be a topological space. A scalar valued function f on X belongs to $\mathcal{C}_0(X)$ if it is continuous and vanishes at infinity (that is, if there exists a compact subset of X outside of which f is zero).

6.3.29. Proposition. *If X is a locally compact Hausdorff space, then, under pointwise algebraic operations and the uniform norm, $\mathcal{C}_0(X)$ is a Banach space.*

We recall one of the most important theorems from real analysis.

6.3.30. Theorem (Riesz representation)**.** *Let X be a locally compact Hausdorff space. If ϕ is a bounded linear functional on $\mathcal{C}_0(X)$, then there exists a unique real regular Borel measure μ on X such that*

$$\phi(f) = \int_X f \, d\mu$$

for all $f \in \mathcal{C}_0(X)$ *and* $\|\mu\| = \|\phi\|$.

Proof. This theorem and the following corollary are nicely presented in Appendix C of Conway's functional analysis text [7]. See also [17], theorem 7.17; [24], theorems 20.47 and 20.48; [34], theorem 9.16;and [41], theorem 6.19.

6.3.31. Corollary. *If* X *is a locally compact Hausdorff space, then* $\big(\mathcal{C}_0(X)\big)^*$ *is isometrically isomorphic to* $\boldsymbol{M}(X)$.

6.3.32. Corollary. *If* X *is a locally compact Hausdorff space, then* $\boldsymbol{M}(X)$ *is a Banach space.*

6.4. The Open Mapping Theorem

A question one can ask of any concrete category is whether bijective morphisms are necessarily isomorphisms. In some categories the answer is trivially *yes* (for example, in **VEC**). In other categories the answer is trivially *no* (in **TOP**, for example, consider the identity map from the reals with the discrete topology to the reals with their usual topology). Frequently in categories which arise in the study of analysis, where, typically, algebra is tugging enthusiastically in one direction while topology is pulling vigorously in the opposite direction, the question can often be quite fascinating—and the answer definitely nontrivial. In the category $\mathbf{BAN}_\infty$ of Banach spaces and continuous linear maps, the question turns out to be intriguingly deep. The answer, which is affirmative, is given by the celebrated *open mapping theorem*. Notice as you work through the proof that completeness of both domain and codomain of the relevant map is essential—and for quite different reasons.

6.4.1. Definition. A mapping $f\colon X \to Y$ between topological spaces is OPEN if it takes open sets to open sets; that is, if $f^{\to}(U)$ is open in Y whenever U is open in X.

There is one technical detail that makes the proof of the *open mapping theorem* a bit tricky. Suppose that T is a continuous linear map from a Banach space B into a normed linear space V. Let R be the image under T of an open ball around the origin in B. It must be verified that if the closure of R contains an open ball about the origin in V, then R itself contains the same ball. Let's separate this out as a separate result.

6.4.2. Proposition. *Let* $T\colon B \to V$ *be a continuous linear map from a Banach space into a normed linear space. Denote by* $(B)_r$ *the open ball in* B *of radius* r *centered at the origin and by* $(V)_r$ *the open ball of radius* r *centered at the origin in* V*. If* $r,\ s > 0$ *and* $(V)_s \subseteq \overline{T\big((B)_r\big)}$*, then* $(V)_s \subseteq T\big((B)_r\big)$.

Hint for proof. First verify that without loss of generality we may assume that $r = s = 1$. Thus our hypothesis is that $(V)_1 \subseteq \overline{T\big((B)_1\big)}$.

Let $z \in (V)_1$. The proof is complete if we can show that $z \in T\big((B)_1\big)$. Choose $\delta > 0$ such that $\|z\| < 1 - \delta$ and let $y = (1-\delta)^{-1}z$. Construct a sequence $(v_k)_{k=0}^{\infty}$ of vectors in V which satisfy

(i) $\|y - v_k\| < \delta^k$ for $k = 0, 1, 2, \dots$ and
(ii) $v_k - v_{k-1} \in T\big(\delta^{k-1}(B)_1\big)$ for $k = 1, 2, 3, \dots$.

To this end, start with $v_0 = \mathbf{0}$. Use the hypothesis to conclude that $y - v_0 \in \overline{T\big((B)_1\big)}$ and that therefore there is a vector $w_0 \in T\big((B)_1\big)$ such that $\|(y - v_0) - w_o\| < \delta$. Let $v_1 = v_0 + w_0$ and check that (i) and (ii) hold for $k = 1$.

Then find $w_1 \in T\big(\delta(B)_1\big)$ so that $\|(y - v_1) - w_1\| < \delta^2$ and let $v_2 = v_1 + w_1$. Check that (i) and (ii) hold for $k = 2$. Proceed inductively.

For each $n \in \mathbb{N}$ choose a vector $u_n \in \delta^{n-1}(B)_1$ such that $T(u_n) = v_n - v_{n-1}$. Show that the sequence (u_n) is summable and let $x = \sum_{n=1}^{\infty} u_n$. Conclude the proof by showing that $Tx = y$ and that $z \in T\big((B)_1\big)$.

6.4.3. Theorem (Open mapping theorem)**.** *Every bounded linear surjection between Banach spaces is an open mapping.*

6.4.4. Corollary. *Every bounded linear bijection between Banach spaces is an isomorphism.*

6.4.5. Corollary. *Every bounded linear surjection between Banach spaces is a quotient map in* $\mathbf{BAN}_\infty$.

6.4.6. Example. Let $\mathcal{C}_u$ be the set of continuous real valued functions on $[0,1]$ under the uniform norm and $\mathcal{C}_1$ be the same set of functions under the L_1 norm ($\|f\|_1 = \int_0^1 |f|\,d\lambda$). Then the identity map $I\colon \mathcal{C}_u \to \mathcal{C}_1$ is a continuous linear bijection but is not open. (Why does this not contradict the *open mapping theorem*?)

6.4.7. Example. Let l be the set of sequences of real numbers that are eventually zero, V be l equipped with the l_1 norm, and W be l with the uniform norm. The identity map $I\colon V \to W$ is a bounded linear bijection but is not an isomorphism.

6.4.8. Example. The mapping $T\colon \mathbb{R}^2 \to \mathbb{R}\colon (x,y) \mapsto x$ shows that although bounded linear surjections between Banach spaces map open sets to open sets they need not map closed sets to closed sets.

6.4.9. Proposition. *Let V be a vector space and suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on V whose corresponding topologies are $\mathfrak{T}_1$ and $\mathfrak{T}_2$. If V is complete with respect to both norms and if $\mathfrak{T}_1 \supseteq \mathfrak{T}_2$, then $\mathfrak{T}_1 = \mathfrak{T}_2$.*

6.4.10. Exercise. Does there exist a sequence (a_n) of complex numbers satisfying the following condition: a sequence (x_n) in $\mathbb{C}$ is absolutely summable if and only if $(a_n x_n)$ is bounded? *Hint.* Consider $T\colon l_\infty \to l_1\colon (x_n) \mapsto (x_n/a_n)$.

6.4.11. Example. Let $\mathcal{C}^2([a,b])$ be the family of all twice continuously differentiable real valued functions on the interval $[a,b]$. Regard it as a vector space in the usual fashion. For each $f \in \mathcal{C}^2([a,b])$ define

$$\|f\| = \|f\|_u + \|f'\|_u + \|f''\|_u\,.$$

This is in fact a norm and under this norm $\mathcal{C}^2([a,b])$ is a Banach space.

6.4.12. Example. Let $\mathcal{C}^2([a,b])$ be the Banach space given in exercise 6.4.11, and p_0 and p_1 be members of $\mathcal{C}([a,b])$. Define

$$T\colon C^2([a,b]) \to C([a,b])\colon f \mapsto f'' + p_1 f' + p_0 f.$$

Then T is a bounded linear map.

6.4.13. Exercise. Consider a differential equation of the form

$$y'' + p_1\,y' + p_0\,y = q \tag{$*$}$$

where p_0, p_1, and q are (fixed) continuous real valued functions on the interval $[a,b]$. Make precise, and prove, the assertion that *the solutions to* $(*)$ *depend continuously on the initial values.* You may use without proof a standard theorem: For every point c in $[a,b]$ and every pair of real numbers a_0 and a_1, there exists a unique solution of $(*)$ such that $y(c) = a_0$ and $y'(c) = a_1$. *Hint.* For a fixed $c \in [a,b]$ consider the map

$$S\colon \mathcal{C}^2([a,b]) \to \mathcal{C}([a,b]) \times \mathbb{R}^2\colon f \mapsto \big(Tf, f(c), f'(c)\big)$$

where $\mathcal{C}^2([a,b])$ is the Banach space of example 6.4.11 and T is the bounded linear map of example 6.4.12.

6.4.14. Definition. An bounded linear map $T\colon V \to W$ between normed linear spaces is BOUNDED AWAY FROM ZERO (or BOUNDED BELOW) if there exists a number $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for all $x \in V$. (Recall that the word "bounded" means one thing when modifying a linear map and and something quite different when applied to a general function; so does the expression "bounded away from zero". See definition 5.2.26.)

6.4.15. Proposition. *A bounded linear map between Banach spaces is bounded away from zero if and only if it has closed range and zero kernel.*

6.4.16. Proposition. *If J is a closed ideal in a Banach algebra, then A/J is a Banach algebra.*

6.4.17. Theorem (Fundamental quotient theorem for **BALG**)**.** *Let A and B be Banach algebras and J be a proper closed ideal in A. If ϕ is a homomorphism from A to B and $\ker\phi \supseteq J$, then there exists a unique homomorphism $\widetilde{\phi}\colon A/J \to B$ which makes the following diagram commute.*

$$\begin{tikzcd}
A \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \\
A/I \arrow[r, "\widetilde{\phi}"'] & B
\end{tikzcd}$$

Furthermore, $\widetilde{\phi}$ is injective if and only if $\ker\phi = J$; and $\widetilde{\phi}$ is surjective if and only if ϕ is.

6.4.18. Proposition. *Let I be a proper closed ideal in a unital commutative Banach algebra A. Then I is maximal if and only if A/I is a field.*

6.5. The Closed Graph Theorem

6.5.1. Theorem (Closed graph theorem)**.** *Let $T\colon B \to C$ be a linear map between Banach spaces. If the graph of T is closed in $B \oplus C$, then T is bounded.*

Hint for proof. Let $G = \{(x, Tx)\colon x \in B\}$ be the graph of T. Apply (corollary 6.4.4 of) the *open mapping theorem* to the map

$$\pi\colon G \to B\colon (x, Tx) \mapsto x\,.$$

6.5.2. Proposition. *Let $T\colon B \to C$ be a linear mapping between Banach spaces. Then T is continuous if and only if the following condition is satisfied:*

$$\textit{if } x_n \to 0 \textit{ in } B \textit{ and } Tx_n \to c \textit{ in } C, \textit{ then } c = 0.$$

6.5.3. Proposition. *Suppose $T\colon B \to C$ is a linear map between Banach spaces such that if $x_n \to 0$ in B then $(g \circ T)(x_n) \to 0$ for every $g \in C^*$. Then T is bounded.*

6.5.4. Proposition. *Let $T\colon B \to C$ be a linear function between Banach spaces. Define $T^*f = f \circ T$ for every $f \in C^*$. If T^* maps C^* into B^*, then T is continuous.*

6.5.5. Definition. A bounded linear map from a Banach space into itself is a projection if it is idempotent. If P is such a map, we say that it is a projection along $\ker P$ onto $\operatorname{ran} P$.

6.5.6. Proposition. *An idempotent linear map from a Banach space into itself is a projection if and only if it has closed kernel and range.*

6.5.7. Proposition. *Let H be a Hilbert space and S and T be functions from H into itself such that*

$$\langle Sx, y\rangle = \langle x, Ty\rangle$$

for all x, $y \in H$. Then S and T are bounded linear operators. (Recall that in this case the operator T is the (Hilbert space) adjoint *of S and $T = S^*$.)*

6.5.8. Definition. Let $a < b$ in $\mathbb{R}$. A function $f\colon [a, b] \to \mathbb{R}$ is continuously differentiable if it is differentiable on (an open set containing) $[a, b]$ and its derivative f' is continuous on $[a, b]$. The set of all continuously differentiable real valued functions on $[a, b]$ is denoted by $\mathcal{C}^1(\,[a, b]\,)$.

6.5.9. Example. Let $D\colon \mathcal{C}^1([0,1]) \to C([0,1])\colon f \mapsto f'$ where both $C^1([0,1])$ and $C([0,1])$ are equipped with the uniform norm. The mapping D is linear and has closed graph but is not continuous. (Why does this not contradict the *closed graph theorem*?)

6.6. Projections and Complemented Subspaces

6.6.1. Definition. Let M be a closed linear subspace of a Banach space B. If there exists a closed linear subspace N of B such that $B = M \oplus N$, then we say that M is a COMPLEMENTED subspace, that N is its (BANACH SPACE) COMPLEMENT, and that the subspaces M and N are COMPLEMENTARY.

6.6.2. Proposition. *Let M and N be complementary subspaces of a Banach space B. Clearly, each element in B can be written uniquely in the form $m+n$ for some $m \in M$ and $n \in N$. Define a mapping $P\colon B \to B$ by $P(m+n) = m$ for $m \in M$ and $n \in N$. Then P is the projection along N onto M.*

6.6.3. Proposition. *If P is a projection on a Banach space B, then its kernel and range are complementary subspaces of B.*

6.6.4. Proposition. *Let B be a Banach space. If $P\colon B \to B$ is a projection onto a subspace M along a subspace N, then $I - P$ is a projection onto N along M.*

6.6.5. Proposition. *If M and N are complementary subspaces of a Banach space B, then B/M and N are isomorphic.*

6.6.6. Proposition. *A bounded linear map $T\colon B \to C$ between Banach spaces has a left inverse if and only if T is injective and $\operatorname{ran} T$ is complemented.*

6.6.7. Proposition. *Let $T\colon B \to C$ and $S\colon C \to B$ be bounded linear maps between Banach spaces. If $ST = I$, then*

(a) S is surjective;
(b) T is injective;
(c) TS is a continuous projection along $\ker S$ onto $\operatorname{ran} T$; and
(d) $C = \operatorname{ran} T \oplus \ker S$.

6.6.8. Exercise. Let $S\colon C \to B$ be a bounded linear map between Banach spaces. Find a necessary and sufficient condition for S to have a right inverse.

6.6.9. Proposition. *Let B be a Banach space and $E\colon B \to B$ be an idempotent linear map. Then E is continuous if and only if $\ker T$ and $\operatorname{ran} T$ are closed.*

6.7. The Principle of Uniform Boundedness

6.7.1. Definition. Let S be a set and V be a normed linear space. A family $\mathcal{F}$ of functions from S into V is POINTWISE BOUNDED if for every $x \in S$ there exists a constant $M_x > 0$ such that $\|f(x)\| \le M_x$ for every $f \in \mathcal{F}$.

6.7.2. Definition. Let V and W be normed linear spaces. A family $\mathcal{T}$ of bounded linear maps from V into W is UNIFORMLY BOUNDED if there exists $M > 0$ such that $\|T\| \le M$ for every $T \in \mathcal{T}$.

6.7.3. Proposition. *Let V and W be normed linear spaces. If a family $\mathcal{T}$ of bounded linear maps from V into W is uniformly bounded, then it is pointwise bounded.*

The *principle of uniform boundedness* is the assertion that the converse of 6.7.3 holds whenever V is complete. To prove this we will make use of a "local" analog of this for continuous functions on a complete metric space.

6.7.4. Proposition. *Let M be a complete metric space and $\mathcal{F} \subseteq \mathcal{C}(M,\mathbb{R})$. If for every $x \in M$ there exists a constant $N_x > 0$ such that $|f(x)| \le N_x$ for every $f \in \mathcal{F}$, then there exists a nonempty open set U in M and a number $N > 0$ such that $|f(u)| \le N$ for every $f \in \mathcal{F}$ and every $u \in U$.*

Hint for proof. For each natural number k let $A_k = \bigcap\{f^{\leftarrow}\big([-k,k]\big) : f \in \mathcal{F}\}$. Conclude from the *Baire category theorem* that at least one A_N has nonempty interior. Let $U = {A_N}^o$.

6.7.5. Theorem (Principle of uniform boundedness)**.** *Let B be a Banach space and W be a normed linear space. If a family $\mathfrak{T}$ of bounded linear maps from B into W is pointwise bounded, then it is uniformly bounded.*

Hint for proof. For every $T \in \mathfrak{T}$ define

$$f_T \colon B \to \mathbb{R} \colon x \mapsto \|Tx\|.$$

Use the preceding proposition to show that there exist a nonempty open subset U of B and a constant $M > 0$ such that $f_T(x) \le M$ for every $T \in \mathfrak{T}$ and every $x \in U$. Choose a point $a \in B$ and a number $r > 0$ so that $\overline{B_r(a)} \subseteq U$. Then verify that

$$\|Ty\| \le r^{-1}\big(\, f_T(a+ry) + f_T(a)\,\big) \le 2Mr^{-1}$$

for every $T \in \mathfrak{T}$ and every $y \in B$ such that $\|y\| \le 1$.

6.7.6. Theorem (Banach-Steinhaus)**.** *Let B be a Banach space, W be a normed linear space, and (T_n) be a sequence of bounded linear maps from B into W. If the pointwise limit $\lim_{n\to\infty} T_n x$ exists for every x in B, then the map $S \colon B \to W \colon x \mapsto \lim_{n\to\infty} T_n x$ is a bounded linear transformation.*

6.7.7. Definition. A subset A of a normed linear space V is WEAKLY BOUNDED (w-bounded) if $f^{\to}(A)$ is a bounded subset of the scalar field of V for every $f \in V^*$.

6.7.8. Proposition. *A subset of a normed linear space is bounded if and only if it is weakly bounded.*

6.7.9. Exercise. Your good friend Fred R. Dimm needs help again. This time he is worried about the assertion (see 6.7.8) that a subset of a normed linear space is bounded if and only if it is weakly bounded. He is considering the sequence (a_n) in the Hilbert space l^2 defined by $a_n = \sqrt{n}\,\mathbf{e}^n$ for each $n \in \mathbb{N}$, where $\{\mathbf{e}^n : n \in \mathbb{N}\}$ is the usual orthonormal basis for l_2 (see example 4.5.7). He sees that it is obviously not bounded in the usual (norm) topology on l^2. But it looks to him as if it is weakly bounded. He argues that there is no sequence $x \in l^2$ such that the set $\{|\langle a_n, x\rangle| : n \in \mathbb{N}\}$ is unbounded, because such a sequence would have to decrease more slowly than the sequence $\big(n^{-1/2}\big)$, which already decreases too slowly to belong to l^2. Put Fred's mind to rest by finding him a suitable sequence.

6.7.10. Definition. A sequence (x_n) in a normed linear space V is WEAKLY CAUCHY if the sequence $(f(x_n))$ is Cauchy for every f in V^*. The sequence CONVERGES WEAKLY to a point $a \in V$ if $f(x_n) \to f(a)$ for every $f \in V^*$ (that is, if it converges in the weak topology induced by the members of V^*). In this case we write $x_n \xrightarrow{\;w\;} a$. The space V is WEAKLY SEQUENTIALLY COMPLETE if every weakly Cauchy sequence in V converges weakly.

6.7.11. Proposition. *In a normed linear space every weakly Cauchy sequence is bounded.*

6.7.12. Proposition. *Every reflexive Banach space is weakly sequentially complete.*

6.7.13. Example. Let $f_n(t) = \left\{ \begin{array}{ll} 1 - nt, & \text{if } 0 \le t \le \frac{1}{n}; \\ 0, & \text{if } \frac{1}{n} < t \le 1. \end{array} \right\}$ for every $n \in \mathbb{N}$. The sequence (f_n) of functions shows that the Banach space $\mathcal{C}([0,1])$ is not w-sequentially complete.

6.7.14. Corollary. *The Banach space $\mathcal{C}([0,1])$ is not reflexive.*

6.7.15. Proposition. *Let $\mathcal{E}$ be an orthonormal basis for a Hilbert space H and (x_n) be a sequence in H. Then $x_n \xrightarrow{\;w\;} \mathbf{0}$ if and only if (x_n) is bounded and $\langle x_n, e\rangle \to 0$ for every $e \in \mathcal{E}$.*

6.7.16. Definition. Let H be a Hilbert space. A net (T_λ) in $\mathfrak{B}(H)$ CONVERGES WEAKLY (or CONVERGES IN THE WEAK OPERATOR TOPOLOGY) to $S \in \mathfrak{B}(H)$ if

$$\langle T_\lambda x, y\rangle \to \langle Sx, y\rangle$$

for every x, $y \in H$. In this case we write $T_\lambda \xrightarrow{\mathbf{WOT}} S$.

A net (T_λ) in $\mathfrak{B}(H)$ CONVERGES STRONGLY (or CONVERGES IN THE STRONG OPERATOR TOPOLOGY) to $S \in \mathfrak{B}(H)$ if

$$T_\lambda x \to Sx$$

for every $x \in H$. In this case we write $T_\lambda \xrightarrow{\mathbf{SOT}} S$.

The usual norm convergence of operators in $\mathfrak{B}(H)$ is often referred to as UNIFORM CONVERGENCE (or CONVERGENCE IN THE UNIFORM OPERATOR TOPOLOGY). That is, we write $T_\lambda \to S$ if $\|S - T_\lambda\| \to 0$.

A family $\mathfrak{T} \subseteq \mathfrak{B}(H,K)$ is WEAKLY BOUNDED (or BOUNDED IN THE WEAK OPERATOR TOPOLOGY) if for every $x \in H$ and every $y \in K$ there exists a positive constant $\alpha_{x,y}$ such that

$$|\langle Tx, y\rangle| \le \alpha_{x,y}$$

for every $T \in \mathfrak{T}$.

6.7.17. Proposition. *Let H and K be Hilbert spaces, (T_n) be a sequence in $\mathfrak{B}(H,K)$, and $B \in \mathfrak{B}(H,K)$. then the following implications hold:*

$$T_n \to B \quad \Longrightarrow \quad T_n \xrightarrow{\textbf{\textit{SOT}}} B \quad \Longrightarrow \quad T_n \xrightarrow{\textbf{\textit{WOT}}} B.$$

6.7.18. Proposition. *Let (S_n) and (T_n) be sequences of operators on a Hilbert space H. If $S_n \xrightarrow{\textbf{\textit{WOT}}} A$ and $T_n \xrightarrow{\textbf{\textit{SOT}}} B$, then $S_nT_n \xrightarrow{\textbf{\textit{WOT}}} AB$.*

Hint for proof. You may find it helpful to prove first that every weakly convergent sequence of Hilbert space operators is bounded.

6.7.19. Example. In the preceding proposition the hypothesis $T_n \xrightarrow{\mathbf{SOT}} B$ cannot be replaced by $T_n \xrightarrow{\mathbf{WOT}} B$.

CHAPTER 7

COMPACT OPERATORS

7.1. Definition and Elementary Properties

We will pick up where we left off at the end of chapter 5, studying classes of operators. Our context will be slightly more general; we will be looking at operators on Banach spaces. First, let us recall a few definitions and elementary facts from advanced calculus (or a first course in analysis).

7.1.1. Definition. A subset A of a metric space M is TOTALLY BOUNDED if for every $\epsilon > 0$ there exists a finite subset F of A such that for every $a \in A$ there is a point $x \in F$ such that $d(x, a) < \epsilon$. For normed linear spaces this definition may be rephrased: A subset A of a normed linear space V is TOTALLY BOUNDED if for every open ball B about zero in V there exists a finite subset F of A such that $A \subseteq F + B$. This has a more or less standard paraphrase: A space is totally bounded if it can be kept under surveillance by a finite number of arbitrarily near-sighted policemen. (Some authors call this property *precompact*.)

7.1.2. Proposition. *A metric space is compact if and only if its is complete and totally bounded.*

7.1.3. Proposition. *A metric space is compact if and only if it is complete and totally bounded.*

7.1.4. Definition. A subset A of a topological space X is RELATIVELY COMPACT if its closure is compact.

7.1.5. Proposition. *In a metric space every relatively compact set is totally bounded.*

In a complete metric space the converse is true.

7.1.6. Proposition. *In a complete metric space every totally bounded set is relatively compact.*

7.1.7. Proposition. *In a metric space every totally bounded set is separable.*

7.1.8. Theorem (Arzelà-Ascoli Theorem)**.** *Let X be a compact Hausdorff space. A subset $\mathcal{F}$ of $\mathcal{C}(X)$ is totally bounded whenever it is (uniformly) bounded and equicontinuous.*

In the following we will be contrasting the *weak* and *strong* topologies on normed linear spaces. The term *weak topology* will refer to the usual weak topology determined by the bounded linear functionals on the space as defined in 6.3.5, while *strong topology* will refer to the norm topology on the space.

7.1.9. Proposition. *Every weakly convergent sequence in a normed linear space is strongly bounded.*

Hint for proof. Use the *principle of uniform boundedness* 6.7.5.

7.1.10. Definition. We say that a linear map $T\colon V \to W$ is WEAKLY CONTINUOUS if it takes weakly convergent nets to weakly convergent nets.

7.1.11. Proposition. *Every strongly continuous linear map between normed linear spaces is weakly continuous.*

7.1.12. Definition. A linear map $T\colon V \to W$ between normed linear spaces is COMPACT if it takes bounded sets to relatively compact sets. Equivalently, T is compact if it maps the closed unit ball in V onto a relatively compact set in W. The family of all compact linear maps from V to W is denoted by $\mathfrak{K}(V, W)$; and we write $\mathfrak{K}(V)$ for $\mathfrak{K}(V, V)$.

7.1.13. Proposition. *Every compact linear map is bounded.*

7.1.14. Proposition. *A linear map $T \colon V \to W$ between normed linear spaces is compact if and only if it maps every bounded sequence in V to a sequence in W which has a convergent subsequence.*

7.1.15. Example. Every finite rank operator on a Banach space is compact.

7.1.16. Example. Every bounded linear functional on a Banach space B is a compact linear map from B to $\mathbb{K}$.

7.1.17. Example. Let C be the Banach space $\mathcal{C}([0,1])$ and $k \in \mathcal{C}([0,1] \times [0,1])$. For each $f \in C$ define a function Kf on $[0,1]$ by

$$Kf(x) = \int_0^1 k(x,y) f(y)\, dy\,.$$

Then the integral operator K is a compact operator on C.

Hint for proof. Use the *Arzelà-Ascoli theorem* 7.1.8.

7.1.18. Example. Let H be the Hilbert space $\mathrm{L}_2([0,1])$ of (equivalence classes of) functions which are square-integrable on $[0,1]$ with respect to Lebesgue measure and k be square-integrable function on $[0,1] \times [0,1]$. For each $f \in H$ define a function Kf on $[0,1]$ by

$$Kf(x) = \int_0^1 k(x,y) f(y)\, dy\,.$$

Then the integral operator K is a compact operator on H.

7.1.19. Proposition. *An operator on a Hilbert space is compact if and only if it maps the closed unit ball in the space onto a compact set.*

7.1.20. Example. If B is a Banach space the family $\mathfrak{B}(B)$ of operators on B is a unital Banach algebra and the family $\mathfrak{K}(B)$ of compact operators on B is a closed ideal in $\mathfrak{B}(B)$. If B is infinite dimensional the ideal $\mathfrak{K}(H)$ is proper.

7.1.21. Proposition. *Let $T \colon B \to C$ be a bounded linear map between Banach spaces. Then T is compact if and only if T^* is.*

7.1.22. Definition. A C^*-ALGEBRA is a Banach algebra A with involution which satisfies

$$\|a^*a\| = \|a\|^2$$

for every $a \in A$. This property of the norm is usually referred to as *the C^*-condition*. An algebra norm satisfying this condition is a C^*-NORM. A C^*-SUBALGEBRA of a C^*-algebra A is a closed $*$-subalgebra of A.

We denote by **CSA** the category of C^*-algebras and $*$-homomorphisms. It may seem odd at first that in this category, whose objects have both algebraic and topological properties, that the morphisms should be required to preserve only algebraic structure. Remarkably enough, it turns out that every morphism in **CSA** is automatically continuous—in fact, contractive (see proposition 12.3.24)—and that every injective morphism is an isometry (see proposition 12.3.25). It is clear that (as in definition 5.3.1) every bijective morphism in this category is an isomorphism. And thus every bijective $*$-homomorphism is an isometric $*$-isomorphism.

7.1.23. Example. The vector space $\mathbb{C}$ of complex numbers with the usual multiplication of complex numbers and complex conjugation $z \mapsto \overline{z}$ as involution is a unital commutative C^*-algebra.

7.1.24. Example. If X is a compact Hausdorff space, the algebra $\mathcal{C}(X)$ of continuous complex valued functions on X is a unital commutative C^*-algebra when involution is taken to be complex conjugation.

7.1.25. Example. If X is a locally compact Hausdorff space, the Banach algebra $\mathcal{C}_0(X) = \mathcal{C}_0(X, \mathbb{C})$ of continuous complex valued functions on X which vanish at infinity is a (not necessarily unital) commutative C^*-algebra when involution is taken to be complex conjugation.

7.1.26. Example. If (X, μ) is a measure space, the algebra $\mathrm{L}_\infty(X, \mu)$ of essentially bounded measurable complex valued functions on X (again with complex conjugation as involution) is a C^*-algebra. (Technically, of course, the members of $\mathrm{L}_\infty(X, \mu)$ are equivalence classes of functions which differ on sets of measure zero.)

7.1.27. Example. The algebra $\mathfrak{B}(H)$ of bounded linear operators on a Hilbert space H is a unital C^*-algebra. The family $\mathfrak{K}(H)$ of compact operators is a closed $*$-ideal in the algebra.

7.1.28. Example. As a special case of the preceding example note that the set $\mathbf{M}_n$ of $n \times n$ matrices of complex numbers is a unital C^*-algebra.

7.1.29. Proposition. *In a Hilbert space H the ideal $\mathfrak{K}(H)$ of compact operators is the closure of the ideal $\mathfrak{FR}(H)$ of finite rank operators.*

In the decade or so before the Second World War, some leading Polish mathematicians gathered regularly in coffee houses in Lwow to discuss results and propose problems. First they met in the *Café Roma* and later in the *Scottish Café*. In 1935 Stefan Banach produced a bound notebook in which they could write down the problems. This was the origin of the now famous *Scottish Book* [33]. On November 6, 1936 Stanislaw Mazur wrote down Problem 153, known as the *approximation problem*, which asked, essentially, if the preceding proposition holds for Banach spaces. Can every compact Banach space operator be approximated in norm by finite rank operators? Mazur offered as the prize for solving this problem a live goose (one of the more generous prizes promised in the *Scottish Book*). The problem remained open for decades, until 1972 when it was solved negatively by Per Enflo, who succeeded in finding a separable reflexive Banach space in which such an approximation fails. In a ceremony held that year in Warsaw, Enflo received his live goose from Mazur himself. The result was published in Acta Mathematica [12] in 1973.

7.1.30. Example. The diagonal operator $\operatorname{diag}(1, \frac{1}{2}, \frac{1}{3}, \dots)$ is a compact operator on the Hilbert space l_2.

7.2. Trace Class Operators

Let's start by reviewing some of the standard properties of the *trace* of an $n \times n$ matrix.

7.2.1. Definition. Let $a = [a_{ij}] \in \mathbf{M}_n$. Define $\operatorname{tr} a$, the TRACE of a by

$$\operatorname{tr} a = \sum_{i=1}^{n} a_{ii}.$$

7.2.2. Proposition. *The function* $\operatorname{tr}\colon \mathbf{M}_n \to \mathbb{C}$ *is linear.*

7.2.3. Proposition. *If a, $b \in \mathbf{M}_n$, then* $\operatorname{tr}(ab) = \operatorname{tr}(ba)$.

The trace is invariant under similarity. Two $n \times n$ matrices a and b are SIMILAR if there exists an invertible matrix s such that $b = sas^{-1}$.

7.2.4. Proposition. *If matrices in $\mathbf{M}_n$ are similar, then they have the same trace.*

7.2.5. Definition. Let H be a separable Hilbert space, (e_n) be an orthonormal basis for H, and A be a positive operator on H. Define the TRACE of T by

$$\operatorname{tr} T := \sum_{k=1}^{\infty} \langle Te_k, e_k \rangle.$$

(The trace is not necessarily finite.)

7.2.6. Proposition. *If S and T are positive operators on a separable Hilbert space and $\alpha \in \mathbb{K}$, then*

$$\operatorname{tr}(S+T) = \operatorname{tr} S + \operatorname{tr} T$$

and

$$\operatorname{tr}(\alpha T) = \alpha \operatorname{tr} T.$$

7.2.7. Proposition. *If T is an operator on a separable Hilbert space then $\operatorname{tr}(T^*T) = \operatorname{tr}(TT^*)$.*

To show that the definition given above of the trace of a positive operator does not depend on the orthonormal basis chosen for the Hilbert space we will need a result (given next) which depends on a fact which appears later in these notes: every positive Hilbert space operator has a (positive) square root (see example 11.5.7).

7.2.8. Proposition. *On the family of positive operators on a separable Hilbert space H the trace is invariant under unitary equivalence; that is, if T is positive and U is unitary, then $\operatorname{tr} T = \operatorname{tr}(UTU^*)$.*

Hint for proof. In the preceding proposition let $S = UT^{\frac{1}{2}}$.

7.2.9. Proposition. *The definition of the* trace *7.2.5 of a positive operator on a separable Hilbert space is independent of the choice of orthonormal basis for the space.*

Hint for proof. Let (e^k) and (f^k) be orthonormal bases for the space and T be a positive operator on the space. Take U to be the unique unitary operator with the property that $e^k = Tf^k$ for each $k \in \mathbb{N}$.

7.2.10. Definition. Let V be a vector space. A subset C of V is a CONE in V is $\alpha C \subseteq C$ for every $\alpha \geq 0$. A cone C in V is PROPER is $C \cap (-C) = \{\mathbf{0}\}$.

7.2.11. Proposition. *A cone C in a vector space is convex if and only if $C + C \subseteq C$.*

7.2.12. Example. On a separable Hilbert space the family of all positive operators with finite trace forms a proper convex cone in the vector space $\mathfrak{B}(H)$.

7.2.13. Definition. In a separable Hilbert space H the linear subspace of $\mathfrak{B}(H)$ spanned by the positive operators with finite trace is the family $\mathfrak{T}(H)$ of TRACE CLASS operators on H.

7.2.14. Proposition. *Let H be a separable Hilbert space with orthonormal basis (e_n). If T is a trace class operator on H, then*

$$\operatorname{tr} T = \sum_{k=1}^{\infty} \langle Te_k, e_k \rangle$$

and the sum on the right is absolutely convergent.

7.3. Hilbert-Schmidt Operators

7.3.1. Definition. An operator A on a separable Hilbert space H is a HILBERT-SCHMIDT operator if A^*A is of trace class; that is, if $\sum_{k=1}^{\infty} \|Ae_k\|^2 < \infty$, where (e_k) is an orthonormal basis for H. The family of all Hilbert-Schmidt operators on H is denoted by $\mathfrak{HS}(H)$.

7.3.2. Proposition. *If H is a separable Hilbert space, then the family of Hilbert-Schmidt operators on H is a two-sided ideal in $\mathfrak{B}(H)$.*

7.3.3. Example. The family of Hilbert-Schmidt operators on a Hilbert space H can be made into an inner product space.

Hint for proof. Polarization. If A, $B \in \mathfrak{HS}(H)$, consider $\sum_{k=0}^{3} i^k \big(A + i^k B\big)^*(A + i^k B)$.

7.3.4. Example. The inner product you defined in the preceding example makes $\mathfrak{HS}(H)$ into a Hilbert space.

7.3.5. Proposition. *On a separable Hilbert space every Hilbert-Schmidt operator is compact.*

Hint for proof. If (e_k) is an orthonormal basis for the Hilbert space, consider the operators $F_n := AP_n$ where P_n is the projection onto the span of $\{e_1 \dots, e_n\}$.

7.3.6. Example. The integral operator defined in example 7.1.18 is a Hilbert-Schmidt operator.

7.3.7. Example. The Volterra operator defined in example 5.2.21 is a Hilbert-Schmidt operator.

CHAPTER 8

SOME SPECTRAL THEORY

In this chapter we will, for the first time, assume, unless otherwise stated, that vector spaces and algebras have complex, rather than real, scalars. A good part of the reason for this is the necessity of using *Liouville's theorem* 8.1.40 to prove proposition 8.1.41.

8.1. The Spectrum

8.1.1. Definition. An element a of a unital algebra A is LEFT INVERTIBLE if there exists an element a_l in A (called a LEFT INVERSE of a) such that $a_l a = \mathbf{1}$ and is RIGHT INVERTIBLE if there exists an element a_r in A (called a RIGHT INVERSE of a)such that $aa_r = \mathbf{1}$. The element is INVERTIBLE if it is both left invertible and right invertible. The set of all invertible elements of A is denoted by $\operatorname{inv} A$.

An element of a unital algebra can have at most one multiplicative inverse. In fact, more is true.

8.1.2. Proposition. *If an element of a unital algebra has both a left inverse and a right inverse, then these two inverses are equal (and so the element is invertible).*

When an element a of a unital algebra is invertible its (unique) inverse is denoted by a^{-1}.

8.1.3. Proposition. *If a is an invertible element of a unital algebra, then a^{-1} is also invertible and*

$$\left(a^{-1}\right)^{-1} = a\,.$$

8.1.4. Proposition. *If a and b are invertible elements of a unital algebra, then their product ab is also invertible and*

$$(ab)^{-1} = b^{-1}a^{-1}\,.$$

8.1.5. Proposition. *If a and b are invertible elements of a unital algebra, then*

$$a^{-1} - b^{-1} = a^{-1}(b-a)b^{-1}\,.$$

8.1.6. Proposition. *Let a and b be elements of a unital algebra. If both ab and ba are invertible, then so are a and b.*

Hint for proof. Use proposition 8.1.2.

8.1.7. Corollary. *Let a be an element of a unital $*$-algebra. Then a is invertible if and only if a^*a and aa^* are invertible, in which case*

$$a^{-1} = \left(a^*a\right)^{-1}a^* = a^*\left(aa^*\right)^{-1}\,.$$

8.1.8. Proposition. *Let a and b be elements of a unital algebra. Then $\mathbf{1} - ab$ is invertible if and only if $\mathbf{1} - ba$ is.*

PROOF. If $\mathbf{1} - ab$ is invertible, then $\mathbf{1} + b(\mathbf{1} - ab)^{-1}a$ is the inverse of $\mathbf{1} - ba$. $\square$

8.1.9. Example. The simplest example of a complex algebra is the family $\mathbb{C}$ of complex numbers. It is both unital and commutative.

8.1.10. Example. If X is a nonempty topological space, then the family $\mathcal{C}(X)$ of all continuous complex valued functions on X is an algebra under the usual pointwise operations of addition, scalar multiplication, and multiplication. It is both unital and commutative.

8.1.11. Example. If X is a locally compact Hausdorff space which is not compact, then (under pointwise operations) the family $\mathcal{C}_0(X)$ of all continuous complex valued functions on X which vanish at infinity is a commutative algebra. However, it is *not* a unital algebra.

8.1.12. Example. The family $\mathbf{M}_n$ of $n \times n$ matrices of complex numbers is a unital algebra under the usual matrix operations of addition, scalar multiplication, and multiplication. It is not commutative when $n > 1$.

8.1.13. Example. Let A be an algebra. Make the family $\mathbf{M}_n(A)$ of $n \times n$ matrices of elements of A into an algebra by using the same rules for matrix operations that are used for $\mathbf{M}_n$. Thus $\mathbf{M}_n$ is just $\mathbf{M}_n(\mathbb{C})$. The algebra $\mathbf{M}_n(A)$ is unital if and only if A is.

8.1.14. Example. If V is a normed linear space, then $\mathfrak{B}(V)$ is a unital algebra. If $\dim V > 1$, the algebra is not commutative.

8.1.15. Definition. Let a be an element of a unital algebra A. The SPECTRUM of a, denoted by $\sigma_A(a)$ or just $\sigma(a)$, is the set of all complex numbers λ such that $a - \lambda\mathbf{1}$ is not invertible.

8.1.16. Example. If z is an element of the algebra $\mathbb{C}$ of complex numbers, then $\sigma(z) = \{z\}$.

8.1.17. Example. Let X be a compact Hausdorff space. If f is an element of the algebra $\mathcal{C}(X)$ of continuous complex valued functions on X, then the spectrum of f is its range.

8.1.18. Example. Let X be an arbitrary topological space. If f is an element of the algebra $\mathcal{C}_b(X)$ of bounded continuous complex valued functions on X, then the spectrum of f is the closure of its range.

8.1.19. Example. Let S be a positive measure space and $f \in \mathrm{L}_\infty(S)$ be an (equivalence class of) essentially bounded function(s) on S. Then the spectrum of f is its essential range.

8.1.20. Example. The family $\mathbf{M}_3$ of 3×3 matrices of complex numbers is a unital algebra under the usual matrix operations. The spectrum of the matrix $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ is $\{1, 2\}$.

8.1.21. Proposition. *Let a be an element of a unital algebra such that $a^2 = \mathbf{1}$. Then either*

(a) *$a = \mathbf{1}$, in which case $\sigma(a) = \{1\}$, or*
(b) *$a = -\mathbf{1}$, in which case $\sigma(a) = \{-1\}$, or*
(c) *$\sigma(a) = \{-1, 1\}$.*

Hint for proof. In (c) to prove $\sigma(a) \subseteq \{-1, 1\}$, consider $\dfrac{1}{1-\lambda^2}(a + \lambda\mathbf{1})$.

8.1.22. Proposition. *Recall that an element a of an algebra is* IDEMPOTENT *if $a^2 = a$. Let a be an idempotent element of a unital algebra. Then either*

(a) *$a = \mathbf{1}$, in which case $\sigma(a) = \{1\}$, or*
(b) *$a = \mathbf{0}$, in which case $\sigma(a) = \{0\}$, or*
(c) *$\sigma(a) = \{0, 1\}$.*

Hint for proof. In (c) to prove $\sigma(a) \subseteq \{0, 1\}$, consider $\dfrac{1}{\lambda-\lambda^2}\big(a + (\lambda - 1)\mathbf{1}\big)$.

8.1.23. Proposition. *Let a be an invertible element of a unital algebra. Then a complex number λ belongs to the spectrum of a if and only if $1/\lambda$ belongs to the spectrum of a^{-1}.*

The next proposition is a simple corollary of proposition 8.1.8.

8.1.24. Proposition. *If a and b are elements of a unital algebra, then, except possibly for 0, the spectra of ab and ba are the same.*

8.1.25. Definition. A map $f\colon A \to B$ between Banach algebras is a (Banach algebra) homomorphism if it is both an algebraic homomorphism and a continuous map between A and B. We denote by **BALG** the category of Banach algebras and continuous algebra homomorphisms.

8.1.26. Proposition. *Let A and B be Banach algebras. The* direct sum *of A and B, denoted by $A \oplus B$, is defined to be the set $A \times B$ equipped with pointwise operations:*

(a) $(a, b) + (a', b') = (a + a', b + b')$;
(b) $(a, b)\,(a', b') = (aa', bb')$;
(c) $\alpha(a, b) = (\alpha a, \alpha b)$

for a,$a' \in A$, b, $b' \in B$, and $\alpha \in \mathbb{C}$. As in 3.5.12 define $\|(a, b)\| = \|a\| + \|b\|$. This makes $A \oplus B$ into a Banach algebra.

8.1.27. Exercise. Identify products and coproducts (if they exist) in the category **BALG** of Banach algebras and continuous algebra homomorphisms and in the category of Banach algebras and contractive algebra homomorphisms. (Here *contractive* means $\|Tx\| \le \|x\|$ for all x.)

8.1.28. Proposition. *In a Banach algebra the operation of multiplication (regarded as maps from $A \oplus A$ to A) is continuous.*

8.1.29. Proposition (The Neumann series)**.** *Let a be an element of a unital Banach algebra. If $\|a\| < 1$, then $\mathbf{1} - a \in \operatorname{inv} A$ and $(\mathbf{1} - a)^{-1} = \sum_{k=0}^{\infty} a^k$.*

Hint for proof. Show that the sequence $(\mathbf{1}, a, a^2, \dots)$ is summable. (In a unital algebra we take a^0 to mean $\mathbf{1}$.)

8.1.30. Corollary. *If a is an element of a unital Banach algebra with $\|\mathbf{1} - a\| < 1$, then a is invertible and*

$$\|a^{-1}\| \le \frac{1}{1 - \|\mathbf{1} - a\|}\,.$$

8.1.31. Corollary. *Suppose that a is an invertible element of a unital Banach algebra A, that $b \in A$, and that $\|a - b\| < \|a^{-1}\|^{-1}$. Then b is invertible in A.*

Hint for proof. Use the preceding corollary to show that $a^{-1}b$ is invertible.

8.1.32. Corollary. *If a belongs to a unital Banach algebra and $\|a\| < 1$, then*

$$\|(\mathbf{1} - a)^{-1} - \mathbf{1}\| \le \frac{\|a\|}{1 - \|a\|}\,.$$

8.1.33. Proposition. *If A is a unital Banach algebra, then $\operatorname{inv} A \overset{\circ}{\subseteq} A$.*

Hint for proof. Let $a \in \operatorname{inv} A$. Show, for sufficiently small h, that $\mathbf{1} - a^{-1}h$ is invertible.

8.1.34. Proposition. *Let A be a unital Banach algebra. The map $a \mapsto a^{-1}$ from $\operatorname{inv} A$ into itself is continuous.*

8.1.35. Proposition. *Let A be a unital Banach algebra. The map $r\colon a \mapsto a^{-1}$ from $\operatorname{inv} A$ into itself is differentiable and at each invertible element a, we have $dr_a(h) = -a^{-1}ha^{-1}$ for all $h \in A$.*

Hint for proof. For a quick introduction to differentiation on infinite dimensional spaces, see Chapter 13 of my notes [13] on Real Analysis.

8.1.36. Proposition. *Let a be an element of a unital Banach algebra A. Then the spectrum of a is compact and $|\lambda| \le \|a\|$ for every $\lambda \in \sigma(a)$.*

Hint for proof. Use the *Heine-Borel theorem.* To prove that the spectrum is closed notice that $(\sigma(a))^c = f^{\leftarrow}(\operatorname{inv} A)$ where $f(\lambda) = a - \lambda\mathbf{1}$ for every complex number λ. Also show that if $|\lambda| > \|a\|$, then $\mathbf{1} - \lambda^{-1}a$ is invertible.

8.1.37. Definition. Let a be an element of a unital Banach algebra. The RESOLVENT MAPPING for a is defined by

$$R_a \colon \mathbb{C} \setminus \sigma(a) \to A \colon \lambda \mapsto (a - \lambda\mathbf{1})^{-1}.$$

8.1.38. Definition. Let $U \overset{\circ}{\subseteq} \mathbb{C}$ and A be a unital Banach algebra. A function $f \colon U \to A$ is ANALYTIC on U if

$$f'(a) := \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists for every $a \in U$. A complex valued function which is analytic on all of $\mathbb{C}$ is an ENTIRE function.

8.1.39. Proposition. *For a an element of a unital Banach algebra A and ϕ a bounded linear functional on A let $f := \phi \circ R_a \colon \mathbb{C} \setminus \sigma(a) \to \mathbb{C}$. Then*

(a) *f is analytic on its domain, and*
(b) *$f(\lambda) \to 0$ as $|\lambda| \to \infty$.*

Hint for proof. For (i) notice that in a unital algebra $a^{-1} - b^{-1} = a^{-1}(b - a)b^{-1}$.

8.1.40. Theorem (Liouville)**.** *Every bounded entire function on $\mathbb{C}$ is constant.*

Comment on proof. The proof of this theorem requires a little background in the theory of analytic functions, material that is not covered in these notes. The theorem would surely be covered in any first course in complex variables. For example, you can find a nice proof in [41], theorem 10.23.

8.1.41. Proposition. *The spectrum of every element of a unital Banach algebra is nonempty.*

Hint for proof. Argue by contradiction. Use *Liouville's theorem* 8.1.40 to show that $\phi \circ R_a$ is constant for every bounded linear functional ϕ on A. Then use (a version of) the *Hahn-Banach theorem* to prove that R_a is constant. Why must this constant be 0?

Note. It is important to keep in mind that we are working only with *complex* algebras. This result is *false* for real Banach algebras. An easy counterexample is the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ regarded as an element of the (real) Banach algebra M_2 of all 2×2 matrices of real numbers.

8.1.42. Theorem (Gelfand-Mazur)**.** *If A is a unital Banach algebra in which every nonzero element is invertible, then A is isometrically isomorphic to $\mathbb{C}$.*

Hint for proof. Let $B = \{\lambda\mathbf{1} \colon \lambda \in \mathbb{C}\}$. Use the preceding proposition 8.1.41) to show that $B = A$.

The following is an immediate consequence of the *Gelfand-Mazur theorem* 8.1.42 and proposition 6.4.18.

8.1.43. Corollary. *If I is a maximal ideal in a commutative unital Banach algebra A, then A/I is isometrically isomorphic to $\mathbb{C}$.*

8.1.44. Proposition. *Let a be an element of a unital algebra. Then*

$$\sigma(a^n) = [\sigma(a)]^n$$

for every $n \in \mathbb{N}$. (The notation $[\sigma(a)]^n$ means $\{\lambda^n \colon \lambda \in \sigma(a)\}$.)

8.1.45. Definition. Let a be an element of a unital algebra. The SPECTRAL RADIUS of a, denoted by $\rho(a)$, is defined to be $\sup\{|\lambda| \colon \lambda \in \sigma(a)\}$.

8.1.46. Proposition. *Let $\phi\colon A \to B$ be a unital homomorphism between unital algebras. If a is an invertible element of A, then $\phi(a)$ is invertible in B and its inverse is $\phi(a^{-1})$.*

8.1.47. Corollary. *If $\phi\colon A \to B$ is a unital homomorphism between unital algebras, then*

$$\sigma(\phi(a)) \subseteq \sigma(a)$$

for every $a \in A$.

8.1.48. Corollary. *If $\phi\colon A \to B$ is a unital homomorphism between unital algebras, then*

$$\rho(\phi(a)) \leq \rho(a)$$

for every $a \in A$.

8.1.49. Theorem (Spectral radius formula)**.** *If a is an element of a unital Banach algebra, then*

$$\rho(a) = \inf\bigl\{\|a^n\|^{1/n} \colon n \in \mathbb{N}\bigr\} = \lim_{n\to\infty} \|a^n\|^{1/n} .$$

8.1.50. Proposition. *Let A be a unital Banach algebra and B a closed subalgebra containing $\mathbf{1}_A$. Then*

(a) $\operatorname{inv} B$ *is both open and closed in $B \cap \operatorname{inv} A$;*
(b) $\sigma_A(b) \subseteq \sigma_B(b)$ *for every $b \in B$; and*
(c) *if $b \in B$ and $\sigma_A(b)$ has no holes (that is, if its complement in $\mathbb{C}$ is connected), then $\sigma_A(b) = \sigma_B(b)$.*

Hint for proof. Consider the function $f_b\colon (\sigma_A(b))^c \to B \cap \operatorname{inv} A\colon \lambda \mapsto b - \lambda\mathbf{1}$.

8.2. Spectra of Hilbert Space Operators

8.2.1. Notation. If A is a set of the complex numbers we define $A^* := \{\overline{\lambda}\colon \lambda \in A\}$. We do this to avoid confusion with the closure of the set A.

8.2.2. Proposition. *If T is a Banach space operator then $\sigma(T^*) = \sigma(T)$, while if it is a Hilbert space operator then $\sigma(T^*) = (\sigma(T))^*$.*

In a finite dimensional space there is only one way that a complex number λ can end up in the spectrum of an operator T. The reason for this is that there is only one way an operator, in this case $T - \lambda I$, on a finite dimensional space can fail to be invertible (see the paragraph before definition 1.1.14). On the other hand there are several ways for λ to get into the spectrum of an operator on an infinite dimensional space—because an operator has several ways in which it may fail to be invertible. (See, for example, proposition 5.2.29.) Historically this has led to a number of proposed taxonomies of the spectrum of an operator.

8.2.3. Definition. Let T be an operator on a Hilbert (or Banach) space H. A complex number λ belongs to the RESOLVENT SET of T if $T - \lambda I$ is invertible. The resolvent set is the complement of the *spectrum* of T. The set of complex numbers λ for which $T - \lambda I$ fails to be injective is the POINT SPECTRUM of T and is denoted by $\sigma_p(T)$. Members of the point spectrum of T are EIGENVALUES of T. The set of complex numbers λ for which $T - \lambda I$ is not bounded away from zero is the APPROXIMATE POINT SPECTRUM of T and is denoted by $\sigma_{ap}(T)$. The set of all complex numbers λ such that the closure of the range of $T - \lambda I$ is a proper subspace of H is the COMPRESSION SPECTRUM of T and is denoted by $\sigma_c(T)$. The RESIDUAL SPECTRUM of T, which we denote by $\sigma_r(T)$, is $\sigma_c(T) \setminus \sigma_p(T)$.

8.2.4. Proposition. *If T is a Hilbert space operator, then $\sigma_p(T) \subseteq \sigma_{ap}(T) \subseteq \sigma(T)$.*

8.2.5. Proposition. *If T is a Hilbert space operator, then $\sigma(T) = \sigma_{ap}(T) \cup \sigma_c(T)$.*

8.2.6. Proposition. *If T is a Hilbert space operator, then $\sigma_p(T^*) = (\sigma_c(T))^*$.*

8.2.7. Corollary. *If T is a Hilbert space operator, then $\sigma(T) = \sigma_{ap}(T) \cup (\sigma_{ap}(T^*))^*$.*

8.2.8. Theorem. *Spectral Mapping Theorem If T is a Hilbert space operator and p is a polynomial, then*

$$\sigma(p(T)) = p(\sigma(T)).$$

The preceding theorem is true more generally for any analytic function defined in a neighborhood of the spectrum of T. (For a proof see [7], chapter VII, theorem 4.10.) The result also holds for parts of the spectrum.

8.2.9. Proposition. *If T is a Hilbert space operator and p is a polynomial, then*

(a) $\sigma_p(p(T)) = p(\sigma_p(T))$,
(b) $\sigma_{ap}(p(T)) = p(\sigma_{ap}(T))$, *and*
(c) $\sigma_c(p(T)) = p(\sigma_c(T))$,

8.2.10. Proposition. *If T is an invertible operator on a Hilbert space, then $\sigma(T^{-1}) = (\sigma(T))^{-1}$.*

8.2.11. Definition. As with vector spaces, two operators R and T on a Hilbert (or Banach) space H are SIMILAR if there exists an invertible operator S on H such that $R = STS^{-1}$.

8.2.12. Proposition. *If S and T are similar operators on a Hilbert space, then $\sigma(S) = \sigma(T)$, $\sigma_p(S) = \sigma_p(T)$, $\sigma_{ap}(S) = \sigma_{ap}(T)$, and $\sigma_c(S) = \sigma_c(T)$.*

8.2.13. Proposition. *If T is a normal operator on a Hilbert space, then $\sigma_c(T) = \sigma_p(T)$, so that $\sigma_r(T) = \emptyset$.*

8.2.14. Proposition. *If T is an operator on a Hilbert space, then $\sigma_{ap}(T)$ is closed.*

8.2.15. Example. Let $D = \operatorname{diag}(a_1, a_2, a_3, \dots)$ be a diagonal operator on a Hilbert space and let $A = \cup_{k=1}^{\infty} a_k$. Then

(a) $\sigma_p(D) = \sigma_c(D) = A$,
(b) $\sigma_{ap}(D) = \overline{A}$, and
(c) $\sigma_r(D) = \emptyset$.

8.2.16. Example. Let S be the unilateral shift operator on l_2 (see example 5.2.11). Then

(a) $\sigma(S) = D$,
(b) $\sigma_p(S) = \emptyset$,
(c) $\sigma_{ap}(S) = C$,
(d) $\sigma_c(S) = B$,
(e) $\sigma(S^*) = D$,
(f) $\sigma_p(S^*) = B$,
(g) $\sigma_{ap}(S^*) = D$, and
(h) $\sigma_c(S^*) = \emptyset$.

where B is the open unit disk, D is the closed unit disk, and C is the unit circle in the complex plane.

8.2.17. Example. Let $(S, \mathfrak{A}, \mu)$ be a σ-finite positive measure space, $\phi \in L_\infty(S, \mu)$, and M_ϕ be the corresponding multiplication operator on the Hilbert space $\mathrm{L}_2(S, \mu)$ (see example 5.2.15). Then the spectrum and the approximate point spectrum of M_ϕ are the essential range of ϕ, while the point spectrum and compression spectrum of M_ϕ are $\{\lambda \in \mathbb{C} \colon \mu(\phi^{\leftarrow}(\{\lambda\})) > 0\}$.

CHAPTER 9

TOPOLOGICAL VECTOR SPACES

9.1. Balanced Sets and Absorbing Sets

9.1.1. Definition. A set S in a vector space is BALANCED (or CIRCLED) if $\mathbb{D}S \subseteq S$.

9.1.2. Notation. Let $\mathbb{D}$ denote the closed unit disk in the complex plane $\{z \in \mathbb{C} \colon |z| \leq 1\}$.

9.1.3. Proposition. *Let B be a balanced subset of a vector space. If α and β are scalars such that $|\alpha| \leq |\beta|$, then $\alpha B \subseteq \beta B$.*

9.1.4. Corollary. *If B is a balanced subset of a vector space and $|\lambda| = 1$, then $\lambda B = B$.*

9.1.5. Definition. If S is a subset of a vector space, then the set $\mathbb{D}S$ is the BALANCED HULL of S.

9.1.6. Proposition. *Let S be a subset of a vector space V. Then the balanced hull of S is the intersection of all the balanced subsets of V which contain S; thus it is the smallest balanced subset of V containing S.*

9.1.7. Definition. A subset A of a vector space V ABSORBS a subset $B \subseteq V$ if there exists $M > 0$ such that $B \subseteq tA$ whenever $|t| \geq M$. The set A is ABSORBING (or RADIAL) if it absorbs every one-element set (equivalently, every finite set).

Thus, speaking loosely, a set is absorbing if any vector in the space can be covered by an appropriate dilation of the set.

9.1.8. Example. Let B be the union of the x- and y-axes in the real vector space $\mathbb{R}^2$. Then B is balanced but not convex.

9.1.9. Example. In the vector space $\mathbb{C}$ an ellipse with one focus at the origin is an example of an absorbing set which is not balanced.

9.1.10. Example. In the vector space $\mathbb{C}^2$ the set $\mathbb{D} \times \{0\}$ is balanced but not absorbing.

9.1.11. Proposition. *A balanced set B in a vector space V is absorbing if and only if for every $x \in V$ there exists a scalar $\alpha \neq 0$ such that $\alpha x \in B$.*

9.2. Filters

In the study of topological vector spaces it is convenient to use the language of *filters* to characterize topological properties of the spaces. It is a tool, alternative, but largely equivalent, to nets, useful especially in (nonmetric) situations where sequences are no longer helpful.

Roughly, a *filter* is a nonempty family of nonempty sets which is closed under finite intersections and supersets.

9.2.1. Definition. A nonempty family $\mathfrak{F}$ of nonempty subsets of a set S is a FILTER on S if

(a) A, $B \in \mathfrak{F}$ implies $A \cap B \in \mathfrak{F}$, and
(b) if $A \in \mathfrak{F}$ and $A \subseteq B$, then $B \in \mathfrak{F}$.

9.2.2. Example. A simple (and, for our work, the most important) example of a filter is the family of neighborhoods of a point a in a topological space. This is the NEIGHBORHOOD FILTER at a. It will be denoted by $\mathfrak{N}_a$.

9.2.3. Definition. A nonempty family $\mathfrak{B}$ of nonempty sets of a set S is a FILTERBASE on S if for every A, $B \in \mathfrak{B}$ there exists $C \in \mathfrak{B}$ such that $C \subseteq A \cap B$.

We say that a subfamily $\mathfrak{B}$ of a filter $\mathfrak{F}$ is a FILTERBASE FOR $\mathfrak{F}$ if every member of $\mathfrak{F}$ contains a member of $\mathfrak{B}$. (It is important to check that any such subfamily really is a filterbase.)

9.2.4. Example. Every filter is a filterbase.

9.2.5. Example. Let a be a point in a topological space. Any neighborhood base at a is a filterbase for the filter $\mathfrak{N}_a$ of all neighborhoods of a. In particular, the family of all open neighborhoods of a is a filterbase for $\mathfrak{N}_a$.

9.2.6. Example. Let $\mathfrak{B}$ be a filterbase in a set S. Then the family $\mathfrak{F}$ of all supersets of members of $\mathfrak{B}$ is a filter in S and $\mathfrak{B}$ is a filterbase for $\mathfrak{F}$. We will call this filter the FILTER GENERATED BY $\mathfrak{B}$.

9.2.7. Definition. A filterbase (in particular, a filter) $\mathfrak{B}$ on a topological space X CONVERGES to a point $a \in X$ if every member N of $\mathfrak{N}_a$, the neighborhood filterbase at a, contains a member of $\mathfrak{B}$. In this case we write $\mathfrak{B} \to a$.

9.2.8. Proposition. *A filterbase on a Hausdorff topological space converges to at most one point.*

9.2.9. Example. Let (x_λ) be a net in a set S and $\mathfrak{F}$ be the family of all $A \subseteq S$ for which the net (x_λ) is eventually in A. Then $\mathfrak{F}$ is a filter in S. We will call this the filter GENERATED BY the net (x_λ).

9.2.10. Proposition. *Suppose that (x_λ) is a net in a topological space X, that $\mathfrak{F}$ is the filter generated by (x_λ), and $a \in X$. Then $\mathfrak{F} \to a$ if and only if $x_\lambda \to a$.*

9.2.11. Example. Let $\mathfrak{B}$ be a filter base on a set S. Let D be the set of all ordered pairs (b, B) such that $b \in B \in \mathfrak{B}$. For (b_1, B_1), $(b_2, B_2) \in D$ define $(b_1, B_1) \leq (b_2, B_2)$ if $B_2 \subseteq B_1$. This makes D into a directed set. Now create a net in S by defining

$$x \colon D \to S \colon (b, B) \mapsto b$$

that is, let $x_\lambda = b$ for all $\lambda = (b, B) \in D$. Then $\big(x_\lambda\big)_{\lambda \in D}$ is a net in S. It is the net GENERATED BY (or BASED ON) the filter base $\mathfrak{B}$.

9.2.12. Proposition. *Let $\mathfrak{B}$ be a filter base in a topological space X, let (x_λ) the the net generated by $\mathfrak{B}$, and let $a \in X$. Then $x_\lambda \to a$ if and only if $\mathfrak{B} \to a$.*

9.3. Compatible Topologies

9.3.1. Definition. Let $\mathfrak{T}$ be a topology on a vector space X. We say that $\mathfrak{T}$ is COMPATIBLE with the linear structure on X if the operations of addition $A \colon X \times X \to X \colon (x, y) \mapsto x + y$ and scalar multiplication $M \colon \mathbb{K} \times X \colon (\alpha, x) \mapsto \alpha x$ are continuous. (Here we understand $X \times X$ and $\mathbb{K} \times X$ to be equipped with the product topology.)

If X is a vector space on which has been defined a topology $\mathfrak{T}$ compatible with its linear structure, we say that the pair $(X, \mathfrak{T})$ is a TOPOLOGICAL VECTOR SPACE. (As you doubtless expect, we will almost always continue using standard illogical abridgements such as "Let X be a topological vector space ...".) We denote by **TVS** the category of topological vector space and continuous linear maps.

9.3.2. Example. Let X be a nonzero vector space. The discrete topology on X is not compatible with the linear structure on X.

9.3.3. Proposition. *Let X be a topological vector space, $a \in X$, and $\alpha \in \mathbb{K}$.*

(a) *The mapping $T_a \colon x \mapsto x + a$ (TRANSLATION by a) of X into itself is a homeomorphism.*
(b) *If $\alpha \neq 0$, the mapping $x \mapsto \alpha x$ of X into itself is a homeomorphism.*

9.3.4. Corollary. *A set U is a neighborhood of a point x in a topological vector space if and only if $-x+U$ is a neighborhood of $\mathbf{0}$.*

9.3.5. Corollary. *Let $\mathfrak{U}$ be a filterbase for the neighborhood filter at the origin in a topological vector space X. A subset V of X is open if and only if for every $x \in V$ there exists $U \in \mathfrak{U}$ such that $x+U \subseteq V$.*

What the preceding corollary tells us is that the topology of a topological vector space is completely determined by a filterbase for the neighborhood filter at the origin of the space. We call such a filterbase a LOCAL BASE (for the topology).

9.3.6. Proposition. *If U is a neighborhood of the origin in a topological vector space and $0 \neq \lambda \in \mathbb{K}$, then λU is a neighborhood of the origin.*

9.3.7. Proposition. *In a topological vector space every neighborhood of $\mathbf{0}$ contains a balanced neighborhood of $\mathbf{0}$.*

9.3.8. Proposition. *In a topological vector space every neighborhood of $\mathbf{0}$ every neighborhood of the origin is absorbing.*

9.3.9. Proposition. *If V is a neighborhood of $\mathbf{0}$ in a topological vector space, then there exists a neighborhood U of $\mathbf{0}$ such that $U+U \subseteq V$.*

The next proposition is (almost) a converse of the preceding three.

9.3.10. Proposition. *Let X be a vector space and $\mathfrak{U}$ be a filterbase on X which satisfies the following conditions:*

(a) *every member of $\mathcal{U}$ is balanced;*
(b) *every member of $\mathcal{U}$ is absorbing; and*
(c) *for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ such that $U+U \subseteq V$.*

Then there exists a topology $\mathfrak{T}$ on X under which X is a topological vector space and $\mathfrak{U}$ is a local base for $\mathfrak{T}$.

9.3.11. Example. In a normed linear space the family $\{C_r(\mathbf{0}) \colon r > 0\}$ of closed balls about the origin is a local base for the topology on the space.

9.3.12. Example. Let Ω be a nonempty open subset of $\mathbb{R}^n$. For every compact subset $K \subseteq \Omega$ and every $\epsilon > 0$ let $U_{K,\epsilon} = \{f \in \mathcal{C}(\Omega) \colon |f(x)| \leq \epsilon \text{ whenever } x \in K\}$. Then the family of all such sets $U_{K,\epsilon}$ is a local base for a topology on $\mathcal{C}(\Omega)$. This is the TOPOLOGY OF UNIFORM CONVERGENCE ON COMPACT SETS.

9.3.13. Proposition. *Let A be a subset of a topological vector space and $0 \neq \alpha \in \mathbb{K}$. Then $\alpha\overline{A} = \overline{\alpha A}$.*

9.3.14. Proposition. *Let S be a set in a topological vector space.*

(a) *If S is balanced, so is its closure.*
(b) *The balanced hull of S may fail to be closed even if S itself is closed.*

Hint for proof. For (b) let S be the hyperbola in $\mathbb{R}^2$ whose equation is $xy = 1$.

9.3.15. Proposition. *If M is a vector subspace of a topological vector space, then so is $\overline{M}$.*

It is clear that every absorbing set in a topological vector space must contain the origin. The next example shows, however, that it need not contain a neighborhood of the origin.

9.3.16. Example (Schatz's apple)**.** In the real vector space $\mathbb{R}^2$ the set

$$C_1\big((-1,0)\big) \cup C_1\big((1,0)\big) \cup \big(\{0\} \times [-1,1]\big)$$

is absorbing, but is not a neighborhood of the origin.

The next result tells us that in a topological vector space compact sets and closed sets can be separated by open sets if they are disjoint.

9.3.17. Proposition. *If K and C are nonempty disjoint subsets of a topological space X with K compact and C closed, then there is a neighborhood V of the origin such that $(K+V)\cap(C+V) = \emptyset$.*

9.3.18. Corollary. *If K is a compact subset of a Hausdorff topological vector space X and C is a closed subset of X, then $K + C$ is closed in X.*

Hint for proof. Let $x \in (K + C)^c$. Consider the sets $x - K$ and C.

9.3.19. Corollary. *Every member of a local base $\mathfrak{B}$ for a topological vector space contains the closure of a member of $\mathfrak{B}$.*

9.3.20. Proposition. *Let X be a topological vector space. Then the following are equivalent.*

(a) *$\{\mathbf{0}\}$ is a closed set in X.*
(b) *$\{x\}$ is a closed set for every $x \in X$.*
(c) *For every $x \neq 0$ in X there exists a neighborhood of the origin to which x does not belong.*
(d) *$\bigcap \mathfrak{U} = \{0\}$ where $\mathfrak{U}$ is a local base for X.*
(e) *X is Hausdorff*

9.3.21. Definition. A topological space is REGULAR if points and closed sets can be separated by open sets; that is, if C is a closed subset of the space and x is a point of the space not in C, then there exist disjoint open sets U and V such that $C \subseteq U$ and $x \in V$.

9.3.22. Proposition. *Every Hausdorff topological vector space is regular.*

9.3.23. Corollary. *If X is a topological space in which points are closed sets, then X is regular.*

9.3.24. Proposition. *Let X be a Hausdorff topological vector space. If M is a subspace of X and F is a finite dimensional vector subspace of X, then $M + F$ is closed in X. (In particular, F is closed.)*

9.3.25. Definition. A subset B of a topological vector space is BOUNDED if for every neighborhood U of $\mathbf{0}$ in the space there exists $\alpha > 0$ such that $B \subseteq \alpha U$.

9.3.26. Proposition. *A subset of a topological vector space is bounded if and only if it is absorbed by every neighborhood of $\mathbf{0}$.*

9.3.27. Proposition. *A subset B of a topological vector space is bounded if and only if $\alpha_n x_n \to \mathbf{0}$ whenever (x_n) is a sequence of vectors in B and (α_n) is a sequence of scalars in c_0.*

9.3.28. Proposition. *If a subset of a topological vector space is bounded, then so is its closure.*

9.3.29. Proposition. *If subsets A and B of a topological vector space are bounded, then so is $A \cup B$.*

9.3.30. Proposition. *If subsets A and B of a topological vector space are bounded, then so is $A + B$.*

9.3.31. Proposition. *Every compact subset of a topological vector space is bounded.*

As is the case with normed linear spaces, we say that linear maps are *bounded* if they take bounded sets to bounded sets.

9.3.32. Definition. A linear map $T\colon X \to Y$ between topological vector spaces is BOUNDED if $T^{\rightarrow}(B)$ is a bounded subset of Y whenever B is a bounded subset of X.

9.3.33. Proposition. *Every continuous linear map between topological vector spaces is bounded.*

9.3.34. Definition. A filter $\mathfrak{F}$ on a subset A of a topological vector space X is a CAUCHY FILTER if for every neighborhood U of the origin in X there exists $B \subseteq A$ such that

$$B - B \subseteq U .$$

9.3.35. Proposition. *On a topological vector space every convergent filter is Cauchy.*

9.3.36. Definition. A subset A of a topological vector space X is COMPLETE if every Cauchy filter on A converges to a point of A.

9.3.37. Proposition. *Every complete subset of a Hausdorff topological vector space is closed.*

9.3.38. Proposition. *If C is a closed subset of a complete set in a topological vector space, then C is complete.*

9.3.39. Proposition. *A linear map $T\colon X \to Y$ between topological vector spaces, is continuous if and only if it is continuous at zero.*

9.4. Quotients

9.4.1. Convention. As is the case with Hilbert and Banach spaces the word "subspace" in the context of topological vector spaces means *closed vector subspace*.

9.4.2. Definition. Let M be a vector subspace of a topological vector space X and $\pi\colon X \to X/M \colon x \mapsto [x]$ be the usual quotient map. On the quotient vector space X/M we define a topology, which we call the QUOTIENT TOPOLOGY, by specifying the neighborhood filter at the origin in X/M to be the image under π of the neighborhood filter at the origin in X. That is, we declare a subset V of X/M to be a neighborhood of $\mathbf{0}$ when, and only when, there exists a neighborhood U of the origin in X such that $V = \pi^{\to}(U)$.

9.4.3. Proposition. *Let M be a vector subspace of a topological vector space X. When X/M is given the quotient topology the quotient map $\pi\colon X \to X/M$ is both an open map and continuous.*

9.4.4. Example. Let M be the y-axis in the topological vector space $\mathbb{R}^2$ (with its usual topology). Identify $\mathbb{R}^2/M$ with the x-axis. Although the quotient map takes open sets to open sets, the hyperbola $\{(x,y) \in \mathbb{R}^2 \colon xy = 1\}$ shows that π need not take closed sets to closed sets.

9.4.5. Proposition. *Let M be a vector subspace of a topological vector space X. The quotient topology on X/M is the largest topology under which the quotient map is continuous.*

The next proposition assures us that the quotient topology, as defined above, makes the quotient vector space into a topological vector space.

9.4.6. Proposition. *Let M be a vector subspace of a topological vector space X. The quotient topology on X/M is compatible with the vector space operations on that space.*

9.4.7. Proposition. *If M is a vector subspace of a topological vector space X, then the quotient space X/M is a Hausdorff if and only if M is closed in X.*

9.4.8. Exercise. Explain also how the preceding proposition enables us to associate with a non-Hausdorff topological vector space another space that is Hausdorff.

9.4.9. Proposition. *Let $T\colon X \to Y$ be a linear map between topological vector spaces and M be a subspace of X. If $M \subseteq \ker T$, then there exists a unique function $\widetilde{T}\colon X/M \to Y$ such that $T = \widetilde{T} \circ \pi$ (where π is the quotient map). The function $\widetilde{T}$ is linear; it is continuous if and only if T is; it is open if and only if T is.*

9.5. Locally Convex Spaces and Seminorms

9.5.1. Proposition. *In a topological vector space the closure of a convex set is convex.*

9.5.2. Proposition. *In a topological vector space every convex neighborhood of* $\mathbf{0}$ *contains a balanced convex neighborhood of* $\mathbf{0}$*.*

9.5.3. Proposition. *A closed subset* C *of a topological vector space is convex if and only if* $\frac{1}{2}(x+y)$ *belongs to* C *whenever* x *and* y *do.*

9.5.4. Proposition. *If a subset of a topological vector space is convex so are its closure and its interior.*

9.5.5. Definition. A topological vector space is LOCALLY CONVEX if it has a local base consisting of convex sets. For brevity we will refer to a locally convex topological vector space simply as a LOCALLY CONVEX SPACE.

9.5.6. Proposition. *If* p *is a seminorm on a vector space, then* $p(\mathbf{0}) = 0$*.*

9.5.7. Proposition. *If* p *is a seminorm on a vector space* V*, then* $|p(x) - p(y)| \leq p(x-y)$ *for all* x*,* $y \in V$*.*

9.5.8. Corollary. *If* p *is a seminorm on a vector space* V*, then* $p(x) \geq 0$ *for all* $x \in V$*.*

9.5.9. Proposition. *If* p *is a seminorm on a vector space* V*, then* $p^{\leftarrow}(\{0\})$ *is a vector subspace of* V*.*

9.5.10. Definition. Let p be a seminorm on a vector space V and $\epsilon > 0$. We will call the set $B_{p,\epsilon} := p^{\leftarrow}\big([0,\epsilon)\big)$ the OPEN SEMIBALL of radius ϵ at the origin determined by p. Similarly, the set $C_{p,\epsilon} := p^{\leftarrow}\big([0,\epsilon]\big)$ is the CLOSED SEMIBALL of radius ϵ at the origin determined by p. For the open and closed semiballs of radius one determined by p use the notations B_p and C_p, respectively.

9.5.11. Proposition. *Let* $\mathcal{P}$ *be a family of seminorms on a vector space* V*. The family* $\mathfrak{U}$ *of all finite intersections of open semiballs at the origin of* V *induces a (not necessarily Hausdorff) topology on* V *which is compatible with the linear structure of* V *and for which* $\mathfrak{U}$ *is a local base.*

Hint for proof. Proposition 9.3.10.

9.5.12. Proposition. *Let* p *be a seminorm on a topological vector space* X*. Then the following are equivalent:*

(a) B_p *is open in* X*;*
(b) p *is continuous at the origin; and*
(c) p *is continuous.*

9.5.13. Proposition. *If* p *and* q *are continuous seminorms on a topological vector space, then so are* $p+q$ *and* $\max\{p,q\}$*.*

9.5.14. Example. Let N be an absorbing, balanced, convex subset of a vector space V. Define

$$\mu_N \colon V \to \mathbb{R} \colon x \mapsto \inf\{\lambda > 0 \colon x \in \lambda N\}.$$

The function μ_N is a seminorm on V. It is called the MINKOWSKI FUNCTIONAL of N.

9.5.15. Proposition. *If* N *is an absorbing balanced convex set in a vector space* V *and* μ_N *is the Minkowski functional of* N*, then*

$$B_{\mu_N} \subseteq N \subseteq C_{\mu_N}$$

and the Minkowski functionals generated by the three sets B_{μ_N}*,* N*, and* C_{μ_N} *are all identical.*

9.5.16. Proposition. *If* N *is an absorbing balanced convex set in a vector space* V *and* μ_N *is the Minkowski functional of* N*, then* μ_N *is continuous if and only if* N *is a neighborhood of zero, in which case*

$$B_{\mu_N} = N^{\circ} \quad \textit{and} \quad C_{\mu_N} = \overline{N}.$$

9.5.17. Corollary. *Let N be an absorbing, balanced, convex, closed subset of a topological vector space X. Then the Minkowski functional μ_N is the unique seminorm on X such that $N = C_{\mu_N}$.*

9.5.18. Proposition. *Let p be a seminorm on a vector space V. Then the set B_p is absorbing, balanced, and convex. Furthermore, $p = \mu_{B_p}$.*

9.5.19. Definition. A family $\mathcal{P}$ of seminorms on a vector space V is SEPARATING if for every $x \in X$ there exists $p \in \mathcal{P}$ such that $p(x) \neq 0$.

9.5.20. Proposition. *In a topological vector space let $\mathfrak{B}$ be a local base consisting of balanced convex open sets. Then $\{\mu_N : N \in \mathfrak{B}\}$ is a separating family of continuous seminorms on the space such that $N = B_{\mu_N}$ for every $N \in \mathfrak{B}$.*

The next proposition demonstrates the way in which many of the most important locally convex spaces arise—from a separating family of seminorms.

9.5.21. Proposition. *Let $\mathcal{P}$ be a separating family of seminorms on a vector space V. For each $p \in \mathcal{P}$ and each $n \in \mathbb{N}$ let $U_{p,n} = p^{\leftarrow}\big([0, \frac{1}{n})\big)$. Then the family of all finite intersections of the sets $U_{p,n}$ comprises a balanced convex local base for a topology on V under which V becomes a locally convex space and every seminorm in $\mathcal{P}$ is continuous.*

9.5.22. Example. Let X be a locally compact Hausdorff space. For every nonempty compact subset K of X define

$$p_K \colon \mathcal{C}(X) \to \mathbb{K} \colon f \mapsto \sup\{|f(x)| \colon x \in K\}.$$

Then the family $\mathcal{P}$ of all p_K where K is a nonempty compact subset of X is a family of seminorms under which $\mathcal{C}(X)$ becomes a locally convex space.

9.5.23. Proposition. *A linear map $T \colon V \to W$ from a topological vector space to a locally convex spaces is continuous if and only if $p \circ T$ is a continuous seminorm on V for every continuous seminorm p on W.*

9.5.24. Proposition. *Let p be a continuous seminorm on a subspace M of a locally convex space X. Then p can be extended to a continuous linear functional on all of X.*

PROOF. See [7], proposition IV.5.13.

9.6. Fréchet Spaces

9.6.1. Definition. A topological space $(X, \mathfrak{T})$ is METRIZABLE if there exists a metric d on $X \times X$ such that the topology generated by d is $\mathfrak{T}$.

9.6.2. Definition. A metric d on a vector space V is TRANSLATION INVARIANT if

$$d(x, y) = d(x + a, y + a)$$

for all x, y, and a in V.

9.6.3. Proposition. *Let $(X, \mathfrak{T})$ be a Hausdorff topological vector space with a countable local base. Then there exists a translation invariant metric d on V which generates the topology $\mathfrak{T}$ and whose open balls at the origin are balanced. If X is a locally convex space then d may be chosen so that, additionally, its open balls are convex.*

PROOF. See [42], theorem 1.24.

9.6.4. Example. Let $\{p_n \colon n \in \mathbb{N}\}$ be a a countable separating family of seminorms on a vector space X and $\mathfrak{T}$ be the topology on X guaranteed by proposition 9.5.21. In this case we can explicitly define a translation invariant metric on X which induces the topology $\mathfrak{T}$. For an arbitrary sequence $(\alpha_k) \in c_0$ a metric with the desired properties is given by

$$d(x, y) = \max_{k \in \mathbb{N}} \left\{ \frac{\alpha_k p_k(x - y)}{1 + p_k(x - y)} \right\}.$$

PROOF. See [42], remark 1.38(c).

9.6.5. Definition. A FRÉCHET SPACE is a complete metrizable locally convex space.

9.6.6. Notation. Let A and B be subsets of a topological space. We write $A \prec B$ if $\overline{A} \subseteq B^{\circ}$.

9.6.7. Proposition. *The following hold for subsets A, B, and C of a topological space:*

(a) if $A \prec B$ and $B \prec C$, then $A \prec C$;
(b) if $A \prec C$ and $B \prec C$, then $A \cup B \prec C$; and
(c) if $A \prec B$ and $A \prec C$, then $A \prec B \cap C$.

Here is a standard result from elementary topology.

9.6.8. Proposition. *Let Ω be a nonempty open subset of $\mathbb{R}^n$. The there exists a sequence (K_n) of nonempty compact subsets of Ω such that $K_i \prec K_j$ whenever $i < j$ and $\bigcup_{i=1}^{\infty} K_i = \Omega$.*

PROOF. See [44], lemma 10.1.

9.6.9. Example. Let Ω be a nonempty open subset of some Euclidean space $\mathbb{R}^n$ and $\mathcal{C}(\Omega)$ be the family of all continuous scalar valued functions on Ω. By the preceding proposition we can write Ω as the union of a countable collection of nonempty compact sets K_1, K_2, ... such that $K_i \prec K_j$ whenever $i < j$. For each $n \in \mathbb{N}$ and $f \in \mathcal{C}(\Omega)$ let $p_n(f) = \sup\{|f(x)| : x \in K_n\}$ as in example 9.5.22. According to proposition 9.5.21 this family of seminorms makes $\mathcal{C}(\Omega)$ into a locally convex space. Let $\mathfrak{T}$ be the resulting topology on this space. By example 9.6.4 the function

$$d(f,g) = \max_{k \in \mathbb{N}} \frac{2^{-k} p_k(f-g)}{1 + p_k(f-g)}$$

defines a metric on $\mathcal{C}(\Omega)$ which induces the topology $\mathfrak{T}$. Under this metric $\mathcal{C}(\Omega)$ is a Fréchet space.

9.6.10. Notation (Multi-index notation)**.** Let Ω be an open subset of a Euclidean space $\mathbb{R}^n$. We will consider infinitely differentiable scalar valued functions on Ω; that is, functions on Ω which have derivatives of all orders at each point of Ω. We will refer to such functions as SMOOTH functions on Ω. The class of all smooth functions on Ω is denoted by $\mathcal{C}^{\infty}(\Omega)$.

Another important class we will consider is $\mathcal{C}_c^{\infty}(\Omega)$, the family of all infinitely differentiable scalar valued functions on Ω which have compact support; that is, which vanish outside of some compact subset of Ω. The functions in this family are frequently called TEST FUNCTIONS on Ω. Another, shorter, notation in common use for this family is $\mathcal{D}(\Omega)$.

We will be interested in differential operators on the spaces $\mathcal{D}(\Omega)$. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ be a MULTI-INDEX, that is, an n-tuple of integers where each $\alpha_k \geq 0$, and let

$$D^{\boldsymbol{\alpha}} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

The ORDER of the differential operator $D^{\boldsymbol{\alpha}}$ is $|\boldsymbol{\alpha}| := \sum_{k=1}^{n} \alpha_k$. (When $|\boldsymbol{\alpha}| = 0$ we understand $D^{\boldsymbol{\alpha}}(f) = f$ for each function f.) An alternative notation for $D^{\boldsymbol{\alpha}} f$ is $f^{(\boldsymbol{\alpha})}$.

9.6.11. Example. On the family $\mathcal{C}^{\infty}(\Omega)$ of smooth functions on a open subset Ω of a Euclidean space $\mathbb{R}^n$ define a family $\{p_{K,j}\}$ of seminorms. For each compact subset K of Ω and each $j \in \mathbb{N}$ let

$$p_{K,j}(f) = \sup\{|D^{\boldsymbol{\alpha}} f(x)| : x \in K \text{ and } |\boldsymbol{\alpha}| \leq j\}.$$

This family of seminorms makes $\mathcal{C}^{\infty}(\Omega)$ into a Fréchet space. The resulting topology on $\mathcal{C}^{\infty}(\Omega)$ is sometimes called the *topology of uniform convergence on compacta of the functions and all their derivatives.*

Hint for proof. To see that the space is metrizable use example 9.6.4 and proposition 9.6.8.

9.6.12. Example. Let $n \in N$. Denote by $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ the space of all smooth functions f defined on all of $\mathbb{R}^n$ which satisfy

$$\lim_{\|x\| \to \infty} \|x\|^k |D^{\boldsymbol{\alpha}} f(x)| = 0$$

for all multi-indices $\boldsymbol{\alpha} = (\alpha_1 \ldots, \alpha_n) \in \mathbb{N}^n$ and all integers $k \geq 0$. Such functions are frequently referred to as *functions with rapidly decreasing derivatives* or even more simply as *rapidly decreasing functions*. On $\mathcal{S}$ we place the topology defined by the seminorms

$$p_{m,k} := \sup_{|\boldsymbol{\alpha}| \leq m} \Big\{ \sup_{x \in \mathbb{R}^n} \{ (1 + \|x\|^2)^m |(D^{\boldsymbol{\alpha}} f)(x)| \} \Big\}$$

for k, $m = 0, 1, 2, \ldots$. The resulting space is a Fréchet space called the SCHWARTZ SPACE of $\mathbb{R}^n$.

Hint for proof. For a proof of completeness see [44], Example IV, pages 92–94.

9.6.13. Proposition. *A test function ψ on $\mathbb{R}$ is the derivative of another test function if and only if $\int_{-\infty}^{\infty} \psi = 0$.*

9.6.14. Proposition. *Let F be a Fréchet space whose topology is induced by a translation invariant metric. Then every bounded subset of F a has finite diameter.*

9.6.15. Example. Consider the Fréchet space $\mathbb{R}$ with the topology induced by the translation invariant metric $d(x, y) = \dfrac{|x - y|}{1 + |x - y|}$. Then the set $\mathbb{N}$ of natural numbers has finite diameter but is not bounded.

CHAPTER 10

DISTRIBUTIONS

10.1. Inductive Limits

10.1.1. Definition. Let D be a directed set under a partial ordering $\leq$ and let $\{A_i : i \in D\}$ be a family of objects in some category $\mathbf{C}$. Suppose that for each i, $j \in D$ with $i \leq j$ there exists a morphism $\phi_{j,i} \colon A_i \to A_j$ satisfying $\phi_{ki} = \phi_{kj}\phi_{ji}$ whenever $i \leq j \leq k$ in D. Suppose also that ϕ_{ii} is the identity mapping on A_i for each $i \in D$. Then the indexed family $\mathbf{A} = \big(A_i\big)_{i \in D}$ of objects together with the indexed family $\boldsymbol{\phi} = \big(\phi_{ji}\big)$ of *connecting morphisms* is called a DIRECTED SYSTEM in $\mathbf{C}$.

10.1.2. Definition. Let the pair $(\mathbf{A}, \boldsymbol{\phi})$ be a directed system (as above) in a category $\mathbf{C}$ whose underlying directed set is D. An INDUCTIVE LIMIT (or DIRECT LIMIT) of the system $(\mathbf{A}, \boldsymbol{\phi})$ is a pair $(L, \boldsymbol{\mu})$ where L is an object in $\mathbf{C}$ and $\boldsymbol{\mu} = \big(\mu_i\big)_{i \in D}$ is a family of morphisms $\mu_j \colon A_j \to L$ in $\mathbf{C}$ which satisfy

(a) $\mu_i = \mu_j \phi_{ji}$ whenever $i \leq j$ in D, and
(b) if $(M, \boldsymbol{\lambda})$ is a pair where M is an object in $\mathbf{C}$ and $\boldsymbol{\lambda} = \big(\lambda_i\big)_{i \in D}$ is an indexed family of morphisms $\lambda_j \colon A_j \to M$ in $\mathbf{C}$ satisfying $\lambda_i = \lambda_j \phi_{ji}$ whenever $i \leq j$ in D, then there exists a unique morphism $\psi \colon L \to M$ such that $\lambda_i = \psi \mu_i$ for each $i \in \mathbb{N}$.

Abusing language in a standard fashion we usually say that L is the inductive limit of the system $\mathbf{A} = (A_i)$ and write $L = \varinjlim A_i$.

$$\begin{tikzcd}
& & & & L \arrow[dd, dashed, "\psi"] \\
\cdots \arrow[r] & A_i \arrow[r, "\phi_{ji}"] \arrow[urrr, "\mu_i"] \arrow[drrr, "\lambda_i"'] & A_j \arrow[r] \arrow[urr, "\mu_j"'] \arrow[drr, "\lambda_j"] & \cdots & \\
& & & & M
\end{tikzcd}$$

In an arbitrary (concrete) category inductive limits need not exist. However, if they do they are unique.

10.1.3. Proposition. *Inductive limits (if they exist in a category) are unique (up to isomorphism).*

10.1.4. Example. Let S be a nonempty object in the category **SET**. Consider the family $\mathfrak{P}(S)$ of all subsets of S directed by the partial ordering $\subseteq$. Whenever $A \subseteq B \subseteq S$, take the connecting morphism ι_{BA} to be the inclusion mapping from A into B. Then $\mathfrak{P}(S)$, together with the family of connecting morphisms, form a directed system. The inductive limit if this system is S.

10.1.5. Example. Direct limits exist in the category **VEC** of vector spaces and linear maps.

Hint for proof. Let $(\mathbf{V}, \boldsymbol{\phi})$ be a directed system in **VEC** (as in definition 10.1.1) and D be its underlying directed set. For the inductive limit of this system try $\prod_{i\in D} V_i \Big/ \bigoplus_{i\in D} V_i$ (see examples 3.5.10 and 3.6.10).

It is helpful to keep in mind that two elements $\mathbf{x}$ and $\mathbf{y}$ in the quotient space lie in the same equivalence class if their difference has finite support, that is, if x_i and y_i are eventually equal. You may find it helpful to make use of the functions $\nu_i \colon V_i \to \prod_{k\in D} V_k$ (where $i \in D$) defined by
$\nu_i(u) = \begin{cases} \phi_{ji}(u), & \text{if } i \le j\,; \\ 0, & \text{otherwise.} \end{cases}$

10.1.6. Example. The sequence $\mathbb{Z} \xrightarrow{\mathbf{1}} \mathbb{Z} \xrightarrow{\mathbf{2}} \mathbb{Z} \xrightarrow{\mathbf{3}} \mathbb{Z} \xrightarrow{\mathbf{4}} \dots$ (where $\mathbf{n} \colon \mathbb{Z} \to \mathbb{Z}$ satisfies $\mathbf{n}(1) = n$) is an inductive sequence of Abelian groups whose inductive limit is the set $\mathbb{Q}$ of rational numbers.

10.1.7. Example. The sequence $\mathbb{Z} \xrightarrow{\mathbf{2}} \mathbb{Z} \xrightarrow{\mathbf{2}} \mathbb{Z} \xrightarrow{\mathbf{2}} \mathbb{Z} \xrightarrow{\mathbf{2}} \dots$ is an inductive sequence of Abelian groups whose inductive limit is the set of dyadic rational numbers.

10.2. LF-spaces

Since our principal goal in this chapter is an introduction to the theory of distributions, we will from now on restrict our attention to an especially simple type of inductive limit—the *strict inductive limit of an inductive sequence*. In particular, we will interested in strict inductive limits of increasing sequences of Fréchet spaces.

Recall that a weak topology on a set S is induced by a family of functions $f_\alpha \colon S \to X_\alpha$ mapping into topological spaces. It is the weakest topology on S which makes all these functions continuous (see exercise 4.7.2). A dual notion is that of a STRONG TOPOLOGY, a topology induced by a family of functions mapping *from* topological spaces X_α *into* S. It is the strongest topology under which all these functions are continuous. One strong topology which we have already encountered is the *quotient topology* (see proposition 9.4.5). Another important example, which we introduce next, is the *inductive limit topology*.

10.2.1. Definition. A STRICT INDUCTIVE SEQUENCE is a sequence (X_i) of locally convex spaces such that for each $i \in \mathbb{N}$

(i) X_i is a closed subspace of X_{i+1} and
(ii) the topology $\mathfrak{T}_i$ on the space X_i is the restriction to X_i of the sets belonging to $\mathfrak{T}_{i+1}$.

This is, of course, a directed system in which the connecting morphisms ϕji are just the inclusion maps of X_i into X_j for $i < j$.

The STRICT INDUCTIVE LIMIT of such a sequence of spaces is their union $L = \bigcup_{i=1}^{\infty} X_i$. We give L the largest locally convex topology under which all the inclusion maps $\psi_i \colon X_i \to L$ are continuous. This is the INDUCTIVE LIMIT TOPOLOGY on L.

10.2.2. Proposition. *The inductive limit topology defined above exists.*

10.2.3. Definition. The strict inductive limit of a sequence of Fréchet spaces is an LF-SPACE.

10.2.4. Example. The space $\mathcal{D}(\Omega) = \mathcal{C}_c^\infty(\Omega)$ of test functions on an open subset Ω of a Euclidean space is an LF-space (see notation 9.6.10).

10.2.5. Proposition. *Every LF-space is complete.*

PROOF. See [44], theorem 13.1.

10.2.6. Proposition. *Let $T\colon L \to Y$ be a linear map from an LF-space to a locally convex space. Suppose that $L = \varinjlim X_k$ where (X_k) is a strict inductive sequence of Fréchet spaces. Then T is continuous if and only if its restriction $T\big|_{X_k}$ to each X_k is continuous.*

PROOF. See [44], proposition 13.1, or [7], proposition IV.5.7, or [19], theorem B.18(c).

CAUTION. François Treves, in his beautifully written book on *Topological Vector Spaces, Distributions, and Kernels*, warns his readers against a particularly insidious error: it is very easy to assume that if M is a closed vector subspace of an LF-space $L = \varinjlim X_k$, then the topology which M inherits from L must be the same as the inductive limit topology it gets from the strict inductive sequence of Fréchet spaces $M \cap X_k$. This turns out *not* to be true in general.

10.2.7. Proposition. *Let $T\colon L \to Y$ be a linear map from an LF-space to a locally convex space. Then T is continuous if and only if it is sequentially continuous.*

10.3. Distributions

10.3.1. Definition. Let Ω be a nonempty open subset of $\mathbb{R}^n$. A function $f\colon \Omega \to \mathbb{R}$ is LOCALLY INTEGRABLE if $\int_K |f|\,d\mu < \infty$ for every compact subset K of Ω. (Here μ is the usual n-dimensional Lebesgue measure.) The family of all such functions on Ω is denoted by $\mathrm{L}_1^{\mathrm{loc}}(\Omega)$.

10.3.2. Example. The constant function $\mathbf{1}$ on the real line is locally integrable, even though it is certainly not integrable.

A locally integrable function need not even be continuous. On the real line, for example, the characteristic function of any Lebesgue measurable set is locally integrable. Imagine how convenient it would be if we could speak meaningfully of the *derivative* of any such function and, even better, be permitted to differentiate it as often as we please. — Welcome to the world of distributions. Here

(1) every locally integrable function may be regarded as a distribution, and
(2) every distribution is infinitely differentiable, and furthermore,
(3) all the familiar differentiation rules of beginning calculus hold.

10.3.3. Definition. Let Ω be a nonempty open subset of $\mathbb{R}^n$. A DISTRIBUTION is a continuous linear functional on the space $\mathcal{D}(\Omega) = \mathcal{C}_c^\infty(\Omega)$ of test functions on Ω. Although such a functional is, technically, defined on a space of functions on Ω, we will follow the usual sloppy practice and refer to it as a *distribution on* Ω. The set of all distributions on Ω, the dual space of $\mathcal{D}(\Omega)$ will be denoted by $\mathcal{D}^*(\Omega)$.

10.3.4. Proposition. *Let Ω be an open subset of $\mathbb{R}^n$. A linear functional L on the space $\mathcal{D}(\Omega)$ of test functions on Ω is a distribution if and only if $L(\phi_k) \to 0$ whenever (ϕ_k) is a sequence of test functions such that $\phi_k^{(\boldsymbol{\alpha})} \to 0$ uniformly for every multi-index $\boldsymbol{\alpha}$ and there exists a compact set $K \subseteq \Omega$ which contains the support of each ϕ_k.*

PROOF. See [7], proposition IV.5.21 or [44], proposition 21.1(b).

10.3.5. Example. If Ω is a nonempty open subset of $\mathbb{R}^n$ and f is a locally integrable function on Ω, then the map

$$L_f\colon \mathcal{D}(\Omega) \to \mathbb{K}\colon \phi \mapsto \int f\phi\,d\lambda$$

is a distribution. Such a distribution is called a REGULAR distribution. A distribution which is not regular is called a SINGULAR distribution.

Note. Let u be a distribution and ϕ be a test function on Ω. Standard notations for $u(\phi)$ are $\langle u, \phi\rangle$ and $\langle \phi, u\rangle$. Also we use the notation $\tilde f$ for L_f. Thus in the preceding example

$$\langle \tilde f, \phi\rangle = \langle \phi, \tilde f\rangle = \tilde f(\phi) = \langle L_f, \phi\rangle = \langle \phi, L_f\rangle = L_f(\phi)$$

for $f \in \mathrm{L}_1^{\mathrm{loc}}(\Omega)$.

10.3.6. Example. If μ is a regular Borel measure on a nonempty open subset Ω of $\mathbb{R}^n$ and f is a locally integrable function on Ω, then the map

$$L_\mu \colon \mathcal{D}(\Omega) \to \mathbb{R} \colon \phi \mapsto \int_\Omega f\phi \, d\mu$$

is a distribution on Ω.

10.3.7. Notation. Let $a \in \mathbb{R}$. Define μ_a, DIRAC MEASURE concentrated at a, on the Borel sets of $\mathbb{R}$ by

$$\mu_a(B) = \begin{cases} 1, & \text{if } a \in B \\ 0, & \text{if } a \notin B. \end{cases}$$

We denote the corresponding distribution by δ_a and call it the DIRAC DELTA DISTRIBUTION AT a. Thus $\delta_a = L_{\mu_a}$. If $a = 0$, we write δ for δ_0. Historically the distribution δ has been afflicted with a notoriously misleading (although not technically incorrect) name: the *Dirac delta function*.

10.3.8. Example. If δ_a is the *Dirac delta distribution* at a point $a \in \mathbb{R}$ and ϕ is a test function on $\mathbb{R}$, then $\delta_a(\phi) = \phi(a)$.

10.3.9. Notation. Let H denote the characteristic function of the interval $[0, \infty)$. This is the HEAVISIDE FUNCTION. Its corresponding distribution $\widetilde{H} = L_H$ is the HEAVISIDE DISTRIBUTION.

10.3.10. Example. If $\widetilde{H}$ is the *Heaviside distribution* and $\phi \in \mathcal{D}(\mathbb{R})$, then $\widetilde{H}(\phi) = \int_0^\infty \phi$.

10.3.11. Example. Let f be a differentiable function on $\mathbb{R}$ whose derivative is locally integrable. Then

$$L_{f'}(\phi) = -L_f(\phi') \tag{10.1}$$

for every $\phi \in \mathcal{D}(\mathbb{R})$.

Hint for proof. Integration by parts.

Now an important question arises: How can we define the "derivative" of a distribution? It is certainly natural to want a regular distribution to behave in much the same way as the function from which it arises. That is, we would like the derivative $(L_f)'$ of a regular distribution L_f to correspond with the regular distribution arising from the derivative of the function f. That is to say, what we want is $(L_f)' = L_{f'}$. This suggests, according to the preceding example, that for differentiable functions f with locally integrable derivatives we should define

$$(L_f)'(\phi) = -L_f(\phi') \tag{10.2}$$

for every $\phi \in \mathcal{D}(\mathbb{R})$. The observation that the right side of the equation (10.2) makes sense for *any distribution whatever* motivates the following definition.

10.3.12. Definition. Let u be a distribution on a nonempty open subset Ω of $\mathbb{R}$. We define u', the DERIVATIVE of u, by

$$u'(\phi) := -u(\phi')$$

for every test function ϕ on Ω. We also use the notation Du for the derivative of u.

10.3.13. Example. The *Dirac distribution* δ is the derivative of the *Heaviside distribution* $\widetilde{H}$.

10.3.14. Exercise. Let S be the signum (or sign) function on $\mathbb{R}$; that is, let $S = 2H - 1$, where H is the Heaviside function. Also let g be the function on $\mathbb{R}$ such that $g \colon x \mapsto -x - 3$ for $x < 0$ and $g \colon x \mapsto x + 5$ for $x \geq 0$.

(a) Find constants α and β such that $\tilde{g}' = \alpha\widetilde{S} + \beta\delta$.
(b) Find a function a on $\mathbb{R}$ such that $\tilde{a}' = \widetilde{S}$.

10.3.15. Exercise. Let $f(x) = \begin{cases} 1-|x|, & \text{if } |x| \le 1; \\ 0, & \text{otherwise.} \end{cases}$ By calculating both sides of equation (10.1) separately, show that it is classically correct for every test functions whose support is contained in the interval $[-1,1]$.

10.3.16. Exercise. Let $f(x) = \chi_{(0,1)}$ for $x \in \mathbb{R}$ and $\phi \in \mathcal{D}\big((-1,1)\big)$. Compare the values of $L_{f\,'}(\phi)$ and $\big(L_f\big)'(\phi)$.

10.3.17. Exercise. Let $f(x) = \begin{cases} x, & \text{if } 0 \le x \le 1; \\ x+2, & \text{if } 1 < x \le 2; \\ 0, & \text{otherwise.} \end{cases}$ Compare the values of $L_{f\,'}(\phi)$ and $\big(L_f\big)'(\phi)$ for a test function ϕ.

10.3.18. Exercise. Let $h(x) = \begin{cases} 2x, & \text{if } x < -1; \\ 1, & \text{if } -1 \le x < 1; \\ 0, & \text{if } x \ge 1. \end{cases}$ Find a locally integrable function f such that $\tilde{f}\,' = \tilde{h} + 3\delta_1 - \delta_{-1}$.

No difficulty is encountered in extending definition 10.3.12 to partial derivatives of distributions.

10.3.19. Definition. Let u be a distribution on some nonempty open subset Ω of $\mathbb{R}^n$ for some $n \ge 2$ and $\boldsymbol{\alpha}$ be a multi-index. We define the DIFFERENTIAL OPERATOR $D^{\boldsymbol{\alpha}} = \big(\frac{\partial}{\partial x_1}\big)^{\alpha_1} \dots \big(\frac{\partial}{\partial x_n}\big)^{\alpha_n}$ of order $|\boldsymbol{\alpha}|$ acting on u by

$$D^{\boldsymbol{\alpha}} u(\phi) := (-1)^{|\boldsymbol{\alpha}|} u\big(\phi^{(\boldsymbol{\alpha})}\big)$$

for every test function ϕ on Ω. An alternative notation for $D^{\boldsymbol{\alpha}} u$ is $u^{(\boldsymbol{\alpha})}$.

10.3.20. Exercise. A DIPOLE (of electric moment 1 at the origin in $\mathbb{R}$) may be thought of as the "limit" as $\epsilon \to 0^+$ of a system T_ϵ of two charges $-\frac{1}{\epsilon}$ and $\frac{1}{\epsilon}$ placed at 0 and ϵ, respectively. Show how this might lead one to define a dipole mathematically as $-\delta\,'$. *Hint.* Think of T_ϵ as the distribution $\frac{1}{\epsilon}\delta_\epsilon - \frac{1}{\epsilon}\delta$.

10.3.21. Definition. Let X be a Hausdorff locally convex space and X^* be its dual space, that is, the space of all continuous linear functionals on X. For every $f \in X^*$ define

$$p_f \colon X \to \mathbb{K} \colon x \mapsto |f(x)|.$$

Clearly each p_f is a seminorm on X. Just as in the case of normed linear spaces we define the WEAK TOPOLOGY on X, denoted by $\sigma(X, X^*)$, to be the weak topology generated the family $\{p_f \colon f \in X^*\}$ of seminorms on X. Similarly, for each $x \in X$ define

$$p_x \colon X^* \to \mathbb{K} \colon f \mapsto |f(x)|.$$

Again each p_x is a seminorm on X^* and we define the w^*-TOPOLOGY on X^*, denoted by $\sigma(X^*, X)$, to be the weak topology generated by the family $\{p_x \colon x \in X\}$ of seminorms on X^*.

10.3.22. Proposition. *If the space of distributions $\mathcal{D}^*(\Omega)$ on an open subset Ω of $\mathbb{R}^n$ is given the w^*-topology, then a net (u_λ) of distributions on Ω converges to a distribution v on Ω if and only if $\langle u_\lambda, \phi\rangle \to \langle v, \phi\rangle$ for every test function ϕ on Ω.*

10.3.23. Proposition. *Suppose that a net (u_λ) of distributions converges to a distribution v. Then ${u_\lambda}^{(\alpha)} \to v^{(\alpha)}$ for every multi-index α.*

PROOF. See [42], Theorem 6.17; or [49], Chapter II, Section 3, Theorem; or [19], Theorem 3.9; or [26], Chapter 8, Proposition 2.4. ☐

10.3.24. Proposition. *If g, $f_k \in \mathrm{L}_1^{\mathrm{loc}}(\Omega)$ for every k (where $\Omega \overset{\circ}{\subseteq} \mathbb{R}^n$) and $f_k \to g$ uniformly on every compact subset of Ω, then $\tilde{f}_k \to \tilde{g}$.*

10.3.25. Proposition. *If $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, then $(L_f)^{(\boldsymbol{\alpha})} = L_{f^{(\boldsymbol{\alpha})}}$ for any multi-index $\boldsymbol{\alpha}$.*

10.3.26. Exercise. If δ isn't a function on $\mathbb{R}$, then what do people think they mean when they write

$$\lim_{k\to\infty} \frac{k}{\pi(1+k^2x^2)} = \delta(x) \quad ?$$

Show that, properly interpreted (as an assertion about distributions), it is even correct. *Hint.* If $s_k(x) = k\,\pi^{-1}(1+k^2x^2)^{-1}$ and ϕ is a test function on $\mathbb{R}$, then $\int_{-\infty}^{\infty} s_k\phi = \int_{-\infty}^{\infty} s_k\phi(0) + \int_{-\infty}^{\infty} s_k\eta$ where $\eta(x) = \phi(x) - \phi(0)$. Show that $\int_{-\infty}^{\infty} s_k\eta \to 0$ as $k \to \infty$.

10.3.27. Exercise. Let $f_n(x) = \sin nx$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$. The sequence (f_n) does not converge classically; but it does converge distributionally (to 0). Make this assertion precise, and prove it. *Hint.* Antidifferentiate.

10.3.28. Proposition. *If ϕ is an infinitely differentiable function on $\mathbb{R}$ with compact support, then*

$$\lim_{n\to\infty} \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{n^3x}{\left(1+n^2x^2\right)^2}\, \phi(x)\,dx = \phi'(0)\,.$$

10.3.29. Exercise. Make sense of the expression

$$2\pi \sum_{n=-\infty}^{\infty} \delta(x-2\pi n) = 1 + 2\sum_{n=1}^{\infty} \cos nx\,,$$

and show that, properly interpreted, it is correct. *Hint.* Anyone depraved enough to believe that $\delta(x)$ *means* something would be likely to interpret $\delta(x-a)$ as being the same thing as $\delta_a(x)$. You might start the process of rationalization by calculating the Fourier series of the function g defined by $g(x) = \frac{1}{4\pi}x^2 - \frac{1}{2}x$ on $[0, 2\pi]$ and then extended periodically to $\mathbb{R}$. You may need to look up theorems on pointwise and uniform convergence of Fourier series and on term-by-term differentiation of such series.

10.3.30. Definition. Let Ω be an open subset of $\mathbb{R}^n$, $u \in \mathcal{D}^*(\Omega)$, and $f \in \mathcal{C}^\infty(\Omega)$. Define

$$(fu)(\phi) := u(f\phi)$$

for all test functions ϕ on Ω. Equivalently, we may write $\langle fu, \phi\rangle = \langle u, f\phi\rangle$.

10.3.31. Proposition. *The function fu in the preceding definition is a distribution on Ω.*

PROOF. See [42], page 159, section 6.15.

10.4. Convolution

This section and section 10.6 provide a *very* brief introduction to the convolution and Fourier transforms of distributions. For many of the proofs, and a more detailed exposition, the reader is referred to Walter Rudin's elegant *Functional Analysis* text [42].

To avoid the appearance in formulas of an excessive number of factors of the form $\sqrt{2\pi}$ and its reciprocal, we will for the remainder of this chapter integrate with respect to NORMALIZED LEBESGUE MEASURE m on $\mathbb{R}$. It is defined by

$$m(A) := \frac{1}{\sqrt{2\pi}}\lambda(A)$$

for every Lebesgue measurable subset A of $\mathbb{R}$ (where λ is ordinary Lebesgue measure on $\mathbb{R}$).

We review a few elementary facts from real analysis concerning convolution of scalar valued functions. Recall that a complex or extended real valued function on $\mathbb{R}$ is said to be *Lebesgue integrable* if $\int_{\mathbb{R}} |f|\,dm < \infty$ (where m is normalized Lebesgue measure on $\mathbb{R}$). We denote by $L_1(\mathbb{R})$ the Banach space of all equivalence classes of Lebesgue integrable functions on $\mathbb{R}$, two functions being *equivalent* if the Lebesgue measure of the set on which they differ is zero. The norm on this Banach space is given by

$$\|f\|_1 := \int_R |f|\,dm.$$

10.4.1. Proposition. *Let f and g be Lebesgue integrable functions on $\mathbb{R}$. The function*

$$y \mapsto f(x-y)g(y)$$

belongs to $L_1(\mathbb{R})$ for almost all x. Define a function $f * g$ by

$$(f*g)(x) := \int_{\mathbb{R}} f(x-y)g(y)\,dm(y)$$

wherever the right hand side makes sense. Then the function $f * g$ belongs to $L_1(\mathbb{R})$.

Proof. See [24], Theorem 21.31.

10.4.2. Definition. The function $f * g$ defined above is the CONVOLUTION of f and g.

10.4.3. Proposition. *The Banach space $L_1(\mathbb{R})$ under convolution is a commutative Banach algebra. It is, however,* not *unital.*

Proof. See [24], Theorems 21.34 and 21.35.

10.4.4. Definition. For each f in $L_1(\mathbb{R})$ define a function $\hat{f}$ by

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-itx}\,dm(t).$$

The function $\hat{f}$ is the FOURIER TRANSFORM of f. Sometimes we write $\mathfrak{F}f$ for $\hat{f}$.

10.4.5. Theorem (Riemann-Lebesgue Lemma)**.** *If $f \in L_1(\mathbb{R})$, then $\hat{f} \in \mathcal{C}_0(\mathbb{R})$.*

Proof. See [24], 21.39.

10.4.6. Proposition. *The Fourier transform*

$$\mathfrak{F} \colon L_1(\mathbb{R}) \to \mathcal{C}_0(\mathbb{R}) \colon f \mapsto \hat{f}$$

is a homomorphism of Banach algebras. Neither algebra is unital.

For applications the most useful aspect of the preceding proposition is that the Fourier transform converts convolution in $L_1(\mathbb{R})$ to pointwise multiplication in $\mathcal{C}_0(\mathbb{R})$; that is, $\widehat{fg} = \hat{f}\hat{g}$.

We next define what it means to take the convolution of a distribution and a test function.

10.4.7. Notation. For a point $x \in \mathbb{R}$ and a scalar valued function ϕ on $\mathbb{R}$ define

$$\phi_x \colon \mathbb{R} \to \mathbb{R} \colon y \mapsto \phi(x-y).$$

10.4.8. Proposition. *If ϕ and ψ are scalar valued functions on $\mathbb{R}$ and $x \in \mathbb{R}$, then*

$$(\phi+\psi)_x = \phi_x + \psi_x\,.$$

10.4.9. Proposition. *For $u \in \mathcal{D}^*(\mathbb{R})$ and $\phi \in \mathcal{D}(\mathbb{R})$, let*

$$(u*\phi)(x) := \langle u, \phi_x\rangle$$

for all $x \in \mathbb{R}$. Then $u * \phi \in \mathcal{C}^\infty(\mathbb{R})$.

Proof. See [42], Theorem 6.30(b).

10.4.10. Definition. When u and ϕ are as in the preceding proposition, the function $u * \phi$ is the CONVOLUTION of u and ϕ.

10.4.11. Proposition. *If u and v are distributions on $\mathbb{R}$ and ϕ is a test function on $\mathbb{R}$, then*

$$(u+v)*\phi = (u*\phi) + (v*\phi)\,.$$

10.4.12. Proposition. *If u is a distribution on $\mathbb{R}$ and ϕ and ψ are test functions on $\mathbb{R}$, then*

$$u * (\phi + \psi) = (u * \phi) + (u * \psi)\,.$$

10.4.13. Example. If H is the Heaviside function and ϕ is a test function on $\mathbb{R}$, then

$$(\widetilde{H} * \phi)(x) = \int_{-\infty}^{x} \phi(t)\,dt.$$

10.4.14. Proposition. *If $u \in \mathcal{D}^*(\mathbb{R})$ and $\phi \in \mathcal{D}(\mathbb{R})$, then*

$$D^{\boldsymbol{\alpha}}(u * \phi) = (D^{\boldsymbol{\alpha}}u) * \phi = u * \big(D^{\boldsymbol{\alpha}}(\phi)\big).$$

Proof. See [42], Theorem 6.30(b).

10.4.15. Proposition. *If $u \in \mathcal{D}^*(\mathbb{R})$ and ϕ, $\psi \in \mathcal{D}(\mathbb{R})$, then*

$$u * (\phi * \psi) = (u * \phi) * \psi.$$

Proof. See [42], Theorem 6.30(c).

10.4.16. Proposition. *Let u, $v \in \mathcal{D}^*(\mathbb{R})$. If*

$$u * \phi = v * \phi$$

for all $\phi \in \mathcal{D}(\mathbb{R})$, then $u = v$.

10.4.17. Definition. Let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathcal{D}^*(\mathbb{R}^n)$. We say that "$u = 0$ on Ω" if $u(\phi) = 0$ for every $\phi \in \mathcal{D}(\Omega)$. In the same vein we say that "two distributions u and v are equal on Ω" if $u - v = 0$ on Ω. A point $x \in \mathbb{R}^n$ belongs to the SUPPORT of u if there is *no* open neighborhood of x on which $u = 0$. Denote the support of u by $\operatorname{supp} u$.

10.4.18. Example. The support of the Dirac delta distribution δ is $\{0\}$.

10.4.19. Example. The support of the Heaviside distribution $\widetilde{H}$ is $[0, \infty)$.

10.4.20. Remark. Suppose that a distribution u on $\mathbb{R}^n$ has compact support. Then it can be shown that u has a unique extension to a continuous linear functional on $\mathcal{C}^\infty(\mathbb{R}^n)$, the family of all smooth functions on $\mathbb{R}^n$. Under these conditions the conclusion of proposition 10.4.9 remains true and the following definition applies. If $u \in \mathcal{D}^*(\mathbb{R}^n)$ has compact support and $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$, then the conclusion of proposition 10.4.14 remains true. And if, additionally, ψ is a test function on $\mathbb{R}^n$, then $u * \psi$ is a test function on $\mathbb{R}^n$ and the conclusion of 10.4.15 continues to hold. For a thorough discussion and verification of these claims see [42], Theorems 6.24 and 6.35.

10.4.21. Proposition. *If u and v are distributions on $\mathbb{R}^n$ and at least one of them has compact support, then there exists a unique distribution $u * v$ such that*

$$(u * v) * \phi = u * (v * \phi)$$

for all test functions ϕ on $\mathbb{R}^n$.

Proof. See [42], Definition 6.36*ff.*

10.4.22. Definition. In the preceding proposition the distribution $u * v$ is the CONVOLUTION of u and v.

10.4.23. Example. If δ is the Dirac delta distribution, then

$$\tilde{1} * \delta' = \tilde{0}\,.$$

10.4.24. Example. If δ is the Dirac delta distribution and H is the Heaviside function, then

$$\delta' * \widetilde{H} = \delta\,.$$

10.4.25. Proposition. *If u and v are distributions on $\mathbb{R}^n$ and at least one of them has compact support, then $u * v = v * u$.*

PROOF. See [42], Theorem 6.37(a).

10.4.26. Exercise. Prove the preceding proposition under the assumption that *both* u and v have compact support. *Hint.* Work with $(u * v) * (\phi * \psi)$ where ϕ and ψ are test functions, keeping in mind that convolution of functions is commutative.

10.4.27. Proposition. *If u and v are distributions on $\mathbb{R}^n$ and at least one of them has compact support, then*

$$\operatorname{supp}(u * v) \subseteq \operatorname{supp} u + \operatorname{supp} v.$$

PROOF. See [42], Theorem 6.37(b).

10.4.28. Proposition. *If u, v, and w are distributions on $\mathbb{R}^n$ and at least two of them have compact support, then*

$$u * (v * w) = (u * v) * w.$$

10.4.29. Example. The expression $\tilde{1} * \delta\,' * \widetilde{H}$ is ambiguous. Thus the hypothesis concerning the support of the distributions in the preceding proposition cannot be omitted.

10.4.30. Proposition. *Under addition, scalar multiplication, and convolution the family of all distributions on $\mathbb{R}$ with compact support is a unital commutative algebra.*

10.5. Distributional Solutions to Ordinary Differential Equations

10.5.1. Notation. Let Ω be a nonempty open subset of $\mathbb{R}$, $n \in \mathbb{N}$, and a_0, a_1, $\dots a_n \in \mathcal{C}^\infty(\Omega)$. We consider the (ordinary) differential operator

$$L = \sum_{k=0}^{n} a_k(x) \frac{d^k}{dx^k}$$

and its associated (ordinary) differential equation

$$Lu = v \tag{10.3}$$

where u and v are distributions on Ω. In connection with equation (10.3) one can also consider two variants: the equation

$$\langle Lu, \phi \rangle = \langle v, \phi \rangle \tag{10.4}$$

where ϕ is a test function, and the equation

$$\langle u, L^*\phi \rangle = \langle v, \phi \rangle \tag{10.5}$$

where ϕ is a test function and $L^* := \displaystyle\sum_{k=0}^{n} (-1)^k \frac{d^k(a_k\phi)}{dx^k}$ is the FORMAL ADJOINT of L.

10.5.2. Definition. Suppose that in the equations above the distribution v is given. If there is a regular distribution u which satisfies (10.3), we say that u is a CLASSICAL solution to the differential equation. If u is a regular distribution corresponding to a function which does *not* satisfy (10.3) in the classical sense but which *does* satisfy (10.4) for every test function ϕ, then we say that u is a WEAK solution to the differential equation. And if u is a singular distribution which satisfies (10.5) for every test function ϕ, we say that u is a DISTRIBUTIONAL solution to the differential equation. The family of GENERALIZED solutions comprises all classical, weak, and distributional solutions.

10.5.3. Example. The only generalized solutions to the equation $\dfrac{du}{dx} = 0$ on $\mathbb{R}$ are the constant distributions (that is, the regular distributions arising from constant functions). Thus every generalized solution is classical.

Hint for proof. Suppose that u is a generalized solution to the equation. Choose a test function ϕ_0 such that $\int_{\mathbb{R}} \phi_0 = 1$. Let ϕ be an arbitrary test function on $\mathbb{R}$. Define

$$\psi(x) = \phi(x) - \phi_0(x) \int_R \phi$$

for all $x \in \mathbb{R}$. Observe that $\langle u, \psi \rangle = 0$ and compute $\langle u, \phi \rangle$.

10.5.4. Example. The Heaviside distribution is a weak solution to the differential equation $x\dfrac{du}{dx} = 0$ on $\mathbb{R}$. Of course, the constant distributions are classical solutions.

10.5.5. Example. The Dirac δ distribution is a distributional solution to the differential equation $x^2\dfrac{du}{dx} = 0$ on $\mathbb{R}$. (The Heaviside distribution is a weak solution; and constant distributions are classical solutions.)

10.5.6. Example. For $p = 0, 1, 2, \dots$ the p^{th} derivative $\dfrac{d^P\delta}{dx^p}$ of the Dirac delta distribution is a solution to the differential equation $x^{p+2}\dfrac{du}{dx} = 0$ on $\mathbb{R}$.

10.5.7. Exercise. Find a distributional solution to the differential equation $x\dfrac{du}{dx} + u = 0$ on $\mathbb{R}$.

10.6. The Fourier Transform

In 10.4.4 we defined the Fourier transform of an $L_1(\mathbb{R})$ function. An equivalent way of expressing this definition is in terms of convolution:

$$(\mathfrak{F}f)(x) = (f * \varepsilon^x)(0)$$

for $x \in \mathbb{R}$, where ε^x is the function $t \mapsto e^{itx}$.

10.6.1. Definition. We wish to extend this helpful transform to distributions. However the space $\mathcal{D}^*(\mathbb{R})$ turns out to be much too large to support a reasonable definition for it. By starting with a larger set of "test functions", those in the Schwartz space $\mathcal{S}(\mathbb{R})$ (in which $\mathcal{D}(\mathbb{R})$ is dense), we get a smaller space of continuous linear functionals, on which, happily, we can define in a natural way a *Fourier transform.*

Before defining the Fourier transform on distributions, let us recall two important facts from real analysis concerning the Fourier transform acting on smooth integrable functions.

10.6.2. Proposition (Fourier Inversion Formula). *If $f \in \mathcal{S}(\mathbb{R})$, then*

$$f(x) = \int_{\mathbb{R}} \hat{f}\varepsilon^x \, dm \,.$$

10.6.3. Proposition. *The Fourier transform*

$$\mathfrak{F}\colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})\colon f \mapsto \hat{f}$$

is a Fréchet space isomorphism (that is, it is a bijective continuous linear map with a continuous inverse) and for all $f \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$ we have $\mathfrak{F}^2 f(x) = f(-x)$ and therefore $\mathfrak{F}^4 f = f$.

Proof. Proofs of the two preceding results, together with much other information about this fascinating transform and its many applications to both ordinary and partial differential equations, can be found in a great variety of sources. A few that come readily to mind are [6], Chapter 7; [28], Chapter 4, Section 11; [34], Section 11.3; [42], Chapter 7; [44], Chapter 25; and [49], Chapter VI.

10.6.4. Definition. The inclusion function $\iota\colon \mathcal{D}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is continuous. Then the mapping between their dual spaces

$$u\colon \mathcal{S}^*(\mathbb{R}) \to \mathcal{D}^*(\mathbb{R})\colon \tau \mapsto u_\tau = \tau \circ \iota$$

is a vector space isomorphism from $\mathcal{S}^*(\mathbb{R})$ onto a space of distributions on $\mathbb{R}$, which we call the space of TEMPERED DISTRIBUTIONS (some authors prefer the phrase TEMPERATE DISTRIBUTIONS). It is conventional to identify linear functionals τ and u_τ above. Thus $\mathcal{S}^*(\mathbb{R})$ itself is regarded as the space of tempered distributions.

If you have trouble verifying any of the assertions in the preceding definition, look at Theorem 7.10*ff* in [**42**].

10.6.5. Example. Every distribution with compact support is tempered.

10.6.6. Definition. Define the FOURIER TRANSFORM $\hat{u}$ of a tempered distribution $u \in \mathcal{S}^*(\mathbb{R})$ by

$$\hat{u}(\phi) = u(\hat{\phi})$$

for all $\phi \in \mathcal{S}(\mathbb{R})$. The notation $\mathfrak{F}u$ is an alternative to $\hat{u}$.

10.6.7. Proposition. *If $f \in \mathrm{L}_1(\mathbb{R})$, then we may treat $\tilde{f}$ as a tempered distribution and in that case*

$$\hat{\tilde{f}} = \tilde{\hat{f}}\,.$$

10.6.8. Example. The Fourier transform of the Dirac delta function is given by

$$\hat{\delta} = \tilde{\mathbf{1}}\,.$$

10.6.9. Example. The Fourier transform of the regular distribution $\tilde{\mathbf{1}}$ is given by

$$\hat{\tilde{\mathbf{1}}} = \delta\,.$$

Proposition 10.6.3 has a gratifying generalization to tempered distributions.

10.6.10. Proposition. *The Fourier transform $\mathfrak{F}\colon \mathcal{S}^*(\mathbb{R}) \to \mathcal{S}^*(\mathbb{R})$ is a bijective continuous linear map whose inverse is also continuous. Furthermore, $\mathfrak{F}^4$ is the identity map on $\mathcal{S}^*(\mathbb{R})$.*

We will return briefly to our discussion of the Fourier transform in the next chapter.

CHAPTER 11

THE GELFAND-NAIMARK THEOREM

As in Chapter 8 we will in this chapter—and for the rest of these notes—assume that our field of scalars is the field $\mathbb{C}$ of complex numbers.

11.1. Maximal Ideals in $\mathcal{C}(X)$

11.1.1. Proposition. *If J is a proper ideal in a unital Banach algebra, then so is its closure.*

11.1.2. Corollary. *Every maximal ideal in a unital Banach algebra is closed.*

11.1.3. Example. For every subset C of a topological space X the set

$$J_C := \big\{ f \in \mathcal{C}(X) \colon f^{\to}(C) = \{0\} \big\}$$

is an ideal in $\mathcal{C}(X)$. Furthermore, $J_C \supseteq J_D$ whenever $C \subseteq D \subseteq X$. (In the following we will write J_x for the ideal $J_{\{x\}}$.)

11.1.4. Proposition. *Let X be a compact topological space and I be a proper ideal in $\mathcal{C}(X)$. Then there exists $x \in X$ such that $I \subseteq J_x$.*

11.1.5. Proposition. *Let x and y be points in a compact Hausdorff space. If $J_x \subseteq J_y$, then $x = y$.*

11.1.6. Proposition. *Let X be a compact Hausdorff space. A subset I of $\mathcal{C}(X)$ is a maximal ideal in $\mathcal{C}(X)$ if and only if $I = J_x$ for some $x \in X$.*

11.1.7. Corollary. *If X is a compact Hausdorff space, then the map $x \mapsto J_x$ from X to* $\operatorname{Max} \mathcal{C}(X)$ *is bijective.*

Compactness is an important ingredient in proposition 11.1.6.

11.1.8. Example. In the Banach algebra $\mathcal{C}_b\big((0,1)\big)$ of bounded continuous functions on the interval $(0,1)$ there exists a maximal ideal I such that for *no* point $x \in (0,1)$ is $I = J_x$. Let I be a maximal ideal containing the ideal S of all functions f in $\mathcal{C}_b\big((0,1)\big)$ for which there exists a neighborhood U_f of 0 in $\mathbb{R}$ such that $f(x) = 0$ for all $x \in U_f \cap (0,1)$.

11.2. The Character Space

11.2.1. Definition. A CHARACTER (or NONZERO MULTIPLICATIVE LINEAR FUNCTIONAL) on an algebra A is a nonzero homomorphism from A into $\mathbb{C}$. The set of all characters on A is denoted by ΔA.

11.2.2. Proposition. *Let A be a unital algebra and $\phi \in \Delta A$. Then*

(a) $\phi(\mathbf{1}) = 1$*;*
(b) *if $a \in \operatorname{inv} A$, then $\phi(a) \neq 0$;*
(c) *if a is* NILPOTENT *(that is, if $a^n = 0$ for some $n \in \mathbb{N}$), then $\phi(a) = 0$;*
(d) *if a is* IDEMPOTENT *(that is, if $a^2 = a$), then $\phi(a)$ is 0 or 1; and*
(e) *$\phi(a) \in \sigma(a)$ for every $a \in A$.*

We note in passing that part (e) of the preceding proposition does not give us an easy way of showing that the spectrum $\sigma(a)$ of an algebra element is nonempty. This would depend on knowing that $\Delta(A)$ is nonempty.

11.2.3. Example. The identity map is the only character on the algebra $\mathbb{C}$.

11.2.4. Example. Let A be the algebra of 2×2 matrices $a = \begin{bmatrix} a_{ij} \end{bmatrix}$ such that $a_{12} = 0$. This algebra has exactly two characters $\phi(a) = a_{11}$ and $\psi(a) = a_{22}$.

Hint for proof. Write $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ as a linear combination of the matrices $u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $v = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $w = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Use proposition 11.2.2.

11.2.5. Example. The algebra of all 2×2 matrices of complex numbers has no characters.

11.2.6. Proposition. *Let A be a unital algebra and ϕ be a linear functional on A. Then $\phi \in \Delta A$ if and only if* $\ker\phi$ *is closed under multiplication and $\phi(\mathbf{1}) = 1$.*

Hint for proof. For the converse apply ϕ to the product of $a - \phi(a)\mathbf{1}$ and $b - \phi(b)\mathbf{1}$ for $a, b \in A$.

11.2.7. Proposition. *Every multiplicative linear functional on a unital Banach algebra A is continuous. In fact, if $\phi \in \Delta(A)$, then ϕ is contractive and $\|\phi\| = 1$.*

11.2.8. Example. Let X be a topological space and $x \in X$. We define the EVALUATION FUNCTIONAL AT x, denoted by $E_X x$, by

$$E_X x \colon \mathcal{C}(X) \to \mathbb{C} \colon f \mapsto f(x)\,.$$

This functional is a character on $\mathcal{C}(X)$ and its kernel is J_x. When there is only one topological space under discussion we simplify the notation from $E_X x$ to E_x. Thus, in particular, for $f \in \mathcal{C}(X)$ we often write $E_x(f)$ for the more cumbersome $E_X x(f)$.

11.2.9. Definition. In proposition 11.2.7 we discovered that every character on a unital Banach algebra A lives on the unit sphere of the dual A^*. Thus we may give the set ΔA of characters on A the relative w^*-topology it inherits from A^*. This is the GELFAND TOPOLOGY on ΔA and the resulting topological space we call the CHARACTER SPACE (or *carrier space* or *structure space* or *maximal ideal space* or *spectrum*) of A.

11.2.10. Proposition. *The character space of a unital Banach algebra is a compact Hausdorff space.*

11.2.11. Example. Let $l_1(\mathbb{Z})$ be the family of all bilateral sequences

$$(\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots)$$

which are ABSOLUTELY SUMMABLE; that is, such that $\sum_{k=-\infty}^{\infty} |a_k| < \infty$. This is a Banach space under pointwise operations of addition and scalar multiplication and norm given by

$$\|a\| = \sum_{k=-\infty}^{\infty} |a_k|\,.$$

For $a, b \in l_1(\mathbb{Z})$ define $a * b$ to be the sequence whose n^{th} entry is given by

$$(a * b)_n = \sum_{k=-\infty}^{\infty} a_{n-k}\, b_k\,.$$

The operation $*$ is called CONVOLUTION. (To see where the definition comes from try multiplying the power series $\sum_{-\infty}^{\infty} a_k z^k$ and $\sum_{-\infty}^{\infty} b_k z^k$.) With this additional operation $l_1(\mathbb{Z})$ becomes a unital commutative Banach algebra.

The maximal ideal space of this Banach algebra is (homeomorphic to) the unit circle $\mathbb{T}$.

Hint for proof. For each $z \in \mathbb{T}$ define

$$\psi_z \colon l_1(\mathbb{Z}) \to \mathbb{C} \colon a \mapsto \sum_{k=-\infty}^{\infty} a_k z^k\,.$$

Show that $\psi_z \in \Delta l_1(\mathbb{Z})$. Then show that the map

$$\psi\colon \mathbb{T} \to \Delta l_1(\mathbb{Z})\colon z \mapsto \psi_z$$

is a homeomorphism.

11.2.12. Proposition. *If $\phi \in \Delta A$ where A is a unital algebra, then $\ker \phi$ is a maximal ideal in A.*

Hint for proof. To show maximality, suppose I is an ideal in A which properly contains $\ker \phi$. Choose $z \in I \setminus \ker \phi$. Consider the element $x - \big(\phi(x)/\phi(z)\big)\, z$ where x is an arbitrary element of A.

11.2.13. Proposition. *A character on a unital algebra is completely determined by its kernel.*

Hint for proof. Let a be an element of the algebra and ϕ be a character. For how many complex numbers λ can $a^2 - \lambda a$ belong to the kernel of ϕ?

11.2.14. Corollary. *If A is a unital algebra, then the map $\phi \mapsto \ker \phi$ from ΔA to $\operatorname{Max} A$ is injective.*

11.2.15. Proposition. *Let I be a maximal ideal in a unital commutative Banach algebra A. Then there exists a character on A whose kernel is I.*

Hint for proof. Use corollary 8.1.43 Why can we think of the quotient map as a character?

11.2.16. Corollary. *If A is a unital commutative Banach algebra, then the map $\phi \mapsto \ker \phi$ is a bijection from ΔA onto $\operatorname{Max} A$.*

11.2.17. Definition. Let A be a unital commutative Banach algebra. In light of the preceding corollary we can give $\operatorname{Max} A$ a topology under which it is homeomorphic to the character space ΔA. This is the MAXIMAL IDEAL SPACE of A. Since ΔA and $\operatorname{Max} A$ are homeomorphic it is common practice to identify them and so ΔA is often called the *maximal ideal space* of A.

11.2.18. Definition. Let X be a compact Hausdorff space and $x \in X$. Recall that in example 11.2.8 we defined $E_X x$, the *evaluation functional* at x by

$$E_X x(f) := f(x)$$

for every $f \in \mathcal{C}(X)$. The map

$$E_X\colon X \to \Delta\mathcal{C}(X)\colon x \mapsto E_X x$$

is the EVALUATION MAP on X. As was mentioned earlier when only one topological space is being considered we usually shorten E_X to E and $E_X x$ to E_x.

11.2.19. Notation. To indicate that two topological spaces X and Y are homeomorphic we write $X \approx Y$.

11.2.20. Proposition. *Let X be a compact Hausdorff space. Then the evaluation map on X*

$$E_X\colon X \to \Delta\mathcal{C}(X)\colon x \mapsto E_X x$$

is a homeomorphism. Thus we have

$$X \approx \Delta\mathcal{C}(X) \approx \operatorname{Max}\mathcal{C}(X)\,.$$

More is true: not only is each E_X a homeomorphism between compact Hausdorff spaces, but E itself is a natural equivalence *between functors—the identity functor and the $\Delta\mathcal{C}$ functor.*

The identification between a compact Hausdorff space X and its character space and its maximal ideal space is so strong that many people speak of them as if they were actually equal. It is very common to hear, for example, that “the maximal ideals in $\mathcal{C}(X)$ are just the points of X”. Although not literally true, it does sound a bit less intimidating than “the maximal ideals of $\mathcal{C}(X)$ are the kernels of the evaluation functionals at points of X”.

11.2.21. Proposition. *Let X and Y be compact Hausdorff spaces and $F\colon X \to Y$ be continuous. Recall that in example 2.2.9 we defined $\mathcal{C}(F)$ on $\mathcal{C}(Y)$ by*

$$\mathcal{C}(F)(g) = g \circ F$$

for all $g \in \mathcal{C}(Y)$. Then

(a) *$\mathcal{C}(F)$ maps $\mathcal{C}(Y)$ into $\mathcal{C}(X)$.*
(b) *The map $\mathcal{C}(F)$ is a contractive unital Banach algebra homomorphism.*
(c) *$\mathcal{C}(F)$ is injective if and only if F is surjective.*
(d) *$\mathcal{C}(F)$ is surjective if and only if F is injective.*
(e) *If X is homeomorphic to Y, then $\mathcal{C}(X)$ is isometrically isomorphic to $\mathcal{C}(Y)$.*

11.2.22. Proposition. *Let A and B be unital commutative Banach algebras and $T\colon A \to B$ be a unital algebra homomorphism. Define ΔT on ΔB by*

$$\Delta T(\psi) = \psi \circ T$$

for all $\psi \in \Delta B$. Then

(a) *ΔT maps ΔB into ΔA.*
(b) *The map $\Delta T\colon \Delta B \to \Delta A$ is continuous.*
(c) *If T is surjective, then ΔT is injective.*
(d) *If T is an (algebra) isomorphism, then ΔT is a homeomorphism.*
(e) *If A and B are (algebraically) isomorphic, then ΔA and ΔB are homeomorphic.*

11.2.23. Corollary. *Let X and Y be compact Hausdorff spaces. If $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are algebraically isomorphic, then X and Y are homeomorphic.*

11.2.24. Corollary. *Two compact Hausdorff spaces are homeomorphic if and only if their algebras of continuous complex valued functions are (algebraically) isomorphic.*

Corollary 11.2.24 gives rise to quite an array of intriguing problems. The corollary suggests that for each topological property of a compact Hausdorff space X there ought to be a corresponding algebraic property of the algebra $\mathcal{C}(X)$, and *vice versa*. But it does not provide a usable recipe for finding this correspondence. One very simple example of this type of result is given below as proposition 11.2.26. For a beautiful discussion of this question see Semadeni's monograph [43], in particular, see the table on pages 132–133.

11.2.25. Corollary. *Let X and Y be compact Hausdorff spaces. If $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are algebraically isomorphic, then they are isometrically isomorphic.*

11.2.26. Proposition. *A topological space X is connected if and only if the algebra $\mathcal{C}(X)$ contains no nontrivial idempotents.*

(The *trivial idempotents* of an algebra are $\mathbf{0}$ and $\mathbf{1}$.)

11.3. The Gelfand Transform

11.3.1. Definition. Let A be a commutative Banach algebra and $a \in A$. Define

$$\Gamma_A a\colon \Delta A \to \mathbb{C}\colon \phi \mapsto \phi(a)$$

for every $\phi \in \Delta(A)$. (Alternative notations: when no confusion seems likely we frequently write Γa or $\widehat{a}$ for $\Gamma_A a$.) The map Γ_A is the GELFAND TRANSFORM ON A.

Since $\Delta A \subseteq A^*$ it is clear that $\Gamma_A a$ is just the restriction of a^{**} to the character space of A. Furthermore the Gelfand topology on ΔA is the relative w^*-topology, the weakest topology under which a^{**} is continuous on ΔA for each $a \in A$, so $\Gamma_A a$ is a continuous function on ΔA. Thus $\Gamma_A\colon A \to \mathcal{C}(\Delta A)$.

As a matter of brevity and convenience the element $\Gamma_A a = \Gamma a = \widehat{a}$ is usually called just the *Gelfand transform of a*—because the phrase *the Gelfand transform on A evaluated at a* is awkward.

11.3.2. Definition. We say that a family $\mathcal{F}$ of functions on a set S SEPARATES POINTS of S (or is a SEPARATING FAMILY of functions on S) if for every pair of distinct elements x and y of S there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.

11.3.3. Proposition. *Let X be a compact topological space. Then $\mathcal{C}(X)$ is separating if and only if X is Hausdorff.*

11.3.4. Proposition. *If A is a commutative Banach algebra, then $\Gamma_A \colon A \to \mathcal{C}(\Delta A)$ is a contractive algebra homomorphism and the range of Γ_A is a separating subalgebra of $\mathcal{C}(\Delta A)$. If, in addition, A is unital, then Γ has norm one, its range is a unital subalgebra of $\mathcal{C}(\Delta A)$, and it is a unital homomorphism.*

11.3.5. Proposition. *Let a be an element of a unital commutative Banach algebra A. Then a is invertible in A if and only if $\hat{a}$ is invertible in $\mathcal{C}(\Delta A)$.*

11.3.6. Proposition. *Let A be a unital commutative Banach algebra and a be an element of A. Then $\operatorname{ran}\hat{a} = \sigma(a)$ and $\|\hat{a}\|_u = \rho(a)$.*

11.3.7. Definition. An element a of a Banach algebra is QUASINILPOTENT if $\lim_{n\to\infty} \|a^n\|^{1/n} = 0$.

11.3.8. Proposition. *Let a be an element of a unital commutative Banach algebra A. Then the following are equivalent:*

(a) *a is quasinilpotent;*
(b) *$\rho(a) = 0$;*
(c) *$\sigma(a) = \{0\}$;*
(d) *$\Gamma a = 0$;*
(e) *$\phi(a) = 0$ for every $\phi \in \Delta A$;*
(f) *$a \in \bigcap \operatorname{Max} A$.*

11.3.9. Definition. A Banach algebra is SEMISIMPLE if it has no nonzero quasinilpotent elements.

11.3.10. Proposition. *Let A be a unital commutative Banach algebra. Then the following are equivalent:*

(a) *A is semisimple;*
(b) *if $\rho(a) = 0$, then $a = 0$;*
(c) *if $\sigma(a) = \{0\}$, then $a = 0$;*
(d) *the Gelfand transform Γ_A is a monomorphism (that is, an injective homomorphism);*
(e) *if $\phi(a) = 0$ for every $\phi \in \Delta A$, then $a = 0$;*
(f) *$\bigcap \operatorname{Max} A = \{0\}$.*

11.3.11. Proposition. *Let A be a unital commutative Banach algebra. Then the following are equivalent:*

(a) *$\|a^2\| = \|a\|^2$ for all $a \in A$;*
(b) *$\rho(a) = \|a\|$ for all $a \in A$; and*
(c) *the Gelfand transform is an isometry; that is, $\|\widehat{a}\|_u = \|a\|$ for all $a \in A$.*

11.3.12. Example. The Gelfand transform on $l_1(\mathbb{Z})$ is not an isometry.

Recall from proposition 11.2.20 that when X is a compact Hausdorff space the evaluation mapping E_X identifies the space X with the maximal ideal space of the Banach algebra $\mathcal{C}(X)$. Thus according to proposition 11.2.21 the mapping $\mathcal{C}E_X$ identifies the algebra $\mathcal{C}(\Delta(\mathcal{C}(X)))$ of continuous functions on this maximal ideal space with the algebra $\mathcal{C}(X)$ itself. It turns out that the Gelfand transform on the algebra $\mathcal{C}(X)$ is just the inverse of this identification map.

11.3.13. Example. Let X be a compact Hausdorff space. Then the Gelfand transform on the Banach algebra $\mathcal{C}(X)$ is an isometric isomorphism. In fact, on $\mathcal{C}(X)$ the Gelfand transform Γ_X is $(\mathcal{C}E_X)^{-1}$.

11.3.14. Example. The maximal ideal space of the Banach algebra $L_1(\mathbb{R})$ is $\mathbb{R}$ itself and the Gelfand transform on $L_1(\mathbb{R})$ is the Fourier transform.

PROOF. See [1], pages 169–171.

The function $t \mapsto e^{it}$ is a bijection from the interval $[-\pi, \pi)$ to the unit circle $\mathbb{T}$ in the complex plane. One consequence of this is that we need not distinguish between

(a) 2π-periodic functions on $\mathbb{R}$,
(b) all functions on $[-\pi, \pi)$,
(c) functions f on $[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$, and
(d) functions on $\mathbb{T}$.

In the sequel we will frequently without further explanation identify these classes of functions.

Another convenient identification is the one between the unit circle $\mathbb{T}$ in $\mathbb{C}$ and the maximal ideal space of the algebra $l_1(\mathbb{Z})$. The homeomorphism ψ between these two compact Hausdorff space was defined in example 11.2.11. It is often technically more convenient in working with the Gelfand transform Γ_a of an element $a \in l_1(\mathbb{Z})$ to treat it as a function, let's call it G_a, whose domain is $\mathbb{T}$ as the following diagram suggests.

$$\begin{tikzcd}
\mathbb{T} \arrow[d, "\psi"'] \arrow[dr, "G_a"] & \\
\Delta l_1(\mathbb{Z}) \arrow[r, "\Gamma_a"'] & \mathbb{C}
\end{tikzcd}$$

Thus for $a \in l_1(\mathbb{Z})$ and $z \in \mathbb{T}$ we have

$$G_a(z) = \Gamma_a(\psi_z) = \psi_z(a) = \sum_{k=-\infty}^{\infty} a_n z^n \,.$$

11.3.15. Definition. If $f \in \mathcal{L}_1([-\pi, \pi))$, the FOURIER SERIES for f is the series

$$\sum_{n=-\infty}^{\infty} \widetilde{f}(n) \exp(int) \qquad -\pi \le t \le \pi$$

where the sequence $\widetilde{f}$ is defined by

$$\widetilde{f}(n) = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \exp(-int)\, dt$$

for all $n \in \mathbb{Z}$. The doubly infinite sequence $\widetilde{f}$ is the FOURIER TRANSFORM of f, and the number $\widetilde{f}(n)$ is the n^{th} FOURIER COEFFICIENT of f. If $\widetilde{f} \in l_1(\mathbb{Z})$ we say that f has an ABSOLUTELY CONVERGENT FOURIER SERIES. The set of all continuous functions on $\mathbb{T}$ with absolutely convergent Fourier series is denoted by $\mathcal{AC}(\mathbb{T})$.

11.3.16. Proposition. *If f is a continuous 2π-periodic function on $\mathbb{R}$ whose Fourier transform is zero, then $f = 0$.*

11.3.17. Corollary. *The Fourier transform on $\mathcal{C}(\mathbb{T})$ is injective.*

11.3.18. Proposition. *The Fourier transform on $\mathcal{C}(\mathbb{T})$ is a left inverse of the Gelfand transform on $l_1(\mathbb{Z})$.*

11.3.19. Proposition. *The range of the Gelfand transform on $l_1(\mathbb{Z})$ is the unital commutative Banach algebra $\mathcal{AC}(\mathbb{T})$.*

11.3.20. Proposition. *There are continuous functions whose Fourier series diverge at 0.*

PROOF. See, for example, [24], exercise 18.45.)

What does the preceding result say about the Gelfand transform $\Gamma\colon l_1(\mathbb{Z}) \to \mathcal{C}(\mathbb{T})$?

Suppose a function f belongs to $\mathcal{AC}(\mathbb{T})$ and is never zero. Then $1/f$ is certainly continuous on $\mathbb{T}$, but does it have an absolutely convergent Fourier series? One of the first triumphs of the abstract study of Banach algebras was a very simple proof of the answer to this question given originally by Norbert Wiener. Wiener's original proof by comparison was quite difficult.

11.3.21. Theorem (Wiener's theorem)**.** *Let f be a continuous function on $\mathbb{T}$ which is never zero. If f has an absolutely convergent Fourier series, then so does its reciprocal $1/f$.*

11.3.22. Example. The Laplace transform can also be viewed as a special case of the Gelfand transform. For details see [1], pages 173–175.

11.4. Unital C^*-algebras

11.4.1. Proposition. *If a is an element of a $*$-algebra, then $\sigma(a^*) = \overline{\sigma(a)}$.*

11.4.2. Proposition. *In every C^*-algebra involution is an isometry. That is, $\|a^*\| = \|a\|$ for every element a in the algebra.*

In definition 3.5.23 of *normed algebra* we made the special requirement that the identity element of a unital normed algebra have norm one. In C^*-algebras this requirement is redundant.

11.4.3. Corollary. *In a unital C^*-algebra $\|\mathbf{1}\| = 1$.*

11.4.4. Corollary. *Every unitary element in a unital C^*-algebra has norm one.*

11.4.5. Corollary. *If a is an element of a C^*-algebra A such that $ab = \mathbf{0}$ for every $b \in A$, then $a = \mathbf{0}$.*

11.4.6. Proposition. *Let a be a normal element of a C^*-algebra. Then $\|a^2\| = \|a\|^2$. And if the algebra is unital, then $\rho(a) = \|a\|$.*

11.4.7. Corollary. *If p is a nonzero projection in a C^*-algebra, then $\|p\| = 1$.*

11.4.8. Corollary. *Let $a \in A$ where A is a commutative C^*-algebra. Then $\|a^2\| = \|a\|^2$ and if additionally A is unital, then $\rho(a) = \|a\|$.*

11.4.9. Corollary. *On a unital commutative C^*-algebra A the Gelfand transform Γ is an isometry; that is, $\|\Gamma_a\|_u = \|\widehat{a}\|_u = \|a\|$ for every $a \in A$.*

11.4.10. Corollary. *The norm of a unital C^*-algebra is unique in the sense that given a unital algebra A with involution there is at most one norm which makes A into a C^*-algebra.*

11.4.11. Proposition. *If a is a self-adjoint element of a unital C^*-algebra, then $\sigma(a) \subseteq \mathbb{R}$.*

Hint for proof. Write $\lambda \in \sigma(a)$ in the form $\alpha + i\beta$ where α, $\beta \in \mathbb{R}$. For every $n \in \mathbb{N}$ let $b_n := a - \alpha\mathbf{1} + in\beta\mathbf{1}$. Show that for every n the scalar $i(n+1)\beta$ belongs to the spectrum of b_n and that therefore

$$|i(n+1)\beta|^2 \le \|a - \alpha\mathbf{1}\|^2 + n^2\beta^2 \,.$$

11.4.12. Corollary. *Let A be a unital commutative C^*-algebra. If $a \in A$ is self-adjoint, then its Gelfand transform $\widehat{a}$ is real valued.*

11.4.13. Corollary. *Every character on a unital C^*-algebra is a $*$-homomorphism.*

11.4.14. Corollary. *The Gelfand transform on a unital commutative C^*-algebra A is a unital $*$-homomorphism into $\mathcal{C}(\Delta A)$.*

11.4.15. Corollary. *The range of the Gelfand transform on a unital commutative C^*-algebra A is a unital self-adjoint separating subalgebra of $\mathcal{C}(\Delta A)$.*

11.4.16. Proposition. *If u is a unitary element of a unital C^*-algebra, then $\sigma(u) \subseteq \mathbb{T}$.*

11.4.17. Proposition. *The Gelfand transform* Γ *is a natural equivalence between the identity functor and the* $\mathcal{C}\Delta$ *functor on the category of commutative unital* C^**-algebras and unital* $*$*-homomorphisms.*

11.5. The Gelfand-Naimark Theorem

We recall a standard theorem from real analysis.

11.5.1. Theorem (Stone-Weierstrass Theorem)**.** *Let* X *be a compact Hausdorff space. Every unital separating* $*$*-subalgebra of* $\mathcal{C}(X)$ *is dense.*

Proof. See [11], theorem 2.40.

The first version of the *Gelfand-Naimark theorem* says that any unital commutative C^*-algebra is the algebra of all continuous functions on some compact Hausdorff space. Of course the word *is* in the preceding sentence means *is isometrically* $*$*-isomorphic to.* The compact Hausdorff space referred to is the character space of the algebra.

11.5.2. Theorem (Gelfand-Naimark theorem I)**.** *Let* A *be a unital commutative* C^**-algebra. Then the Gelfand transform* $\Gamma_A\colon a \mapsto \widehat{a}$ *is an isometric* $*$*-isomorphism of* A *onto* $\mathcal{C}(\Delta A)$*.*

11.5.3. Definition. Let S be a nonempty subset of a C^*-algebra A. The intersection of the family of all C^*-subalgebras of A which contain S is the C^*-subalgebra generated by S. We denote it by $C^*(S)$. (It is easy to see that the intersection of a family of C^*-subalgebras really is a C^*-algebra.) In some cases we shorten the notation slightly: for example, if $a \in A$ we write $C^*(a)$ for $C^*(\{a\})$ and $C^*(\mathbf{1}, a)$ for $C^*(\{\mathbf{1}, a\})$.

11.5.4. Proposition. *Let* S *be a nonempty subset of a* C^**-algebra* A*. For each natural number* n *define the set* W_n *to be the set of all elements* a *of* A *for which there exist* $x_1, x_2, \dots, x_n$ *in* $S \cup S^*$ *such that* $a = x_1 x_2 \cdots x_n$*. Let* $W = \bigcup_{n=1}^{\infty} W_n$*. Then*

$$C^*(S) = \overline{\operatorname{span} W}\,.$$

11.5.5. Theorem (Abstract Spectral Theorem)**.** *If* a *is a normal element of a unital* C^**-algebra* A*, then the* C^**-algebra* $\mathcal{C}(\sigma(a))$ *is isometrically* $*$*-isomorphic to* $C^*(\mathbf{1}, a)$*.*

Hint for proof. Use the Gelfand transform of a to identify the maximal ideal space of $C^*(\mathbf{1}, a)$ with the spectrum of a. Apply the functor $\mathcal{C}$. Compose the resulting map with Γ^{-1} where Γ is the Gelfand transform on the C^*-algebra $C^*(\mathbf{1}, a)$.

If $\psi\colon \mathcal{C}(\sigma(a)) \to C^*(\mathbf{1}, a)$ is the $*$-isomorphism in the preceding theorem and f is a continuous function on the spectrum of a, then the element $\psi(f)$ in the algebra A is usually denoted by $f(a)$. This operation of associating with the continuous function f an element $f(a)$ in A is referred to as the *functional calculus* associated with a.

11.5.6. Example. Suppose that in the preceding theorem $\psi\colon \mathcal{C}(\sigma(a)) \to C^*(\mathbf{1}, a)$ implements the isometric $*$-isomorphism. Then the image under ψ of the constant function $\mathbf{1}$ on the spectrum of a is $\mathbf{1}_A$ and the image under ψ of the identity function $\lambda \mapsto \lambda$ on the spectrum of a is a.

11.5.7. Example. Let T be a normal operator on a Hilbert space H whose spectrum is contained in $[0, \infty)$. Suppose that $\psi\colon \mathcal{C}(\sigma(T)) \to C^*(I, T)$ implements the isometric *-isomorphism between these two C^*-algebras. Then there is at least one operator $\sqrt{T}$ whose square is T. Indeed, whenever f is a continuous function on the spectrum of a normal operator T, we may meaningfully speak of the operator $f(T)$.

11.5.8. Example. If N is a normal Hilbert space operator whose spectrum is $\{0, 1\}$, then N is an orthogonal projection.

11.5.9. Example. If N is a normal Hilbert space operator whose spectrum is is contained in the unit circle $\mathbb{T}$, then N is a unitary operator.

11.5.10. Proposition (Spectral mapping theorem)**.** *Let a be a self-adjoint element in a unital C^*-algebra A and f be a continuous complex valued function on the spectrum of a. Then*

$$\sigma(f(a)) = f^{\rightarrow}(\sigma(a))\,.$$

The functional calculus sends continuous functions to continuous functions in the following sense.

11.5.11. Proposition. *Suppose A is a unital C^*-algebra, K is a nonempty compact subset of $\mathbb{R}$, and $f\colon K \to \mathbb{C}$ is a continuous function. Let $\mathfrak{H}_K$ be the set of all self-adjoint elements of A whose spectra are subsets of K. Then the function $f\colon \mathfrak{H}_K \to A\colon h \mapsto f(h)$ is continuous.*

Hint for proof. Approximate f uniformly by a polynomial and use an "ϵ over 3" argument and the *spectral mapping theorem* 11.5.10.

CHAPTER 12

MULTIPLICATIVE IDENTITIES AND THEIR ALTERNATIVES

The definition of the *spectrum* of an element in an algebra given in 8.1.15 makes use of the multiplicative identity of the algebra. What do we do if our algebra, call it A, has no such identity? The simple answer is: *we add one*. The resulting algebra $\widetilde{A}$ we call the *unitization* of A. Simple as this answer appears, it can lead to confusion. For example, in [40], Exercise 1.3(vi), one is told that if A is itself unital, then $\widetilde{A}$ is isomorphic to $A \oplus \mathbb{C}$. But in [45], Example 2.1.2, we read, "If A is unital then the only unitization of A is A itself." Now surely both these statements cannot be correct. In fact, neither author is making an error here; they are simply using different definitions of "unitization". Some writers find it convenient to have unitizations properly contain (isomorphic images of) the algebras they started with; others do not. In these notes unitizations will always contain new elements.

A similar (and closely related) issue arises in topology. Do we define the one-point compactification of a space which is already compact? And, if so, does it contain a new point? Our answer in these notes will be *yes* to both questions.

12.1. Quasi-inverses

12.1.1. Definition. Let A be an algebra. Define $A \bowtie \mathbb{C}$ to be the set $A \times \mathbb{C}$ on which addition and scalar multiplication are defined pointwise and multiplication is defined by

$$(a, \lambda) \cdot (b, \mu) = (ab + \mu a + \lambda b, \lambda\mu).$$

If the algebra A is equipped with an involution, define an involution on $A \bowtie \mathbb{C}$ pointwise; that is, $(a, \lambda)^* := (a^*, \overline{\lambda})$. (The notation $A \bowtie \mathbb{C}$ is not standard.)

12.1.2. Proposition. *If A is an algebra, then $A \bowtie \mathbb{C}$ is a unital algebra in which A is embedded as (that is, isomorphic to) an ideal such that $(A \bowtie \mathbb{C})/A \cong \mathbb{C}$. The identity of $A \bowtie \mathbb{C}$ is $(0, 1)$. If A is a $*$-algebra, then $A \bowtie \mathbb{C}$ is a unital $*$-algebra in which A is a $*$-ideal such that $(A \bowtie \mathbb{C})/A \cong \mathbb{C}$.*

12.1.3. Definition. The algebra $A \bowtie \mathbb{C}$ is the UNITIZATION of the algebra (or $*$-algebra) A.

12.1.4. Notation. In the preceding construction elements of the algebra $A \bowtie \mathbb{C}$ are technically ordered pairs (a, λ). They are usually written differently. Let $\iota\colon A \to A \bowtie \mathbb{C}\colon a \mapsto (a, 0)$ and $\pi\colon A \bowtie \mathbb{C} \to \mathbb{C}\colon (a, \lambda) \mapsto \lambda$. It follows, since $(0, 1)$ is the identity in $A \bowtie \mathbb{C}$, that

$$\begin{aligned}(a, \lambda) &= (a, 0) + (\mathbf{0}, \lambda)\\ &= \iota(a) + \lambda\mathbf{1}_{A\bowtie\mathbb{C}}\end{aligned}$$

It is conventional to treat ι as an inclusion mapping. Thus it is reasonable to write (a, λ) as $a + \lambda\mathbf{1}_{A\bowtie\mathbb{C}}$ or simply as $a + \lambda\mathbf{1}$. No ambiguity seems to follow from omitting reference to the multiplicative identity, so a standard notation for the pair (a, λ) is $a + \lambda$.

12.1.5. Definition. Let A be a *nonunital* algebra. Define the SPECTRUM of an element $a \in A$ to be the spectrum of a regarded as an element of $A \bowtie \mathbb{C}$; that is, $\sigma_A(a) := \sigma_{A\bowtie\mathbb{C}}(a)$.

12.1.6. Definition. Let A be a normed algebra (with or without involution). On the unitization $A \bowtie \mathbb{C}$ of A define $\|(a, \lambda)\| := \|a\| + |\lambda|$.

12.1.7. Proposition. *Let A be a normed algebra. The mapping $(a,\lambda) \mapsto \|a\| + |\lambda|$ defined above is a norm under which $A \bowtie \mathbb{C}$ is a normed algebra. If A is a Banach algebra (respectively, a Banach $*$-algebra), then $A \bowtie \mathbb{C}$ is a unital Banach algebra (respectively, a Banach $*$-algebra). The resulting Banach algebra (or Banach $*$-algebra) is the* UNITIZATION *of A and will be denoted by $\widetilde{A}$.*

With this expanded definition of *spectrum* many of the earlier facts for unital Banach algebras remain true in the more general setting. In particular, for future reference we restate items 8.1.36, 8.1.41, 8.1.44, and 8.1.49.

12.1.8. Proposition. *Let a be an element of a Banach algebra A. Then the spectrum of a is compact and $|\lambda| \le \|a\|$ for every $\lambda \in \sigma(a)$.*

12.1.9. Proposition. *The spectrum of every element of a Banach algebra is nonempty.*

12.1.10. Proposition. *Let a be an element of an algebra. Then*

$$\sigma(a^n) = [\sigma(a)]^n$$

for every $n \in \mathbb{N}$.

12.1.11. Theorem (Spectral radius formula)**.** *If a is an element of a Banach algebra, then*

$$\rho(a) = \inf\{\|a^n\|^{1/n} \colon n \in \mathbb{N}\} = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

12.1.12. Definition. An element b of an algebra A is a LEFT QUASI-INVERSE for $a \in A$ if $ba = a+b$. It is a RIGHT QUASI-INVERSE for a if $ab = a + b$. If b is both a left and a right quasi-inverse for a it is a QUASI-INVERSE for a. When a has a quasi-inverse denote it by a'.

12.1.13. Proposition. *If b is a left quasi-inverse for a in an algebra A and c is a right quasi-inverse for a in A, then $b = c$.*

12.1.14. Proposition. *Let A be a unital algebra and let a, $b \in A$. Then*

(a) a' exists if and only if $(\mathbf{1} - a)^{-1}$ exists; and
(b) b^{-1} exists if and only if $(\mathbf{1} - b)'$ exists.

Hint for proof. For the reverse direction in (a) consider $a + c - ac$ where $c = \mathbf{1} - (\mathbf{1} - a)^{-1}$.

12.1.15. Proposition. *Let A be a Banach algebra and $a \in A$. If $\rho(a) < 1$, then a' exists and $a' = -\sum_{k=1}^{\infty} a^k$.*

12.1.16. Proposition. *Let A be a Banach algebra and $a \in A$. If $\|a\| < 1$, then a' exists and*

$$\frac{\|a\|}{1 + \|a\|} \le \|a'\| \le \frac{\|a\|}{1 - \|a\|}.$$

12.1.17. Proposition. *Let A be a unital Banach algebra and $a \in A$. If $\rho(\mathbf{1} - a) < 1$, then a is invertible in A.*

12.1.18. Proposition. *Let A be a Banach algebra and $a, b \in A$. If b' exists and $\|a\| < (1+\|b'\|)^{-1}$, then $(a + b)'$ exists and*

$$\|(a+b)' - b'\| \le \frac{\|a\|\,(1 + \|b'\|)^2}{1 - \|a\|\,(1 + \|b'\|)}\,.$$

Hint for proof. Show first that $u = (a - b'a)'$ exists and that $u + b' - ub'$ is a left quasi-inverse for $a\ +\ b$.

Compare the next result with propositions 8.1.33 and 8.1.34

12.1.19. Proposition. *The set Q_A of quasi-invertible elements of a Banach algebra A is open in A, and the map $a \mapsto a'$ is a homeomorphism of Q_A onto itself.*

12.1.20. Notation. If a and b are elements in an algebra we define $a \circ b := a + b - ab$.

12.1.21. Proposition. *If A is an algebra, then Q_A is a group under $\circ$.*

12.1.22. Proposition. *If A is a unital Banach algebra and* $\operatorname{inv} A$ *is the set of its invertible elements, then*

$$\Psi\colon Q_A \to \operatorname{inv} A\colon a \mapsto \mathbf{1} - a$$

is an isomorphism.

12.1.23. Definition. If A is a algebra and $a \in A$, we define the Q-SPECTRUM of a in A by

$$\breve{\sigma}_A(a) := \{\lambda \in \mathbb{C}\colon \lambda \neq 0 \text{ and } \frac{a}{\lambda} \notin Q_A\} \cup \{0\}.$$

In a unital algebra, the preceding definition is "almost" the usual one.

12.1.24. Proposition. *Let A be a unital algebra and $a \in A$. Then for all $\lambda \neq 0$, we have $\lambda \in \sigma(a)$ if and only if $\lambda \in \breve{\sigma}(a)$.*

12.1.25. Proposition. *If an algebra A is not unital and $a \in A$, then $\breve{\sigma}_A(a) = \breve{\sigma}_{\widetilde{A}}(a)$, where $\widetilde{A}$ is the unitization of A.*

12.1.26. Proposition. *An element a of a commutative Banach algebra is quasi-invertible if and only if for every $\phi \in \Delta A$ we have $\widehat{a}(\phi) \neq 1$.*

12.2. Modular Ideals

12.2.1. Definition. Let A be an algebra. A left ideal J in A is a MODULAR (or REGULAR) LEFT IDEAL if there exists an element u in A such that $au - a \in J$ for every $a \in A$. Such an element u is called a RIGHT IDENTITY WITH RESPECT TO J (or MODULO J). Similarly, a right ideal J in A is a MODULAR (or REGULAR) RIGHT IDEAL if there exists an element v in A such that $va - a \in J$ for every $a \in A$. Such an element v is called a LEFT IDENTITY WITH RESPECT TO J (or MODULO J). A two-sided ideal J is a MODULAR IDEAL if there exists an element e which is both a left and a right identity with respect to J.

12.2.2. Proposition. *An ideal J in an algebra is modular if and only if it is both left modular and right modular.*

Hint for proof. Show that if u is a right identity with respect to J and v is a left identity with respect to J, then vu is both a right and left identity with respect to J.

12.2.3. Proposition. *An ideal J in an algebra A is modular if and only if the quotient algebra A/J is unital.*

12.2.4. Example. Let X be a locally compact Hausdorff space. For every $x \in X$ the ideal J_x is a maximal modular ideal in the C^*-algebra $\mathcal{C}_0(X)$ of continuous complex valued functions on X.

PROOF. By the locally compact Hausdorff space version of *Urysohn's lemma* (see, for example, [13], theorem 17.2.10) there exists a function $g \in \mathcal{C}_0(X)$ such that $g(x) = 1$. Thus J_x is modular because g is an identity with respect to J_x. Since $\mathcal{C}_0(X) = J_x \oplus \operatorname{span}\{g\}$ the ideal J_x has codimension 1 and is therefore maximal. □

12.2.5. Proposition. *If J is a proper modular ideal in a Banach algebra, then so is its closure.*

12.2.6. Corollary. *Every maximal modular ideal in a Banach algebra is closed.*

12.2.7. Proposition. *Every maximal modular ideal in a Banach algebra is a maximal ideal.*

12.2.8. Proposition. *Every proper modular ideal in a Banach algebra is contained in a maximal modular ideal.*

12.2.9. Proposition. *An element a of a Banach algebra A has a left (right) quasi-inverse if and only if it is not a right (left) identity with respect to any maximal modular ideal in A.*

12.2.10. Proposition. *Let A be a commutative Banach algebra and $\phi \in \Delta A$. Then $\ker \phi$ is a maximal modular ideal in A and $A/\ker \phi$ is a field. Furthermore, every maximal modular ideal is the kernel of exactly one character in ΔA.*

12.2.11. Proposition. *If ϕ is a character on a Banach algebra, then $\|\phi\| \le 1$.*

The preceding proposition tells us that, just as in the case of unital Banach algebras, the characters of any Banach algebra A live in the closed unit ball of the dual space of A. Thus we may give the set ΔA of all characters on A the relative w^*-topology it inherits from A^*. As before this topology is known as the *Gelfand topology* on ΔA.

12.2.12. Example. Let $\widetilde{A}$ be the unitization of a commutative Banach algebra A (see proposition 12.1.7). Define

$$\phi_\infty \colon \widetilde{A} \to \mathbb{C} \colon (a, \lambda) \mapsto \lambda\,.$$

Then ϕ_∞ is a character on $\widetilde{A}$.

12.2.13. Proposition. *Every character ϕ on a commutative Banach algebra A has a unique extension to a character $\widetilde{\phi}$ on the unitization $\widetilde{A}$ of A. And the restriction to A of any character on $\widetilde{A}$, with the obvious exception of ϕ_∞, is a character on A.*

12.2.14. Proposition. *If A is a commutative Banach algebra, then*

(a) *ΔA is a locally compact Hausdorff space,*
(b) *$\Delta\widetilde{A} = \Delta A \cup \{\phi_\infty\}$,*
(c) *$\Delta\widetilde{A}$ is the one-point compactification of ΔA, and*
(d) *the map $\phi \mapsto \widetilde{\phi}$ is a homeomorphism from ΔA onto $\Delta\widetilde{A} \setminus \{\phi_\infty\}$.*

If A is unital (so that ΔA is compact), then ϕ_∞ is an isolated point of $\Delta\widetilde{A}$.

12.2.15. Theorem. *If A is a commutative Banach algebra without identity, then the Gelfand transform*

$$\Gamma = \Gamma_A \colon \mathcal{C}_0(\Delta A) \colon a \mapsto \widehat{a} = \Gamma_a$$

is a contractive algebra homomorphism and $\rho(a) = \|\widehat{a}\|_u$. Furthermore, if A is not *unital, then $\sigma(a) = (\operatorname{ran} \widehat{a}) \cup \{0\}$ for every $a \in A$.*

12.3. Unitization of C^*-algebras

The rather simple procedure for the unitization of Banach $*$-algebras (see 12.1.1 and 12.1.6) does not carry over to C^*-algebras. The norm defined in 12.1.6 does not satisfy the C^*-condition (definition 7.1.22). It turns out that the unitization of C^*-algebras is a bit more complicated.

12.3.1. Proposition. *The kernel of a $*$-homomorphism $\phi \colon A \to B$ between C^*-algebras is a closed $*$-ideal in A and its range is a $*$-subalgebra of B.*

12.3.2. Proposition. *If a is an element in a C^*-algebra, then*

$$\begin{aligned} \|a\| &= \sup\{\|xa\| \colon \|x\| \le 1\} \\ &= \sup\{\|ax\| \colon \|x\| \le 1\}. \end{aligned}$$

12.3.3. Corollary. *If a is an element of a C^*-algebra A, the operator L_a, called* LEFT MULTIPLICATION BY *a and defined by*

$$L_a \colon A \to A \colon x \mapsto ax$$

is a (bounded linear) operator on A. Furthermore, the map

$$L \colon A \to \mathfrak{B}(A) \colon a \mapsto L_a$$

is both an isometry and an algebra homomorphism.

12.3.4. Definition. We say that a short exact sequence of C^*-algebras and $*$-homomorphisms

$$0 \longrightarrow A \xrightarrow{\ \phi\ } E \xrightarrow{\ \psi\ } B \longrightarrow 0 \tag{12.1}$$

is SPLIT EXACT if there exists a $*$-homomorphism $\xi\colon B \to E$ such that $\psi \circ \xi = \mathrm{id}_B$.

Often in the context of C^*-algebras an exact sequence such as (12.1) is referred to as an EXTENSION. Some authors refer to it as an *extension of A by B* (for example, [45] and [9]) while others say it is an *extension of B by A* ([35], [16], and [2]). In [16] and [2] the *extension* is defined to be the sequence (12.1); in [45] it is defined to be the ordered triple (ϕ, E, ψ); and in [35] and [9] it is defined to be the C^*-algebra E itself. Regardless of the formal definitions it is common to say that *E is an extension of A by B (or of B by A).*

12.3.5. Definition. Let A and B be C^*-algebras. We define the (EXTERNAL) DIRECT SUM of A and B, denoted by $A \oplus B$, to be the Cartesian product $A \times B$ with pointwise defined algebraic operations and norm given by

$$\|(a, b)\| = \max\{\|a\|, \|b\|\}$$

for all $a \in A$ and $b \in B$. An alternative notation for the element (a, b) in $A \oplus B$ is $a \oplus b$.

12.3.6. Example. Let A and B be C^*-algebras. Then the direct sum of A and B is a C^*-algebra and the following sequence is split short exact:

$$\mathbf{0} \longrightarrow A \xrightarrow{\ \iota_1\ } A \oplus B \xrightarrow{\ \pi_2\ } B \longrightarrow \mathbf{0}$$

The indicated maps in the preceding are the obvious ones:

$$\iota_1\colon A \to A \oplus B\colon a \mapsto (a, \mathbf{0}) \qquad \text{and} \qquad \pi_2\colon A \oplus B \to B : (a, b) \mapsto b\,.$$

This sequence is frequently called the DIRECT SUM EXTENSION.

12.3.7. Proposition. *If A and B are nonzero C^*-algebras, then their direct sum $A \oplus B$ is a product in the category* **CSA** *of C^*-algebras and $*$-homomorphisms. The direct sum is unital if and only if both A and B are.*

12.3.8. Definition. Let A and B be C^*-algebras and E and E' be extensions of A by B. These extensions are STRONGLY EQUIVALENT if there exists a *-isomorphism $\theta\colon E \to E'$ that makes the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\theta"] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & E' \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

commute.

12.3.9. Proposition. *In the preceding definition it is enough to require θ to be a *-homomorphism.*

12.3.10. Proposition. *Let A and B be C^*-algebras. An extension*

$$\mathbf{0} \longrightarrow A \xrightarrow{\ \phi\ } E \xrightarrow{\ \psi\ } B \longrightarrow \mathbf{0}$$

is strongly equivalent to the direct sum extension $A \oplus B$ if and only if there exists a $*$-homomorphism $\nu\colon E \to A$ such that $\nu \circ \phi = \mathrm{id}_A$.

12.3.11. Proposition. *If the sequences of C^*-algebras*

$$0 \longrightarrow A \xrightarrow{\ \phi\ } E \xrightarrow{\ \psi\ } B \longrightarrow 0 \tag{12.2}$$

and

$$0 \longrightarrow A \xrightarrow{\ \phi'\ } E' \xrightarrow{\ \psi'\ } B \longrightarrow 0 \tag{12.3}$$

are strongly equivalent and (12.2) splits, then so does (12.3).

12.3.12. Proposition. *Let A be a C^*-algebra. Then there exists a unital C^*-algebra $\widetilde{A}$ in which A is embedded as an ideal such that the sequence*

$$\mathbf{0} \longrightarrow A \longrightarrow \widetilde{A} \longrightarrow \mathbb{C} \longrightarrow \mathbf{0} \tag{12.4}$$

is split exact. If A is unital then the sequence (12.4) is strongly equivalent to the direct sum extension, so that $\widetilde{A} \cong A \oplus \mathbb{C}$. If A is not *unital, then $\widetilde{A}$ is not isomorphic to $A \oplus \mathbb{C}$.*

Hint for proof. The proof this result is a little complicated. Everyone should go through all the details at least once in his/her life. What follows is an outline of a proof.

Notice that we speak of the unitization of C^*-algebra A *whether or not A already has a unit (multiplicative identity).* We divide the argument into two cases.

Case 1: the algebra A is unital.

(1) On the unital algebra $A \bowtie \mathbb{C}$ define

$$\|(a,\lambda)\| := \max\{\|a + \lambda\mathbf{1}_A\|, |\lambda|\}$$

and let $\widetilde{A} := A \bowtie \mathbb{C}$ together with this norm.

(2) Prove that the map $(a,\lambda) \mapsto \|(a,\lambda)\|$ is a norm on $A \bowtie \mathbb{C}$.

(3) Prove that this norm is an algebra norm.

(4) Show that it is, in fact, a C^*-norm on $A \bowtie \mathbb{C}$.

(5) Observe that it is an extension of the norm on A.

(6) Prove that $A \bowtie \mathbb{C}$ is a unital C^*-algebra by verifying completeness of the metric space induced by the preceding norm.

(7) Prove that the sequence

$$\begin{tikzcd} \mathbf{0} \arrow[r] & A \arrow[r, "\iota"] & \widetilde{A} \arrow[r, shift left, "Q"] & \mathbb{C} \arrow[l, shift left, "\psi"] \arrow[r] & \mathbf{0} \end{tikzcd}$$

is split exact (where $\iota\colon a \mapsto (a,0)$, $Q\colon (a,\lambda) \mapsto \lambda$, and $\psi\colon \lambda \mapsto (0,\lambda)$).

(8) Prove that $\widetilde{A} = A \oplus \mathbb{C}\mathbf{j}$ where $\mathbf{j} := \mathbf{1}_{\widetilde{A}} - \mathbf{1}_A$.

CAUTION. In (8) above make sure you understand why it is correct to write $\widetilde{A} = A \oplus \mathbb{C}\mathbf{j}$ or $\widetilde{A} \overset{*}{\cong} A \oplus \mathbb{C}$, but *not* correct to write $\widetilde{A} = A \oplus \mathbb{C}$.

Case 2: the algebra A is *not* unital.

(9) Prove the following simple fact.

12.3.13. Lemma. *Let A be an algebra and B be a normed algebra. If $\phi\colon A \to B$ is an algebra homomorphism, the function $a \mapsto \|a\| := \|\phi(a)\|$ is a seminorm on A. The function is a norm if ϕ is injective.*

(10) Recall that we defined the operator L_a, *left multiplication by a*, in 12.3.3. Now let

$$A^\sharp := \{L_a + \lambda I_A \in \mathfrak{B}(A) \colon a \in A \text{ and } \lambda \in \mathbb{C}\}$$

and show that $A^\sharp$ is a normed algebra.

(11) Make $A^\sharp$ into a $*$-algebra by defining

$$(L_a + \lambda I_A)^* := L_{a^*} + \overline{\lambda} I_A$$

for all $a \in A$ and $\lambda \in \mathbb{C}$.

(12) Define

$$\phi\colon A \bowtie \mathbb{C} \to A^\sharp\colon (a,\lambda) \mapsto L_a + \lambda I_A$$

and verify that ϕ is a $*$-homomorphism.

(13) Prove that ϕ is injective.

(14) Use (9) to endow $A \bowtie \mathbb{C}$ with a norm which makes it into a unital normed algebra. Let $\widetilde{A} := A \bowtie \mathbb{C}$ with the norm pulled back by ϕ from $A^\sharp$

(15) Verify the following facts.
  (a) The map $\phi\colon \widetilde{A} \to A^\sharp$ is an isometric isomorphism.
  (b) $\operatorname{ran} L$ is a closed subalgebra of $\operatorname{ran} \phi = A^\sharp \subseteq \mathfrak{B}(A)$.
  (c) $I_A \notin \operatorname{ran} L$.

(16) Prove that the norm on $\widetilde{A}$ satisfies the C^*-condition.

(17) Prove that $\widetilde{A}$ is a unital C^*-algebra. (To show that $\widetilde{A}$ is complete we need only show that $A^\sharp$ is complete. To this end let $\big(\phi(a_n, \lambda_n)\big)_{n=1}^\infty$ be a Cauchy sequence in $A^\sharp$. To show that this sequence converges it suffices to show that it has a convergent subsequence. Showing that the sequence (λ_n) is bounded allows us to extract from it a convergent subsequence $\big(\lambda_{n_k}\big)$. Prove that $(L_{a_{n_k}})$ converges.)

(18) Prove that the sequence

$$0 \longrightarrow A \longrightarrow \widetilde{A} \longrightarrow \mathbb{C} \longrightarrow 0 \tag{12.5}$$

is split exact.

(19) The C^*-algebra $\widetilde{A}$ is *not* strongly equivalent to $A \oplus \mathbb{C}$.

12.3.14. Definition. The C^*-algebra $\widetilde{A}$ constructed in the preceding proposition is the UNITIZATION of A.

Note that the expanded definition of *spectrum* given in 12.1.5 applies to C^*-algebras since the added identity is purely an algebraic matter and is the same for C^*-algebras as it is for general Banach algebras. Thus many of the earlier facts stated for unital C^*-algebras remain true. In particular, for future reference we restate items 11.4.6, 11.4.8, 11.4.9, 11.4.10, 11.4.11, 11.4.12, and 11.4.14.

12.3.15. Proposition. *Let a be a normal element of a C^*-algebra. Then $\|a^2\| = \|a\|^2$ and therefore $\rho(a) = \|a\|$.*

12.3.16. Corollary. *If A is a commutative C^*-algebra, then $\|a^2\| = \|a\|^2$ and $\rho(a) = \|a\|$ for every $a \in A$.*

12.3.17. Corollary. *On a commutative C^*-algebra A the Gelfand transform Γ is an isometry; that is, $\|\Gamma_a\|_u = \|\widehat{a}\|_u = \|a\|$ for every $a \in A$.*

12.3.18. Corollary. *The norm of a C^*-algebra is unique in the sense that given a algebra A with involution there is at most one norm which makes A into a C^*-algebra.*

12.3.19. Proposition. *If h is a self-adjoint element of a C^*-algebra, then $\sigma(h) \subseteq \mathbb{R}$.*

12.3.20. Proposition. *If a is a self-adjoint element in a C^*-algebra, then its Gelfand transform $\widehat{a}$ is real valued.*

12.3.21. Proposition. *Every character on a C^*-algebra A preserves involution, thus the Gelfand transform Γ_A is a $*$-homomorphism.*

An immediate result of the preceding results is the second version of the *Gelfand-Naimark theorem*, which says that any commutative C^*-algebra is (isometrically unitally $*$-isomorphic to) the algebra of all those continuous functions on some locally compact Hausdorff space which vanish at infinity. As was the case with the first version of this theorem (see 11.5.2) the locally compact Hausdorff space referred to is the character space of the algebra.

12.3.22. Theorem (Gelfand-Naimark Theorem II)**.** *Let A be a commutative C^*-algebra. Then the Gelfand transform $\Gamma_A\colon a \mapsto \widehat{a}$ is an isometric unital $*$-isomorphism of A onto $\mathcal{C}_0(\Delta A)$.*

It follows from the next proposition that the unitization process is functorial.

12.3.23. Proposition. *Every $*$-homomorphism $\phi\colon A \to B$ between C^*-algebras has a unique extension to a unital $*$-homomorphism $\widetilde{\phi}\colon \widetilde{A} \to \widetilde{B}$ between their unitizations.*

In contrast to the situation in general Banach algebras there is no distinction between topological and geometric categories of C^*-algebras. One of the most remarkable aspects of C^*-algebra theory is that $*$-homomorphisms between such algebras are automatically continuous—in fact, contractive. It follows that if two C^*-algebras are algebraically $*$-isomorphic, then they are isometrically isomorphic.

12.3.24. Proposition. *Every $*$-homomorphism between C^*-algebras is contractive.*

12.3.25. Proposition. *Every injective $*$-homomorphism between C^*-algebras is an isometry.*

12.3.26. Proposition. *Let X be a locally compact Hausdorff space and $\widetilde{X} = X \cup \{\infty\}$ be its one-point compactification. Define*

$$\iota\colon \mathcal{C}_0(X) \to \mathcal{C}(\widetilde{X})\colon f \mapsto \widetilde{f}$$

where

$$\widetilde{f}(x) = \begin{cases} f(x), & \text{if } x \in X; \\ 0, & \text{if } x = \infty. \end{cases}$$

Also let E_∞ be defined on $\mathcal{C}(\widetilde{X})$ by $E_\infty(g) = g(\infty)$. Then the sequence

$$\mathbf{0} \longrightarrow \mathcal{C}_0(X) \xrightarrow{\ \iota\ } \mathcal{C}(\widetilde{X}) \xrightarrow{\ E_\infty\ } \mathbb{C} \longrightarrow \mathbf{0}$$

is exact.

In the preceding proposition we refer to $\widetilde{X}$ as the one-point compactification of X even in the case that X is compact to begin with. Most definitions of *compactification* require a space to be dense in any compactification. (See my remarks in the beginning of section 17.3 of [13].) We have previously adopted the convention that the unitization of a unital algebra gets a new multiplicative identity. In the spirit of consistency with this choice we will in the sequel subscribe to the convention that the one-point compactification of a compact space gets an additional (isolated) point.

From the point of view of the Gelfand-Naimark theorem (12.3.22) the fundamental insight prompted by the next proposition is that the unitization of a commutative C^*-algebra is, in some sense, the "same thing" as the one-point compactification of a locally compact Hausdorff space.

12.3.27. Proposition. *If X is a locally compact Hausdorff space, then the unital C^*-algebras $\big(\mathcal{C}_0(X)\big)^\sim$ and $\mathcal{C}(\widetilde{X})$ are isometrically $*$-isomorphic.*

Proof. Define

$$\theta\colon \big(\mathcal{C}_0(X)\big)^\sim \to \mathcal{C}(\widetilde{X})\colon (f,\lambda) \mapsto \widetilde{f} + \lambda \mathbf{1}_{\widetilde{X}}$$

(where $\mathbf{1}_{\widetilde{X}}$ is the constant function 1 on $\widetilde{X}$). Then consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}_0(X) \arrow[r] \arrow[d, equal] & \big(\mathcal{C}_0(X)\big)^\sim \arrow[r] \arrow[d, "\theta"] & \mathbb{C} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{C}_0(X) \arrow[r, "\iota"] & \mathcal{C}(\widetilde{X}) \arrow[r, "E_\infty"] & \mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

The top row is exact by proposition 12.3.12, the bottom row is exact by proposition 12.3.26, and the diagram obviously commutes. It is routine to check that θ is a $*$-homomorphism. Therefore θ is an isometric $*$-isomorphism by proposition 12.3.9 and corollary 12.3.25. $\square$

12.4. Positive Elements in a C^*-Algebra

12.4.1. Definition. A self-adjoint element a of a C^*-algebra A is POSITIVE if $\sigma(a) \subseteq [0, \infty)$. In this case we write $a \geq \mathbf{0}$. We denote the set of all positive elements of A by A^+. This is the POSITIVE CONE of A. For any subset B of A let $B^+ = B \cap A^+$. We will use the positive cone to induce a partial ordering on A: we write $a \leq b$ when $b - a \in A^+$.

12.4.2. Definition. Let $\leq$ be a relation on a nonempty set S. If the relation $\leq$ is reflexive and transitive, it is a PREORDERING. If $\leq$ is a preordering and is also antisymmetric, it is a PARTIAL ORDERING.

A partial ordering $\leq$ on a real vector space V is COMPATIBLE with (or RESPECTS) the operations (addition and scalar multiplication) on V if for all x, y, $z \in V$

(a) $x \leq y$ implies $x + z \leq y + z$, and
(b) $x \leq y$, $\alpha \geq 0$ imply $\alpha x \leq \alpha y$.

A real vector space equipped with a partial ordering which is compatible with the vector space operations is an ORDERED VECTOR SPACE.

12.4.3. Definition. Let V be a vector space. A subset C of V is a CONE in V if $\alpha C \subseteq C$ for every $\alpha \geq 0$. A cone C in V is PROPER if $C \cap (-C) = \{\mathbf{0}\}$.

12.4.4. Example. If V is an ordered vector space, then the set

$$V^+ := \{x \in V : x \geq \mathbf{0}\}$$

is a proper convex cone in V. This is the POSITIVE CONE of V and its members are the POSITIVE ELEMENTS of V.

12.4.5. Proposition. *Let V be a real vector space and C be a proper convex cone in V. Define $x \leq y$ if $y - x \in C$. Then the relation $\leq$ is a partial ordering on V and is compatible with the vector space operations on V. This relation is the* PARTIAL ORDERING INDUCED BY *the cone C. The positive cone V^+ of the resulting ordered vector space is just C itself.*

12.4.6. Proposition. *If a is a self-adjoint element of a unital C^*-algebra and $t \in \mathbb{R}$, then*

(i) $a \geq \mathbf{0}$ whenever $\|a - t\mathbf{1}\| \leq t$; and
(ii) $\|a - t\mathbf{1}\| \leq t$ whenever $\|a\| \leq t$ and $a \geq \mathbf{0}$.

12.4.7. Example. The positive cone of a C^*-algebra A is a closed proper convex cone in the real vector space $\mathfrak{H}(A)$.

12.4.8. Proposition. *If a and b are positive elements of a C^*-algebra and $ab = ba$, then ab is positive.*

12.4.9. Proposition. *Every positive element of a C^*-algebra A has a unique positive n^{th} root ($n \in \mathbb{N}$). That is, if $a \in A^+$, then there exists a unique $b \in A^+$ such that $b^n = a$.*

PROOF. *Hint.* The existence part is a simple application of the C^*-functional calculus (that is, the *abstract spectral theorem* 11.5.5). The element b given by the functional calculus is positive in the algebra $C^*(\mathbf{1}, a)$. Explain why it is also positive in A. The uniqueness argument deserves considerable care.

12.4.10. Theorem (Jordan Decomposition)**.** *If c is a self-adjoint element of a C^*-algebra A, then there exist unique positive elements c^+ and c^- of A such that $c = c^+ - c^-$ and $c^+c^- = \mathbf{0}$.*

12.4.11. Lemma. *If c is an element of a C^*-algebra such that $-c^*c \geq \mathbf{0}$, then $c = \mathbf{0}$.*

Hint for proof. Write $c = h + ik$ where h and k are self-adjoint. Show that $c^*c + cc^* \geq 0$. Conclude that $\sigma(c^*c) = \{0\}$.

12.4.12. Proposition. *If a is an element of a C^*-algebra, then $a^*a \geq \mathbf{0}$.*

Hint for proof. Let $b = a^*a$ and $c = ab^-$. Use the preceding lemma.

12.4.13. Proposition. *If c is an element of a C^*-algebra, then the following are equivalent:*

(i) $c \geq \mathbf{0}$;
(ii) there exists $b \geq \mathbf{0}$ *such that* $c = b^2$; *and*
(iii) there exists $a \in A$ *such that* $c = a^*a$,

12.4.14. Example. If T is an operator on a Hilbert space H, then T is a positive member of the C^*-algebra $\mathfrak{B}(H)$ if and only if $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

Hint for proof. Showing that if $T \geq \mathbf{0}$ in $\mathfrak{B}(H)$, then $\langle Tx, x\rangle \geq 0$ for all $x \in H$ is easy: use proposition 12.4.13 to write T as S^*S for some operator S.

For the converse suppose that $\langle Tx, x\rangle \geq 0$ for all $x \in H$. It is easy to see that this implies that T is self-adjoint. Use the *Jordan decomposition theorem*12.4.10 to write T as $T^+ - T^-$. For arbitrary $u \in H$ let $x = T^-u$ and verify that $0 \leq \langle Tx, x\rangle = -\langle (T^-)^3u, u\rangle$. Now $(T^-)^3$ is a positive element of $\mathfrak{B}(H)$. (Why?) Conclude that $(T^-)^3 = \mathbf{0}$ and therefore $T^- = \mathbf{0}$. (For additional detail see [10], page 37.)

12.4.15. Definition. For an arbitrary element a of a C^*-algebra we define $|a|$ to be $\sqrt{a^*a}$.

12.4.16. Proposition. *If a is a self-adjoint element of a C^*-algebra, then*

$$|a| = a^+ + a^- .$$

12.4.17. Example. The absolute value in a C^*-algebra need not be subadditive; that is, $|a + b|$ need not be less than $|a| + |b|$. For example, in $\mathbf{M}_2(\mathbb{C})$ take $a = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$. Then $|a| = a$, $|b| = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, and $|a + b| = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$. If $|a + b| - |a| - |b|$ were positive, then, according to example 12.4.14, $\langle\, (|a + b| - |a| - |b|)\, x\, ,\, x\,\rangle$ would be positive for every vector $x \in \mathbb{C}^2$. But this is not true for $x = (1, 0)$.

12.4.18. Proposition. *If $\phi\colon A \to B$ is a $*$-homomorphism between C^*-algebras, then $\phi(a) \in B^+$ whenever $a \in A^+$. If ϕ is a $*$-isomorphism, then $\phi(a) \in B^+$ if and only if $a \in A^+$.*

12.4.19. Proposition. *Let a be a self-adjoint element of a C^*-algebra A and f a continuous complex valued function on the spectrum of a. Then $f \geq \mathbf{0}$ in $\mathcal{C}(\sigma(a))$ if and only if $f(a) \geq \mathbf{0}$ in A.*

12.4.20. Proposition. *If a is a self-adjoint element of a C^*-algebra A, then $\|a\|\mathbf{1}_A \pm a \geq \mathbf{0}$.*

12.4.21. Proposition. *If a and b are self-adjoint elements of a C^*-algebra A and $a \leq b$, then $x^*ax \leq x^*bx$ for every $x \in A$.*

12.4.22. Proposition. *If a and b are elements of a C^*-algebra with $0 \leq a \leq b$, then $\|a\| \leq \|b\|$.*

12.4.23. Proposition. *Let A be a unital C^*-algebra and $c \in A^+$. Then c is invertible if and only if $c \geq \epsilon\mathbf{1}$ for some $\epsilon > 0$.*

12.4.24. Proposition. *Let A be a unital C^*-algebra and $c \in A$. If $c \geq \mathbf{1}$, then c is invertible and $\mathbf{0} \leq c^{-1} \leq \mathbf{1}$.*

12.4.25. Proposition. *If a is a positive invertible element in a unital C^*-algebra, then a^{-1} is positive.*

Next we show that the notation $a^{-\frac{1}{2}}$ is unambiguous.

12.4.26. Proposition. *Let $a \in A^+$ where A is a unital C^*-algebra. If a is invertible, so is $a^{\frac{1}{2}}$ and $\big(a^{\frac{1}{2}}\big)^{-1} = \big(a^{-1}\big)^{\frac{1}{2}}$.*

12.4.27. Corollary. *If a is an invertible element in a unital C^*-algebra, then so is $|a|$.*

12.4.28. Proposition. *Let a and b be elements of a C^*-algebra. If $\mathbf{0} \le a \le b$ and a is invertible, then b is invertible and $b^{-1} \le a^{-1}$.*

12.4.29. Proposition. *If a and b are elements of a C^*-algebra and $\mathbf{0} \le a \le b$, then $\sqrt{a} \le \sqrt{b}$.*

12.4.30. Example. Let a and b be elements of a C^*-algebra with $\mathbf{0} \le a \le b$. It is *not* necessarily the case that $a^2 \le b^2$.

Hint for proof. In the C^*-algebra $\mathbf{M}_2$ let $a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $b = a + \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

12.5. Approximate Identities

12.5.1. Definition. An APPROXIMATE IDENTITY (or APPROXIMATE UNIT) in a C^*-algebra A is an increasing net $(e_\lambda)_{\lambda\in\Lambda}$ of positive elements of A such that $\|e_\lambda\| \le 1$ and $ae_\lambda \to a$ (equivalently, $e_\lambda a \to a$) for every $a \in A$. If such a net is in fact a sequence, we have a SEQUENTIAL APPROXIMATE IDENTITY. In the literature be careful of varying definitions: many authors omit the requirements that the net be increasing and/or that it be bounded.

12.5.2. Example. Let $A = \mathcal{C}_0(\mathbb{R})$. For each $n \in \mathbb{N}$ let $U_n = (-n, n)$ and let $e_n \colon \mathbb{R} \to [0, 1]$ be a function in A whose support is contained in U_{n+1} and such that $e_n(x) = 1$ for every $x \in U_n$. Then (e_n) is a (sequential) approximate identity for A.

12.5.3. Proposition. *If A is a C^*-algebra, then the set*

$$\Lambda := \{a \in A^+ \colon \|a\| < 1\}$$

is a directed set (under the ordering it inherits from $\mathfrak{H}(A)$) and is an approximate identity for A.

PROOF. See [2], Proposition II.4.1.3; [3], proposition 2.2.18; [9], Theorem I.4.8; [10], proposition 13.1; [16], Theorem 2.5.2; [35], Theorem 3.1.1; [36], Theorem 1.4.2; and [48], Section VI.3, item 3.3.

12.5.4. Corollary. *Every C^*-algebra A has an approximate identity. If A is separable then it has a sequential approximate identity.*

12.5.5. Proposition. *Every closed ideal in a C^*-algebra is self-adjoint.*

12.5.6. Proposition. *If J is a closed ideal in a C^*-algebra A, then A/J is a C^*-algebra.*

PROOF. See [15], volume 1, chapter VI, proposition 8.7; [25], theorem 1.7.4; or [9], pages 13–14; or [16], theorem 2.5.4.

12.5.7. Example. If J is a closed ideal in a C^*-algebra A, then the sequence

$$\mathbf{0} \longrightarrow J \longrightarrow A \xrightarrow{\ \pi\ } A/J \longrightarrow \mathbf{0}$$

is short exact.

12.5.8. Theorem. *Let A and B be C^*-algebras and J be a closed ideal in A. If ϕ is a $*$-homomorphism from A to B and $\ker\phi \supseteq J$, then there exists a unique $*$-homomorphism $\widetilde{\phi} \colon A/J \to B$ which makes the following diagram commute.*

$$\begin{tikzcd} A \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \\ A/J \arrow[r, "\widetilde{\phi}"'] & B \end{tikzcd}$$

Furthermore, $\widetilde{\phi}$ is injective if and only if $\ker\phi = J$; and $\widetilde{\phi}$ is surjective if and only if ϕ is.

12.5.9. Proposition. *The range of a $*$-homomorphism between C^*-algebras is closed (and therefore itself a C^*-algebra).*

12.5.10. Definition. A C^*-subalgebra B of a C^*-algebra A is HEREDITARY if $a \in B$ whenever $a \in A$, $b \in B$, and $\mathbf{0} \le a \le b$.

12.5.11. Proposition. *Suppose $x^*x \le a$ in a C^*-algebra A. Then there exists $b \in A$ such that $x = ba^{\frac{1}{4}}$ and $\|b\| \le \|a\|^{\frac{1}{4}}$.*

PROOF. See [9], page 13.

12.5.12. Proposition. *Suppose J is a closed ideal in a C^*-algebra A, $j \in J^+$, and $a^*a \le j$. Then $a \in J$. Thus closed ideals in C^*-algebras are hereditary.*

12.5.13. Corollary. *If $\mathbf{0} \longrightarrow A \xrightarrow{\phi} E \longrightarrow B \longrightarrow \mathbf{0}$ is a short exact sequence of C^*-algebras, then $E/\operatorname{ran}\phi$ and B are isometrically $*$-isomorphic.*

12.5.14. Corollary. *Every C^*-algebra A has codimension one in its unitization $\widetilde{A}$; that is,*

$$\dim \widetilde{A}/A \;=\; 1\,.$$

12.5.15. Proposition. *Let A be a C^*-algebra, B be a C^*-subalgebra of A, and J be a closed ideal in A. Then*

$$B/(B \cap J) \cong (B + J)/J\,.$$

Let B be a unital subalgebra of an arbitrary algebra A. It is clear that if an element $b \in B$ is invertible in B, then it is also invertible in A. The converse turns out to be true in C^*-algebras: if b is invertible in A, then its inverse lies in B. This is usually expressed by saying that *every unital C^*-subalgebra of a C^*-algebra is* INVERSE CLOSED.

12.5.16. Proposition. *Let B be a unital C^*-subalgebra of a C^*-algebra A. If $b \in \operatorname{inv}(A)$, then $b^{-1} \in B$.*

12.5.17. Corollary. *Let B be a unital C^*-subalgebra of a C^*-algebra A and $b \in B$. Then*

$$\sigma_B(b) = \sigma_A(b)\,.$$

12.5.18. Corollary. *Let $\phi\colon A \to B$ be a unital C^*-monomorphism of a C^*-algebra A and $a \in A$. Then*

$$\sigma(a) = \sigma(\phi(a))\,.$$

CHAPTER 13

THE GELFAND-NAIMARK-SEGAL CONSTRUCTION

13.1. Positive Linear Functionals

13.1.1. Definition. Let A be an algebra with involution. For each linear functional τ on A and each $a \in A$ define $\tau^\star(a) = \overline{\tau(a^*)}$. We say that τ is HERMITIAN if $\tau^\star = \tau$. Notice that a linear functional $\tau\colon A \to \mathbb{C}$ is Hermitian if and only if it preserves involution; that is, $\tau^\star = \tau$ if and only if $\tau(a^*) = \overline{\tau(a)}$ for all $a \in A$.

CAUTION. The $\tau^\star$ defined above should not be confused with the usual adjoint mapping $\tau^*\colon \mathbb{C}^* \to A^*$. Context (or use of a magnifying glass) should make it clear which is intended.

13.1.2. Proposition. *A linear functional τ on a C^*-algebra A is Hermitian if and only if $\tau(a) \in \mathbb{R}$ whenever a is self-adjoint.*

As is always the case with maps between ordered vector spaces, *positive* maps are the ones that take positive elements to positive elements.

13.1.3. Definition. A linear functional τ on a C^*-algebra A is POSITIVE if $\tau(a) \geq 0$ whenever $a \geq \mathbf{0}$ in A for all $a \in A$.

13.1.4. Proposition. *Every positive linear functional on a C^*-algebra is Hermitian.*

13.1.5. Proposition. *The family of positive linear functionals is a proper convex cone in the real vector space of all Hermitian linear functionals on a C^*-algebra. The cone induces a partial ordering on the vector space: $\tau_1 \leq \tau_2$ whenever $\tau_2 - \tau_1$ is positive.*

13.1.6. Definition. A STATE of a C^*-algebra A is a positive linear functional τ on A such that $\tau(\mathbf{1}) = 1$.

13.1.7. Example. Let x be a vector in a Hilbert space H. Define

$$\omega_x : \mathfrak{B}(H) \to \mathbb{C}\colon T \mapsto \langle Tx, x\rangle\,.$$

Then ω_x is a positive linear functional on $\mathfrak{B}(H)$. If x is a unit vector, then ω_x is a state of $\mathfrak{B}(H)$. A state τ is a VECTOR STATE if $\tau = \omega_x$ for some unit vector x.

13.1.8. Proposition (Schwarz inequality)**.** *If τ is a positive linear functional on a C^*-algebra A, then*

$$|\tau(b^*a)|^2 \leq \tau(a^*a)\tau(b^*b)$$

for all a, $b \in A$.

13.1.9. Proposition. *A linear functional τ on a C^*-algebra A is positive if and only if it is bounded and $\|\tau\| = \tau(\mathbf{1}_A)$.*

PROOF. See [29], pages 256–257.

13.2. Representations

13.2.1. Definition. Let A be a C^*-algebra. A REPRESENTATION of A is a pair (π, H) where H is a Hilbert space and $\pi\colon A \to \mathfrak{B}(H)$ is a $*$-homomorphism. Usually one says simply that π is a representation of A. When we wish to emphasize the role of the particular Hilbert space we say that π *is a representation of A on H*. Depending on context we may write either π_a or $\pi(a)$ for the Hilbert space operator which is the image of the algebra element a under π. A representation π of A on H is NONDEGENERATE if $\pi(A)H$ is dense in H.

13.2.2. Convention. We add to the preceding definition the following requirement: if the C^*-algebra A is unital, then a representation of A must be a *unital* $*$-homomorphism.

13.2.3. Definition. A representation π of a C^*-algebra A on a Hilbert space H is FAITHFUL if it is injective. If there exists a vector $x \in H$ such that $\pi^{\rightarrow}(A)x = \{\pi_a(x) \colon a \in A\}$ is dense in H, then we say that the representation π is CYCLIC and that x is a CYCLIC VECTOR for π.

13.2.4. Example. Let $(S, \mathfrak{A}, \mu)$ be a σ-finite measure space and $L_\infty = L_\infty(S, \mathfrak{A}, \mu)$ be the C^*-algebra of essentially bounded μ-measurable functions on S. As we saw in example 5.2.15 for each $\phi \in L_\infty$ the corresponding multiplication operator M_ϕ is an operator on the Hilbert space $L_2 = L_2(S, \mathfrak{A}, \mu)$. The mapping $M \colon L_\infty \to \mathfrak{B}(L_2) \colon \phi \mapsto M_\phi$ is a faithful representation of the C^*-algebra L_∞ on the Hilbert space L_2.

13.2.5. Example. Let $\mathcal{C}([0,1])$ be the C^*-algebra of continuous functions on the interval $[0,1]$. For each $\phi \in \mathcal{C}([0,1])$ the corresponding multiplication operator M_ϕ is an operator on the Hilbert space $L_2 = L_2([0,1])$ of functions on $[0,1]$ which are square-integrable with respect to Lebesgue measure. The mapping $M \colon \mathcal{C}([0,1]) \to \mathfrak{B}(L_2) \colon \phi \mapsto M_\phi$ is a faithful representation of the C^*-algebra $\mathcal{C}([0,1])$ on the Hilbert space L_2.

13.2.6. Example. Suppose that π is a representation of a unital C^*-algebra A on a Hilbert space H and x is a unit vector in H. If ω_x is the corresponding vector state of $\mathfrak{B}(H)$, then $\omega_x \circ \pi$ is a state of A.

13.2.7. Exercise. Let X be a locally compact Hausdorff space. Find an isometric (therefore faithful) representation (π, H) of the C^*-algebra $C_0(X)$ on some Hilbert space H..

13.2.8. Definition. Let ρ be a state of a C^*-algebra A. Then

$$L_\rho := \{a \in A \colon \rho(a^*a) = 0\}$$

is called the LEFT KERNEL of ρ.

Recall that as part of the proof of *Schwarz inequality* 13.1.8 for positive linear functionals we verified the following result.

13.2.9. Proposition. *If ρ is a state of a C^*-algebra A, then $\langle a, b\rangle_0 := \rho(b^*a)$ defines a semi-inner product on A.*

13.2.10. Corollary. *If ρ is a state of a C^*-algebra A, then its left kernel L_ρ is a vector subspace of A and $\langle\, [a], [b]\,\rangle := \langle a, b\rangle_0$ defines an inner product on the quotient vector space A/L_ρ.*

13.2.11. Proposition. *Let ρ be a state of a C^*-algebra A and $a \in L_\rho$. Then $\rho(b^*a) = 0$ for every $b \in A$.*

13.2.12. Proposition. *If ρ is a state of a C^*-algebra A, then its left kernel L_ρ is a closed left ideal in A.*

13.3. The GNS-Construction and the Third Gelfand-Naimark Theorem

The following theorem is known as the *Gelfand-Naimark-Segal construction* (the *GNS-construction*).

13.3.1. Theorem (GNS-construction)**.** *Let ρ be a state of a C^*-algebra A. Then there exists a cyclic representation π_ρ of A on a Hilbert space H_ρ and a unit cyclic vector x_ρ for π_ρ such that $\rho = \omega_{x_\rho} \circ \pi_\rho$.*

13.3.2. Notation. In the following material π_ρ, H_ρ, and x_ρ are the cyclic representation, the Hilbert space, and the unit cyclic vector guaranteed by the GNS-construction starting with a given state ρ of a C^*-algebra

13.3.3. Proposition. *Let ρ be a state of a C^*-algebra A and π be a cyclic representation of A on a Hilbert space H such that $\rho = \omega_x \circ \pi$ for some unit cyclic vector x for π. Then there exists a unitary map U from H_ρ to H such that $x = Ux_\rho$ and $\pi(a) = U\pi_\rho(a)U^*$ for all $a \in A$.*

13.3.4. Definition. Let $\{H_\lambda \colon \lambda \in \Lambda\}$ be a family of Hilbert spaces. Denote by $\bigoplus\limits_{\lambda\in\Lambda} H_\lambda$ the set of all functions $x \colon \Lambda \to \bigcup\limits_{\lambda\in\Lambda} H_\lambda \colon \lambda \mapsto x_\lambda$ such that $x_\lambda \in H_\lambda$ for each $\lambda \in \Lambda$ and $\sum\limits_{\lambda\in\Lambda} \|x_\lambda\|^2 < \infty$. On $\bigoplus\limits_{\lambda\in\Lambda} H_\lambda$ define addition and scalar multiplication pointwise; that is. $(x+y)_\lambda = x_\lambda + y_\lambda$ and $(\alpha x)_\lambda = \alpha x_\lambda$ for all $\lambda \in \Lambda$, and define an inner product by $\langle x, y\rangle = \sum_{\lambda\in\Lambda}\langle x_\lambda, y_\lambda\rangle$. These operations (are well defined and) make $\bigoplus\limits_{\lambda\in\Lambda} H_\lambda$ into a Hilbert space. It is the DIRECT SUM of the Hilbert spaces H_λ. Various notations for elements of this direct sum occur in the literature: x, (x_λ), $(x_\lambda)_{\lambda\in\Lambda}$, and $\oplus_\lambda x_\lambda$ are common.

Now suppose that $\{T_\lambda \colon \lambda \in \Lambda\}$ is a family of Hilbert space operators where $T_\lambda \in \mathfrak{B}(H_\lambda)$ for each $\lambda \in \Lambda$. Suppose further that $\sup\{\|T_\lambda\| \colon \lambda \in \Lambda\} < \infty$. Then $T(x_\lambda)_{\lambda\in\Lambda} = (T_\lambda x_\lambda)_{\lambda\in\Lambda}$ defines an operator on the Hilbert space $\bigoplus_\lambda H_\lambda$. The operator T is usually denoted by $\bigoplus_\lambda T_\lambda$ and is called the DIRECT SUM of the operators T_λ.

13.3.5. Proposition. *The claims made in the preceding definition that $\bigoplus\limits_{\lambda\in\Lambda} H_\lambda$ is a Hilbert space and $\bigoplus_\lambda T_\lambda$ is an operator on $\bigoplus\limits_{\lambda\in\Lambda} H_\lambda$ are correct.*

13.3.6. Example. Let A be a C^*-algebra and $\{\pi_\lambda \colon \lambda \in \Lambda\}$ be a family of representations of A on Hilbert spaces H_λ so that $\pi_\lambda(a) \in \mathfrak{B}(H_\lambda)$ for each $\lambda \in \Lambda$ and each $a \in A$. Then

$$\pi = \bigoplus\nolimits_\lambda \pi_\lambda \colon A \to \mathfrak{B}\big(\bigoplus\nolimits_\lambda H_\lambda\big) \colon a \mapsto \bigoplus\nolimits_\lambda \pi_\lambda(a)$$

is a representation of A on the Hilbert space $\bigoplus_\lambda H_\lambda$. It is the DIRECT SUM of the representations π_λ.

13.3.7. Theorem. *Every C^*-algebra has a faithful representation.*

PROOF. See [**2**], Corollary II.6.4.10; [**7**], page 253; [**10**], Theorem 19.1; [**16**], Theorem 5.4.1; [**29**], page 281; or [**35**], Theorem 3.4.1.

An obvious restatement of the preceding theorem is a third version of the *Gelfand-Naimark theorem*, which says that every C^*-algebra is (essentially) an algebra of Hilbert space operators.

13.3.8. Corollary (Gelfand-Naimark Theorem III)**.** *Every C^*-algebra is isometrically $*$-isomorphic to a C^*-subalgebra of $\mathfrak{B}(H)$ for some Hilbert space H.*

CHAPTER 14

MULTIPLIER ALGEBRAS

14.1. Hilbert Modules

14.1.1. Notation. The inner products that occur previously in these notes and that one encounters in standard textbooks and monographs on Hilbert spaces, functional analysis, and so on, are linear in the first variable and conjugate linear in the second. Most contemporary operator algebraists have chosen to work with objects called *right* Hilbert A-modules (A being a C^*-algebra). For such modules it turns out to be more convenient to have "inner products" that are linear in the second variable and conjugate linear in the first. While this switch in conventions may provoke some slight irritation, it is, mathematically speaking, of little consequence. Of course, we would like Hilbert spaces to be examples of Hilbert $\mathbb{C}$-modules. To make this possible we equip a Hilbert space, whose inner product is denoted by $\langle\ ,\ \rangle$ with a new "inner product" defined by $\langle x \,|\, y\rangle := \langle y, x\rangle$. This "inner product" is linear in the second variable and conjugate linear in the first. I will try to be consistent in using $\langle\ ,\ \rangle$ for the standard inner product and $\langle\ \,|\ \rangle$ for the one which is linear in its second variable.

Another (very common) solution to this problem is to insist that an inner product is always linear in its second variable and "correct" standard texts, monographs, and papers accordingly.

14.1.2. Convention. In light of the preceding remarks we will in the sequel use the word "sesquilinear" to mean either *linear in the first variable and conjugate linear in the second* or *linear in the second variable and conjugate linear in the first*.

14.1.3. Definition. Let A be a nonzero C^*-algebra. A vector space V is an A-MODULE if there is a bilinear map

$$B\colon V \times A \to V\colon (x, a) \mapsto xa$$

such that $x(ab) = (xa)b$ holds for all $x \in V$ and $a, b \in A$. We also require that $x\mathbf{1}_A = x$ for every $x \in V$ if A is unital. (A function of two variables is BILINEAR if it is linear in both of its variables.)

14.1.4. Definition. Let $(A, +, M, \cdot\,)$ be a (complex) algebra. Then A^{op} is the algebra $(A, +, M, *)$ where $a * b = b \cdot a$ for all $a, b \in A$. This is the OPPOSITE ALGEBRA of A. It is just A with the order of multiplication reversed. If A and B are algebras, then any function $f\colon A \to B$ induces in an obvious fashion a function from A into B^{op} (or from A^{op} into B, or from A^{op} into B^{op}). We will denote all these functions simply by f.

14.1.5. Definition. Let A and B be algebras. A function $\phi\colon A \to B$ is an ANTIHOMOMORPHISM if the function $f\colon A \to B^{\mathrm{op}}$ is a homomorphism. A bijective antihomomorphism is an ANTI-ISOMORPHISM. An anti-isomorphism should not be regarded something terribly different from an isomorphism, but actually as something nearly as good.

The notion of an A-module (where A is an algebra) was defined in 14.1.3. You may prefer the following alternative definition, which is more in line with the definition of *vector space* given in 1.1.2.

14.1.6. Definition. Let A be an algebra. An A-MODULE is an ordered quadruple $(V, +, M, \Phi)$ where $(V, +, M)$ is a vector space and $\Phi\colon A \to \mathfrak{L}(V)$ is an algebra homomorphism. If A is unital we require also that Φ be unital.

14.1.7. Exercise. Check that the definitions of *A-module* given in 14.1.3 and 14.1.6 are equivalent.

We now say precisely what it means,when A is a C^*-algebra, to give an A-module an A-valued inner product.

14.1.8. Definition. Let A be a C^*-algebra. A SEMI-INNER PRODUCT A-MODULE is an A-module V together with a mapping

$$\beta \colon V \times V \to A \colon (x, y) \mapsto \langle x|\, y\rangle$$

which is linear in its second variable and satisfies

(i) $\langle x|\, ya\rangle = \langle x|\, y\rangle a$,
(ii) $\langle x|\, y\rangle = \langle y|\, x\rangle^*$, and
(iii) $\langle x|\, x\rangle \geq \mathbf{0}$

for all x, $y \in V$ and $a \in A$. It is an INNER PRODUCT A-MODULE (or a PRE-HILBERT A-MODULE) if additionally

(iv) $\langle x|\, x\rangle = \mathbf{0}$ implies that $x = \mathbf{0}$

when $x \in V$. We will refer to the mapping β as an A-VALUED (SEMI-)INNER PRODUCT on V.

14.1.9. Example. Every inner product space is an inner product $\mathbb{C}$-module.

14.1.10. Proposition. *Let A be a C^*-algebra and V be a semi-inner product A-module. The semi-inner product $\langle\ |\ \rangle$ is conjugate linear in its first variable both literally and in the sense that $\langle va|\, w\rangle = a^*\langle v|\, w\rangle$ for all v, $w \in V$ and $a \in A$.*

14.1.11. Proposition (Schwarz inequality—for inner product A-modules)**.** *Let V be an inner product A-module where A is a C^*-algebra. Then*

$$\langle x|\, y\rangle^*\langle x|\, y\rangle \leq \|\langle x|\, x\rangle\|\, \langle y|\, y\rangle$$

for all x, $y \in V$.

Hint for proof. Show that no generality is lost in assuming that $\|\langle x|\, x\rangle\| = 1$. Consider the positive element $\langle xa - y|\, xa - y\rangle$ where $a = \langle x|\, y\rangle$. Use propositions 12.4.20 and 12.4.21.

14.1.12. Definition. For every element v of an inner product A-module (where A is a C^*-algebra) define

$$\|v\| := \|\langle v|\, v\rangle\|^{1/2}.$$

14.1.13. Proposition (Yet another Schwarz inequality)**.** *Let A be a C^*-algebra and V be an inner product A-module. Then for all v, $w \in V$*

$$\|\langle v|\, w\rangle\| \leq \|v\|\, \|w\|.$$

14.1.14. Corollary. *If v and w are elements of an inner product A-module (where A is a C^*-algebra), then $\|v + w\| \leq \|v\| + \|w\|$ and the map $x \mapsto \|x\|$ is a norm on V.*

14.1.15. Proposition. *If A is a C^*-algebra and V is an inner product A-module, then*

$$\|va\| \leq \|v\|\, \|a\|$$

for all $v \in V$ and $a \in A$.

14.1.16. Definition. Let A be a C^*-algebra and V be an inner product A-module. If V is complete with respect to (the metric induced by) the norm defined in 14.1.12, then V is a HILBERT A-MODULE.

14.1.17. Example. For a and b in a C^*-algebra A define

$$\langle a|\, b\rangle := a^*b\,.$$

Then A is itself a Hilbert A-module. Any closed right ideal in A is also a Hilbert A-module.

14.1.18. Definition. Let V and W be Hilbert A-modules where A is a C^*-algebra. A mapping $T \colon V \to W$ is A-LINEAR if it is linear and if $T(va) = T(v)a$ holds for all $v \in V$ and $a \in A$. The mapping T is a HILBERT A-MODULE MORPHISM if it is bounded and A-linear.

Recall from definition 5.2.10 that every Hilbert space operator has an adjoint. This is not true for Hilbert A-modules

14.1.19. Definition. Let V and W be Hilbert A-modules where A is a C^*-algebra. A function $T\colon V \to W$ is ADJOINTABLE if there exists a function $T^*\colon W \to V$ satisfying

$$\langle Tv|\, w\rangle = \langle v|\, T^*w\rangle$$

for all $v \in V$ and $w \in W$. The function T^*, if it exists, is the ADJOINT of T. Denote by $\mathfrak{L}(V,W)$ the family of adjointable maps from V to W. We shorten $\mathfrak{L}(V,V)$ to $\mathfrak{L}(V)$.

14.1.20. Proposition. *Let V and W be Hilbert A-modules where A is a C^*-algebra. If a function $T\colon V \to W$ is adjointable, then it is a Hilbert A-module morphism. Furthermore if T is adjointable, then so is its adjoint and $T^{**} = T$.*

14.1.21. Example. Let X be the unit interval $[0,1]$ and let $Y = \{0\}$. With its usual topology X is a compact Hausdorff space and Y is a subspace of X. Let A be the C^*=algebra $\mathcal{C}(X)$ and J_0 be the ideal $\{f \in A\colon f(0) = 0\}$ (see proposition 11.1.6). Regard $V = A$ and $W = J_0$ as Hilbert A-modules (see example 14.1.17). Then the inclusion map $\iota\colon V \to W$ is a Hilbert A-module morphism which is not adjointable.

14.1.22. Proposition. *Let A be a C^*-algebra. The pair of maps $V \mapsto V$, $T \mapsto T^*$ is a contravariant functor from the category of Hilbert A-modules and adjointable maps into itself.*

14.1.23. Proposition. *Let A be a C^*-algebra and V be a Hilbert A-module. Then $\mathfrak{L}(V)$ is a unital C^*-algebra.*

14.1.24. Notation. Let V and W be Hilbert A-modules where A is a C^*-algebra. For $v \in V$ and $w \in W$ let

$$\Theta_{v,w}\colon W \to V \colon x \mapsto v\langle w|\, x\rangle\,.$$

(Compare this with example 5.7.2.)

14.1.25. Proposition. *The map Θ defined above is sesquilinear.*

14.1.26. Proposition. *Let V and W be Hilbert A-modules where A is a C^*-algebra. For every $v \in V$ and $w \in W$ the map $\Theta_{v,w}$ is adjointable and $(\Theta_{v,w})^* = \Theta_{w,v}$.*

The next proposition generalizes propositions 5.7.7 and 5.7.8.

14.1.27. Proposition. *Let U, V, W, and Z be Hilbert A-modules where A is a C^*-algebra. If $S \in \mathfrak{L}(Z,W)$ and $T \in \mathfrak{L}(V,U)$, then*

$$T\Theta_{v,w} = \Theta_{Tv,w} \qquad \textit{and} \qquad \Theta_{v,w}S = \Theta_{v,S^*w}$$

for all $v \in V$ and $w \in W$.

$$Z \xrightarrow{\ S\ } W \xrightarrow{\ \Theta_{v,w}\ } V \xrightarrow{\ T\ } U$$

14.1.28. Proposition. *Let U, V, and W be Hilbert A-modules where A is a C^*-algebra. Suppose $u \in U$; v, $v' \in V$; and $w \in W$. Then*

$$\Theta_{u,v}\Theta_{v',w} = \Theta_{u\langle v|\, v'\rangle,w} = \Theta_{u,w\langle v'|\, v\rangle}\,.$$

14.1.29. Notation. Let A be a C^*-algebra and V and W be Hilbert A-modules. We denote by $\mathfrak{K}(W,V)$ the closed linear span of $\{\Theta_{v,w}\colon v \in V \text{ and } w \in W\}$. As usual we shorten $\mathfrak{K}(V,V)$ to $\mathfrak{K}(V)$.

14.1.30. Proposition. *If A is a C^*-algebra and V is a Hilbert A-module, then $\mathfrak{K}(V)$ is an ideal in the C^*-algebra $\mathfrak{L}(V)$.*

The next example is intended as justification for the standard practice of identifying a C^*-algebra A with $\mathfrak{K}(A)$.

14.1.31. Example. If we regard a C^*-algebra A as an A-module (see example 14.1.17), then $\mathfrak{K}(A) \overset{*}{\cong} A$.

Hint for proof. As in corollary 12.3.3 define for each $a \in A$ the left multiplication operator $L_a \colon A \to A \colon x \mapsto ax$. Show that each such operator is adjointable and that the map $L \colon A \to \mathfrak{L}(A) \colon a \mapsto L_a$ is a $*$-isomorphism onto a C^*-subalgebra of $\mathfrak{L}(A)$. Then verify that $\mathfrak{K}(A)$ is the closure of the image under L of the span of products of elements of A.

14.1.32. Example. Let H be a Hilbert space regarded as a $\mathbb{C}$-module. Then $\mathfrak{K}(H)$ (as defined in 14.1.29) is the ideal of compact operators on H (see proposition 7.1.27).

Proof. See [**38**], example 2.27.

The preceding example has lead many researchers, when dealing with an arbitrary Hilbert module V, to refer to members of $\mathfrak{K}(V)$ as *compact operators*. This is dubious terminology since such operators certainly need not be compact. (For example, if we regard an infinite dimensional unital C^*-algebra A as an A-module, then $\Theta_{\mathbf{1},\mathbf{1}} = I_A$, but the identity operator on A is not compact.)

The fact that in these notes rather limited use is made of Hilbert C^*-modules should not lead you to think that their study is specialized and/or of marginal interest. To the contrary, it is currently an important and vigorous research area having applications to fields as diverse as K-theory, graph C^*-algebras, quantum groups, quantum probability, vector bundles, non-commutative geometry, algebraic and geometric topology, operator spaces and algebras, and wavelets. Take a look at Michael Frank's webpage [**18**], *Hilbert C^*-modules and related subjects—a guided reference overview*, where he lists 1531 references (as of his 11.09.10 update) to books, papers, and theses dealing with such modules and categorizes them by application. There is an interesting graphic (on page 9) illustrating the growth of this field of mathematics. It covers material from the pioneering efforts in the 50's and early 60's (0–2 papers per year) to the time of this writing (about 100 papers per year).

14.2. Essential Ideals

14.2.1. Example. If A and B are C^*-algebras, then A (more precisely, $A \oplus \{\mathbf{0}\}$) is an ideal in $A \oplus B$.

14.2.2. Convention. As the preceding example suggests, it is conventional to regard A as a subset of $A \oplus B$. In the sequel we will do this without further mention.

14.2.3. Notation. For an element c of an algebra A let

$$I_c := \left\{ a_0c + cb_0 + \sum_{k=1}^{p} a_k c b_k \colon p \in \mathbb{N},\, a_0, \dots, a_p,\, b_0, \dots, b_p \in A \right\}$$

14.2.4. Proposition. *If c is an element of an algebra A, then I_c is an (algebraic) ideal in A.*

Notice that in the preceding proposition no claim is made that the algebraic ideal I_c must be proper. It may well be the case that $I_c = A$ (as, for example, when c is an invertible element of a unital algebra).

14.2.5. Definition. Let c be an element of a C^*-algebra A. Define J_c, the PRINCIPAL IDEAL containing c, to be the intersection of the family of all (closed) ideals of A which contain c. Clearly, J_c is the smallest ideal containing c.

14.2.6. Proposition. *In a C^*-algebra the closure of an algebraic ideal is an ideal.*

14.2.7. Example. The closure of a proper algebraic ideal in a C^*-algebra need not be a proper ideal. For example, l_c, the set of sequences of complex numbers which are eventually zero, is dense in the C^*-algebra $l_0 = \mathcal{C}_0(\mathbb{N})$. (But recall proposition 11.1.1.)

14.2.8. Proposition. *If c is an element of a C^*-algebra, then $J_c = \overline{I_c}$.*

14.2.9. Notation. We adopt a standard notational convention. If A and B are nonempty subsets of an algebra. By AB we mean the linear span of products of elements in A and elements in B; that is, $AB = \operatorname{span}\{ab \colon a \in A \text{ and } b \in B\}$. (Note that in definition 5.7.9 it makes no difference whether we take AJ to mean the set of products of elements in A with elements in J or the span of that set.)

14.2.10. Proposition. *If I and J are ideals in a C^*-algebra, then $\overline{IJ} = I \cap J$.*

A nonunital C^*-algebra A can be embedded as an ideal in a unital C^*-algebra in different ways. The smallest unital C^*-algebra containing A is its unitization $\widetilde{A}$ (see proposition 12.3.12). Of course there is no largest unital C^*-algebra in which A can be embedded as an ideal because if A is embedded as an ideal in a unital C^*-algebra B and C is *any* unital C^*-algebra, then A is an ideal in the still larger unital C^*-algebra $B \oplus C$. The reason this larger unitization is not of much interest is that the intersection of the ideal C with A is $\{\mathbf{0}\}$. This motivates the following definition.

14.2.11. Definition. An ideal J in a C^*-algebra A is ESSENTIAL if and only if $I \cap J \neq \mathbf{0}$ for every nonzero ideal I in A.

14.2.12. Example. A C^*-algebra A is an essential ideal in its unitization $\widetilde{A}$ if and only if A is *not* unital.

14.2.13. Example. If H is a Hilbert space the ideal of compact operators $\mathfrak{K}(H)$ is an essential ideal in the C^*-algebra $\mathfrak{B}(H)$.

14.2.14. Definition. Let J be an ideal in a C^*-algebra A. Then we define $J^{\perp}$, the ANNIHILATOR of J, to be $\big\{a \in A \colon Ja = \{\mathbf{0}\}\big\}$.

14.2.15. Proposition. *If J is an ideal in a C^*-algebra, then so is $J^{\perp}$.*

14.2.16. Proposition. *An ideal J in a C^*-algebra A is essential if and only if $J^{\perp} = \{\mathbf{0}\}$.*

14.2.17. Proposition. *If J is an ideal in a C^*-algebra, then $\big(J \oplus J^{\perp}\big)^{\perp} = \{\mathbf{0}\}$.*

14.2.18. Notation. Let f be a (real or) complex valued function on a set S. Then

$$Z_f := \{s \in S \colon f(s) = 0\}.$$

This is the ZERO SET of f.

Suppose that A is a nonunital commutative C^*-algebra. By the second *Gelfand-Naimark theorem* 12.3.22 there exists a noncompact locally compact Hausdorff space X such that $A = \mathcal{C}_0(X)$. (Here, of course, we are permitting ourselves a conventional abuse of language: for literal correctness the indicated equality should be an isometric $*$-isomorphism.) Now let Y be a compact Hausdorff space in which X is an open subset and let $B = \mathcal{C}(Y)$. Then B is a unital commutative C^*-algebra. Regard A as embedded as an ideal in B by means of the map $\iota \colon A \to B \colon f \mapsto \widetilde{f}$ where

$$\widetilde{f}(y) = \begin{cases} f(y), & \text{if } y \in X; \\ 0, & \text{otherwise.} \end{cases}$$

Notice that the closed set X^c is $\bigcap\{Z_{\widetilde{f}} \colon f \in \mathcal{C}_0(X)\}$.

14.2.19. Proposition. *Let the notation be as in the preceding paragraph. Then the ideal A is essential in B if and only if the open subset X is dense in Y.*

Thus the property of an ideal being essential in the context of unitizations of nonunital commutative C^*-algebras corresponds exactly with the property of an open subspace being dense in the context of compactifications of noncompact locally compact Hausdorff spaces.

14.3. Compactifications and Unitizations

In definition 12.3.14 we called the object whose existence was proved in the preceding proposition 12.3.12 "the" unitization of a C^*-algebra. The definite article there is definitely misleading. Just as a topological space may have many different compactifications, a C^*-algebra may have many unitizations. If A is a nonunital commutative C^*-algebra, it is clear from corollary 12.5.14 that the unitization $\widetilde{A}$ is the smallest possible unitization of A. Similarly, in topology, if X is a noncompact locally compact Hausdorff space, then its one-point compactification is obviously the smallest possible compactification of X. Recall that in proposition 12.3.27 we established the fact that constructing the smallest unitization of A is "essentially" the same thing as constructing the smallest compactification of X.

It is sometimes convenient (as, for example, in the preceding paragraph) to take a "unitization" of an algebra that is already unital and sometimes convenient to take a "compactification" of a space that is already compact. Since there appears to be no universally accepted terminology, I introduce the following (definitely nonstandard, but I hope helpful) language.

14.3.1. Definition. Let A and B be C^*-algebras and X and Y be Hausdorff topological spaces. We will say that

(1) B is a UNITIZATION of A if B is unital and A is ($*$-isomorphic to) a C^*-subalgebra of B;
(2) B is an ESSENTIAL UNITIZATION of A if B is unital and A is ($*$-isomorphic to) an essential ideal of B;
(3) Y is a COMPACTIFICATION of X if it is compact and X is (homeomorphic to) a subspace of Y; and
(4) Y is an ESSENTIAL COMPACTIFICATION of X if it is compact and X is (homeomorphic to) a dense subspace of Y

Perhaps a few words are in order concerning the bits in parentheses in the preceding definition. It is seldom the case that a topological space X is literally a *subset* of a particular compactification of X or that a C^*-algebra A is a *subset* of a particular unitization of A. While certainly true that it is frequently convenient to regard one C^*-algebra as a subset of another when in fact the first is merely $*$-isomorphic to a subset of the second, there are also occasions when it clarifies matters to specify the actual embeddings involved. If the details of these distinctions are not entirely familiar, the next two definitions are intended to help.

14.3.2. Definition. Let A and B be C^*-algebras. We say that A is EMBEDDED in B if there exists an injective $*$-homomorphism $\iota\colon A \to B$; that is, if A is $*$-isomorphic to a C^*-subalgebra of B (see propositions 12.5.9 and 12.3.25). The injective $*$-homomorphism ι is an EMBEDDING of A into B. In this situation it is common practice to treat A and the range of ι as identical C^*-algebras. The pair (B, ι) is a UNITIZATION of A if B is a unital C^*-algebra and $\iota\colon A \to B$ is an embedding. The unitization (B, ι) is ESSENTIAL if the range of ι is an essential ideal in B.

14.3.3. Definition. Let X and Y be Hausdorff topological spaces. We say that X is EMBEDDED in Y if there exists a homeomorphism j from X to a subspace of Y. The homeomorphism j is a EMBEDDING of X into Y. As in C^*-algebras it is common practice to identify the range of j with the space X. The pair (Y, j) is a COMPACTIFICATION of X if Y is a compact Hausdorff space and $j\colon X \to Y$ is an embedding. The compactification (Y, j) is ESSENTIAL if the range of j is dense in Y.

We have discussed the smallest unitization of a C^*-algebra and the smallest compactification of a locally compact Hausdorff space. Now what about a largest, or even maximal, unital algebra containing A? Clearly there is no such thing, for if B is a unital algebra containing A, then so is $B \oplus C$ where C is any unital C^*-algebra. Similarly, there is no largest compact space containing X: if Y is a compact space containing X, then so is the topological disjoint union $Y \uplus K$ where K is any nonempty compact space. However, it does make sense to ask whether there is a maximal essential unitization of a C^*-algebra or a maximal essential compactification of a locally compact Hausdorff space. The answer is *yes* in both cases. The well-known *Stone-Čech compactification* $\beta(X)$ is maximal among essential compactifications of a noncompact locally compact Hausdorff space X. Details can be found in any good topology text. One readable standard treatment is [47], items 19.3–19.12. More sophisticated approaches make use of some functional analysis—see, for example, [7], chapter V, section 6. There turns out also to be a maximal essential unitization of a nonunital C^*-algebra A—it is called the *multiplier algebra* of A.

We say that an essential unitization M of a C^*-algebra A is *maximal* if any C^*-algebra that contains A as an essential ideal embeds in M. Here is a more formal statement.

14.3.4. Definition. An essential unitization (M, j) of a C^*-algebra A is said to be MAXIMAL if for every embedding $\iota\colon A \to B$ whose range is an essential ideal in B there exists a $*$-homomorphism $\phi\colon B \to M$ such that $\phi \circ \iota = j$.

14.3.5. Proposition. *In the preceding definition the $*$-homomorphism ϕ, if it exists must be injective.*

14.3.6. Proposition. *In the preceding definition the $*$-homomorphism ϕ, if it exists must be unique.*

Compare the following definition with 13.2.1.

14.3.7. Definition. Let A and B be C^*-algebras and V be a Hilbert A-module. A $*$-homomorphism $\phi\colon B \to \mathfrak{L}(V)$ is NONDEGENERATE if $\phi^{\rightarrow}(B)\,V$ is dense in V.

14.3.8. Proposition. *Let A, B, and J be C^*-algebras, V be a Hilbert B-module, and $\iota\colon J \to A$ be an injective $*$-homomorphism whose range is an ideal in A. If $\phi\colon J \to \mathfrak{L}(V)$ is a nondegenerate $*$-homomorphism, then there exists a unique extension of ϕ to a $*$-homomorphism $\overline{\phi}\colon A \to \mathfrak{L}(V)$ which satisfies $\overline{\phi} \circ \iota = \phi$.*

14.3.9. Proposition. *If A is a nonzero C^*-algebra, then $(\mathfrak{L}(A), L)$ is a maximal essential unitization of A. It is unique in the sense that if (M, j) is another maximal essential unitization of A, then there exists a $*$-isomorphism $\phi\colon M \to \mathfrak{L}(A)$ such that $\phi \circ j = L$.*

14.3.10. Definition. Let A be a C^*-algebra. We define the MULTIPLIER ALGEBRA of A, to be the family $\mathfrak{L}(A)$ of adjointable operators on A. From now on we denote this family by $M(A)$.

CHAPTER 15

THE K_0-FUNCTOR

We now take a brief look at the so-called K-theory of C^*-algebras. Perhaps the best introduction to the subject is [40], some of the more elementary parts of which these notes follow rather closely. Another very readable introduction is [45].

Given a C^*-algebra A we will be interested primarily in two groups known as $K_0(A)$ and $K_1(A)$. The present chapter examines the first of these, a group of projections in A under addition. Of course if we wish to add arbitrary projections, we have already encountered a serious difficulty. We showed earlier (in proposition 5.5.6) that in order to add projections they must commute! The solution to this dilemma is typical of the thinking that goes in K-theory in general. If two projections don't commute, remove the obstruction that prevents them from doing so. If there isn't enough space for them to get past each other, give them more room. Don't insist on regarding them as creatures trying to live in a hopelessly narrow world of 1×1 matrices of elements of A. Allow them, for example, to be roam about the much roomier world of 2×2 matrices. To accomplish this technically we search for an equivalence relation $\sim$ that identifies a projection p in A with the matrices $\begin{bmatrix} p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & p \end{bmatrix}$. Then the problem is solved: if p and q are projections in A, then

$$pq \sim \begin{bmatrix} p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & q \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & q \end{bmatrix} \begin{bmatrix} p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \sim qp\,.$$

Now p and q commute modulo the equivalence relation $\sim$ and we may define addition by something like $p \oplus q = \begin{bmatrix} p & \mathbf{0} \\ \mathbf{0} & q \end{bmatrix}$.

Pretty clearly, the preceding simple-minded device is not going to produce anything like an Abelian group, but it's a start. If you suspect that, in the end, the construction of $K_0(A)$ will be a bit complicated, you are entirely correct.

15.1. Partial Isometries

15.1.1. Definition. An element v of a C^*-algebra A is a PARTIAL ISOMETRY if v^*v is a projection in A. (Since v^*v is always self-adjoint, it is enough to require that v^*v be idempotent.) The element v^*v is the INITIAL (or SUPPORT) PROJECTION of v and vv^* is the FINAL (or RANGE) PROJECTION of v. (It is an obvious consequence of the next proposition that if v is a partial isometry, then vv^* is in fact a projection.)

15.1.2. Proposition. *If v is a partial isometry in a C^*-algebra, then*

(a) $vv^*v = v$ *; and*
(b) v^* *is a partial isometry.*

Hint for proof. Let $z = v - vv^*v$ and consider z^*z.

15.1.3. Proposition. *Let v be a partial isometry in a C^*-algebra A. Then its initial projection p is the smallest projection (with respect to the partial ordering $\preceq$ on $\mathcal{P}(A)$) such that $vp = v$ and its final projection q is the smallest projection such that $qv = v$.*

15.1.4. Proposition. *If V is a partial isometry on a Hilbert space H (that is, if V is a partial isometry in the C^*-algebra $\mathfrak{B}(H)$), then the initial projection V^*V is the projection of H onto $(\ker V)^\perp$ and the final projection VV^* is the projection of H onto $\operatorname{ran} V$.*

Because of the preceding result $(\ker V)^\perp$ is called the INITIAL (or SUPPORT) SPACE of V and $\operatorname{ran} V$ is sometimes called the FINAL SPACE of V.)

15.1.5. Proposition. *An operator on a Hilbert space is a partial isometry if and only if it is an isometry on the orthogonal complement of its kernel.*

15.2. Equivalence Relations on Projections

We define several equivalence relations, all of which are appropriate to projections in a unital C^*-algebra.

15.2.1. Definition. In a unital algebra elements a and b are SIMILAR if there exists an invertible element s such that $b = sas^{-1}$. In this case we write $a \sim_s b$.

15.2.2. Proposition. *The relation $\sim_s$ of similarity defined above is an equivalence relation on the elements of a unital algebra.*

15.2.3. Definition. In a C^*-algebra elements a and b are UNITARILY EQUIVALENT if there exists an unitary element $u \in \widetilde{A}$ such that $b = uau^*$. In this case we write $a \sim_u b$.

15.2.4. Proposition. *The relation $\sim_u$ of unitary equivalence defined above is an equivalence relation on the elements of a C^*-algebra.*

15.2.5. Proposition. *Elements a and b of a unital C^*-algebra A are unitarily equivalent if and only if there exists an element $u \in \mathfrak{U}(A)$ such that $b = uau^*$.*

Hint for proof. Suppose there exists $v \in \mathfrak{U}(\widetilde{A})$ such that $b = vav^*$. By proposition 12.3.12 there exist $u \in A$ and $\alpha \in \mathbb{C}$ such that $v = u + \alpha\mathbf{j}$ (where $\mathbf{j} = \mathbf{1}_{\widetilde{A}} - \mathbf{1}_A$). Show that $|\alpha| = 1$, u is unitary, and $b = uau^*$.

For the converse suppose that there exists $u \in \mathfrak{U}(A)$ such that $b = uau^*$. Let $v = u + \mathbf{j}$.

15.2.6. Definition. A PATH in a topological space X is a continuous map from the interval $[0,1]$ into X. Two points p and q in X are said to be CONNECTED BY A PATH (or HOMOTOPIC) in X if there exists a path $f\colon [0,1] \to X$ in X such that $f(0) = p$ and $f(1) = q$. In this case we write $p \sim_h q$ in X (or just $p \sim_h q$ when the space X is clear from context).

15.2.7. Example. Let a be an invertible element in a unital C^*-algebra A and b be an element of A such that $\|a - b\| < \|a^{-1}\|^{-1}$. Then $a \sim_h b$ in $\operatorname{inv}(A)$.

Hint for proof. Apply corollary 8.1.31 to points in the closed segment $[a, b]$.

15.2.8. Proposition. *The relation $\sim_h$ of homotopy equivalence defined above is an equivalence relation on the set of points of a topological space.*

15.2.9. Definition. If X is a topological space and $\sim_h$ is the relation of homotopy equivalence, then the resulting equivalence classes are the PATH COMPONENTS of X.

15.2.10. Proposition (Polar decomposition)**.** *For every invertible element s of a unital C^*-algebra, then there exists a unitary element $\omega(s)$ in the algebra such that*

$$s = \omega(s)|s|\,.$$

Hint for proof. Use Corollary 12.4.27.

15.2.11. Proposition. *If A is a unital C^*-algebra, then the function $\omega\colon \operatorname{inv} A \to \mathfrak{U}(A)$ defined in the preceding proposition is continuous.*

Hint for proof. Use propositions 8.1.34 and 11.5.11 .

15.2.12. Proposition. *Let $s = u|s|$ be the polar decomposition of an invertible element s in a unital C^*-algebra A. Then $u \sim_h s$ in $\operatorname{inv} A$.*

Hint for proof. For $0 \le t \le 1$ let $c_t = u(t|s| + (1-t)\mathbf{1})$. Conclude from proposition 12.4.23 that there exists $\epsilon \in (0,1]$ such that $|s| \ge \epsilon\mathbf{1}$. Use the same proposition to show that $t|s| + (1-t)\mathbf{1}$ is invertible for every $t \in [0,1]$.

15.2.13. Proposition. *Let u and v be unitary elements in a unital C^*-algebra A. If $u \sim_h v$ in $\operatorname{inv} A$, then $u \sim_h v$ in $\mathfrak{U}(A)$.*

15.2.14. Proposition. *Let u_1, u_2, u_3, and u_4 be unitary elements in a unital C^*-algebra A. If $u_1 \sim_h u_2$ in $\mathfrak{U}(A)$ and $u_3 \sim_h u_4$ in $\mathfrak{U}(A)$, then $u_1u_3 \sim_h u_2u_4$ in $\mathfrak{U}(A)$.*

15.2.15. Example. Let h be a self-adjoint element of a unital C^*-algebra A. Then $\exp(ih)$ is unitary and homotopic to $\mathbf{1}$ in $\mathfrak{U}(A)$.

Hint for proof. Consider the path $c\colon [0,1] \to \mathbb{T}\colon t \mapsto \exp(ith)$. Use proposition 11.5.11.

15.2.16. Notation. If p and q are projections in a C^*-algebra A, we write $p \sim q$ (p is MURRAY-VON NEUMANN EQUIVALENT to q) if there exists an element $v \in A$ such that $v^*v = p$ and $vv^* = q$. Note that such a v is automatically a partial isometry. We will refer to it as a *partial isometry that implements the equivalence.*

15.2.17. Proposition. *The relation $\sim$ of Murray-von Neumann equivalence is an equivalence relation on the family $\mathcal{P}(A)$ of projections in a C^*-algebra A.*

15.2.18. Proposition. *Let a and b be self-adjoint elements of a unital C^*-algebra. If $a \sim_s b$, then $a \sim_u b$. In fact, if $b = sas^{-1}$ and $s = u|s|$ is the polar decomposition of s, then $b = uau^*$.*

Hint for proof. Suppose there is an invertible element s such that $b = sas^{-1}$. Let $s = u|s|$ be the polar decomposition of s. Show that a commutes with $|s|^2$ and therefore with anything in the algebra $C^*(\mathbf{1}, |s|^2)$. In particular, a commutes with $|s|^{-1}$. From this it follows that $uau^* = b$.

15.2.19. Proposition. *If p and q are projections in a C^*-algebra, then*

$$p \sim_h q \implies p \sim_u q \implies p \sim q\,.$$

Hint for proof. In this hint $\mathbf{1} = \mathbf{1}_{\widetilde{A}}$. For the first implication show that there is no loss of generality in supposing that $\|p - q\| < \frac{1}{2}$. Let $s = pq + (\mathbf{1} - p)(\mathbf{1} - q)$. Prove that $p \sim_s q$. To this end write $s - \mathbf{1}$ as $p(q - p) + (\mathbf{1} - p)((\mathbf{1} - q) - (\mathbf{1} - p))$ and use corollary 8.1.30. Then use proposition 15.2.18.

To prove the second implication notice that if $upu^* = q$ for some unitary element in $\widetilde{A}$, then up is a partial isometry in A.

In general, the converse of the second implication, $p \sim_u q \implies p \sim q$, in the preceding proposition does not hold (see example 15.2.23 below). However, for projections p and q in a unital C^*-algebra, if we have both $p \sim q$ and $\mathbf{1} - p \sim \mathbf{1} - q$, then we can conclude $p \sim_u q$.

15.2.20. Proposition. *Let p and q be projections in a unital C^*-algebra A. Then $p \sim_u q$ if and only if $p \sim q$ and $\mathbf{1} - p \sim \mathbf{1} - q$.*

Hint for proof. For the converse suppose that a partial isometry v implements the equivalence $p \sim q$ and w implements $\mathbf{1} - p \sim \mathbf{1} - q$. Consider the element $u = v + w + \mathbf{j}$ in $\widetilde{A}$.

15.2.21. Definition. An element s of a unital C^*-algebra is an ISOMETRY if $s^*s = \mathbf{1}$.

15.2.22. Exercise. Explain why the terminology in the preceding definition is reasonable.

It is not difficult to see that the converse of the second implication in proposition 15.2.19 fails in general.

15.2.23. Example. If p and q are projections in a C^*-algebra, then

$$p \sim q \;\not\Longrightarrow\; p \sim_u q\,.$$

For example, if s is a nonunitary isometry (such as the unilateral shift), then $s^*s \sim ss^*$, but $s^*s \nsim_u ss^*$.

Hint for proof. Prove, and keep in mind, that no nonzero projection can be Murray-von Neumann equivalent to the zero projection.

15.2.24. Remark. The converse of the first implication, $p \sim_h q \implies p \sim_u q$, in proposition 15.2.19 also does not hold in general for projections in a C^*-algebra. However, an example illustrating this phenomenon is not easy to come by. To see what is involved consult [40], examples 2.2.9 and 11.3.4.

It would be nice if he implications in proposition 15.2.19 were reversible. But as we have seen, they are not. One way of dealing with recalcitrant facts, as was noted at the beginning of this chapter, is to give the mathematical objects we are dealing with more room to move around in. Pass to matrices. Propositions 15.2.26 and 15.2.32 are examples of how this technique works. We can, in a sense, get the implications in 15.2.19 to reverse.

15.2.25. Notation. If a_1, a_2, $\dots a_n$ are elements of a C^*-algebra A, then $\operatorname{diag}(a_1, a_2, \dots, a_n)$ is the diagonal matrix in $\mathbf{M}_n(A)$ whose main diagonal consists of the elements a_1, $\dots$, a_n. We also use this notation for block matrices. For example if a is an $m \times m$ matrix and b is an $n \times n$ matrix, then $\operatorname{diag}(a, b)$ is the $(m+n) \times (m+n)$ matrix $\begin{bmatrix} a & \mathbf{0} \\ \mathbf{0} & b \end{bmatrix}$. (Of course, the $\mathbf{0}$ in the upper right corner of this matrix is the $m \times n$ matrix all of whose entries are zero and the $\mathbf{0}$ in the lower left corner is the $n \times m$ matrix with each entry zero.)

15.2.26. Proposition. *Let p and q be projections in a C^*-algebra A. Then*

$$p \sim q \implies \operatorname{diag}(p, \mathbf{0}) \sim_u \operatorname{diag}(q, \mathbf{0}) \text{ in } \mathbf{M}_2(A).$$

Hint for proof. Let v be a partial isometry in A that implements the Murray-von Neumann equivalence $p \sim q$. Consider the matrix $u = \begin{bmatrix} v & \mathbf{1}_{\widetilde{A}} - q \\ \mathbf{1}_{\widetilde{A}} - p & v^* \end{bmatrix}$.

Recall from corollary 11.4.4 that the spectrum of any unitary element of a unital C^*-algebra lies in the unit circle. Remarkably, if its spectrum is not the entire circle, then it is homotopic to $\mathbf{1}$ in $\mathfrak{U}(A)$.

15.2.27. Proposition. *Let u be a unitary element in a unital C^*-algebra. If $\sigma(u) \neq \mathbb{T}$, then $u \sim_h \mathbf{1}$ in $\mathfrak{U}(A)$.*

Hint for proof. Choose $\theta \in \mathbb{R}$ such that $\exp(i\theta) \notin \sigma(u)$. Then there is a (unique continuous) function

$$\phi \colon \sigma(u) \to (\theta, \theta + 2\pi) \colon \exp(it) \mapsto t\,.$$

Let $h = \phi(u)$ and use example 15.2.15.

15.2.28. Example. If A is a unital C^*-algebra, then $\begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \sim_h \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$ in $\mathfrak{U}(\mathbf{M}_2(A))$.

The matrix $J = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$ is very useful when used in conjunction with proposition 15.2.14 in establishing homotopies between matrices in $\mathfrak{U}(A)$. In verifying the next few examples it is quite helpful. Notice that multiplication of a 2×2 matrix on the right by J interchanges its columns; multiplication on the left by J interchanges rows; and multiplication on both left and right by J interchanges elements on both diagonals.

15.2.29. Example. If u and v are unitary elements in a unital C^*-algebra A, then

$$\operatorname{diag}(u, v) \sim_h \operatorname{diag}(v, u)$$

in $\mathfrak{U}(\mathbf{M}_2(A))$.

15.2.30. Example. If u and v are unitary elements in a unital C^*-algebra A, then

$$\operatorname{diag}(u, v) \sim_h \operatorname{diag}(uv, \mathbf{1})$$

in $\mathfrak{U}(\mathbf{M}_2(A))$.

15.2.31. Example. If u and v are unitary elements in a unital C^*-algebra A, then

$$\operatorname{diag}(uv, \mathbf{1}) \sim_h \operatorname{diag}(vu, \mathbf{1})$$

in $\mathfrak{U}(\mathbf{M}_2(A))$.

15.2.32. Proposition. *Let p and q be projections in a C^*-algebra A. Then*

$$p \sim_u q \implies \operatorname{diag}(p, \mathbf{0}) \sim_h \operatorname{diag}(q, \mathbf{0}) \text{ in } \mathfrak{P}(\mathbf{M}_2(A)).$$

15.2.33. Example. Two projections in $\mathbf{M}_n$ are Murray-von Neumann equivalent if and only if they have equal traces, which is, in turn, equivalent to their having ranges of equal dimension.

Hint for proof. Use propositions 7.2.3–7.2.4 and 15.1.4–15.1.5.

In example 8.1.13 we introduced the $n \times n$ matrix algebra $\mathbf{M}_n(A)$ where A is an algebra. If A is a $*$-algebra and $n \in \mathbb{N}$ we can equip $\mathbf{M}_n(A)$ with an involution in a natural way. It is defined, as you would expect, as the analog of "conjugate transposition": if $\mathbf{a} \in \mathbf{M}_n(A)$, then

$$\mathbf{a}^* = \begin{bmatrix} a_{ij} \end{bmatrix}^* := \begin{bmatrix} {a_{ji}}^* \end{bmatrix}.$$

If A is a C^*-algebra we can introduce a norm on $\mathbf{M}_n(A)$ under which it also is a C^*-algebra. Choose a faithful representation $\phi\colon A \to \mathfrak{B}(H)$ of A, where H is a Hilbert space. For each $n \in \mathbb{N}$ define

$$\phi_n\colon \mathbf{M}_n(A) \to \mathfrak{B}(H^n)\colon \begin{bmatrix} a_{ij} \end{bmatrix} \mapsto \begin{bmatrix} \phi(a_{ij}) \end{bmatrix}$$

where H^n is the n-fold direct sum of H with itself. Then define the norm of an element $\mathbf{a} \in \mathbf{M}_n(A)$ to be the norm of the operator $\phi_n(\mathbf{a}) \in \mathfrak{B}(H^n)$. That is, $\|\mathbf{a}\| := \|\phi_n(\mathbf{a})\|$.

This norm is a C^*-norm on $\mathbf{M}_n(A)$. It does not depend on the particular representation we choose by virtue of the uniqueness of C^*-norms (see corollary 12.3.18).

15.2.34. Proposition. *Let A be a C^*-algebra and $\mathbf{a} \in \mathbf{M}_n(A)$. Then*

$$\max\{\|a_{ij}\|\colon 1 \le i, j \le n\} \le \|\mathbf{a}\| \le \sum_{i,j=1}^{n} \|a_{ij}\|.$$

Hint for proof. Since any faithful representation is an isometry (see proposition 12.3.25) you may as well simplify things by regarding the representation $\phi\colon A \to \mathfrak{B}(H)$ discussed above as an inclusion map.

For the first inequality define, for $v \in H$, the vector $\mathbf{E}_j(v) \in H^n$ to be the n-tuple all of whose entries are zero except for the j^{th} one which is v. Then verify that

$$\|a_{ij}v\| \le \|\mathbf{a}\mathbf{E}_j(v)\| \le \|\mathbf{a}\|$$

whenever $\mathbf{a} \in \mathbf{M}_n(A)$, $v \in H$, and $\|v\| \le 1$.

For the second inequality show that

$$\|\mathbf{a}\mathbf{v}\|^2 \le \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \|a_{ij}\| \right)^2 \le \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \|a_{ij}\| \right)^2$$

whenever $\mathbf{a} \in \mathbf{M}_n(A)$, $\mathbf{v} \in H^n$, and $\|\mathbf{v}\| \le 1$.

15.3. A Semigroup of Projections

15.3.1. Notation. When A is a C^*-algebra and $n \in \mathbb{N}$ we let

$$\mathcal{P}_n(A) = \mathcal{P}(\mathbf{M}_n(A)) \quad \text{and}$$

$$\mathcal{P}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(A).$$

We now extend Murray-von Neumann equivalence to matrices of different sizes.

15.3.2. Notation. If A is a C^*-algebra let $\mathbf{M}_{n,m}(A)$ denote the set of $n \times m$ matrices with entries belonging to A.

15.3.3. Definition. If A is a C^*-algebra, $p \in \mathcal{P}_m(A)$, and $q \in \mathcal{P}_n(A)$, we set $p \sim q$ if there exists $v \in \mathbf{M}_{n,m}(A)$ such that

$$v^*v = p \qquad \text{and} \qquad vv^* = q.$$

Note: if $m = n$, then $\sim$ defined above is just our familiar Murray-von Neumann equivalence.

15.3.4. Proposition. *The relation $\sim$ defined above is an equivalence relation on $\mathcal{P}_\infty(A)$.*

15.3.5. Definition. For each C^*-algebra A we define a binary operation $\oplus$ on $\mathcal{P}_\infty(A)$ by

$$p \oplus q = \operatorname{diag}(p, q)\,.$$

Thus if $p \in \mathcal{P}_m(A)$ and $q \in \mathcal{P}_n(A)$, then $p \oplus q \in \mathcal{P}_{m+n}(A)$.

In the next proposition $\mathbf{0}_n$ is the additive identity in $\mathcal{P}_n(A)$.

15.3.6. Proposition. *Let A be a C^*-algebra and $p \in \mathcal{P}_\infty(A)$. Then $p \sim p \oplus \mathbf{0}_n$ for every $n \in \mathbb{N}$.*

15.3.7. Proposition. *Let A be a C^*-algebra and p, p', q, $q' \in \mathcal{P}_\infty(A)$. If $p \sim p'$ and $q \sim q'$, then $p \oplus q \sim p' \oplus q'$.*

15.3.8. Proposition. *Let A be a C^*-algebra and p, $q \in \mathcal{P}_\infty(A)$. Then $p \oplus q \sim q \oplus p$.*

15.3.9. Proposition. *Let A be a C^*-algebra and p, $q \in \mathcal{P}_n(A)$ for some n. If $p \perp q$, then $p + q$ is a projection in $\mathbf{M}_n(A)$ and $p + q \sim p \oplus q$.*

15.3.10. Proposition. *If A is a C^*-algebra, then $\mathcal{P}_\infty(A)$ is a commutative semigroup under the operation $\oplus$.*

15.3.11. Notation. If A is a C^*-algebra and $p \in \mathcal{P}_\infty(A)$, let $[p]_{\mathcal{D}}$ be the equivalence class containing p determined by the equivalence relation $\sim$. Also let $\mathcal{D}(A) := \{\, [p]_{\mathcal{D}} \colon p \in \mathcal{P}_\infty(A)\,\}$.

15.3.12. Definition. Let A be a C^*-algebra. Define a binary operation $+$ on $\mathcal{D}(A)$ by

$$[p]_{\mathcal{D}} + [q]_{\mathcal{D}} := [p \oplus q]_{\mathcal{D}}$$

where p, $q \in \mathcal{P}_\infty(A)$.

15.3.13. Proposition. *The operation $+$ defined in 15.3.12 is well defined and makes $\mathcal{D}(A)$ into a commutative semigroup.*

15.3.14. Example. The semigroup $\mathcal{D}(\mathbb{C})$ is isomorphic to the additive group of positive integers; that is,

$$\mathcal{D}(\mathbb{C}) \cong \mathbb{Z}^+ = \{0, 1, 2, \dots\}\,.$$

Hint for proof. Use example 15.2.33.

15.3.15. Example. If H is a Hilbert space, then

$$\mathcal{D}(\mathfrak{B}(H)) \cong \mathbb{Z}^+ \cup \{\infty\}.$$

(Use the usual addition on $\mathbb{Z}^+$ and let $n + \infty = \infty + n = \infty$ whenever $n \in \mathbb{Z}^+ \cup \{\infty\}$.)

15.3.16. Example. For the C^*-algebra $\mathbb{C} \oplus \mathbb{C}$ we have

$$\mathcal{D}(\mathbb{C} \oplus \mathbb{C}) \cong \mathbb{Z}^+ \oplus \mathbb{Z}^+\,.$$

15.4. The Grothendieck Construction

In the construction of the field of real numbers we may use the same technique to get from the (additive semigroup of) natural numbers to the (additive group of all) integers as we do to get from the (multiplicative semigroup of) nonzero integers to the (multiplicative group of all) nonzero rational numbers. It is called the *Grothendieck construction*.

15.4.1. Definition. Let $(S,+)$ be a commutative semigroup. Define a relation $\sim$ on $S \times S$ by

$$(a,b) \sim (c,d) \text{ if there exists } k \in S \text{ such that } a+d+k = b+c+k.$$

15.4.2. Proposition. *The relation $\sim$ defined above is an equivalence relation.*

15.4.3. Notation. For the equivalence relation $\sim$ defined in 15.4.1 the equivalence class containing the pair (a,b) will be denote by $\langle a,b\rangle$ rather than by $[(a,b)]$.

15.4.4. Definition. Let $(S,+)$ be a commutative semigroup. On $G(S)$ define a binary operation (also denoted by $+$) by:

$$\langle a,b\rangle + \langle c,d\rangle := \langle a+c, b+d\rangle\,.$$

15.4.5. Proposition. *The operation $+$ defined above is well defined and under this operation $G(S)$ becomes and Abelian group.*

The Abelian group $(G(S),+)$ is called the Grothendieck group of S.

15.4.6. Proposition. *For a semigroup S and an arbitrary $a \in S$ define a mapping*

$$\gamma_S \colon S \to G(S) \colon s \mapsto \langle s+a, a\rangle\,.$$

The mapping γ_S, called the Grothendieck map*, is well defined and is a semigroup homomorphism.*

Saying that the Grothendieck map is well defined means that its definition is independent of the choice of a. We frequently write just γ for γ_S.

15.4.7. Example. Both $\mathbb{N} = \{1,2,3,\dots\}$ and $\mathbb{Z}^+ = \{0,1,2,\dots\}$ are commutative semigroups under addition. They generate the same Grothendieck group

$$G(\mathbb{N}) = G(\mathbb{Z}^+) = \mathbb{Z}\,.$$

Nothing guarantees that the Grothendieck group of an arbitrary semigroup will be of much interest.

15.4.8. Example. Let S be the commutative additive semigroup $\mathbb{Z}^+ \cup \{\infty\}$. Then $G(S) = \{\mathbf{0}\}$.

15.4.9. Example. Let $\mathbb{Z}_0$ be the (commutative) multiplicative semigroup of nonzero integers. Then $G(\mathbb{Z}_0) = \mathbb{Q}_0$, the Abelian multiplicative group of nonzero rational numbers.

15.4.10. Proposition. *If S is a commutative semigroup, then*

$$G(S) = \{\gamma(s) - \gamma(t) \colon s,t \in S\}\,.$$

15.4.11. Proposition. *If r, $s \in S$, where S is a commutative semigroup, then $\gamma(r) = \gamma(s)$ if and only if there exists $t \in S$ such that $r+t = s+t$.*

15.4.12. Definition. A commutative semigroup S has the cancellation property if whenever r, s, $t \in S$ satisfy $r+t = s+t$, then $r = s$.

15.4.13. Corollary. *Let S be a commutative semigroup. The Grothendieck map $\gamma_S \colon S \to G(S)$ is injective if and only if S has the cancellation property.*

The next proposition asserts the universal property of the Grothendieck group.

15.4.14. Proposition. *Let S be a commutative (additive) semigroup and $G(S)$ be its Grothendieck group. If H is an Abelian group and $\phi\colon S \to H$ is an additive map, then there exists a unique group homomorphism $\psi\colon G(S) \to H$ such that the following diagram commutes.*

$$
\begin{tikzcd}
S \arrow[r, "\gamma"] \arrow[dr, "\phi"'] & {|G(S)|} \arrow[d, dashed, "|\psi|"] & G(S) \arrow[d, dashed, "\psi"] \\
& {|H|} & H
\end{tikzcd}
\tag{15.1}
$$

In the preceding diagram $|G(S)|$ and $|H|$ are just $G(S)$ and H regarded as semigroups and $|\psi|$ is the corresponding semigroup homomorphism. In other words, the forgetful functor $|\ \ |$ "forgets" only about identities and inverses but not about the operation of addition. Thus the triangle on the left is a commutative diagram in the category of semigroups and semigroup homomorphisms.

15.4.15. Proposition. *Let $\phi\colon S \to T$ be a homomorphism of commutative semigroups. Then the map $\gamma_T \circ \phi\colon S \to G(T)$ is additive. By proposition 15.4.14 there exists a unique group homomorphism $G(\phi)\colon G(S) \to G(T)$ such that the following diagram commutes.*

$$
\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[d, "\gamma_S"'] & T \arrow[d, "\gamma_T"] \\
G(S) \arrow[r, "G(\phi)"'] & G(T)
\end{tikzcd}
$$

15.4.16. Proposition. *The pair of maps $S \mapsto G(S)$, which takes commutative semigroups to their corresponding Grothendieck groups, and $\phi \mapsto G(\phi)$, which takes semigroup homomorphisms to group homomorphism (as defined in 15.4.15) is a covariant functor from the category of commutative semigroups and semigroup homomorphisms to the category of Abelian groups and group homomorphisms.*

One slight advantage of the rather pedantic inclusion of a forgetful functor in diagram (15.1) is that it makes it possible to regard the Grothendieck map $\gamma\colon S \mapsto \gamma_S$ as a natural transformation of functors.

15.4.17. Corollary (Naturality of the Grothendieck map)**.** *Let $|\ \ |$ be the forgetful functor on Abelian groups which "forgets" about identities and inverses but not the group operation as in 15.4.14. Then $|G(\ \ )|$ is a covariant functor from the category of commutative semigroups and semigroup homomorphisms to itself. Furthermore, the Grothendieck map $\gamma\colon S \mapsto \gamma_S$ is a natural transformation from the identity functor to the functor $|G(\ \ )|$.*

$$
\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[d, "\gamma_S"'] & T \arrow[d, "\gamma_T"] \\
{|G(S)|} \arrow[r, "|G(\phi)|"'] & {|G(T)|}
\end{tikzcd}
$$

15.5. The K_0 Group for Unital C^*-Algebras

15.5.1. Definition. Let A be a unital C^*-algebra. Let $K_0(A) := G(\mathcal{D}(A))$, the Grothendieck group of the semigroup $\mathcal{D}(A)$ defined in 15.3.11 and 15.3.12, and define

$$[\ \ ]\colon \mathcal{P}_\infty(A) \to K_0(A)\colon p \mapsto \gamma_{\mathcal{D}(A)}\big([p]_{\mathcal{D}}\big)\,.$$

The next proposition is an obvious consequence of this definition.

15.5.2. Proposition. *Let A be a unital C^*-algebra and p, $q \in \mathcal{P}_\infty(A)$. If p and q are Murray-von Neumann equivalent, then $[p] = [q]$ in $K_0(A)$.*

15.5.3. Definition. Let A be a unital C^*-algebra and p, $q \in \mathcal{P}_\infty(A)$. We say that p is STABLY EQUIVALENT to q and write $p \sim_{st} q$ if there exists a projection $r \in \mathcal{P}_\infty(A)$ such that $p \oplus r \sim q \oplus r$.

15.5.4. Proposition. *Stable equivalence $\sim_{st}$ is an equivalence relation on $\mathcal{P}_\infty(A)$.*

15.5.5. Proposition. *Let A be a unital C^*-algebra and p, $q \in \mathcal{P}_\infty(A)$. Then $p \sim_{st} q$ if and only if $p \oplus \mathbf{1}_n \sim q \oplus \mathbf{1}_n$ for some $n \in \mathbb{N}$.*

Here, of course, $\mathbf{1}_n$ is the multiplicative identity in $\mathbf{M}_n(A)$.

15.5.6. Proposition (Standard Picture of $K_0(A)$ when A is unital)**.** *If A is a unital C^*-algebra, then*

$$\begin{aligned} K_0(A) &= \{[p] - [q]\colon p, q \in \mathcal{P}_\infty(A)\} \\ &= \{[p] - [q]\colon p, q \in \mathcal{P}_n(A) \text{ for some } n \in \mathbb{N}\}\,. \end{aligned}$$

15.5.7. Proposition. *Let A be a unital C^*-algebra and p, $q \in \mathcal{P}_\infty(A)$. Then $[p \oplus q] = [p] + [q]$.*

15.5.8. Proposition. *Let A be a unital C^*-algebra and p, $q \in \mathcal{P}_n(A)$. If $p \sim_h q$ in $\mathcal{P}_n(A)$, then $[p] = [q]$.*

15.5.9. Proposition. *Let A be a unital C^*-algebra and p, $q \in \mathcal{P}_n(A)$. If $p \perp q$ in $\mathcal{P}_n(A)$, then $p + q \in \mathcal{P}_n(A)$ and $[p + q] = [p] + [q]$.*

15.5.10. Proposition. *Let A be a unital C^*-algebra and p, $q \in \mathcal{P}_\infty(A)$. Then $[p] = [q]$ if and only if $p \sim_{st} q$.*

The next proposition specifies a universal property of $K_0(A)$ when A is unital. In it the forgetful functor $|\ \ |$ is the one described in proposition 15.4.14.

15.5.11. Proposition. *Let A be a unital C^*-algebra, G be an Abelian group, and $\nu\colon \mathcal{P}_\infty(A) \to |G|$ be a semigroup homomorphism that satisfies*

(a) *$\nu(\mathbf{0}_A) = \mathbf{0}_G$ and*
(b) *if $p \sim_h q$ in $\mathcal{P}_n(A)$ for some n, then $\nu(p) = \nu(q)$.*

Then there exists a unique group homomorphism $\widetilde{\nu}\colon K_0(A) \to G$ such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{P}_\infty(A) \arrow[r, "{[\ \ ]}"] \arrow[dr, "\nu"'] & {|K_0(A)|} \arrow[d, dashed, "|\widetilde{\nu}|"] & K_0(A) \arrow[d, dashed, "\widetilde{\nu}"] \\ & {|G|} & G \end{tikzcd}$$

Hint for proof. Let $\tau\colon \mathcal{D}(A) \to K_0(A)\colon [p]_{\mathcal{D}} \mapsto \nu(p)$. Verify that τ is well-defined and that it is a semigroup homomorphism. Then use proposition 15.4.14.

15.5.12. Definition. A $*$-homomorphism $\phi\colon A \to B$ between C^*-algebras extends, for every $n \in \mathbb{N}$, to a $*$-homomorphism $\phi\colon \mathbf{M}_n(A) \to \mathbf{M}_n(B)$ and also (since $*$-homomorphisms take projections to projections) to a $*$-homomorphism ϕ from $\mathcal{P}_\infty(A)$ to $\mathcal{P}_\infty(B)$. For such a $*$-homomorphism ϕ define

$$\nu\colon \mathcal{P}_\infty(A) \to K_0(B)\colon p \mapsto [\phi(p)]\,.$$

Then ν is a semigroup homomorphism satisfying conditions (a) and (b) of proposition 15.5.11 according to which there exists a unique group homomorphism $K_0(\phi)\colon K_0(A) \to K_0(B)$ such that $K_0(\phi)\big([p]\big) = \nu(p)$ for every $p \in \mathcal{P}_\infty(A)$.

15.5.13. Proposition. *The pair of maps $A \mapsto K_0(A)$, $\phi \mapsto K_0(\phi)$ is a covariant functor from the category of unital C^*-algebras and $*$-homomorphisms to the category of Abelian groups and group homomorphisms. Furthermore, for all $*$-homomorphisms $\phi\colon A \to B$ between unital C^*-algebras the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{P}_\infty(A) \arrow[r, "\phi"] \arrow[d, "{[\ ]}"'] & \mathcal{P}_\infty(B) \arrow[d, "{[\ ]}"] \\
K_0(A) \arrow[r, "K_0(\phi)"'] & K_0(B)
\end{tikzcd}$$

15.5.14. Notation. For C^*-algebras A and B let $\operatorname{Hom}(A,B)$ be the family of all $*$-homomorphisms from A to B.

15.5.15. Definition. Let A and B be C^*-algebras and $a \in A$. For ϕ, $\psi \in \operatorname{Hom}(A,B)$ let

$$d_a(\phi,\psi) = \|\phi(a) - \psi(a)\|\,.$$

Then d_a is a pseudometric on $\operatorname{Hom}(A,B)$. The topology generated by the family $\{d_a\colon a \in A\}$ is the POINT-NORM TOPOLOGY on $\operatorname{Hom}(A,B)$. (For each $a \in A$ let $\mathfrak{B}_a$ be the family of open balls in $\operatorname{Hom}(A,B)$ generated by the pseudometric d_a. The family $\bigcup\{\mathfrak{B}_a\colon a \in A\}$ is a subbase for the point-norm topology.)

15.5.16. Definition. Let A and B be C^*-algebras. We say that $*$-homomorphisms ϕ, $\psi\colon A \to B$ are HOMOTOPIC, and write $\phi \sim_h \psi$, if there exists a function

$$c\colon [0,1] \times A \to B\colon (t,a) \mapsto c_t(a)$$

such that

(a) for every $t \in [0,1]$ the map $c_t\colon A \to B\colon a \mapsto c_t(a)$ is a $*$-homomorphism,
(b) for every $a \in A$ the map $c(a)\colon [0,1] \to B\colon t \mapsto c_t(a)$ is a (continuous) path in B,
(c) $c_0 = \phi$, and
(d) $c_1 = \psi$.

15.5.17. Proposition. *Two $*$-homomorphisms ϕ_0, $\phi_1\colon A \to B$ between C^*-algebras are homotopic if and only if there exists a point-norm continuous path from ϕ_0 to ϕ_1 in $\operatorname{Hom}(A,B)$.*

15.5.18. Definition. We say that C^*-algebras are HOMOTOPICALLY EQUIVALENT if there exist $*$-homomorphisms $\phi\colon A \to B$ and $\psi\colon B \to A$ such that $\psi\circ\phi \sim_h \mathrm{id}_A$ and $\phi\circ\psi \sim_h \mathrm{id}_B$. A C^*-algebra is CONTRACTIBLE if it is homotopically equivalent to $\{\mathbf{0}\}$.

15.5.19. Proposition. *Let A and B be unital C^*-algebras. If $\phi \sim_h \psi$ in $\operatorname{Hom}(A,B)$, then $K_0(\phi) = K_0(\psi)$.*

15.5.20. Proposition. *If unital C^*-algebras A and B are homotopically equivalent, then $K_0(A) \cong K_0(B)$.*

15.5.21. Proposition. *If A is a unital C^*-algebra, then the split exact sequence*

$$\mathbf{0} \longrightarrow A \xrightarrow{\iota} \widetilde{A} \underset{\psi}{\overset{Q}{\rightleftarrows}} \mathbb{C} \longrightarrow \mathbf{0} \tag{15.2}$$

(see 12.3.12, Case 1, item (7)) *induces another split exact sequence*

$$\mathbf{0} \longrightarrow K_0(A) \xrightarrow{K_0(\iota)} K_0(\widetilde{A}) \underset{K_0(\psi)}{\overset{K_0(Q)}{\rightleftarrows}} K_0(\mathbb{C}) \longrightarrow \mathbf{0}\,. \tag{15.3}$$

Hint for proof. Define two additional functions

$$\mu\colon \widetilde{A} \to A\colon a + \lambda\mathbf{j} \mapsto a$$

and

$$\psi'\colon \mathbb{C} \to \widetilde{A}\colon \lambda \mapsto \lambda\mathbf{j}\,.$$

Then verify that

(a) $\mu \circ \iota = \mathrm{id}_A$,
(b) $\iota \circ \mu + \psi' \circ Q = \mathrm{id}_{\widetilde{A}}$,
(c) $Q \circ \iota = \mathbf{0}$, and
(d) $Q \circ \psi = \mathrm{id}_{\mathbb{C}}$.

15.5.22. Example. For every $n \in \mathbb{N}$, $K_0(\mathbf{M}_n) \cong \mathbb{Z}$.

15.5.23. Example. If H is a separable infinite dimensional Hilbert space, then $K_0(\mathfrak{B}(H)) \cong \mathbf{0}$.

15.5.24. Definition. Recall that a topological space X is CONTRACTIBLE if there is a point a in the space and a continuous function $f\colon [0,1] \times X \to X$ such that $f(1,x) = x$ and $f(0,x) = a$ for every $x \in X$.

15.5.25. Example. If X is a contractible compact Hausdorff space, then $K_0(\mathcal{C}(X)) \cong \mathbb{Z}$.

15.6. $K_0(A)$—the Nonunital Case

15.6.1. Definition. Let A be a nonunital C^*-algebra. Recall that the split exact sequence (15.2) for the unitization of A induces a split exact sequence (15.3) between the corresponding K_0 groups. Define

$$K_0(A) = \ker(K_0(\pi))\,.$$

15.6.2. Proposition. *For a nonunital C^*-algebra A the mapping $[\ \ ]\colon \mathcal{P}_\infty(A) \to K_0(\widetilde{A})$ may be regarded as a mapping from $\mathcal{P}_\infty(A)$ into $K_0(A)$.*

15.6.3. Proposition. *For both unital and nonunital C^*-algebras the sequence*

$$\mathbf{0} \longrightarrow K_0(A) \longrightarrow K_0(\widetilde{A}) \longrightarrow K_0(\mathbb{C}) \longrightarrow \mathbf{0}$$

is exact.

15.6.4. Proposition. *For both unital and nonunital C^*-algebras the group $K_0(A)$ is (isomorphic to) $\ker(K_0(\pi))$.*

15.6.5. Proposition. *If $\phi\colon A \to B$ is a $*$-homomorphism between C^*-algebras, then there exists a unique $*$-homomorphism $K_0(\phi)$ which makes the following diagram commute.*

$$\begin{tikzcd}
K_0(A) \arrow[r] \arrow[d, dotted, "K_0(\phi)"'] & K_0(\widetilde{A}) \arrow[r, "K_0(\pi_A)"] \arrow[d, "K_0(\widetilde{\phi})"] & K_0(\mathbb{C}) \arrow[d, equal] \\
K_0(B) \arrow[r] & K_0(\widetilde{B}) \arrow[r, "K_0(\pi_B)"'] & K_0(\mathbb{C})
\end{tikzcd}$$

15.6.6. Proposition. *The pair of maps $A \mapsto K_0(A)$, $\phi \mapsto K_0(\phi)$ is a covariant functor from the category* **CSA** *of C^*-algebras and $*$-homomorphisms to the category of Abelian groups and group homomorphisms.*

In propositions 15.5.19 and 15.5.20 we asserted the homotopy invariance of the functor K_0 for unital C^*-algebras. We now extend the result to arbitrary C^*-algebras.

15.6.7. Proposition. *Let A and B be C^*-algebras. If $\phi \sim_h \psi$ in $\operatorname{Hom}(A, B)$, then $K_0(\phi) = K_0(\psi)$.*

15.6.8. Proposition. *If C^*-algebras A and B are homotopically equivalent, then $K_0(A) \cong K_0(B)$.*

15.6.9. Definition. Let π and λ be the $*$-homomorphisms in the split exact sequence (15.2) for the unitization of as C^*-algebra A. Define the SCALAR MAPPING $s\colon \widetilde{A} \to \widetilde{A}$ for $\widetilde{A}$ by $s := \lambda \circ \pi$. Every member of $\widetilde{A}$ can be written in the form $a + \alpha \mathbf{1}_{\widetilde{A}}$ for some $a \in A$ and $\alpha \in \mathbb{C}$. Notice that $s(a + \alpha \mathbf{1}_{\widetilde{A}}) = \alpha \mathbf{1}_{\widetilde{A}}$ and that $x - s(x) \in A$ for every $x \in \widetilde{A}$. For each natural number n the scalar mapping induces a corresponding map $s = s_n\colon \mathbf{M}_n(\widetilde{A}) \to \mathbf{M}_n(\widetilde{A})$. An element $x \in \mathbf{M}_n(\widetilde{A})$ is a SCALAR ELEMENT of $\mathbf{M}_n(\widetilde{A})$ if $s(x) = x$.

15.6.10. Proposition (Standard Picture of $K_0(A)$ for arbitrary A)**.** *If A is a C^*-algebra, then*

$$K_0(A) = \{\, [p] - [s(p)] \colon p, q \in \mathcal{P}_\infty(\widetilde{A}) \} \,.$$

15.7. Exactness and Stability Properties of the K_0 Functor

15.7.1. Definition. A covariant functor F from a category **A** to a category **B** is SPLIT EXACT if it takes split exact sequences to split exact sequences. And it is HALF EXACT provided that whenever the sequence

$$\mathbf{0} \longrightarrow A_1 \xrightarrow{\ j\ } A_2 \xrightarrow{\ k\ } A_3 \longrightarrow \mathbf{0}$$

is exact in **A**, then

$$F(A_1) \xrightarrow{\ F(j)\ } F(A_2) \xrightarrow{\ F(k)\ } F(A_3)$$

is exact in **B**.

15.7.2. Proposition. *The functor K_0 is half exact.*

15.7.3. Proposition. *The functor K_0 is split exact.*

15.7.4. Proposition. *The functor K_0 preserves direct sums. That is, if A and B are C^*-algebras, then $K_0(A \oplus B) = K_0(A) \oplus K_0(B)$.*

15.7.5. Example. If A is a C^*-algebra, then $K_0(\widetilde{A}) = K_0(A) \oplus \mathbb{Z}$.

Despite being both split exact and half exact the functor K_0 is not exact. Each of the next two examples is sufficient to demonstrate this.

15.7.6. Example. The sequence

$$\mathbf{0} \longrightarrow \mathcal{C}_0\big((0,1)\big) \xrightarrow{\ \iota\ } \mathcal{C}\big([0,1]\big) \xrightarrow{\ \psi\ } \mathbb{C} \oplus \mathbb{C} \longrightarrow \mathbf{0}$$

where $\psi(f) = \big(f(0), f(1)\big)$, is clearly exact; but $K_0(\psi)$ is not surjective.

15.7.7. Example. If H is a Hilbert space the exact sequence

$$\mathbf{0} \longrightarrow \mathfrak{K}(H) \xrightarrow{\ \iota\ } \mathfrak{B}(H) \xrightarrow{\ \pi\ } \mathfrak{Q}(H) \longrightarrow \mathbf{0}$$

associated with the Calkin algebra $\mathfrak{Q}(H)$ is exact but $K_0(\iota)$ is not injective. (This example requires a fact we have not derived: $K_0(\mathfrak{K}(H)) \cong \mathbb{Z}$, for which see [40], Corollary 6.4.2.)

Next is an important stability property of the functor K_0.

15.7.8. Proposition. *If A is a C^*-algebra, then $K_0(A) \cong K_0(\mathbf{M}_n(A))$.*

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Index