OLS: Theoretical Review

- Assumption 1: Linearity
  - $y = \mathbf{x}'\boldsymbol{\beta} + \varepsilon$ (or $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$)
    (where $\mathbf{x}$, $\boldsymbol{\beta}$: Kx1)
- Assumption 2: Exogeneity
  - $E(\boldsymbol{\varepsilon}|\mathbf{X}) = 0$
  - $E(\mathbf{x}\varepsilon) = \lim_{n\to\infty} \sum_i \mathbf{x}_i\varepsilon_i/n = \lim_{n\to\infty} \mathbf{X}'\boldsymbol{\varepsilon}/n = 0$
  - $Var(\mathbf{x}\varepsilon) = E(\mathbf{x}\mathbf{x}'\varepsilon^2) = \lim_{n\to\infty} \sum_i \mathbf{x}_i\mathbf{x}_i'\varepsilon_i^2/n$

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- Assumption 3: No Multicolinearity
  - rank(**X**) = K
  - $E(\mathbf{x}\mathbf{x}') = \lim_{n\to\infty} \sum_i \mathbf{x}_i\mathbf{x}_i'/n = \lim_{n\to\infty} \mathbf{X}'\mathbf{X}/n$
  - plim $\mathbf{X}'\mathbf{X}/n$ exists and nonsingular

OLS: Theoretical Review

- Non-Spherical Disturbances: General Heteroschedasticity
  - $\text{Var}(\boldsymbol{\varepsilon}) = \boldsymbol{\Omega}$
  - $\text{Var}(\mathbf{x}\varepsilon) = E(\mathbf{x}\mathbf{x}'\varepsilon^2) = \lim_{n\to\infty} \mathbf{X}'\boldsymbol{\Omega}\mathbf{X}/\mathbf{n}$
  - $\mathbf{X}'\boldsymbol{\varepsilon}/\sqrt{n} \to^{d} N(\mathbf{0}, \mathbf{X}'\boldsymbol{\Omega}\mathbf{X}/\mathbf{n})$

$$\Omega = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

OLS: Theoretical Review

- Assumption 4: Homoschedasticity and No Autocorrelation
  - $\mathrm{Var}(\boldsymbol{\varepsilon}) = \sigma^2\mathbf{I}$
  - $\mathrm{Var}(\mathbf{x}\varepsilon) = E(\mathbf{x}\mathbf{x}'\varepsilon^2) = \lim_{n\to\infty} \sigma^2\mathbf{X}'\mathbf{X}/n$
  - $\mathbf{X}'\boldsymbol{\varepsilon}/\sqrt{n} \to^{d} N(\mathbf{0}, \sigma^2\mathbf{X}'\mathbf{X}/n)$

OLS: Theoretical Review

- Least Squares Estimator
  - $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$
- Variance-Covariance Matrix of **b**
  - General Heteroscedasticity: $\text{Var}(\mathbf{b}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Omega}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$
  - Homoschedasticity: $\text{Var}(\mathbf{b}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$

OLS: Theoretical Review

- Least Squares Estimator
  - $\mathbf{b} = (\mathbf{X}'\mathbf{X}/n)^{-1}(\mathbf{X}'\mathbf{y}/n)$
  - $\mathbf{b} = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X}/n)^{-1}(\mathbf{X}'\varepsilon/n)$, $\mathbf{b} \rightarrow^{p} \boldsymbol{\beta}$
  - $\sqrt{\mathbf{n}}(\mathbf{b} - \boldsymbol{\beta}) = (\mathbf{X}'\mathbf{X}/n)^{-1}(\mathbf{X}'\varepsilon/\sqrt{n})$
  - $\sqrt{\mathbf{n}}(\mathbf{b} - \boldsymbol{\beta}) \rightarrow^{d} N(\mathbf{0},\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1})$
    $\mathbf{A} = E(\mathbf{x}\mathbf{x}') = \lim_{n\rightarrow\infty} \mathbf{X}'\mathbf{X}/n$
    $\mathbf{B} = E(\mathbf{x}\mathbf{x}'\varepsilon^2) = \lim_{n\rightarrow\infty} \mathbf{X}'\boldsymbol{\Omega}\mathbf{X}/n$
  - $\sqrt{\mathbf{n}}(\mathbf{b} - \boldsymbol{\beta}) \rightarrow^{d} N(\mathbf{0}, \sigma^2\mathbf{A}^{-1})$ under homoschedasticity

OLS: Theoretical Review

- Asymptotic Normality
  - $\mathbf{b} \sim^a N(\boldsymbol{\beta}, (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Omega}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1})$
  - $\mathbf{b} \sim^a N(\boldsymbol{\beta}, \sigma^2(\mathbf{X}'\mathbf{X})^{-1})$ under homoschedasticity
  - The unknown $\boldsymbol{\Omega}$ or σ^2 needs to be consistently estimated.

OLS: Theoretical Review

- Estimate of Asymptotic Var(b)
  - Under Homoschedasticity

$$\hat{V}ar(\mathbf{b}) = \hat{\sigma}^2 (\mathbf{X}'\mathbf{X})^{-1}$$

$$where\ \hat{\sigma}^2 = \sum_{i=1}^{n} \hat{\boldsymbol{\varepsilon}}_i^2 / (n - K) \rightarrow \sigma^2$$

$$\hat{\varepsilon}_i = y_i - \mathbf{x}_i'\mathbf{b}$$

OLS: Theoretical Review

- Estimate of Asymptotic Var(b)
  - White Estimator (Heteroschedasticity-Consistent Estmate of Asymptotic Covariance Matrix)

$$\hat{V}ar(\mathbf{b}) = (\mathbf{X}'\mathbf{X})^{-1}\left[\frac{n}{n-K}\sum_{i=1}^{n}\hat{\boldsymbol{\varepsilon}}_i^2\mathbf{x}_i\mathbf{x}_i'\right](\mathbf{X}'\mathbf{X})^{-1}$$

$$where \sum_{i=1}^{n}\hat{\boldsymbol{\varepsilon}}_i^2\mathbf{x}_i\mathbf{x}_i' \rightarrow \mathbf{X}'\Omega\mathbf{X}$$

$$\hat{\varepsilon}_i = y_i - \mathbf{x}_i'\mathbf{b}$$

OLS: Theoretical Review

- GLS (Generalized Least Squares)
  - If Ω is known and nonsingular, then
    $\Omega^{-1/2}\mathbf{y} = \Omega^{-1/2}\mathbf{X}\boldsymbol{\beta} + \Omega^{-1/2}\boldsymbol{\varepsilon}$ or
    $\mathbf{y}^* = \mathbf{X}^*\boldsymbol{\beta} + \boldsymbol{\varepsilon}^*$
  - $E(\boldsymbol{\varepsilon}^*|\mathbf{X}^*) = 0$, $E(\boldsymbol{\varepsilon}^*\boldsymbol{\varepsilon}^{*\prime}|\mathbf{X}^*) = \mathbf{I}$
  - $\mathbf{b}_{GLS} = (\mathbf{X}'\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\Omega^{-1}\mathbf{y}$
  - $Var(\mathbf{b}_{GLS}) = (\mathbf{X}'\Omega^{-1}\mathbf{X})^{-1}$
  - $\mathbf{b}_{GLS} \sim^{a} N(\boldsymbol{\beta}, (\mathbf{X}'\Omega^{-1}\mathbf{X})^{-1})$

Example

- U. S. Gasoline Market, 1953-2004
  - EXPG = Total U.S. gasoline expenditure
  - PG = Price index for gasoline
  - Y = Per capita disposable income
  - Pnc = Price index for new cars
  - Puc = Price index for used cars
  - Ppt = Price index for public transportation
  - Pd = Aggregate price index for consumer durables
  - Pn = Aggregate price index for consumer nondurables
  - Ps = Aggregate price index for consumer services
  - Pop = U.S. total population in thousands

Example

- $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$
- $\mathbf{y}$ = G; $\mathbf{X}$ = [1 PG Y]
  where G = (EXPG/PG)/POP
- $\mathbf{y} = ln(G)$; $\mathbf{X} = [1\ ln(PG)\ ln(Y)]$
- Elasticity Interpretation