Time Series Asymptotics

Time series as a stochastic process must satisfy the property of stationarity and ergodicity

Stationarity

- A time series process $\{z_i\}$ is (strongly) stationary if the joint probability distribution of any set of k observations in the sequence $\{z_i, z_{i+1}, \ldots, z_{i+k-1}\}$ is the same regardless of the origin, i, in the time scale.

Stationarity

- A time series process $\{z_i\}$ is covariance stationary or weakly stationary if $E(z_i) = \mu$ is finite and is the same for all i; and if the covariance between any two observations, $Cov(z_i, z_{i-k}) = \gamma_k$, is a finite function only of model parameters and their distance apart in time k, but not the absolute location of either observation on the time scale.
- $\gamma_k = \gamma_{-k}$

Ergodicity

- A stationary time series process $\{z_i\}$ is ergodic if for any two bunded functions that map vectors in the a and b dimensional real vector spaces to real scalars, f: $R^a \rightarrow R$ and g: $R^b \rightarrow R$,

$$\lim_{k\to\infty}\left|E\left[f\left(z_i, z_{i+1}, \ldots, z_{i+a-1}\right)g\left(z_{i+k}, z_{i+k+1}, \ldots, z_{i+k+b-1}\right)\right]\right|$$
$$=\left|E\left[f\left(z_i, z_{i+1}, \ldots, z_{i+a-1}\right)\right]\right|\left|E\left[g\left(z_{i+k}, z_{i+k+1}, \ldots, z_{i+k+b-1}\right)\right]\right|$$

Ergodic Theorem

- If $\{z_i\}$ is a time series process that is stationary and ergodic and $E[|z_i|]$ is a finite constant, and if
$$\bar{z}_n = \frac{1}{n}\sum_{i=1}^{n} z_i, \; then \; \bar{z}_n \rightarrow_{as} \mu, \; where \; \mu = E(z_i)$$

Ergodicity of Functions

- If $\{z_i\}$ is a time series process that is stationary and ergodic and if $y_i = f(z_i)$ is a measurable function in the probability space that defines z_i, then $\{y_i\}$ is also stationary and ergodic.
- Let $\{Z_i\}$ define a Kx1 vector valued stochastic process—each element of the vector $\{z_i\}$ is an ergodic and stationary series, and the characteristics of ergodicity and stationarity apply to the joint distribution of the elements of $\{Z_i\}$. Then, the ergodic theorem applies to functions of $\{Z_i\}$.

Martingale Sequence

- A vector sequence $\{Z_i\}$ is a martingale sequence if $E(Z_i|Z_{i-1},Z_{i-2},\ldots) = Z_i$.
- A vector sequence $\{Z_i\}$ is a martingale difference sequence (m.d.s) if $E(Z_i|Z_{i-1},Z_{i-2},\ldots) = 0$.

Martingale Difference CLT

- If $\{Z_i\}$ is a vector valued stationary and ergodic martingale difference sequence, with $E(Z_i Z_i') = \Sigma$, where Σ is a finite positive definite matrix, and if

$$\bar{Z}_n = \frac{1}{n}\sum_{i=1}^{n} Z_i, \; then \; \sqrt{n}\bar{Z}_n \to_d N(0,\Sigma)$$

Gordin's CLT

- If $\{Z_i\}$ is stationary and ergodic and if the following three conditions are met, then

$$\sqrt{n}\bar{Z}_n \rightarrow_d N(0, \Gamma^*)$$

$$where\,\Gamma^* = \lim_{n\rightarrow\infty} Var\left(\sqrt{n}\bar{Z}_n\right)$$

  – 1. Asymptotic uncorrelatedness: $E(Z_i | Z_{i-k}, Z_{i-k-1}, \ldots)$ converges in mean squares to zero as $k \rightarrow \infty$.

Gordin's CLT

– 2. Summability of autocovariances:
With dependent observations, the following covariance matrix is finite.

$$\lim_{n\to\infty} Var\left(\sqrt{n}\bar{Z}_n\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} Cov(Z_i, Z_j^{'}) = \sum_{k=-\infty}^{\infty} \Gamma_k = \Gamma^*$$

In particular, $E(Z_i Z_j{}') = \Gamma_0$, a finite matrix.

Gordin's CLT

– 3. Asymptotic negligibility of innovations:
  The information eventually becomes negligible as it fades far back in time from the current observation.

$$Let\ r_{ik} = E\left(Z_i \mid Z_{i-k}, Z_{i-k-1}, \ldots\right) - E\left(Z_i \mid Z_{i-k-1}, Z_{i-k-2}, \ldots\right),$$

$$then\ Z_i = \sum_{k=0}^{\infty} r_{ik}\ and\ \sum_{k=0}^{\infty} \sqrt{E(r_{ik}^{'} r_{ik})}\ is\ finite$$

Time Series Asymptotics

- Linear Model: $y_i = \mathbf{x}_i'\beta + \varepsilon_i$ (i=1,2,…,n)
- Least Squares Estimation
  - $\mathbf{b} = (\mathbf{X'X})^{-1}\mathbf{X'y} = \beta + (\mathbf{X'X}/n)^{-1}(\mathbf{X}'\varepsilon/n)$
  - $E(\mathbf{b}) = \beta + (\mathbf{X'X}/n)^{-1} E(\mathbf{X}'\varepsilon/n) = \beta$
  - $Var(\mathbf{b}) = (\mathbf{X'X}/n)^{-1} E[(\mathbf{X}'\varepsilon/n)(\mathbf{X}'\varepsilon/n)'](\mathbf{X'X}/n)^{-1}$
    $= (\mathbf{X'X})^{-1}(\mathbf{X}'\mathbf{\Omega}\mathbf{X})(\mathbf{X'X})^{-1}$ where $\mathbf{\Omega} = E(\varepsilon\varepsilon')$

Time Series Asymptotics

- We assume the stochastic process generating $\{\mathbf{x}_i\}$ is stationary and ergodic, then by ergodic theorem, $\lim_{n\to\infty} \sum_i \mathbf{x}_i\mathbf{x}_i'/n = E(\mathbf{x}_i\,\mathbf{x}_i')$ is finite
- If ε_i is not serially correlated, then $\mathbf{g}_i = \mathbf{x}_i\varepsilon_i$ is a m.d.s, so
  - $\lim_{n\to\infty} \sum_i \mathbf{x}_i\varepsilon_i/n = E(\mathbf{x}_i\varepsilon_i) = E(\mathbf{g}_i)\mathbf{=0}$
  - $\lim_{n\to\infty} \sum_i \mathbf{x}_i\mathbf{x}_j'\varepsilon_i\varepsilon_j/n = E(\mathbf{g}_i\mathbf{g}_j') = Var(\mathbf{g}_i)$ is finite
- Therefore, b is consistent!

Time Series Asymptotics

- If $\{\mathbf{x}_i, \varepsilon_i\}$ are jointly stationary and ergodic, then by ergodic theorems, consistency of the parameter estimates can be established.
- If ε_i is not serially correlated, then by Martingale Difference CLT,

$$\bar{\mathbf{g}}_n = \frac{1}{n}\sum_{i=1}^{n}\mathbf{g}_i, \text{ } then \sqrt{n}\bar{\mathbf{g}}_n \rightarrow_d N(\mathbf{0}, \mathbf{\Sigma}) \text{ } where \text{ } \mathbf{\Sigma} = \mathbf{X}'\mathbf{\Omega}\mathbf{X}$$

- If ε_i is serially correlated and dependence in $\mathbf{x}_i$, then by Godin's CLT,

$$\sqrt{n}\bar{\mathbf{g}}_n \rightarrow_d N(0, \Gamma^*), \text{ } where \text{ } \Gamma^* = \lim_{n\rightarrow\infty} Var\left(\sqrt{n}\bar{\mathbf{g}}_n\right)$$